

Clique-inverse graphs of K_3 -free and K_4 -free graphs

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ABSTRACT

The *clique graph* $K(G)$ of a given graph G is the intersection graph of the collection of maximal cliques of G . Given a family \mathcal{F} of graphs, the *clique-inverse graphs* of \mathcal{F} are the graphs whose clique graphs belong to \mathcal{F} . In this work, we describe characterizations for clique-inverse graphs of K_3 -free and K_4 -free graphs. The characterizations are formulated in terms of forbidden induced subgraphs.

Keywords: intersection graph, clique graph, clique-inverse graph

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1 Introduction

The problem of characterizing and recognizing clique graphs of certain families of graphs has been studied for several cases, e.g. [1, 2, 4, 5, 9]. However, much less is known about the corresponding inverse problem, which can be stated as follows: given a family \mathcal{F} of graphs, characterize those graphs whose clique graphs are members of \mathcal{F} . This class consists of the *clique-inverse graphs* of \mathcal{F} . In this work, we describe characterizations for clique-inverse graphs of K_3 -free and K_4 -free graphs. The characterizations are formulated in terms of forbidden induced subgraphs.

Clique-inverse graphs were the subjects of [6] and [8]. They are also called *roots* (relative to the clique operator), see e.g. [7]. Clique-inverse graphs of complete graphs are called *clique-complete*. A characterization of the minimal clique-complete graphs with no universal vertices has been formulated in [6]. It corresponds to a description of minimal graphs whose maximal cliques do not satisfy the Helly property. On the other hand, [8] contains a study of clique-inverse graphs, which originated the present work.

Let G be a finite undirected graph with no loops or multiple edges. Denote the vertex set of G by VG , and the edge set by EG . A subgraph H of G is a graph where $VH \subseteq VG$ and $EH \subseteq EG$. For a set X of vertices of G , denote by $G[X]$ the *subgraph of G induced by X* , that is, the vertex set of $G[X]$ is X and two vertices are adjacent in it if and only if they are so in G . If $X = \{x_1, \dots, x_k\}$, write $G[x_1, \dots, x_k]$ to mean $G[X]$. A vertex u is a *universal vertex* in the subgraph H of G if $u \in VH$ and u is adjacent to every other vertex of H . The *degree* of a vertex is the number of its neighbors.

A *clique* is a subset of vertices inducing a complete subgraph of G , while a *maximal clique* is one not properly contained in any other. Denote by K_r

the complete graph with r vertices. If G does not contain K_r as a subgraph, say that G is K_r -free.

The *clique graph* $K(G)$ of G is the intersection graph of the collection of maximal cliques of G . If $H = K(G)$, say that G is a *clique-inverse graph* of H . Given a family \mathcal{F} of graphs, the family of graphs whose clique graphs are members of \mathcal{F} is the family of clique-inverse graphs of \mathcal{F} .

2 The characterizations

In this section we describe complete characterizations for the situations in which $K(G)$ is K_3 -free or K_4 -free. The characterizations will be formulated in terms of lists of forbidden subgraphs, in the following way: “ G is a clique-inverse graph of a K_3 -free (K_4 -free) graph if and only if G does not contain any graph of a certain finite list \mathcal{L} as an induced subgraph”.

We begin by analyzing the case in which $K(G)$ is K_3 -free. Clearly, this occurs if and only if G does not contain three distinct pairwise intersecting maximal cliques. Hamelink [3] proves that a graph formed by exactly three pairwise intersecting maximal cliques has a universal vertex. The following theorem extends this result:

Theorem 1 *G is a clique-inverse graph of a K_3 -free graph if and only if G does not contain as an induced subgraph any of the following graphs: $K_{1,3}$, 4-fan, 4-wheel (see Figure 1). \square*

Proof. Assume that G is a clique-inverse graph of a K_3 -free graph. If G contains either a 4-wheel, a 4-fan, or $K_{1,3}$ as an induced subgraph, then there exist (at least) three pairwise intersecting maximal cliques in G , that is,

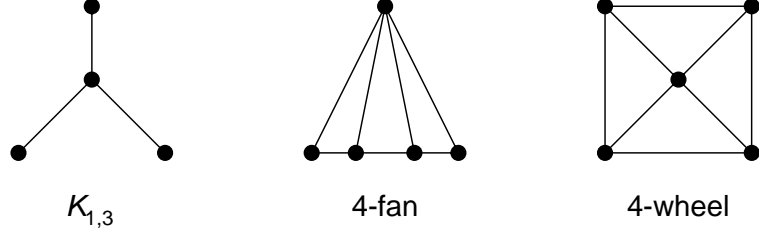


Figure 1: Forbidden subgraphs for clique-inverse graphs of K_3 –free graphs.

$K(G)$ contains K_3 , a contradiction. Thus, the necessity follows. Conversely, let us show that if G is not a clique-inverse graph of a K_3 –free graph, then G necessarily contains as an induced subgraph either a 4-wheel, a 4-fan, or $K_{1,3}$. Assume that $K(G)$ is not K_3 –free, and let M_1 , M_2 , and M_3 be three distinct pairwise intersecting maximal cliques in G . Define $R_{123} = M_1 \cap M_2 \cap M_3$, $R_1 = M_1 - (M_2 \cup M_3)$, $R_2 = M_2 - (M_1 \cup M_3)$, $R_3 = M_3 - (M_1 \cup M_2)$, $R_{12} = (M_1 \cap M_2) - M_3$, $R_{13} = (M_1 \cap M_3) - M_2$, and $R_{23} = (M_2 \cap M_3) - M_1$. Call each of these sets a *region*. See Figure 2. Let us divide the proof into five cases. From now on, we will use the notation $u \sim w$ to denote that u and w are adjacent vertices, and $u \not\sim w$ to denote that they are not.

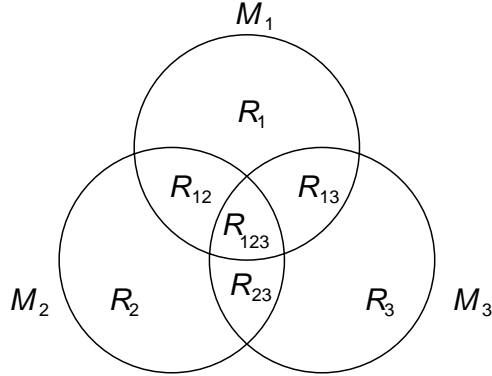


Figure 2: Intersection regions for the maximal cliques M_1 , M_2 , and M_3 .

Case 1: $R_{123} \neq \emptyset$, $R_{12} = R_{13} = R_{23} = \emptyset$.

Let $u \in R_{123}$. Let $u_1 \in R_1$, $u_2 \in R_2$ such that $u_2 \not\sim u_1$, and $u_3 \in R_3$ such that $u_3 \not\sim u_1$ (there are u_2 and u_3 satisfying these conditions, otherwise M_2 and M_3 would not be maximal cliques). If $u_3 \not\sim u_2$, then $G[u, u_1, u_2, u_3] = K_{1,3}$ (Figure 3a). If $u_3 \sim u_2$, let $u'_3 \in R_3$ such that $u'_3 \not\sim u_2$ (there is u'_3 satisfying these conditions, otherwise M_3 would not be a maximal clique). If $u'_3 \sim u_1$, then $G[u, u_1, u_2, u_3, u'_3]$ is a 4-fan (Figure 3b). If $u'_3 \not\sim u_1$, then $G[u, u_1, u_2, u'_3] = K_{1,3}$ (Figure 3c).

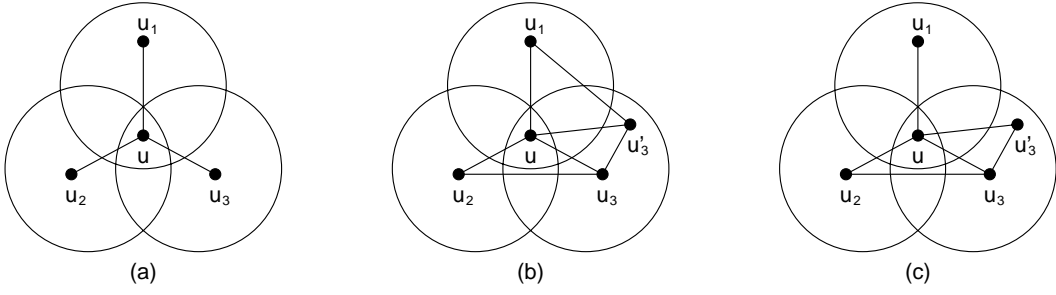


Figure 3: Case 1 of Theorem 1.

Case 2: $R_{123} \neq \emptyset$, $R_{12} \neq \emptyset$, $R_{13} = R_{23} = \emptyset$.

Let $u \in R_{123}$ and $u_{12} \in R_{12}$. By the maximality of the cliques, let $u_1 \in R_1$, $u_2 \in R_2$, and u_3 in R_3 such that $u_2 \not\sim u_1$ and $u_3 \not\sim u_{12}$. If $u_3 \not\sim u_1$ and $u_3 \not\sim u_2$, then $G[u, u_1, u_2, u_3] = K_{1,3}$ (Figure 4a). If $u_3 \sim u_1$ and $u_3 \not\sim u_2$ (or $u_3 \sim u_2$ and $u_3 \not\sim u_1$), then $G[u, u_1, u_{12}, u_2, u_3]$ is a 4-fan (Figure 4b). If $u_3 \sim u_1$ and $u_3 \sim u_2$, then $G[u, u_1, u_{12}, u_2, u_3]$ is a 4-wheel (Figure 4c).

Case 3: $R_{123} \neq \emptyset$, $R_{12} \neq \emptyset$, $R_{13} \neq \emptyset$, $R_{23} = \emptyset$.

Let $u \in R_{123}$, $u_{12} \in R_{12}$ and $u_{13} \in R_{13}$. By the maximality of M_2 and M_3 , there exist $u_2 \in R_2$ and $u_3 \in R_3$ such that $u_2 \not\sim u_{13}$ and $u_3 \not\sim u_{12}$. If

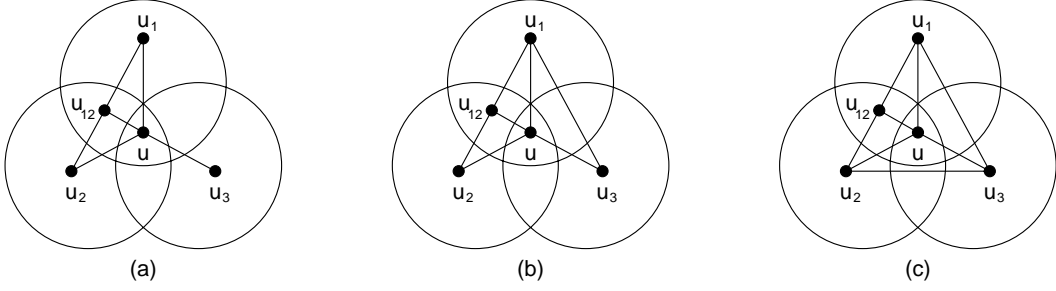


Figure 4: Case 2 of Theorem 1.

$u_2 \not\sim u_3$, then $G[u, u_{12}, u_{13}, u_2, u_3]$ is a 4-fan (Figure 5a). If $u_2 \sim u_3$, then $G[u, u_{12}, u_{13}, u_2, u_3]$ is a 4-wheel (Figure 5b).

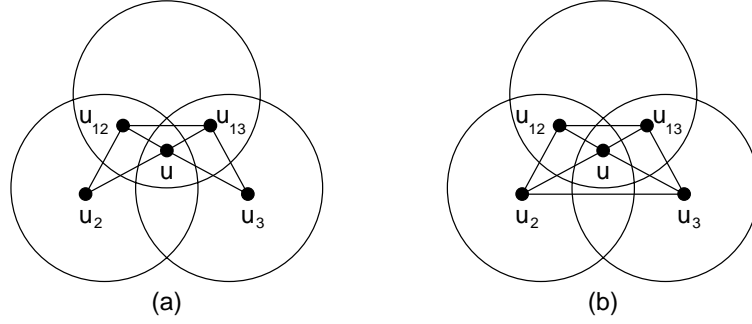


Figure 5: Case 3 of Theorem 1.

Case 4: $R_{123} \neq \emptyset$, $R_{12} \neq \emptyset$, $R_{13} \neq \emptyset$, $R_{23} \neq \emptyset$.

Let $u \in R_{123}$, $u_{12} \in R_{12}$, $u_{13} \in R_{13}$, and $u_{23} \in R_{23}$. Again, by the maximality of M_2 and M_3 , let $u_2 \in R_2$ such that $u_2 \not\sim u_{13}$, and $u_3 \in R_3$ such that $u_3 \not\sim u_{12}$. In fact, this case can be reduced to the previous case, since the existence of u_{23} does not affect the adjacency relations involving u, u_{12}, u_{13}, u_2 , and u_3 . If $u_2 \not\sim u_3$, then $G[u, u_{12}, u_{13}, u_2, u_3]$ is a 4-fan (Figure 6a). If $u_2 \sim u_3$, then $G[u, u_{12}, u_{13}, u_2, u_3]$ is a 4-wheel (Figure 6b).

Case 5: $R_{123} = \emptyset$.

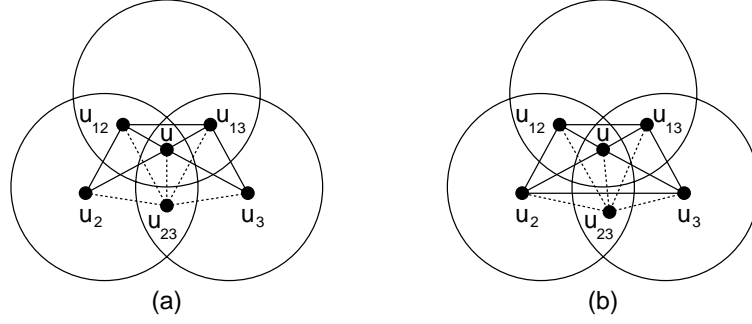


Figure 6: Case 4 of Theorem 1.

In this case, R_{12} , R_{13} and R_{23} are not empty. Let $u_{12} \in R_{12}$, $u_{13} \in R_{13}$, and $u_{23} \in R_{23}$. By the maximality of M_1 and M_2 , there exist $u_1 \in R_1$ and $u_2 \in R_2$ such that $u_1 \not\sim u_{23}$ and $u_2 \not\sim u_{13}$. If $u_1 \not\sim u_2$, then $G[u_1, u_{12}, u_{13}, u_2, u_{23}]$ is a 4-fan (Figure 7a). If $u_1 \sim u_2$, then $G[u_1, u_{12}, u_{13}, u_2, u_{23}]$ is a 4-wheel (Figure 7b).

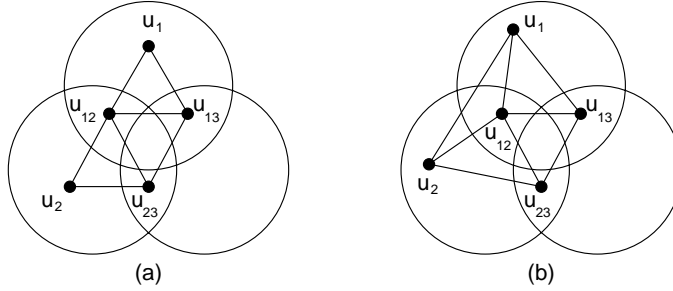


Figure 7: Case 5 of Theorem 1.

In every presented case, we have always obtained either a 4-wheel, a 4-fan, or $K_{1,3}$ as an induced subgraph. Therefore, the theorem is proved. \square

The regions R_S depicted in Figure 2, for $S \subseteq \{1, 2, 3\}$, form a decomposition of the set $M_1 \cup M_2 \cup M_3$ in such a way that:

$$\bigcup_{S \subseteq \{1,2,3\}} R_S = M_1 \cup M_2 \cup M_3 \quad \text{and} \quad R_S \cap R_{S'} = \emptyset \quad \text{for} \quad S \neq S'.$$

Notice that some of the regions may be empty.

We may generalize this decomposition for any collection containing k maximal cliques. Let G be a graph and $\mathcal{C} = \{M_1, M_2, \dots, M_k\}$, $k \geq 1$, be a collection of distinct maximal cliques of G . Let $S \subseteq \{1, 2, \dots, k\}$. Let us define the *region* $R_S^{\mathcal{C}}$ in the following way:

$$R_S^{\mathcal{C}} = \bigcap_{i \in S} M_i - \bigcup_{j \notin S} M_j,$$

where $j \notin S$ means $j \in \{1, 2, \dots, k\} - S$. That is, if $u \in R_S^{\mathcal{C}}$ and i is an index of the set $\{1, \dots, k\}$, then $u \in M_i$ if and only if $i \in S$ (observe that u may belong to maximal cliques of G other than those belonging to \mathcal{C}). If $S = \{i, j, k, l, \dots\}$, simply write $R_{ijkl\dots}$ to mean $R_S^{\mathcal{C}}$ when there is no ambiguity. Notice that each region R_S induces a clique. Moreover, the following lemma is straightforward:

Lemma 2 *Let $S, T \subseteq \{1, 2, \dots, k\}$ such that $S \cap T \neq \emptyset$. If $u \in R_S$ and $w \in R_T$, then u and w are neighbours. \square*

In order to analyze the case in which $K(G)$ is K_4 -free, we need another lemma, which provides a partial characterization for the existence of a collection of pairwise intersecting maximal cliques in G :

Lemma 3 *Let G be a graph and \mathcal{C} be a collection of pairwise intersecting maximal cliques of G . Then, one of the following holds:*

- (i) *the cliques in \mathcal{C} have a vertex in common;*
- (ii) *G contains an extended Hajós graph as an induced subgraph (see Fig. 8).*

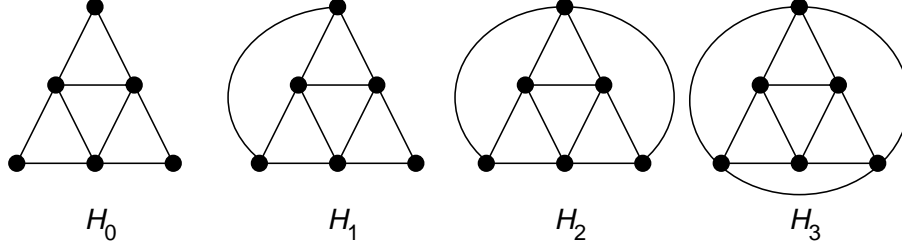


Figure 8: The extended Hajós graphs.

Proof. Assume that (i) does not occur. Let \mathcal{B} be a minimal subcollection of \mathcal{C} such that the cliques in \mathcal{B} do not have a vertex in common. Clearly, \mathcal{B} must contain at least three maximal cliques of \mathcal{C} . Let us write $\mathcal{B} = \{M_1, M_2, \dots, M_k\}$ ($k \geq 3$). Consider the regions R_S , $S \subseteq \{1, \dots, k\}$, relative to \mathcal{B} . Let S_i be the index set $\{1, \dots, k\} - \{i\}$. By the minimality of \mathcal{B} , $R_{S_i} \neq \emptyset$ for every $i \in \{1, 2, \dots, k\}$. Let $u_i \in R_{S_i}$, $i = 1, \dots, k$. By Lemma 2, $\{u_i, 1 \leq i \leq k\}$ is a clique. Moreover, u_i is adjacent to any vertex $w \in \bigcup_{M \in \mathcal{C}} M$ not belonging to R_i . Since $u_i \notin \bigcap_{M \in \mathcal{C}} M$, we have that $u_i \notin M_i$. Thus, $R_i \neq \emptyset$, that is, there exists a vertex $w_i \in R_i$ which is not adjacent to u_i , for every $i \in \{1, 2, \dots, k\}$. Since by Lemma 2 w_i is adjacent to u_j for every $j \neq i$, it follows that the subgraph H induced by the vertices u_1, u_2, u_3, w_1, w_2 , and w_3 is an extended Hajós graph. This subgraph does exist, since $k \geq 3$. The adjacency relations involving w_1, w_2 , and w_3 determine which type of extended Hajós graph H is. If w_1, w_2 , and w_3 induce a subgraph without edges, then $H = H_0$. Otherwise, the other possibilities lead to the construction of H_1, H_2 , or H_3 , respectively. \square

A consequence of the previous lemma is the following result, which provides a first characterization for the situation in which $K(G)$ is K_4 -free:

Corollary 4 *Let G be a graph. Then, G is a clique-inverse graph of a*

K_4 -free graph if and only if G contains neither H_0 as an induced subgraph nor a vertex belonging to four distinct maximal cliques.

Proof. Assume that G is a clique-inverse graph of a K_4 -free graph. If G contains either H_0 as an induced subgraph or a vertex belonging to four distinct maximal cliques, then $K(G)$ contains K_4 , a contradiction. Thus, the necessity follows. Conversely, assume that G contains neither H_0 as an induced subgraph nor a vertex belonging to four distinct maximal cliques, and assume also that $K(G)$ contains K_4 . Let \mathcal{C} be a collection of four distinct pairwise intersecting maximal cliques of G . By Lemma 3 and the above assumptions, G contains either H_1 , H_2 , or H_3 as an induced subgraph. But this is also a contradiction, since any of these three graphs contains a vertex belonging to four distinct maximal cliques. Thus, the result follows. \square

Now, our aim is to refine the above result, in order to establish a characterization in terms of a complete list \mathcal{L} of forbidden subgraphs. We then need to determine which are the vertex-minimal graphs with four maximal cliques having a vertex in common. There are eight graphs with this property, depicted in Figure 9. For any graph in Figure 9, there is a vertex whose removal leaves a graph which contains neither four distinct maximal cliques nor a universal vertex.

Theorem 5 *Let G be a graph. Then, there exists a vertex belonging to four distinct maximal cliques of G if and only if G contains some graph from Figure 9 as an induced subgraph.*

Proof. It is easy to see that if G contains some graph from Figure 9 as an induced subgraph, then there is a vertex belonging to four distinct maximal cliques of G . Conversely, let $u \in VG$ such that u belongs to the distinct

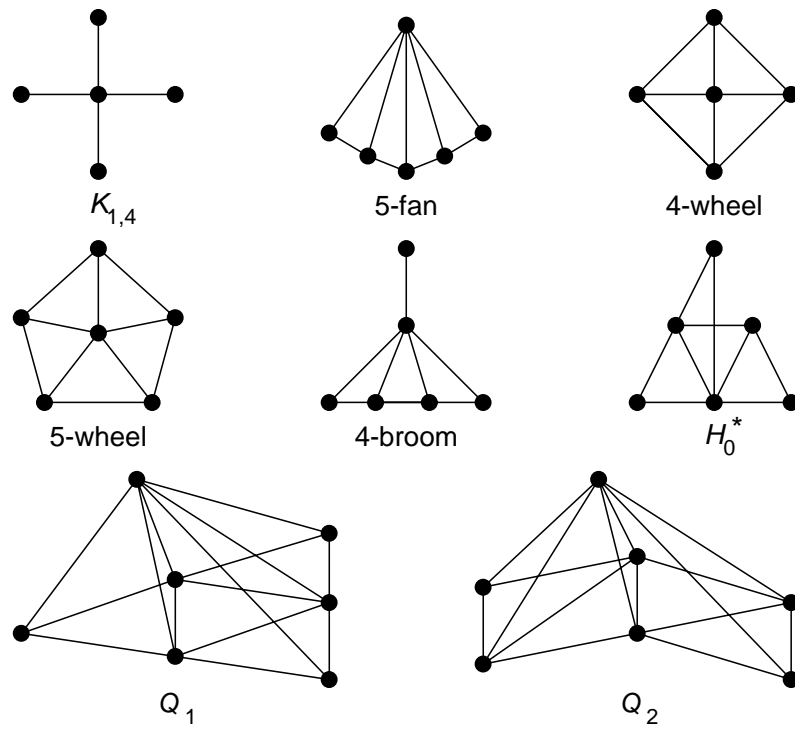


Figure 9: Vertex-minimal graphs with four maximal cliques having a vertex in common.

maximal cliques M_1, M_2, M_3 , and M_4 of G . Since M_1, M_2 , and M_3 contain u , then, by Theorem 1, G contains as an induced subgraph either a 4-wheel, a 4-fan, or $K_{1,3}$. If G contains a 4-wheel as an induced subgraph, nothing remains to prove. Otherwise, let us divide the proof into the two remaining cases.

Case 1: G contains $H = K_{1,3}$ as an induced subgraph.

Recall from Theorem 1 that this situation is illustrated by Figures 3a, 3c, and 4a: there exists a vertex u belonging to $M_1 \cap M_2 \cap M_3$ which is the vertex of degree three in H . Assume that u_1, u_2 , and u_3 are the vertices of degree one in H , where $u_i \in M_i, i = 1, 2, 3$. Notice that at most one from u_1, u_2 , and u_3 may belong to M_4 . Let us analyze the two possible cases.

Case 1.1: u_1, u_2 , and u_3 do not belong to M_4 .

Let $u_4 \in M_4$ be a new vertex such that $u_4 \not\sim u_1$ (such a vertex exists, since $u_1 \notin M_4$). We have that $u_4 \sim u$. Let us analyze the adjacency relations between u_4 and vertices u_2 and u_3 , according to Figure 10 (in all the figures illustrating this proof, dashed lines indicate adjacency relations to be defined subsequently). There are three subcases (1.1.1 to 1.1.3).

Case 1.1.1: $u_4 \not\sim u_2, u_4 \not\sim u_3$. Then, $G[u, u_1, u_2, u_3, u_4] = K_{1,4}$.

Case 1.1.2: u_4 is adjacent to only one from u_2 and u_3 . Assume that $u_4 \sim u_2$ and $u_4 \not\sim u_3$. Since $u_2 \notin M_4$, let $u'_4 \in M_4$ be a new vertex such that $u'_4 \not\sim u_2$. We have that $u'_4 \sim u_4$ and $u'_4 \sim u$. Figure 11 shows this situation, where it still remains to analyze the adjacency relations between u'_4 and vertices u_1 and u_3 . There are three subcases. If $u'_4 \not\sim u_1$ and $u'_4 \not\sim u_3$, then $G[u, u_1, u_2, u_3, u'_4] = K_{1,4}$. If $u'_4 \sim u_1$ and $u'_4 \sim u_3$, then $G[u, u_1, u_2, u_3, u_4, u'_4] = H_0^*$. Finally, if u'_4 is adjacent to only one from u_1 and

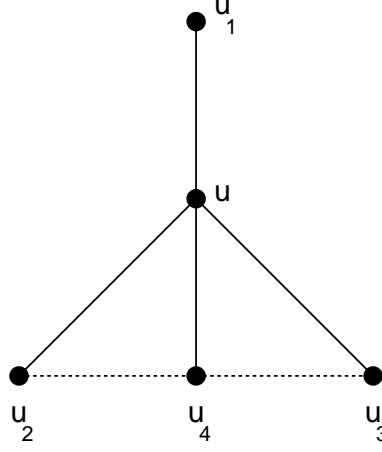


Figure 10: Case 1.1 of Theorem 5.

u_3 , then $G[u, u_1, u_2, u_3, u_4, u'_4]$ is a 4-broom.

Case 1.1.3: $u_4 \sim u_2, u_4 \sim u_3$. Since $u_2 \notin M_4$, let $u'_4 \in M_4$ be again a new vertex, such that $u'_4 \not\sim u_2$. We have that $u'_4 \sim u_4$ and $u'_4 \sim u$. Let us analyze the adjacency relations between u'_4 and vertices u_1 and u_3 , according to Figure 12. There are four subcases (1.1.3.1 to 1.1.3.4).

Case 1.1.3.1: $u'_4 \not\sim u_1, u'_4 \not\sim u_3$. Then, $G[u, u_1, u_2, u_3, u'_4] = K_{1,4}$.

Case 1.1.3.2: $u'_4 \sim u_1, u'_4 \not\sim u_3$. Then, $G[u, u_1, u_2, u_3, u_4, u'_4] = H_0^*$.

Case 1.1.3.3: $u'_4 \not\sim u_1, u'_4 \sim u_3$. Notice that $G[u, u_1, u_2, u_3, u'_4]$ is isomorphic to the graph $G[u, u_1, u_2, u_3, u_4]$ in Case 1.1.2. Thus, this case can be reduced to Case 1.1.2.

Case 1.1.3.4: $u'_4 \sim u_1, u'_4 \sim u_3$. In this situation we can identify only three distinct maximal cliques. Since $u_3 \sim u'_4, u_3 \sim u_4, u_3 \sim u$, and $u_3 \notin M_4$, there exists a new vertex $u''_4 \in M_4$ such that $u''_4 \not\sim u_3$. No-

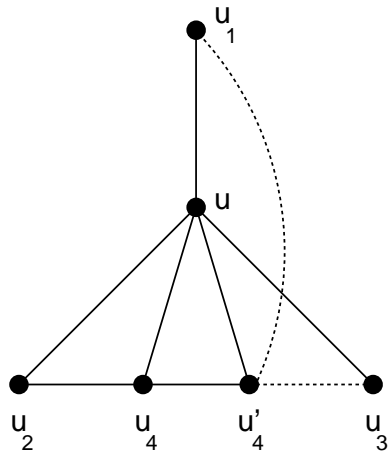


Figure 11: Case 1.1.2 of Theorem 5.

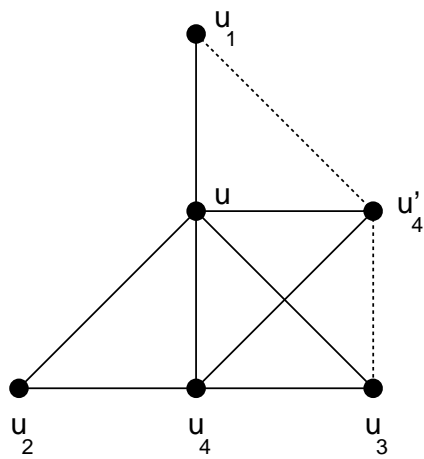


Figure 12: Case 1.1.3 of Theorem 5.

tice that u_4'' is adjacent to u_4' , u_4 , and u , according to Figure 13. We need to analyze the adjacency relations between u_4'' and vertices u_1 and u_2 . If $u_4'' \not\sim u_1$ and $u_4'' \not\sim u_2$, then $G[u, u_1, u_2, u_3, u_4''] = K_{1,4}$. If $u_4'' \not\sim u_1$ and $u_4'' \sim u_2$, then $G[u, u_1, u_2, u_3, u_4', u_4''] = H_0^*$. If $u_4'' \sim u_1$ and $u_4'' \not\sim u_2$, then $G[u, u_1, u_2, u_3, u_4, u_4''] = H_0^*$. Finally, if $u_4'' \sim u_1$ and $u_4'' \sim u_2$, then $G[u, u_1, u_2, u_3, u_4, u_4''] = Q_1$.

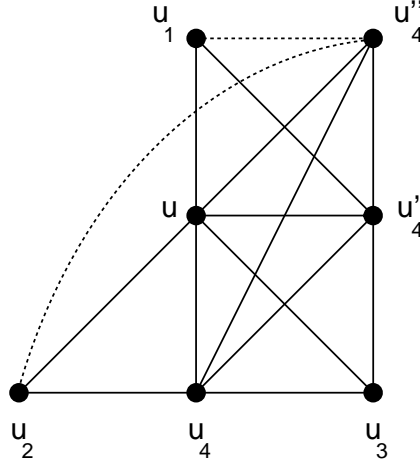


Figure 13: Case 1.1.3.4 of Theorem 5.

Case 1.2: exactly one vertex from u_1 , u_2 , and u_3 belongs to M_4 .

Assume that $u_2 \in M_4$. It is clear that M_4 contains a new vertex u_4 , distinct from u_1 , u_2 , and u_3 . Moreover, we may choose u_4 in such a way that $u_4 \notin M_2$. Since $u_2 \sim u_4$, it remains to verify the adjacency relations between u_4 and vertices u_1 and u_3 , according to Figure 14. There are three subcases (1.2.1 to 1.2.3).

Case 1.2.1: $u_4 \not\sim u_1, u_4 \not\sim u_3$. Notice that $G[u, u_1, u_2, u_3, u_4]$ is isomorphic to

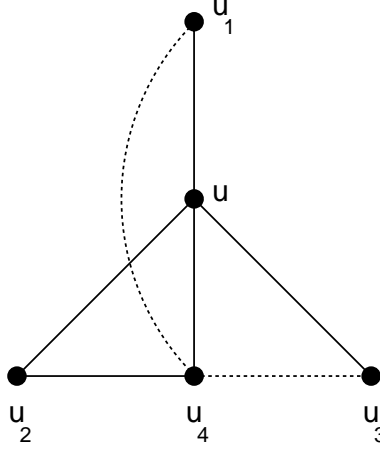


Figure 14: Case 1.2 of Theorem 5.

the graph induced by the same vertices in Case 1.1.2. However, in this case, $u_4 \notin M_2$ (in Case 1.1.2, we had $u_2 \notin M_4$). Thus, this case can be reduced to Case 1.1.2.

Case 1.2.2: u_4 is adjacent to only one from u_1 and u_3 . Assume that $u_4 \not\sim u_1$ and $u_4 \sim u_3$. Take a new vertex $u'_2 \in M_2$ which is adjacent to u_2 but not to u_4 , according to Figure 15. We need to analyze the adjacency relations between u'_2 and vertices u_1 and u_3 . If $u'_2 \sim u_3$, then $G[u, u_2, u'_2, u_3, u_4]$ is a 4-wheel. If $u'_2 \not\sim u_3$ and $u'_2 \not\sim u_1$, then $G[u, u_1, u_2, u'_2, u_3, u_4]$ is a 4-broom. Finally, if $u'_2 \not\sim u_3$ and $u'_2 \sim u_1$, then $G[u, u_1, u_2, u'_2, u_3, u_4]$ is a 5-fan.

Case 1.2.3: $u_4 \sim u_1, u_4 \sim u_3$. Take again a new vertex $u'_2 \in M_2$ which is adjacent to u_2 but not to u_4 , according to Figure 16. We then need to analyze the adjacency relations between u'_2 and vertices u_1 and u_3 . If

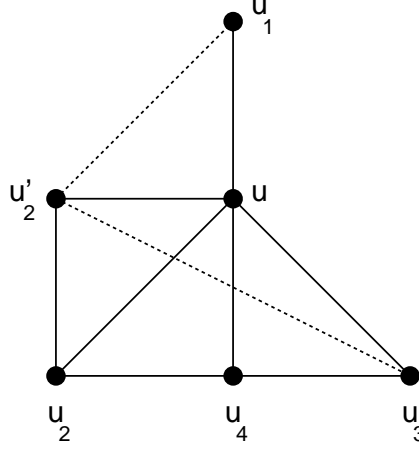


Figure 15: Case 1.2.2 of Theorem 5.

$u'_2 \sim u_1$, then $G[u, u_1, u_2, u'_2, u_4]$ is a 4-wheel. If $u'_2 \not\sim u_1$ and $u'_2 \not\sim u_3$, then $G[u, u_1, u_2, u'_2, u_3, u_4] = H_0^*$. Finally, if $u'_2 \not\sim u_1$ and $u'_2 \sim u_3$, then $G[u, u_2, u'_2, u_3, u_4]$ is a 4-wheel.

Case 2: G contains a 4-fan H as an induced subgraph.

By analyzing the cases of Theorem 1, this situation is illustrated by Figures 3b, 4b, 5a, and 6a: there exists a vertex u which is the vertex of degree four in H . Let x and y be the vertices of degree three in H . Observe that:

- (a) in Figures 3b and 4b, at least one from x and y belongs to exactly one maximal clique from M_1 , M_2 , and M_3 ;
- (b) in Figures 5a and 6a, x and y both belong to exactly two maximal cliques from M_1 , M_2 , and M_3 .

Let us divide the proof of Case 2 into the above subcases.

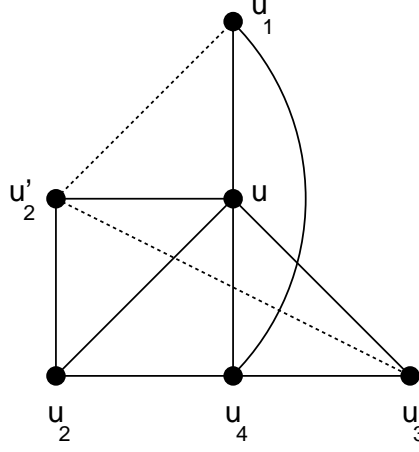


Figure 16: Case 1.2.3 of Theorem 5.

Case 2.1: at least one from x and y belongs to exactly one maximal clique from M_1 , M_2 , and M_3 .

Assume that x is the vertex belonging to exactly one maximal clique from M_1 , M_2 , and M_3 (in Figure 3b, $x = u_3$ or $x = u'_3$; in Figure 4b, $x = u_1$). Without loss of generality, assume also that $x \in M_1$. Let u_2 and u_3 be the vertices of degree two in H . Assume that $u_2 \in M_2$, $u_3 \in M_3$, $x \sim u_2$, and $y \sim u_3$. We then have the following situation: u belongs to M_1 , M_2 , and M_3 ; x belongs to M_1 , but neither to M_2 nor to M_3 ; y belongs to M_1 and perhaps to M_3 , but not to M_2 ; u_2 belongs to M_2 , but neither to M_1 nor to M_3 ; u_3 belongs to M_3 , but neither to M_1 nor to M_2 . Since $x \sim u_2$ and $x \notin M_2$, there exists a new vertex $u'_2 \in M_2$ which is not adjacent to x . It is clear that $u'_2 \sim u$ and $u'_2 \sim u_2$, as in Figure 17. Let us verify the adjacency relations between u'_2 and vertices y and u_3 . If $u'_2 \sim y$, then $G[u, x, y, u_2, u'_2]$ is a 4-wheel. If $u'_2 \not\sim y$ and $u'_2 \not\sim u_3$, then $G[u, x, y, u_2, u'_2, u_3]$ is a 5-fan. If

$u'_2 \not\sim y$ and $u'_2 \sim u_3$, then $G[u, x, y, u_2, u'_2, u_3]$ is a 5-wheel.

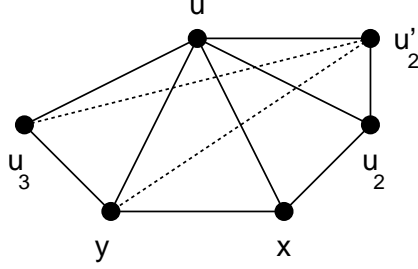


Figure 17: Case 2.1 of Theorem 5.

Case 2.2: x and y both belong to exactly two maximal cliques from M_1 , M_2 , and M_3 .

Observe that x and y correspond to the vertices u_{12} and u_{13} in Figures 5a and 6a. Write $x = u_{12}$ and $y = u_{13}$. Assume that u_2 and u_3 are the vertices of degree two in H , where $u_2 \in M_2$, $u_3 \in M_3$, $u_{12} \sim u_2$, and $u_{13} \sim u_3$. We then have the following situation: u belongs to M_1 , M_2 , and M_3 ; u_{12} belongs to M_1 and M_2 , but not to M_3 ; u_{13} belongs to M_1 and M_3 , but not to M_2 ; u_2 belongs to M_2 , but neither to M_1 nor to M_3 ; u_3 belongs to M_3 , but neither to M_1 nor to M_2 . Notice that u_2 and u_3 cannot both belong to M_4 . We have two subcases.

Case 2.2.1: $u_2, u_3 \notin M_4$. In this case, u_{12} and u_{13} may or may not belong to M_4 . We have two more subcases.

Case 2.2.1.1: at least one from u_{12} and u_{13} does not belong to M_4 . Assume $u_{12} \notin M_4$. Then there exists a new vertex $u_4 \in M_4$ which is not adjacent to u_{12} , according to Figure 18. The adjacency relations between u_4 and vertices u_{13} , u_2 , and u_3 determine the following seven possibilities.

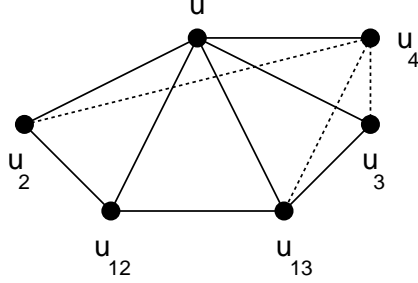


Figure 18: Case 2.2.1.1 of Theorem 5.

- (a) $u_4 \sim u_{13}, u_4 \sim u_2$. Then, $G[u, u_{12}, u_{13}, u_2, u_4]$ is a 4-wheel.
- (b) $u_4 \not\sim u_{13}, u_4 \not\sim u_2, u_4 \not\sim u_3$. Then, $G[u, u_{12}, u_{13}, u_2, u_3, u_4]$ is a 4-broom.
- (c) $u_4 \not\sim u_{13}, u_4 \not\sim u_2, u_4 \sim u_3$. Then, $G[u, u_{12}, u_{13}, u_2, u_3, u_4]$ is a 5-fan.
- (d) $u_4 \not\sim u_{13}, u_4 \sim u_2, u_4 \not\sim u_3$. Then, $G[u, u_{12}, u_{13}, u_2, u_3, u_4]$ is a 5-fan.
- (e) $u_4 \not\sim u_{13}, u_4 \sim u_2, u_4 \sim u_3$. Then, $G[u, u_{12}, u_{13}, u_2, u_3, u_4]$ is a 5-wheel.
- (f) $u_4 \sim u_{13}, u_4 \not\sim u_2, u_4 \not\sim u_3$. Then, $G[u, u_{12}, u_{13}, u_2, u_3, u_4] = H_0^*$.
- (g) $u_4 \sim u_{13}, u_4 \not\sim u_2, u_4 \sim u_3$. In this situation we can identify three distinct maximal cliques only. Since $u_{12}, u_2, u_3 \notin M_4$, there exists a new vertex $u'_4 \in M_4$ which is not adjacent to u_3 . Let us verify the adjacency relations between u'_4 and vertices u_{12}, u_{13} , and u_2 , according to Figure 19. There are seven additional possibilities.

- (g.1) $u'_4 \sim u_{12}, u'_4 \not\sim u_{13}$. Then, $G[u, u_{12}, u_{13}, u_4, u'_4]$ is a 4-wheel.
- (g.2) $u'_4 \not\sim u_{12}, u'_4 \not\sim u_{13}, u'_4 \not\sim u_2$. Then, $G[u, u_{12}, u_{13}, u_2, u_4, u'_4]$ is a 5-fan.
- (g.3) $u'_4 \not\sim u_{12}, u'_4 \not\sim u_{13}, u'_4 \sim u_2$. Then, $G[u, u_{12}, u_{13}, u_2, u_4, u'_4]$ is a 5-wheel.
- (g.4) $u'_4 \not\sim u_{12}, u'_4 \sim u_{13}, u'_4 \not\sim u_2$. Then, $G[u, u_{12}, u_{13}, u_2, u_3, u'_4] = H_0^*$.
- (g.5) $u'_4 \not\sim u_{12}, u'_4 \sim u_{13}, u'_4 \sim u_2$. Then, $G[u, u_{12}, u_{13}, u_2, u'_4]$ is a 4-wheel.
- (g.6) $u'_4 \sim u_{12}, u'_4 \sim u_{13}, u'_4 \not\sim u_2$. Then, $G[u, u_{12}, u_2, u_3, u_4, u'_4]$ is a 5-fan.

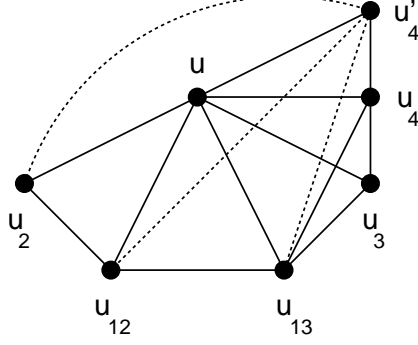


Figure 19: Case 2.2.1.1(g) of Theorem 5.

(g.7) $u'_4 \sim u_{12}, u'_4 \sim u_{13}, u'_4 \sim u_2$. Then, $G[u, u_{12}, u_{13}, u_2, u_3, u_4, u'_4] = Q_2$.

Case 2.2.1.2: u_{12} and u_{13} belong both to M_4 . Thus, there exist new vertices $u'_1 \in M_1$ and $u_4 \in M_4$ such that $u'_1 \not\sim u_4$. These vertices exist because it is necessary to distinguish clique M_1 from clique M_4 . Notice that u'_1 and u_4 are adjacent to u, u_{12} , and u_{13} . See Figure 20. The adjacency relations between u'_1, u_4 and the remaining vertices (u_2 and u_3) determine the following five possibilities. Let $H = G[u'_1, u_4, u_2, u_3]$.

(a) $EH = \emptyset$. Then, $G[u, u'_1, u_2, u_3, u_4] = K_{1,4}$.

(b) $EH = \{e\}$. Then, $G[u, w, u'_1, u_2, u_3, u_4] = H_0^*$, where $w = u_{13}$ if e is incident to u_2 , and $w = u_{12}$ if e is incident to u_3 .

(c) H consists of a path with three vertices together with an isolated vertex w , where $w = u'_1$ or $w = u_4$. Then, $G[u, u_{12}, u_{13}, u'_1, u_2, u_3, u_4] = Q_1$.

(d) H consists of two disjoint edges. Then, $G[u, u_{12}, u_{13}, u'_1, u_2, u_3, u_4] = Q_2$.

(e) there exists a path from u'_1 to u_4 in H . In this case, $G[u, u_{12}, u'_1, u_3, u_4]$ is a 4-wheel if there exists a path from u'_1 to u_4 passing through u_3 . Analogously,

$G[u, u_{13}, u'_1, u_2, u_4]$ is a 4-wheel if there exists a path from u'_1 to u_4 passing through u_2 .

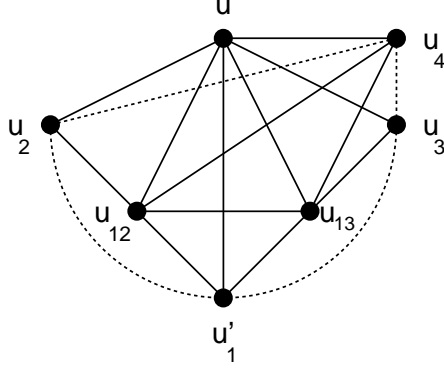


Figure 20: Case 2.2.1.2 of Theorem 5.

Case 2.2.2: one vertex from u_2 and u_3 belongs to M_4 . Assume $u_3 \in M_4$. Thus, $u_{12} \notin M_4$. We have then the following configuration: u belongs to M_1 , M_2 and M_4 ; u_{12} belongs to M_1 and M_2 , but not to M_4 ; u_2 belongs to M_2 , but neither to M_1 nor to M_4 ; u_3 belongs to M_4 , but neither to M_1 nor to M_2 ; u_{13} belongs to M_1 , but not to M_2 . Since u_{13} may or may not belong to M_4 , there are two possibilities:

(a) if $u_{13} \notin M_4$, then we have here the same situation of Case 2.1, where now the clique M_4 plays the same role as the clique M_3 did in the previous analysis. Thus, the present situation can be reduced to Case 2.1.

(b) if $u_{13} \in M_4$, then it follows that u , u_{13} , and u_3 belong to M_3 and M_4 simultaneously (recall from Figures 5a and 6a that u , u_{13} , and u_3 belong to M_3). Since here u_{12} and u_2 belong neither to M_3 nor to M_4 , there exists a new vertex $u'_4 \in M_4$ such that $u'_4 \notin M_3$. By considering the graph induced by

vertices u, u_{12}, u_{13}, u_2 , and u'_4 , we have the same situation as in Case 2.2.1.1, where we need to replace M_3 by M_4 and u_3 by u'_4 . Thus, the present situation can be reduced to Case 2.2.1.1.

In every presented case, we have always obtained one of the graphs in Figure 9 as an induced subgraph. Therefore, the theorem is proved. \square

Note that H_0 (Figure 8) is obtained from Q_1 (Figure 9) by removing the universal vertex of Q_1 . Thus, the results of Corollary 4 and Theorem 5 may be combined in the following way:

Corollary 6 *Let G be a graph. Then, G is a clique-inverse graph of a K_4 -free graph if and only if G does not contain as an induced subgraph any of the following graphs: H_0 , $K_{1,4}$, 4-broom, 4-wheel, 5-wheel, 5-fan, H_0^* , Q_2 . \square*

3 Conclusions

We have described characterizations for clique-inverse graphs of K_3 -free and K_4 -free graphs in terms of forbidden induced subgraphs.

Let H be an induced subgraph of a graph G , and let $r \geq 3$. If $K(G)$ is K_r -free, then $K(H)$ is also K_r -free, that is, the family of clique-inverse graphs of K_r -free graphs is closed under induced subgraphs. Therefore, the problems dealt with in this paper can be extended in the following way: for a fixed $r \geq 3$, find the family \mathcal{F}_r of graphs such that $K(G)$ is K_r -free if and only if G does not contain any graph of \mathcal{F}_r as an induced subgraph.

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