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ABSTRACT

A convergence enhancement technique known as the integral balance approach is employed in combination with the Generalized Integral Transform Technique (GITT) for solving diffusion or convection-diffusion problems in physical domains with subregions of markedly different materials properties and/or spatial scales. GITT is employed in the solution of the differential eigenvalue problem with space variable coefficients, by adopting simpler auxiliary eigenproblems for the eigenfunction representation. The examples provided deal with heat conduction in heterogeneous media and forced convection in a microchannel embedded in a substrate. The convergence characteristics of the proposed novel solution are critically compared against the conventional approach through integral transforms without the integral balance enhancement, with the aid of fully converged results from the available exact solutions.

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1. Introduction

A number of applications in heat and fluid flow offer challenging mathematical formulations in diffusion and convection-diffusion with abrupt and/or multiscale spatial and physical properties variations in the associated partial differential equations' coefficients. These governing parameters' behavior may originate, for instance, from natural or tailored physical properties' variability in heterogeneous media and from abrupt material transitions or multiscale spatial dimensions in multiregion geometries, such as in multilayered media or conjugated problems [1–6].

A few different approaches have been proposed to effectively deal with such spatial variations throughout computational domains [3–11], including attempts at benefiting from semi-analytical solution implementations, towards more robust and cost-effective simulations, for instance aimed at computationally intensive tasks, such as in optimization, inverse problem analysis, and simulation under uncertainty.

One such hybrid numerical-analytical approach, known as the Generalized Integral Transform Technique (GITT) [12–18], has been employed in the solution of diffusion problems in heterogeneous media with spatial variations in thermophysical properties [3–6]. Particularly in connection with conjugated heat transfer problems [5–6], a strategy coined as the single domain formulation has been introduced to rewrite a multiregion problem into one single region with space variable thermophysical properties and source terms, so as to allow for a single integral transformation operation of

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Color versions of one or more of the figures in this article can be found online on at www.tandfonline.com/unht.

Nomenclature

$d(\mathbf{x})$	Dissipation operator coefficient, Eq. (1.a)	α_0	Reference thermal diffusivity in FGM application, Eq. (23.c)
$f(\mathbf{x})$	Initial condition, Eq. (1.a)	β	Coefficient in FGM application, Eq. (23.a,b)
$k(\mathbf{x})$	Diffusion operator coefficient, Eq. (1.a)	λ_n	Eigenvalues of problem (11)
M	Truncation order of the algebraic eigenvalue problem	μ_i	Eigenvalues of problem (5)
N_i	Normalization integral of the eigenvalue problem, Eq. (7)	ψ_i	Eigenfunctions of eigenvalue problem (5)
$P(\mathbf{x}, t, T)$	Nonlinear source term appearing in Eq. (1.a)	Ω	Eigenfunction of the auxiliary problem, Eq. (11)
$T(\mathbf{x}, t)$	Potential	$\phi(\mathbf{x}, t, T)$	Nonlinear source term appearing in Eq. (1.c)
$T_F(\mathbf{x}, t, T^*)$	Filtering solution, Eq. (2).		
t	Time variable		
u	Dependent variable in FGM application, Eq. (25)	***Subscripts & Superscripts:***	
U	Dimensionless fluid velocity in conjugated problem	i, n	Order of eigen quantities
x	Space variable (one-dimensional problem)	–	Integral transform
$\mathbf{x}$	Position vector	~	Normalized eigenfunction
$w(\mathbf{x})$	Transient operator coefficient, Eq. (1.a)	*	Filtered temperature field
$\alpha(\mathbf{x})$, $\beta(\mathbf{x})$	Coefficients for the boundary conditions, Eq. (1.c)	s	Quantity corresponding to the solid region (channel walls)
		f	Quantity corresponding to the fluid flow region

the whole domain. This strategy is appropriate to dealing with complex configurations and irregular regions, as recently illustrated [5–6], and has a potential for further extension aiming at automatic implementations of the integral transform procedure, as illustrated through the so-called Unified Integral Transforms (UNIT) algorithm [19–21].

In this approach, the resulting convection-diffusion equations with space variable coefficients, for each associated potential, are solved by integral transforms through eigenvalue problems that carry the information on the variable coefficients. The eigenvalue problems are themselves handled by the GITT [13, 22, 23], transforming the differential eigenvalue problems into algebraic eigensystems which can be solved very efficiently by widely available subroutines and computational platforms [24]. However, it has been observed in recent developments that the eigenfunction expansions proposed in the solution of eigenvalue problems with abrupt and/or multiscale variations on the governing coefficients can experience a slower convergence rate. Therefore, it is of interest to implement a convergence enhancement technique to allow for computational savings on GITT application in such cases.

Thus, based on previous developments on convergence enhancement of eigenfunction expansions for diffusion problems [25, 26], by employing an integral balance analytical procedure the GITT solution of eigenvalue problems [13, 22] is here revisited so as to derive analytical expansions to accelerate the convergence of eigenfunctions expansion, explicitly accounting for the space variable coefficients of the original problem formulation, within the resulting functional form of the redefined inverse formulae for the eigenfunctions. Starting with successive integration of the original eigenvalue problem from the boundaries to any point within the domain, analytical expressions for the eigenfunction and its derivative are obtained, which depend on the boundary values of both the eigenfunction and the associated derivative, but explicitly account for the space variable coefficients in the eigenfunction representation. Then, by making use of the available boundary conditions, the boundary quantities are eliminated from the newly derived expressions of the eigenfunction, as well as of its derivative. Working expressions are provided for a general one-dimensional Sturm–Liouville problem, and the approach is tested for both a functionally graded material (FGM) heat conduction problem, which involves markedly different orders of magnitude in the thermophysical properties, and a conjugated heat transfer problem, which involves abrupt material transitions and markedly different spatial scales.

2. Formal solution

The formal solution of a general convection-diffusion problem is first presented in brief, readily available from different sources [12–18], so as to lead to the eigenvalue problem that will be handled through the convergence enhancement technique here proposed. The following nonlinear convection-diffusion problem is then analyzed:

$$w(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t} = \nabla\cdot[k(\mathbf{x})\nabla T(\mathbf{x},t)] - d(\mathbf{x})T(\mathbf{x},t) + P(\mathbf{x},t,T), \mathbf{x}\in V, t>0 \tag{1.a}$$

subjected to the following initial and boundary conditions:

$$T(\mathbf{x},0) = f(\mathbf{x}), \mathbf{x}\in V \tag{1.b}$$

$$\alpha(\mathbf{x})T(\mathbf{x},t) + \beta(\mathbf{x})k(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial \mathbf{n}} = \phi(\mathbf{x},t,T), \mathbf{x}\in S \tag{1.c}$$

Any sort of nonlinearities in the equation and boundary condition coefficients can be recast into the corresponding source terms, $P(\mathbf{x}, t, T)$ and $\phi(\mathbf{x}, t, T)$, without loss of generality. In this sense, the linear equation (w, k, d) and boundary condition (α, β) coefficients are essentially characteristic functions that are chosen so as to intrinsically formulate the eigenvalue problem that is adopted as a basis for the eigenfunction expansion solution that now follows.

Before applying the integral transform methodology, it is usually recommended to reduce the importance of the source terms in the original equation and the boundary conditions given by Eq. (1a,c), since these are largely responsible for an eventually slower convergence behavior. A filtering scheme is then applied in a general form that includes both linear and implicit nonlinear filters, given as

$$T(\mathbf{x},t) = T^*(\mathbf{x},t) + T_F(\mathbf{x},t,T^*) \tag{2}$$

where $T_F(\mathbf{x},t,T^*)$ is the proposed filter and $T^*(\mathbf{x},t)$ is the resulting filtered potential to be determined. After introducing Eq. (2) into Eq. (1)

$$w(\mathbf{x})\frac{\partial T^*(\mathbf{x},t)}{\partial t} = \nabla\cdot[k(\mathbf{x})\nabla T^*(\mathbf{x},t)] - d(\mathbf{x})T^*(\mathbf{x},t) + P^*(\mathbf{x},t,T^*),\ \mathbf{x}\in V, t>0 \tag{3a}$$

$$T^*(\mathbf{x},0) = f^*(\mathbf{x}),\ \mathbf{x}\in V \tag{3b}$$

$$\alpha(\mathbf{x})T^*(\mathbf{x},t) + \beta(\mathbf{x})k(\mathbf{x})\frac{\partial T^*(\mathbf{x},t)}{\partial \mathbf{n}} = \phi^*(\mathbf{x},t,T^*),\ \mathbf{x}\in S \tag{3c}$$

where the filtered functions are written as

$$f^*(\mathbf{x}) \equiv f(\mathbf{x}) - T_F(\mathbf{x},0,T^*(\mathbf{x},0)) \tag{4.a}$$

$$P^*(\mathbf{x},t,T^*) = P(\mathbf{x},t,T) - w(\mathbf{x})\frac{\partial T_F(\mathbf{x},t,T^*)}{\partial t} + \nabla\cdot[k(\mathbf{x})\nabla T_F(\mathbf{x},t,T^*)] - d(\mathbf{x})T_F(\mathbf{x},t,T^*) \tag{4.b}$$

$$\phi^*(\mathbf{x},t,T^*) = \phi(\mathbf{x},t,T) - \alpha(\mathbf{x})T_F(\mathbf{x},t,T^*) - \beta(\mathbf{x})k(\mathbf{x})\frac{\partial T_F(\mathbf{x},t,T^*)}{\partial \mathbf{n}},\ \mathbf{x}\in S \tag{4.c}$$

According to Eq. (2), either a linear explicit filter, $T_F(\mathbf{x}, t)$, or a nonlinear implicit filter, $T_F(\mathbf{x},t,T^*)$, can be chosen, with inherently more simple expressions, in the choice of a linear filter, for the filtered initial conditions and source terms obtained from Eq. (4). In any case, it is in general desirable that the chosen filter at least satisfies Eq. (1.c), so as to homogenize the boundary conditions, thus leading to $\phi^*(\mathbf{x},t,T^*) = 0$. Nevertheless, *a posteriori* convergence enhancement techniques [25, 26] can still be employed when a boundary condition source term remains in the unfiltered or filtered problem.

Following the steps in the integral transform approach [12–17], we define an auxiliary eigenvalue problem, which shall provide the basis for the eigenfunction expansions, as

$$\nabla\cdot[k(\mathbf{x})\nabla\psi_i(\mathbf{x})] + [\mu_i^2 w(\mathbf{x}) - d(\mathbf{x})]\psi_i(\mathbf{x}) = 0,\ \mathbf{x}\in V \tag{5.a}$$

$$\alpha(\mathbf{x})\psi_i(\mathbf{x}) + \beta(\mathbf{x})k(\mathbf{x})\frac{\partial\psi_i(\mathbf{x})}{\partial\mathbf{n}} = 0,\ \mathbf{x} \in S \tag{5.b}$$

The eigenvalue problem given by Eq. (5) allows for the definition of the integral transform pair below:

$$\bar{T}_i(t) = \int_V w(\mathbf{x})\psi_i(\mathbf{x})T^*(\mathbf{x},t)dV,\ \text{transform} \tag{6.a}$$

$$T^*(\mathbf{x},t) = \sum_{i=1}^{\infty}\frac{1}{N_i}\psi_i(\mathbf{x})\bar{T}_i(t), \text{inverse} \tag{6.b}$$

and the normalization integral

$$N_i = \int_v w(\mathbf{x})\psi_i^2(\mathbf{x})dV \tag{7}$$

After application of the integral transformation procedure, through the operator $\int_v \psi_i(\mathbf{x})(.)dV$ over Eq. (3.a), and $\int_v w(\mathbf{x})\psi_i(\mathbf{x})(.)dV$ over Eq. (3.b), the resulting ODE system for the transformed potentials, $\overline{T_i}(t)$, is written as

$$\frac{d\bar{T}_i(t)}{dt} + \mu_i^2\,\bar{T}_i(t) = \bar{g}_i(t,\bar{\mathbf{T}}),\ t > 0, i = 1, 2 \tag{8.a}$$

with initial conditions

$$\bar{T}_i(0) = \bar{f}_i \tag{8.b}$$

where

$$\bar{g}_i(t,\bar{\mathbf{T}}) = \int_v \psi_i(\mathbf{x})\ P^*(\mathbf{x},t,T^*)dV + \int_S \phi^*(\mathbf{x},t,T^*)\left(\frac{\psi_i(\mathbf{x}) - k(\mathbf{x})\frac{\partial\psi_i}{\partial\mathbf{n}}}{\alpha(\mathbf{x}) + \beta(\mathbf{x})}\right)dS \tag{8.c}$$

$$\bar{f}_i = \int_v w(\mathbf{x})\psi_i(\mathbf{x})f^*(\mathbf{x})\mathrm{d}V \tag{8.d}$$

$$\bar{\mathbf{T}} = \{\bar{T}_1(t), \bar{T}_2(t), \ldots\}^T \tag{8.e}$$

System (8) is then numerically solved through well-established initial value problem solvers, readily available in scientific subroutine libraries, or directly as a built-in function in mixed symbolic-numerical platforms, such as the function NDSolve of the *Mathematica* system [24], which implements automatic relative error control schemes. After truncation to a sufficiently large finite truncation order N, the desired hybrid numerical-analytical solution is then reconstructed as

$$T(\mathbf{x},t) = \sum_{i=1}^{N}\frac{1}{N_i}\psi_i(\mathbf{x})\bar{T}_i(t) + T_F(\mathbf{x},t,T^*) \tag{9}$$

It should be recalled that the filter problem, in the case of an implicit filtering strategy for nonlinear problems, has to be solved simultaneously with the transformed system, Eq. (8).

3. Eigenvalue problem solution

The eigenvalue problem that provides the basis for the eigenfunction expansion can be efficiently solved through the GITT itself, as proposed in [13, 22] and successfully employed in different applications. The idea is to employ the generalized integral transform technique formalism to reduce the eigenvalue problem described by partial differential equations into standard algebraic eigenvalue problems, which can be solved by existing routines for matrix eigensystem analysis. Therefore, the eigenfuctions of the original auxiliary problem can be expressed by eigenfunction expansions based on a simpler auxiliary eigenvalue problem, for which exact analytic solutions are available.

The solution of problem (5) is thus proposed as an eigenfunction expansion:

$$\psi_i(\mathbf{x}) = \sum_{n=1}^{\infty} \tilde{\Omega}_n(\mathbf{x})\bar{\psi}_{i,n}, \qquad \textit{inverse} \tag{10.a}$$

$$\bar{\psi}_{i,n} = \int_V \hat{w}(\mathbf{x})\, \psi_i(\mathbf{x})\tilde{\Omega}_n(\mathbf{x})dV, \qquad \textit{transform} \tag{10.b}$$

where the normalized auxiliary eigenfunction and its norms are

$$\tilde{\Omega}_n(\mathbf{x}) = \frac{\Omega_n(\mathbf{x})}{\sqrt{N_{\Omega n}}}\ , \text{ with } N_{\Omega_n} = \int_V \hat{w}(\mathbf{x})\Omega_n^2(\mathbf{x})dV \tag{10.c, d}$$

in terms of the simpler auxiliary eigenvalue problem given as

$$\nabla.\hat{k}(\mathbf{x})\nabla\Omega_n(\mathbf{x}) + (\lambda_n^2\hat{w}(\mathbf{x}) - \hat{d}(\mathbf{x}))\Omega_n(\mathbf{x}) = 0,\ \ \mathbf{x} \in V \tag{11.a}$$

with boundary conditions

$$\alpha(\mathbf{x})\Omega_n(\mathbf{x}) + \beta(\mathbf{x})\hat{k}(\mathbf{x})\frac{\partial\Omega_n(\mathbf{x})}{\partial\mathbf{n}} = 0, \quad \mathbf{x} \in S \tag{11.b}$$

where the coefficients, $\hat{w}(\mathbf{x})$, $\hat{k}(\mathbf{x})$, and $\hat{d}(\mathbf{x})$, are simpler forms of the equation coefficients chosen so as to allow for an analytical solution of the auxiliary problem. Thus, the solution of problem (11), which needs to be known in terms of the eigenfunctions $\Omega_n(\mathbf{x})$ and related eigenvalues λ_n, offers a basis for the eigenfunction expansion of the original eigenvalue problem (5). Equation (5a) is now operated on with $\int_V \tilde{\Omega}_i(\mathbf{x})(\cdot)dV$, to yield the transformed algebraic system:

$$(\mathbf{A} + \mathbf{C})\{\bar{\psi}\} = \mu^2\mathbf{B}\{\bar{\psi}\} \tag{12.a}$$

with the elements of the M × M matrices given by

$$\begin{aligned} A_{ij} = &\int_S \frac{\tilde{\Omega}_i(\mathbf{x}) - \hat{k}(\mathbf{x})\frac{\partial\tilde{\Omega}_i(\mathbf{x})}{\partial\mathbf{n}}}{\alpha(\mathbf{x}) + \beta(\mathbf{x})}\left[\beta(\mathbf{x})\left(k(\mathbf{x}) - \hat{k}(\mathbf{x})\right)\frac{\partial\tilde{\Omega}_j(\mathbf{x})}{\partial\mathbf{n}}\right]dS \\ &- \int_S \left(k(\mathbf{x}) - \hat{k}(\mathbf{x})\right)\tilde{\Omega}_i(\mathbf{x})\frac{\partial\tilde{\Omega}_j(\mathbf{x})}{\partial\mathbf{n}}dS + \\ &+ \int_V \left(k(\mathbf{x}) - \hat{k}(\mathbf{x})\right)\nabla\tilde{\Omega}_i(\mathbf{x})\cdot\nabla\tilde{\Omega}_j(\mathbf{x})dV + \int_V \left(d(\mathbf{x}) - \hat{d}(\mathbf{x})\right)\tilde{\Omega}_i(\mathbf{x})\tilde{\Omega}_j(\mathbf{x})dV \end{aligned} \tag{12.b}$$

$$C_{ij} = \lambda_i^2\delta_{ij} \tag{12.c}$$

$$B_{ij} = \int_V w(\mathbf{x})\tilde{\Omega}_i(\mathbf{x})\tilde{\Omega}_j(\mathbf{x})dV \tag{12.d}$$

where δ_{ij} is the Kronecker delta.

Therefore, the eigenvalue problem given by Eq. (5) is reduced to the standard algebraic eigenvalue problem given by Eq. (12), which can be solved with existing software for matrix eigensystem analysis, yielding the eigenvalues μ, whereas the corresponding calculated eigenvectors from this numerical solution, $\bar{\psi}_i$, are to be used in the inversion formula, given by Eq. (10.a), to find the desired eigenfunction. By increasing the number of terms in the truncated expansion, *M*, the eigenfunctions are obtained to within the user prescribed accuracy.

The one-dimensional version of the problem given by Eq. (5.a,b) is reduced to

$$\frac{d}{dx}\left[k_x(x)\frac{dX(x)}{dx}\right]+\left[\gamma^2 w_x(x)-d_x(x)\right]X(x)=0,\;\; x_0\le x\le x_1 \tag{13.a}$$

$$\alpha_{x,0}X(x)-\beta_{x,0}k_x(x)\frac{dX(x)}{dx}=0,\quad x=x_0 \tag{13.b}$$

$$\alpha_{x,1}X(x)+\beta_{x,1}k_x(x)\frac{dX(x)}{dx}=0,\quad x=x_1 \tag{13.c}$$

for which the elements of the M x M matrices in Eq. (12.a) are simplified to

$$\begin{aligned}A_{ij}=&-\frac{\tilde{\Omega}_i(x_0)+\hat{k}_x(x_0)\tilde{\Omega}_i'(x_0)}{\alpha_{x,0}+\beta_{x,0}}\beta_{x,0}(x_0)\Big(k_x(x_0)-\hat{k}_x(x_0)\Big)\tilde{\Omega}_j'(x_0)+\\&+\frac{\tilde{\Omega}_i(x_1)-\hat{k}_x(x_1)\tilde{\Omega}_i'(x_1)}{\alpha_{x,1}+\beta_{x,1}}\beta_{x,1}(x_1)\Big(k_x(x_1)-\hat{k}_x(x_1)\Big)\tilde{\Omega}_j'(x_1)+\\&+\Big(k_x(x_0)-\hat{k}_x(x_0)\Big)\tilde{\Omega}_i(x_0)\tilde{\Omega}_j'(x_0)-\Big(k_x(x_1)-\hat{k}_x(x_1)\Big)\tilde{\Omega}_i(x_1)\tilde{\Omega}_j'(x_1)+\\&+\int_{x_0}^{x_1}\Big(k_x(x)-\hat{k}_x(x)\Big)\tilde{\Omega}_i'(x)\tilde{\Omega}_j'(x)dx+\int_{x_0}^{x_1}\Big(d_x(x)-\hat{d}_x(x)\Big)\tilde{\Omega}_i(x)\tilde{\Omega}_j(x)dx\end{aligned} \tag{14.a}$$

$$C_{ij}=\lambda_i^2\delta_{ij} \tag{14.b}$$

$$B_{ij}=\int_{x_0}^{x_1}w_x(x)\tilde{\Omega}_i(x)\tilde{\Omega}_j(x)dx \tag{14.d}$$

For an improved convergence of the eigenfunction expansion for the original potential, Eq. (6.b), it is of interest to include as much information as possible of the coefficients' spatial behavior in the eigenvalue problem, Eq. (5). This is particularly important when multiple spatial scales and/or very abrupt variations of the coefficients need to be handled. However, when dealing with the GITT solution of this eigenvalue problem with markedly variable spatial coefficients, it is not always possible to employ an auxiliary eigenvalue problem that incorporates even part of this information, as in Eq. (11), since it may not be solvable in analytic explicit form. Therefore, in many cases it is necessary to choose very simple non-informative auxiliary coefficients, $\hat{w}(\mathbf{x})$, $\hat{k}(\mathbf{x})$, and $\hat{d}(\mathbf{x})$, which may lead to slowly converging expansions for the original eigenfunctions, Eq. (10.a). In such cases, an integral balance procedure [25, 26] can be particularly beneficial in accelerating the convergence of such eigenfunction expansions by analytically rewriting Eq. (10.a), while explicitly accounting for the space variable coefficients local variation.

The integral balance procedure employed is a convergence acceleration technique [15, 25, 26] that is here aimed at obtaining eigenfunction expansions of improved convergence behavior for both the eigenfunction and its derivatives, through integration over the spatial domain, thus benefiting from the better convergence characteristics of the integrals of eigenfunction expansions. It consists of the double integration of the original equation that governs the potential for which the convergence improvement is being sought, in this case, the eigenvalue problem itself, Eq. (13.a) for the one-dimensional formulation to be illustrated. Through a single integration of the original equation, an improved expression for the eigenfunction derivative is obtained, and a second integration then offers an improved relation for computation of the eigenfunction itself. However, the problem boundary conditions need to be accounted for, so that the eigenfunctions and respective derivatives at the boundaries can be eliminated.

The first step is thus the integration of Eq. (13.a) with $\int_{x_0}^{x}(.)dx$ to find

$$\frac{dX(x)}{dx}=\frac{1}{k_x(x)}k_x(x_0)\frac{dX(x)}{dx}\bigg|_{x_0}-\frac{1}{k_x(x)}\gamma^2\, Iw_{x_0}(x)+\frac{1}{k_x(x)}\, Id_{x_0}(x) \tag{15.a}$$

where

$$Iw_{x_0}(x)=\int_{x_0}^{x}w_x(x')X(x')dx';\quad Id_{x_0}(x)=\int_{x_0}^{x}d_x(x')X(x')dx' \tag{15.b, c}$$

A second integration over Eq. (15.a) is then performed with $\int_{x}^{x_1}(.)dx$ to yield

$$X(x)=X(x_1)-k_x(x_0)\frac{dX(x)}{dx}\bigg|_{x_0}Ik_{x_1}(x)+\gamma^2\, Iwk_{x_1}(x)-\ Idk_{x_1}(x) \tag{16.a}$$

where

$$\begin{aligned}Ik_{x_1}(x)&=\int_{x}^{x_1}\frac{1}{k_x(x')}dx';\ Iwk_{x_1}(x)=\int_{x}^{x_1}\frac{1}{k_x(x')}Iw_{x_0}(x')dx';\\ Idk_{x_1}(x)&=\int_{x}^{x_1}\frac{1}{k_x(x')}Id_{x_0}(x')dx'\end{aligned} \tag{16.b − d}$$

A similar procedure could be implemented by starting from the integration of Eq. (13.a) with $\int_{x}^{x_1}(.)dx$ to find

$$\frac{dX(x)}{dx}=\frac{1}{k_x(x)}k_x(x_1)\frac{dX(x)}{dx}\bigg|_{x_1}+\frac{1}{k_x(x)}\gamma^2\, Iw_{x_1}(x)-\frac{1}{k_x(x)}\, Id_{x_1}(x) \tag{17.a}$$

where

$$Iw_{x_1}(x)=\int_{x}^{x_1}w_x(x')X(x')dx';\quad Id_{x_1}(x)=\int_{x}^{x_1}d_x(x')X(x')dx' \tag{17.b, c}$$

A second integration over Eq. (17.a) is now performed with $\int_{x_0}^{x}(.)dx$ to yield

$$X(x)=X(x_0)+k_x(x_1)\frac{dX(x)}{dx}\bigg|_{x_1}Ik_{x_0}(x)+\gamma^2\, Iwk_{x_0}(x)-\ Idk_{x_0}(x) \tag{18.a}$$

where

$$\begin{aligned}Ik_{x_0}(x)&=\int_{x_0}^{x}\frac{1}{k_x(x')}dx';\ Iwk_{x_0}(x)=\int_{x_0}^{x}\frac{1}{k_x(x')}Iw_{x_1}(x')dx';\\ Idk_{x_0}(x)&=\int_{x_0}^{x}\frac{1}{k_x(x')}Id_{x_1}(x')dx'\end{aligned} \tag{18.b − d}$$

Either Eq. (15.a) or (17.a) can be used as alternative expressions for the eigenfunction derivatives, while Eq. (16.a) or (18.a) can be adopted for computing the eigenfunctions, once the boundary conditions are employed to eliminate the values of the eigenfunctions and derivatives at the boundaries. Therefore, writing Eq. (15.a) for $x=x_1$ and Eq. (16.a) for $x=x_0$, the following two equations are obtained:

$$k_x(x_1)\frac{dX(x)}{dx}\bigg|_{x_1}=k_x(x_0)\frac{dX(x)}{dx}\bigg|_{x_0}-\gamma^2\, Iw_{x_0}(x_1)+Id_{x_0}(x_1) \tag{19.a}$$

$$X(x_0)=X(x_1)-k_x(x_0)\frac{dX(x)}{dx}\bigg|_{x_0}Ik_{x_1}(x_0)+\gamma^2\, Iwk_{x_1}(x_0)-\ Idk_{x_1}(x_0) \tag{19.a}$$

These two relations, together with the boundary conditions, Eq. (13.b,c), provide four equations for determination of the boundary quantities, $X(x_0)$, $X(x_1)$, $\left.\frac{dX(x)}{dx}\right|_{x_0}$, $\left.\frac{dX(x)}{dx}\right|_{x_1}$, which can be readily solved through symbolic computation in the more general situation of third kind boundary conditions, to yield

$$X(x_0) = -\frac{\beta_0[\beta_1(Id_{x0}(x_1) - \gamma^2 Iw_{x0}(x_1)) + \alpha_1(Idk_{x1}(x_0) - \gamma^2 Iwk_{x1}(x_0))]}{\alpha_0\beta_1 + \alpha_1(\beta_0 + \alpha_0 Ik_{x1}(x_0))} \tag{20.a}$$

$$X(x_1) = \frac{\beta_1\begin{bmatrix}\alpha_0 Idk_{x1}(x_0) - (\beta_0 + \alpha_0 Ik_{x1}(x_0)) \\ (Id_{x0}(x_1) - \gamma^2 Iw_{x0}(x_1)) - \gamma^2\alpha_0 Iwk_{x1}(x_0)\end{bmatrix}}{\alpha_0\beta_1 + \alpha_1(\beta_0 + \alpha_0 Ik_{x1}(x_0))} \tag{20.b}$$

$$\left.\frac{dX}{dx}\right|_{x_0} = -\frac{\alpha_0[\beta_1(Id_{x0}(x_1) - \gamma^2 Iw_{x0}(x_1)) + \alpha_1(Idk_{x1}(x_0) - \gamma^2 Iwk_{x1}(x_0))]}{[\alpha_0\beta_1 + \alpha_1(\beta_0 + \alpha_0 Ik_{x1}(x_0))]k_x(x_0)} \tag{20.c}$$

$$\left.\frac{dX}{dx}\right|_{x_1} = \frac{\alpha_1\begin{bmatrix}-\alpha_0 Idk_{x1}(x_0) + (\beta_0 + \alpha_0 Ik_{x1}(x_0)) \\ (Id_{x0}(x_1) - \gamma^2 Iw_{x0}(x_1)) + \gamma^2\alpha_0 Iwk_{x1}(x_0)\end{bmatrix}}{[\alpha_0\beta_1 + \alpha_1(\beta_0 + \alpha_0 Ik_{x1}(x_0))]k_x(x_1)} \tag{20.d}$$

Substituting Eq. (20) into Eqs. (15.a, 16.a) results in the following general expressions with enhanced convergence for the eigenfunctions and the corresponding derivatives, respectively:

$$\begin{aligned} X(x) = {} & -Idk_{x1}(x) + Iwk_{x1}(x)\gamma^2 + \\ & + \beta_1\left[\frac{\begin{array}{l}Idk_{x1}(x_0)\alpha_0 - Id_{x0}(x_1)(Ik_{x1}(x_0)\alpha_0 + \beta_0) \\ +\gamma^2(-Iwk_{x1}(x_0) + Ik_{x1}(x_0)Iw_{x0}(x_1)\alpha_0 + Iw_{x0}(x_1)\beta_0)\end{array}}{Ik_{x1}(x_0)\alpha_0\alpha_1 + \alpha_1\beta_0 + \alpha_0\beta_1}\right] + \\ & + \frac{Ik_{x1}(x)\alpha_0[Idk_{x1}(x_0)\alpha_1 - Iwk_{x1}(x_0)\gamma^2\alpha_1 + \beta_1(Id_{x0}(x_1) - Iw_{x0}(x_1)\gamma^2)]}{Ik_{x1}(x_0)\alpha_0\alpha_1 + \alpha_1\beta_0 + \alpha_0\beta_1} \end{aligned} \tag{21.a}$$

$$\begin{aligned} \frac{dX(x)}{dx} = \frac{1}{k(x)}\Bigg\{ & Id_{x0}(x) - Iw_{x0}(x)\gamma^2 + \\ & + \left[\frac{-Idk_{x1}(x_0)\alpha_1 - Id_{x0}(x_1)\beta_1 + \gamma^2(Iwk_{x1}(x_0)\alpha_1 + Iw_{x0}(x_1)\beta_1)}{Ik_{x1}(x_0)\alpha_0\alpha_1 + \alpha_1\beta_0 + \alpha_0\beta_1}\right]\Bigg\} \end{aligned} \tag{21.b}$$

which account for the local space variations of the equation and boundary condition coefficients. Alternatively, substituting Eq. (20) into Eqs. (17.a, 18.a) yields equivalent expressions. In the particular cases of either first or second kind boundary conditions, at one or both boundaries, the final expressions become even simpler.

The expressions provided by Eq. (21) can then be employed back into the solution of the eigenvalue problem (13), following the integral transformation procedure above described, yielding the algebraic eigenvalue which provides the eigenvalues γ^2 and the eigenvectors that can be readily substituted back in the inversion formula, Eq. (10.a). This procedure is further detailed in the next section for the two applications that illustrate this work.

4. Applications

In order to demonstrate the methodology here developed, two applications are selected. The chosen problems involve the analysis of two quite different situations. First, an example of variable coefficients with large but continuous scale changes is drawn from the literature related to the heat transfer analysis of FGM [3]. The other example is related to an abrupt variation of thermophysical properties, typical of the transition between two different materials layers, at different length scales, as in conjugated heat transfer problems in microchannels [5].

4.1. Heat conduction in FGMs

The related dimensionless energy equation with initial and boundary conditions for transient heat conduction across a slab of a FGM is written as [3]

$$w(x)\frac{\partial T(x,t)}{\partial t}=\frac{\partial}{\partial x}[k(x)\frac{\partial T(x,t)}{\partial x}],\quad 0<x<1,\quad t>0 \tag{22a}$$

with initial and boundary conditions

$$T(x,0)=f(x),\quad 0<x<1 \tag{22b}$$

$$T(0,t)=0,\quad T(1,t)=0,\quad t>0 \tag{22c, d}$$

where the thermophysical properties are assumed to vary exponentially in the form [3]

$$k(x)=k_0e^{2\beta x},\quad w(x)=w_0e^{2\beta x},\quad \alpha_0=\frac{k_0}{w_0}=const. \tag{23a - c}$$

This particular choice of functional forms leads to a problem formulation that allows for an exact solution via the classical integral transform method [27], yielding a benchmark solution for the variable coefficients case. Thus, after manipulating the coefficients within Eq. (22a), one finds

$$\frac{1}{\alpha_0}\frac{\partial T(x,t)}{\partial t}=\frac{\partial^2 T(x,t)}{\partial x^2}+2\beta\frac{\partial T(x,t)}{\partial x},\quad 0<x<1,\quad t>0 \tag{24}$$

In addition, a dependent variable transformation can recover the usual heat conduction equation form, as

$$T(x,t)=u(x,t)e^{-\beta(x+\beta\alpha_0 t)} \tag{25}$$

Then, the problem for $u(x,\ t)$ becomes

$$\frac{1}{\alpha_0}\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2},\quad 0<x<1,\quad t>0 \tag{26.a}$$

with initial and boundary conditions:

$$u(x,0)=f^*(x)=f(x)e^{\beta x},\quad 0<x<1 \tag{26.b}$$

$$u(0,t)=0,\quad u(1,t)=0,\quad t>0 \tag{26.c, d}$$

This first application was solved for extreme values of the parameter β, with the initial condition given by

$$f(x)=\frac{1-e^{2\beta(1-x)}}{1-e^{2\beta}} \tag{27}$$

which corresponds to the steady-state solution for the case of prescribed temperatures $T(0,\ t)=1$ and $T(1,\ t)=0$. Problem (26) can then be directly solved in analytical form for verification purposes, employing the classical integral transform method [27].

The preferred eigenvalue problem for the solution of problem (22) through the GITT is given by

$$\frac{d}{dx}\left(k(x)\frac{dX(x)}{dx}\right)+\gamma^2w(x)X(x)=0 \tag{28.a}$$

$$X(0)=0,\ X(1)=0 \tag{28.b, c}$$

Then, the expressions for the eigenfunctions and its derivative, as computed from the integral balance approach, given by Eq. (21.a,b) are simplified to

$$X(x) = Iwk_{x1}(x)\gamma^2 - \frac{Ik_{x1}(x)Iwk_{x1}(0)}{Ik_{x1}(0)}\gamma^2 \tag{29.a}$$

$$\frac{dX(x)}{dx} = \frac{\gamma^2}{k(x)}\left[-Iw_{x0}(x) + \frac{Iwk_{x1}(0)}{Ik_{x1}(0)}\right] \tag{29.b}$$

Substituting the corresponding expressions for $Iwk_{x1}(x)$ and $Iw_{x0}(x)$ and employing the inversion formula for the original eigenfunctions appearing on the right hand side of Eq. (29.a,b), one obtains

$$X_i(x) = \gamma_i^{\,2}\sum_n \bar{X}_{in}IB_n(x) - \gamma_i^{\,2}\frac{Ik_{x1}(x)}{Ik_{x1}(0)}\sum_n \bar{X}_{in}IB_n(0) \tag{30.a}$$

$$\frac{dX_i(x)}{dx} = -\frac{\gamma_i^{\,2}}{k(x)}\sum_n \bar{X}_{in}IA_n(x) + \frac{\gamma_i^{\,2}}{k(x)Ik_{x1}(0)}\sum_n \bar{X}_{in}IB_n(0) \tag{30.b}$$

with $IA_n(x)$ and $IB_n(x)$ given by

$$IA_n(x) = \int_0^x w(x')\tilde{\Omega}_n(x')dx' \tag{30.c}$$

$$IB_n(x) = \int_x^1 \frac{1}{k(x')}IA_n(x')dx' \tag{30.d}$$

where the eigenfunctions $\Omega(x)$ are derived from a simpler auxiliary eigenvalue problem with known solution. For this application we have chosen the simplest possible eigenvalue problem with constant unitary coefficients:

$$\frac{d^2\Omega(x)}{dx^2} + \lambda^2\Omega(x) = 0 \tag{31.a}$$

$$\Omega(0) = 0,\ \Omega(1) = 0 \tag{31.b, c}$$

with

$$\tilde{\Omega}_n(x) = \frac{\Omega_n(x)}{N^{1/2}{}_{\Omega n}},\ N_{\Omega n} = \int_0^1 \Omega^2{}_n(x)dX \tag{32.a, b}$$

The integral transformation of the eigenvalue problem (28) can be achieved by operating on Eq. (28.a) with $\int_0^1 (\cdot)\tilde{\Omega}_m(x)dx$ to obtain

$$\int_0^1 \frac{d}{dx}\left(k(x)\frac{dX_i(x)}{dx}\right)\tilde{\Omega}_m(x)dx + \gamma^2\int_0^1 w(x)X_i(x)\tilde{\Omega}_m(x)dx = 0 \tag{33}$$

employing integration by parts on the first term, and making use of the boundary conditions given by Eq. (28.b,c), it can be written more conveniently as

$$-\int_0^1 k(x)\frac{dX_i(x)}{dx}\frac{d\tilde{\Omega}_m(x)}{dx}dx + \gamma^2\int_0^1 w(x)X_i(x)\tilde{\Omega}_m(x)dx = 0 \tag{34}$$

and now substituting the expressions for the eigenfunctions and their derivatives with improved convergence, given by Eq. (30.a,b), into Eq. (34), and truncating the expansions to a finite order

M, results in the following algebraic eigenvalue problem:

$$\left(\mathbf{A}-\gamma^2\mathbf{B}\right)\bar{\mathbf{X}}=0 \tag{35.a}$$

where

$$\mathbf{A}=\mathbf{A}_1-\mathbf{A}_2,\ \mathbf{B}=\mathbf{B}_2-\mathbf{B}_1 \tag{35.b, c}$$

with the matrices coefficients given by

$$A_{1n,m}=\int_0^1 IA_n(x)\frac{d\tilde{\Omega}_m(x)}{dx}dx,\ A_{2n,m}=\frac{IB_n(0)}{Ik_{x1}(0)}\int_0^1\frac{d\tilde{\Omega}_m(x)}{dx}dx \tag{35.d, e}$$

$$B_{1n,m}=\int_0^1 w(x)IB_n(x)\tilde{\Omega}_m(x)dx,\ B_{2n,m}=\frac{IB_n(0)}{Ik_{x1}(0)}\int_0^1 Ik_{x1}(x)w(x)\tilde{\Omega}_m(x)dx \tag{35.f, g}$$

The algebraic problem (35) can be numerically solved to provide results for the eigenvalues γ^2 and eigenvectors $\bar{X}_{in}$, upon truncation to a sufficiently large finite order M, and then employed in Eq. (30.a,b) to provide the desired eigenfunctions and their derivatives with improved convergence behavior. Once these eigenfunctions $X_i(x)$ corresponding to eigenvalue problem (28) are made available, problem (22) becomes completely transformable and the solution for $T(x, t)$ becomes straightforward [3].

4.2. Conjugated heat transfer with single domain formulation

The second example is related to conjugated heat transfer in parallel plate microchannels, when the substrate material participates in the heat transfer process significantly [5], as schematically shown in Figure 1. We assume that the flow is dynamically developed and thermally developing. The formulation of the conjugated problem as a single region model is proposed, accounting for heat transfer phenomena at both the fluid flow and the channel solid wall, by making use of coefficients represented as space variable functions where abrupt transitions occur at the fluid–solid wall interface. The conjugated problem is then given in dimensionless form by the following single domain formulation with space variable coefficients [5]:

$$U(Y)\frac{\partial\theta(Y,Z)}{\partial Z}=\frac{\partial}{\partial Y}\left(K(Y)\frac{\partial\theta}{\partial Y}\right), 0<Y<1, Z>0 \tag{36.a}$$

$$\theta(Y,Z=0)=\theta_{in} \tag{36.b}$$

$$\left.\frac{\partial\theta}{\partial Y}\right|_{Y=0}=0,\ \theta(Y=1,Z)=\theta_w \tag{36.c, d}$$

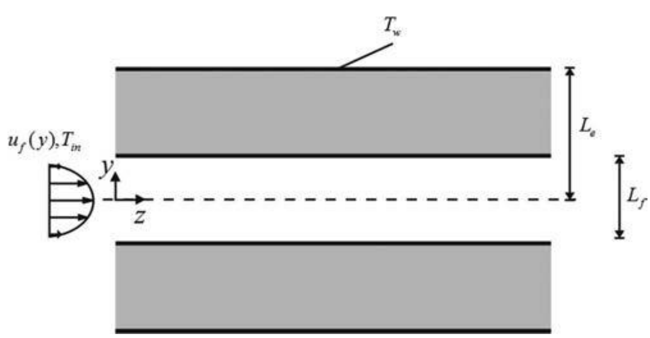

Figure 1. Geometry and coordinates system for conjugated heat transfer in a parallel plates microchannel [5].

where θ_{in} and θ_w are the dimensionless temperatures at the channel inlet (fluid and wall) and at the external face of the channel wall, respectively. The dimensionless space variable coefficients are given by

$$U(Y) = \begin{cases} U_f(Y), & \text{if } 0 < Y < Y_i = y_i/y_w \\ 0, & \text{if } Y_i < Y < 1 \end{cases} \tag{36.e}$$

$$K(Y) = \begin{cases} 1, & \text{if } 0 < Y < Y_i = y_i/y_w \\ k_s/k_f, & \text{if } Y_i < Y < 1 \end{cases} \tag{36.f}$$

Problem (36) has also been selected for illustration of the methodology, since a straightforward exact solution can be readily obtained. For this purpose, the heat transfer problem is then modeled as an internal convective problem for the fluid, coupled at the interface $Y = Y_i$ with the conduction problem for the solid wall. Thus, the problem for the fluid flow region becomes a Graetz type problem with third kind boundary condition:

$$U_f(Y)\frac{\partial\theta_f(Y,Z)}{\partial Z} = \frac{\partial^2\theta_f}{\partial Y^2}, 0 < Y < Y_i, Z > 0 \tag{37.a}$$

$$\theta_f(Y, Z = 0) = 0 \tag{37.b}$$

$$\left.\frac{\partial\theta_f}{\partial Y}\right|_{Y=0} = 0, \qquad \left.\frac{\partial\theta_f}{\partial Y}\right|_{Y=Y_i} + \frac{k_s/k_f}{1-Y_i}\theta_f(Y_i, Z) = \frac{k_s/k_f}{1-Y_i} \tag{37.c, d}$$

Problem (37) has an exact analytical solution readily obtainable by the Classical Integral Transform Technique [27] and then the channel wall region temperature distribution can be directly obtained from its linear profile across the transversal coordinate. The exact solution for the fluid flow region is obtained from the solution of the following eigenvalue problem, formulated by directly applying separation of variables to the homogeneous version of problem (37):

$$\frac{d^2\phi(Y)}{dY^2} + U_f(Y)\gamma^2\phi(Y) = 0 \tag{38.a}$$

$$\left.\frac{d\phi}{dY}\right|_{Y=0} = 0, \left.\frac{d\phi}{dY}\right|_{Y=Y_i} + \frac{k_s/k_f}{1-Y_i}\phi(Y_i) = 0 \tag{38.b, c}$$

which allows for an analytical solution in terms of hypergeometric functions that can be readily obtained using the routine DSolve of the *Mathematica* platform [24]. These results are used as a benchmark solution for the comparisons that follow.

Handling problem (36) through GITT, as previously described in the general solution methodology, yields the following preferred choice of eigenvalue problem:

$$\frac{d}{dx}\left(k(x)\frac{dX(x)}{dx}\right) + \gamma^2 w(x)X(x) = 0 \tag{39.a}$$

$$\left.\frac{dX}{dx}\right|_{x=0} = 0, \; X(1) = 0 \tag{39.b, c}$$

where the variable x in Eqs. (39.a–c) above corresponds to the transversal direction, Y, in problem (36). For this application, the expressions for the eigenfunction and its derivative, as computed from the integral balance scheme given by Eq. (21.a,b), are simplified to

$$X(x) = Iwk_{x1}(x)\gamma^2 \tag{40.a}$$

$$\frac{dX(x)}{dx} = -\frac{\gamma^2}{k(x)}Iw_{x0}(x) \tag{40.b}$$

Substituting the corresponding expressions for $Iwk_{x1}(x)$ and $Iw_{x0}(x)$ and employing the inversion formula for the original eigenfunctions appearing on the rhs of Eq. (40.a, b), one obtains

$$X_i(x) = \gamma_i^2 \sum_n \bar{X}_{in} IB_n(x) \tag{41.a}$$

$$\frac{dX_i(x)}{dx} = -\frac{\gamma_i^2}{k(x)}\sum_n \bar{X}_{in} IA_n(x) \tag{41.b}$$

with $IA_n(x)$ and $IB_n(x)$ already defined in Eq. (30c, d), with the eigenfunctions $\Omega(x)$ calculated from a simpler auxiliary eigenvalue problem, which in this second illustration is again chosen as the simplest possible one:

$$\frac{d^2\Omega(x)}{dx^2} + \lambda^2\Omega(x) = 0 \tag{42.a}$$

$$\left.\frac{d\Omega(x)}{dx}\right|_{x=0} = 0,\ \Omega(1) = 0 \tag{42.b, c}$$

with

$$\tilde{\Omega}_n(x) = \frac{\Omega_n(x)}{N_{\Omega n}^{1/2}},\ N_{\Omega n} = \int_0^1 \Omega^2{}_n(x)dX \tag{42.d, e}$$

The integral transformation of the eigenvalue problem (39) can be achieved by operating on Eq. (39a) with $\int_0^1 (\cdot)\tilde{\Omega}_m(x)dx$ to obtain, after integrating by parts the first term:

$$-\int_0^1 k(x)\frac{dX_i(x)}{dx}\frac{d\tilde{\Omega}_m(x)}{dx}dx + \gamma^2\int_0^1 w(x)X_i(x)\tilde{\Omega}_m(x)dx = 0 \tag{43}$$

and now substituting the expressions for the eigenfunctions and their derivatives with improved convergence, given by Eq. (41.a,b), into Eq. (43), and truncating the expansions to a finite order M, yields the following algebraic eigenvalue problem:

$$\left(\mathbf{A} - \gamma^2\mathbf{B}\right)\bar{\mathbf{X}} = 0 \tag{44.a}$$

where

$$\mathbf{A} = \mathbf{A}_1,\ \ \mathbf{B} = -\mathbf{B}_1 \tag{44.b, c}$$

with the matrices coefficients given by Eqs. (35.d) and (35.f). The algebraic problem (44) can be numerically solved to provide results for the eigenvalues γ^2 and eigenvectors $\bar{X}_{in}$, upon truncation to a sufficiently large finite order M, and then employed in Eq. (41.a,b) to provide the desired eigenfunctions and their derivatives with improved convergence behavior. Once the eigenfunctions $X_i(x)$ corresponding to eigenvalue problem (39) are made available, problem (35) becomes completely transformable and the solution becomes straightforward [5].

5. Results and discussion

5.1. Heat conduction in FGMs

The application related to transient heat conduction across a FGM layer has been previously solved through GITT in [3], for a range of the parameter β within [−3, 3]. At the highest value in this interval, $\beta = 3$, the thermal conductivities at the two boundaries, $k(1)$ and $k(0)$, experience a ratio of about 400. Here, a higher value of $\beta = 4$ is considered, which leads to a ratio $k(1)/k(0)$ of approximately 3000, as can be seen in Figure 2. Therefore, FGM thermal conductivity would, for instance, vary from a dimensional thermal conductivity of 0.1 W/m°C at $x = 0$, typical of insulating materials, up to a thermal conductivity of 300 W/m°C at $x = 1$, typical of some highly conductive metallic materials.

Besides the exact solution from the GITT solution using the traditional inverse formula for the eigenfunctions, the improved eigenfunction expansions obtained from the integral balance approach, as detailed in Section 4.1, were also computed for different truncation orders in both the eigenvalue

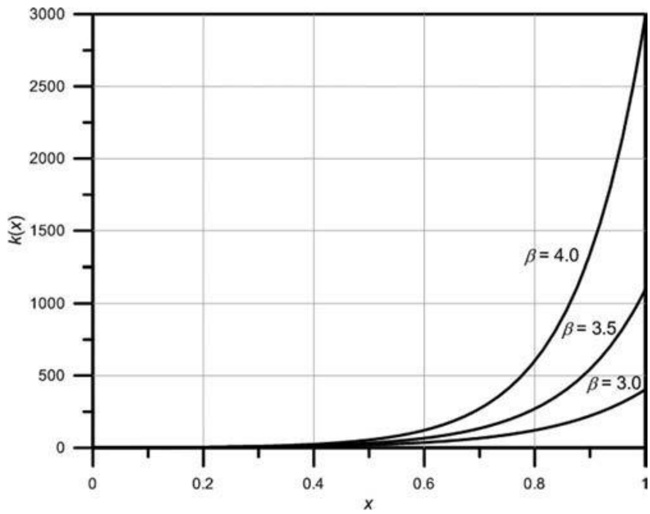

Figure 2. Behavior of FGM dimensionless thermal conductivity [3].

problem (M) and on the temperature expansion (N). In order to illustrate the improved behavior of the proposed eigenfunction expansions, we first present in Tables 1–3 the convergence of the dimensionless temperature at the position $x = 0.1$ and time $t = 0.01$, in the vicinity of the temperature maximum, for increasing values of the truncation order in the temperature expansion, from $N = 5$ to 40, which is sufficient to yield full convergence to the six significant digits here presented. Tables 1–3 correspond to increasing values of the eigenvalue problem truncation order, respectively, $M = 40$, 60, and 80. Besides the exact solution and the two alternative GITT solutions, through the usual inversion formulae and via the integral balance improvement, the relative deviations with respect to the exact solution are also presented. First, it can be verified that all three solutions are fully converged to six significant digits for the range of the temperature expansions truncation orders, $N < 40$. It can be observed that improvement in accuracy is provided by the increase in the eigenvalue problem truncation order (M), which reduces the error in the first 40 eigenvalues and eigenfunctions that are actually employed in the temperature convergence analysis here undertaken. The relative error in the traditional inversion formula drops from about 0.06% at $M = 40$ to about 0.0075% at $M = 80$, while the integral balance approach drops from 0.0007 to 0% relative error for the same values of M. However, more impressive is the improvement allowed for by the integral balance approach in comparison to the plain inverse formula expansion. In general, two additional significant digits of precision are achieved with the use of the approach here proposed, with respect to the inverse formula, which is also fully analytical and readily applicable as here illustrated. This improved convergence behavior is even more evident in the calculation of dimensionless heat fluxes, or temperature derivatives, as shown in Table 4, for the same position and time and increasing truncation orders in the derivative

Table 1. Comparison of the exact solution, GITT via inverse formula, and GITT via integral balance for temperature convergence behavior at $x = 0.1$ and $t = 0.01$, with the eigenvalue problem truncation order of $M = 40$ (heat conduction in FGM).

N	Exact	GITT Inv. Form.	Rel. Error Inv. Form. %—	GITT Int. Balance	Rel. Error Int. Balance %—
5	0.360318	0.360152	−0.0460	0.360316	−0.000555
10	0.456293	0.456017	−0.0604	0.456288	−0.001096
15	0.435866	0.435613	−0.0582	0.435863	−0.000688
20	0.431986	0.431738	−0.0574	0.431983	−0.000694
25	0.432331	0.432078	−0.0586	0.432326	−0.001157
30	0.432365	0.432111	−0.0587	0.432360	−0.001156
35	0.432363	0.432110	−0.0587	0.432360	−0.000694
40	0.432363	0.432110	−0.0587	0.432360	−0.000694

Table 2. Comparison of the exact solution, GITT via inverse formula, and GITT via integral balance for temperature convergence behavior at $x = 0.1$ and $t = 0.01$, with the eigenvalue problem truncation order of $M = 60$ (heat conduction in FGM).

N	Exact	GITT Inv. Form.	Rel. Error Inv. Form. %—	GITT Int. Balance	Rel. Error Int. Balance %—
5	0.360318	0.360269	−0.0136	0.360318	0.0
10	0.456293	0.456211	−0.0179	0.456292	−0.000219
15	0.435866	0.435789	−0.0177	0.435866	0.0
20	0.431986	0.431910	−0.0176	0.431986	0.0
25	0.432331	0.432254	−0.0178	0.432330	−0.000231
30	0.432365	0.432287	−0.0178	0.432364	−0.000231
35	0.432363	0.432286	−0.0178	0.432363	0.0
40	0.432363	0.432286	−0.0178	0.432363	0.0

Table 3. Comparison of the exact solution, GITT via inverse formula, and GITT via integral balance for temperature convergence behavior at $x = 0.1$ and $t = 0.01$, with the eigenvalue problem truncation order of $M = 80$ (heat conduction in FGM).

N	Exact	GITT Inv. Form.	Rel. Error Inv. Form. %—	GITT Int. Balance	Rel. Error Int. Balance %—
5	0.360318	0.360297	−0.00572	0.360318	0.0
10	0.456293	0.456259	−0.00750	0.456293	0.0
15	0.435866	0.435834	−0.00749	0.435866	0.0
20	0.431986	0.431954	−0.00745	0.431986	0.0
25	0.432331	0.432298	−0.00754	0.432331	0.0
30	0.432365	0.432332	−0.00755	0.432364	−0.000231
35	0.432363	0.432331	−0.00755	0.432363	0.0
40	0.432363	0.432331	−0.00755	0.432363	0.0

expansions with a fixed value of $M = 80$. As usual, the derivative expressions are in general slower in convergence than the original potentials, but through the integral balance approach we can observe that at least four significant digits of agreement with the exact solution could be achieved at this truncation order. One should note that the relative error obtained for the derivative employing the integral balance scheme is two orders of magnitude lower than that obtained through the traditional eigenfunction expansion approach. Finally, Table 5 provides an evaluation of the convergence behavior of the temperature distribution by the two approaches throughout the spatial domain, from $x = 0.01$ to 0.9, for fixed truncation orders of $N = 40$ and $M = 60$. Clearly, the integral balance procedure provides final results with four to five significant digits already fully converged and coincident with the exact solution.

5.2. Conjugated heat transfer with single domain formulation

This example comes from the original work that introduced the methodology of combining the single domain reformulation with GITT in solving conjugated problems [5]. The application is motivated by a channel of polyester resin ($k_s = 0.16\,\mathrm{W}/mK$) with water as the working fluid ($k_f = 0.64\,\mathrm{W}/mK$), and it is considered that the channel wall thickness is half of the channel height, so that the interface occurs at $Y_i = 0.5$.

Table 6 shows the dimensionless temperatures at different transversal positions in the fluid flow region at $Z = 0.01$, calculated by solving the eigenvalue problem with space variable coefficients with the traditional inverse formula, as presented in ref [5]., in comparison with the solution employing the integral balance expressions presented in this work. Both cases employed $N = 5$ terms in the temperature expansion, which is enough to achieve convergence of the five significant digits shown. The traditional solution scheme employed $M = 50$ terms in the eigenvalue problem solution, only $M = 5$ terms for the integral balance scheme. The results are quite impressive, demonstrating that with a much lower truncation order in the eigenfunction expansion, the integral balance scheme achieved a much more accurate solution, with four to five digits of agreement with the exact solution provided.

In order to provide more challenging examples, we considered two cases in which the channel wall thickness was 1:1000 with respect to channel height, and accurate local temperatures at both the fluid

Table 4. Comparison of the exact solution, GITT via inverse formula, and GITT via integral balance for temperature derivative convergence behavior at $x = 0.1$ and $t = 0.01$, with the eigenvalue problem truncation order of $M = 80$ (heat conduction in FGM).

N	Exact	GITT Inv. Form.	Rel. Error Inv. Form. %—	GITT Int. Balance	Rel. Error Int. Balance %—
5	0.680294	0.678029	−0.333	0.680289	−0.000735
10	−2.03037	−2.03430	0.193	−2.030400	0.00148
15	−2.78191	−2.78643	0.162	−2.781950	0.00144
20	−2.58533	−2.59013	0.186	−2.585390	0.00232
25	−2.55610	−2.56094	0.189	−2.556160	0.00235
30	−2.55829	−2.56313	0.189	−2.558350	0.00234
35	−2.55844	−2.56329	0.189	−2.558500	0.00234
40	−2.55844	−2.56328	0.189	−2.558500	0.00234

Table 5. Comparison of the exact solution, GITT via inverse formula, and GITT via integral balance for temperature along coordinate x and $t = 0.01$, with temperature expansion with $N = 40$ and the eigenvalue problem truncation order of $M = 60$ (heat conduction in FGM).

x	Exact	GITT Inv. Form.	GITT Int. Balance
0.01	0.134336	0.133847	0.134327
0.1	0.432363	0.432286	0.432363
0.2	0.201625	0.201607	0.201625
0.3	0.0904128	0.0904049	0.0904127
0.4	0.0404403	0.0404360	0.0404402
0.5	0.0179862	0.0179835	0.0179862
0.6	0.00789693	0.00789506	0.0078969
0.7	0.00336353	0.00336225	0.00336351
0.8	0.00132654	0.00132573	0.00132653
0.9	0.000411261	0.000410878	0.000411252

Table 6. Comparison of the exact solution, GITT via inverse formula with $M = 50$, and GITT via integral balance with $M = 5$ for dimensionless temperatures at different transversal positions at the fluid flow region, $Z = 0.01$ (conjugated heat transfer).

Y	Exact sol.	GITT	Relative error GITT %—	GITT Int. Balance	Relative error Int. Balance %—
0.00	0.010413	0.010422	0.086	0.010413	0.0
0.10	0.015230	0.015246	0.11	0.015230	0.0
0.15	0.021439	0.021465	0.12	0.021439	0.0
0.20	0.030396	0.030435	0.13	0.030396	0.0
0.25	0.042192	0.042249	0.13	0.042192	0.0
0.30	0.056776	0.056854	0.14	0.056777	0.0
0.35	0.073900	0.074001	0.14	0.073901	0.0
0.40	0.093122	0.093249	0.14	0.093123	0.0
0.45	0.11384	0.11399	0.13	0.11384	0.0
0.50	0.13534	0.13605	0.53	0.13534	0.0

Table 7. Convergence of temperatures at the interface between the fluid flow region and the channel wall at $Z = 0.05$ (water flow in the polyester resin channel).

M	GITT	Rel. Error* GITT %—	GITT Int. Balance	Rel. Error* Int. Balance %—
5	0.04398	68.9	0.14133	0.100
10	0.05200	63.2	0.14144	0.018
15	0.06208	56.1	0.14146	0.005
20	0.07392	47.8	0.14147	0.0
40	0.11748	17.0		
60	0.12657	10.5		
80	0.12821	9.4		
100	0.13266	6.2		
120	0.13337	5.7		

*Calculated with respect to the exact solution: 0.14147.

Table 8. Comparison of the exact solution, GITT via inverse formula with $M = 120$, and GITT via integral balance with $M = 20$ for dimensionless temperatures at different transversal positions at the fluid flow region, $Z = 0.05$ (water flow in the polyester resin channel).

y	Exact	GITT	Rel. Error GITT %—	GITT Int. Balance	Rel. Error Int. Balance %—
0.1	0.99561	0.99555	0.006	0.99561	0.0
0.2	0.98775	0.98760	0.015	0.98775	0.0
0.3	0.96852	0.96816	0.037	0.96852	0.0
0.4	0.92938	0.92863	0.081	0.92938	0.0
0.5	0.86123	0.85987	0.159	0.86123	0.0
0.6	0.75857	0.75637	0.289	0.75857	0.0
0.7	0.62266	0.61949	0.509	0.62266	0.0
0.8	0.46149	0.45730	0.907	0.46149	0.0
0.9	0.28613	0.28094	1.81	0.28613	0.0
0.99	0.07215	0.07375	2.22	0.07215	0.0

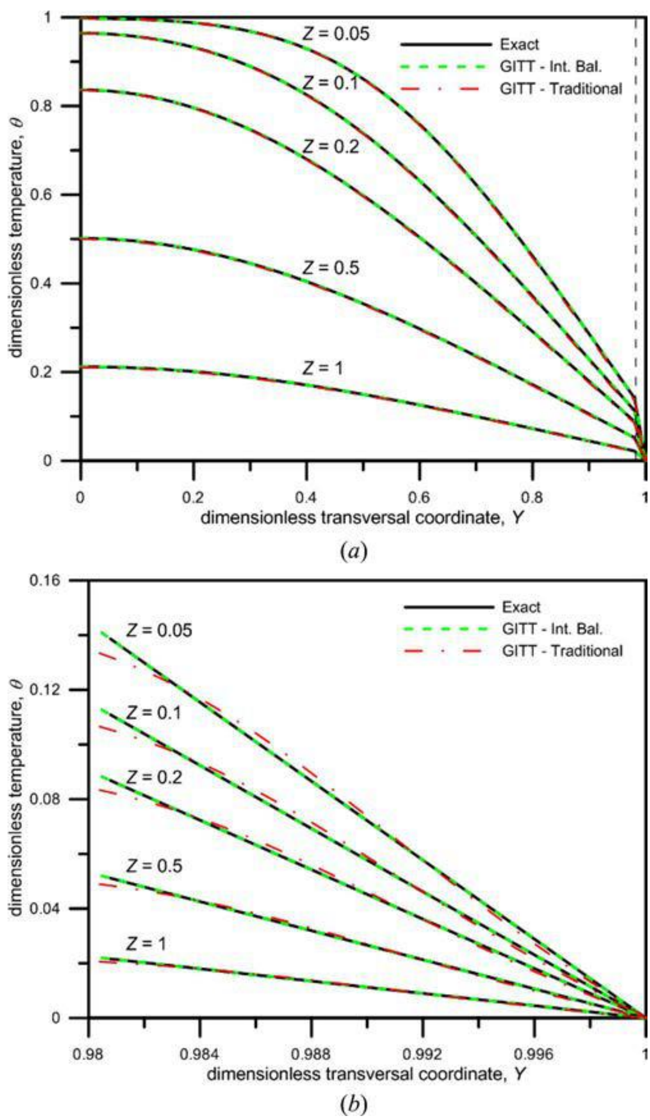

Figure 3. Comparison among transversal temperature profiles at different longitudinal positions obtained with the three solutions (exact, traditional scheme, and integral balance scheme) for conjugated heat transfer of water flow in polyester resin channel substrate: (*a*) full solid–fluid domain; (*b*) wall region.

Table 9. Convergence of temperatures at the interface between the fluid flow region and the channel wall at $Z = 0.05$ (air flow in acrylic channel).

M	GITT	Rel. Error* GITT %—	GITT Int. Balance	Rel. Error* Int. Balance %—
5	0.01835	245.3	0.0053170	0.041
10	0.01183	122.6	0.0053151	0.006
15	0.00904	70.1	0.0053149	0.002
20	0.00766	44.2	0.0053148	0.0
40	0.00670	26.1		
60	0.00623	17.2		
80	0.00596	12.2		
100	0.00590	11.0		
120	0.00575	8.1		

*Calculated with respect to the exact solution: 0.0053148.

and channel wall were sought. In the first case we considered the flow of water in a polyester resin channel, yielding $k_s/k_f = 0.25$. The second case involved the flow of air ($k_f = 0.0271\,\mathrm{W}/mK$) in an acrylic channel ($k_s = 0.2\,\mathrm{W}/mK$), yielding $k_s/k_f = 7.38$.

For the case of water flow in the polyester resin channel, Table 7 presents the calculated temperatures at the interface between the fluid flow region and the channel wall, typically the most critical position for convergence in the single domain formulation, at $Z = 0.05$. The results presented are calculated with $N = 5$ terms in the temperature expansion, which is enough to achieve convergence of the five significant digits shown. The results are presented for increasing truncation orders M in the eigenfunction expansion, and the good convergence behavior achieved by the integral balance scheme is remarkable, yielding a fully converged result of five significant digits, in full agreement with the exact solution, with only $M = 20$ terms, whereas the solution obtained through the traditional inverse formula is slowly converging, clearly not yet fully converged to within $M = 120$ terms.

Table 8 summarizes the results for different transversal positions at $Z = 0.05$, obtained with $M = 20$ terms in the integral balance scheme and $M = 120$ terms in the traditional scheme. The results confirm the good convergence behavior provided by the integral balance scheme at all positions. The traditional scheme provides reasonably good results at the fluid region, but the convergence is noticeably affected for increasing Y values, as the interface is approached. This conclusion is also reached by observing Figures 3*a, b*, which present the comparison among the transversal temperature profiles at different longitudinal positions obtained with the three solutions (exact, traditional scheme, and integral balance scheme). In Figure 3*a* the whole domain is presented, while in Figure 3*b* only the thin solid wall region is presented. It is clearly seen that the exact solution and the GITT with integral balance scheme are fully coincident with the graph scale, while the traditional GITT solution still shows some deviation from the exact solution at the wall region.

Finally, the results of the last case (air flow in acrylic channel) are presented in Tables 9 and 10, and in Figures 4*a, b*. It is observed that the results obtained with the traditional scheme present a worse convergence behavior in comparison with the previous case, for the same truncation orders. This is

Table 10. Comparison of the exact solution, GITT via inverse formula with$M = 120$, and GITT via integral balance with $M = 20$ for dimensionless temperatures at different transversal positions at the fluid flow region, $Z = 0.05$ (air flow in acrylic channel).

Y	Exact	GITT	Rel. Error GITT %—	GITT Int. Balance	Rel. Error Int. Balance %—
0.1	0.99405	0.99402	0.003	0.99405	0.0
0.2	0.98376	0.98367	0.009	0.98376	0.0
0.3	0.95912	0.95892	0.021	0.95912	0.0
0.4	0.91013	0.90971	0.046	0.91013	0.0
0.5	0.82689	0.82616	0.089	0.82689	0.0
0.6	0.70449	0.70336	0.160	0.70449	0.0
0.7	0.54622	0.54466	0.286	0.54622	0.0
0.8	0.36249	0.36052	0.543	0.36249	0.0
0.9	0.16597	0.16366	1.39	0.16597	0.0
0.99	0.0027105	0.00285	5.21	0.0027105	0.0

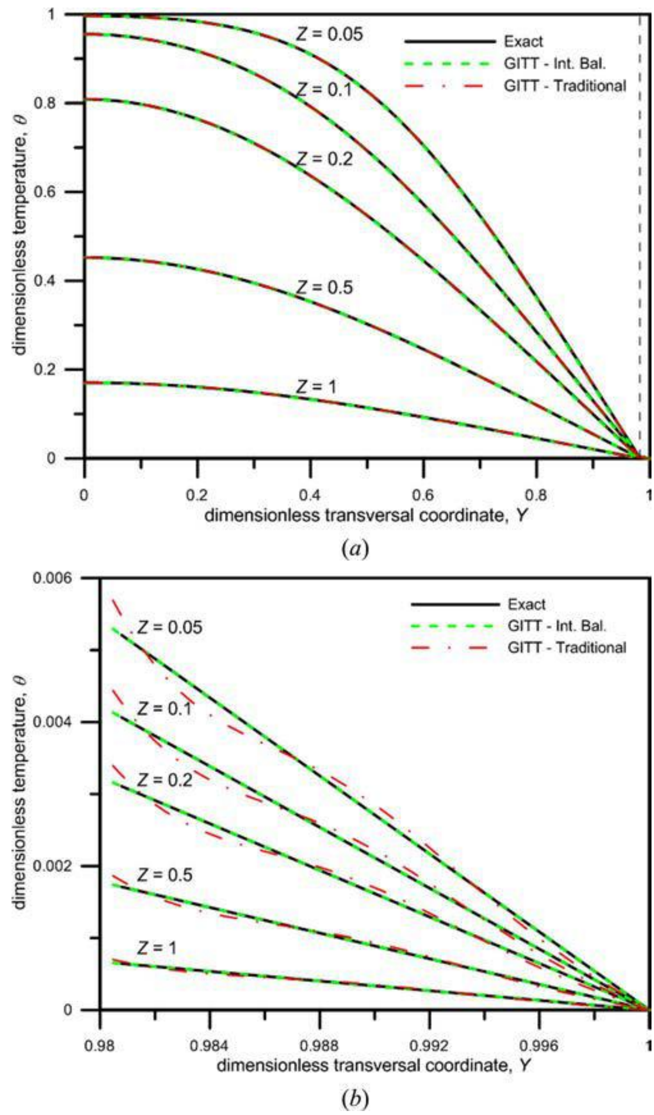

Figure 4. Comparison among the transversal temperature profiles at different longitudinal positions obtained with the three solutions (exact, traditional scheme, and integral balance scheme) for conjugated heat transfer of air flow in acrylic channel substrate: (*a*) full solid–fluid domain; (*b*) wall region.

probably due to the larger variation in the abrupt transition of the thermal conductivity (4:1 in the previous case and 1:7.38 in the current case). On the other hand, the results obtained with the integral balance scheme remain in full agreement with the exact solution, demonstrating the robustness of the approach in dealing with abrupt transitions and multiscale problems.

6. Conclusions

The integral balance approach, as applied to Sturm–Liouville problems, has been derived and employed in generating improved eigenfunction expressions for the integral transform solution of diffusion and convection-diffusion problems with space variable coefficients. The derived expressions are then used in the integral transformation of the eigenvalue problem with space variable coefficients, generating the matrix coefficients in the algebraic eigenvalue problem, thus also improving the accuracy of eigenvalue computation. The proposed approach is aimed at improving the accuracy and convergence rates of eigenfunction expansions when multiscale variations of thermophysical properties and/or geometrical dimensions are present in the problem formulation. The derived expressions explicitly incorporate the space variable coefficients in the final eigenfunction relations, thus providing improved results for the local variations of the eigenfunctions.

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