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Nonlinear eigenvalue problem in the integral transforms solution of convection-diffusion with nonlinear boundary conditions

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Abstract

Purpose – The purpose of this paper is to propose the generalized integral transform technique (GITT) to the solution of convection-diffusion problems with nonlinear boundary conditions by employing the corresponding nonlinear eigenvalue problem in the construction of the expansion basis.
Design/methodology/approach – The original nonlinear boundary condition coefficients in the problem formulation are all incorporated into the adopted eigenvalue problem, which may be itself integral transformed through a representative linear auxiliary problem, yielding a nonlinear algebraic eigenvalue problem for the associated eigenvalues and eigenvectors, to be solved along with the transformed ordinary differential system. The nonlinear eigenvalues computation may also be accomplished by rewriting the corresponding transcendental equation as an ordinary differential system for the eigenvalues, which is then simultaneously solved with the transformed potentials.
Findings – An application on one-dimensional transient diffusion with nonlinear boundary condition coefficients is selected for illustrating some important computational aspects and the convergence behavior of the proposed eigenfunction expansions. For comparison purposes, an alternative solution with a linear eigenvalue problem basis is also presented and implemented.
Originality/value – This novel approach can be further extended to various classes of nonlinear convection-diffusion problems, either already solved by the GITT with a linear coefficients basis, or new challenging applications with more involved nonlinearities.

Keywords Diffusion, Hybrid methods, Integral transforms, Eigenvalue problem, Nonlinear boundary conditions, Nonlinear problems

Paper type Research paper

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Nomenclature

Bi_c	Characteristic convective Biot number, Equation (29e)	$T(\mathbf{x}, t)$	Potential
Bi_r	Characteristic radiative Biot number, Equation (29e)	$T_F(\mathbf{x}, t)$	Filtering solution, Equation (2).
$Bi(T(1, t))$	Nonlinear combined Biot number, Equation (29e)	t	Time variable
$d(\mathbf{x})$	Dissipation operator coefficient, Equations (1a) and (13a)	x	Space variable
$f(\mathbf{x})$	Initial condition, Equations (1a) and (13a)	$\mathbf{x}$	Position vector
$k(\mathbf{x})$	Diffusion operator coefficient, Equations (1a) and (13a)	$w(\mathbf{x})$	Transient operator coefficient, Equations (1a) and (13a)
M	Truncation order of the algebraic eigenvalue problem	α, β	Coefficients for the boundary condition, Equations (1c) and (13c)
N_i	Normalization integral of the eigenvalue problem, Equations (7) and (24)	γ	Parameter in radiative Biot number, Equation (29e)
$P(\mathbf{x}, t, T)$	Nonlinear source term appearing in Equations (1a) and (13a)	μ_i	Eigenvalues of problems (5) and (17)
		ψ_i	Eigenfunctions of problems (5) and (17)
		$\phi(\mathbf{x}, t, T)$	Nonlinear source term appearing in Equations (1c) and (13c)

1. Introduction

Nonlinear diffusion and convection-diffusion problems provide the mathematical formulations of major interest for most applied research in transport phenomena. The numerical solution of nonlinear problems in the realm of discrete approaches is widely documented, and systematically compiled in different sources, for instance (Minkowycz *et al.*, 2006), through the use of well-established computational approaches such as the best known finite differences, finite elements and finite volumes methods. Nevertheless, it has always been desirable to obtain analytical solutions, even if just approximate, for such a wide class of problems, either for verification of purely numerical solutions or for the development of more robust, precise and cost effective computational approaches. Hybrid numerical-analytical computational approaches for nonlinear partial differential equations not only have been calling more attention in recent years, for the reasons discussed above, but have also been pushed forward in dealing with computationally intensive tasks, such as optimization, inverse problem analysis and simulation under uncertainty. Then, the analytical framework can be particularly beneficial in reducing overall computational costs, allowing in addition for the direct mathematical manipulation of the deduced expressions, even if part of the information for constructing the final solution is still obtained by numerical algorithms. In this sense, the combined use of hybrid approaches with modern symbolic computation platforms can be particularly symbiotic.

In this context, the hybrid numerical-analytical solution of nonlinear diffusion problems through integral transforms has been proposed back in 1990 (Cotta, 1990), by extending the ideas in the so called generalized integral transform technique (GITT), as reviewed and compiled in different sources since then (Cotta, 1990, 1993, 1994, 1998; Cotta and Mikhailov, 1997, 2006). The main idea behind the application of the GITT to nonlinear problems (Cotta, 1990), afterwards progressively extended to various classes of problems with nonlinear coefficients including the boundary layer and

Navier-Stokes equations (Serfaty and Cotta, 1990, 1992; Cotta and Serfaty, 1991; Leiroz and Cotta, 1993; Ribeiro and Cotta, 1995; Cotta and Ramos, 1998; Machado and Cotta, 1999; Leal *et al.*, 2000; Macedo *et al.*, 2000; Alves *et al.*, 2001; Cotta *et al.*, 2007; Pontedeiro *et al.*, 2007), is to first of all rewrite the problem formulation, grouping all of the nonlinear information on the equation and boundary conditions operators, into the corresponding source terms within the domain or at the boundary surfaces. Then, the problem is reinterpreted as one of linear differential operators but with nonlinear sources, which naturally leads to a choice of basis for the eigenfunction expansions through the characteristic linear coefficients that were adopted to reformulate the problem. Once the eigenvalue problem with linear coefficients is solved for, the integral transformation procedure is analytically implemented to yield a coupled system of nonlinear ordinary differential equations (ODEs) for the transformed potentials, which is either an initial value problem for parabolic and parabolic-hyperbolic formulations, or a boundary value problem for elliptic formulations. The transformed system, in the case of a general nonlinear situation, is then numerically solved for the transformed potentials, and the inverse formula is then recalled to complete the hybrid solution, which remains analytical in all but one independent variable. Therefore, the major numerical task in the solution, which consists of solving the ODE system for the transformed potentials, is readily available in commercial or public libraries, in most cases providing error controlled solutions and interpolated final results. This approach has been proved very successful along the years, and has led to the establishment of a general purpose algorithm made available in an open source code implemented with symbolic-numerical computation resources, the Unified Integral Transforms (UNIT) algorithm (Cotta *et al.*, 2010, 2013, 2014; Sphaier *et al.*, 2011).

The strategy of incorporating the problem nonlinearities within the source terms may eventually require convergence enhancement techniques (Scofano Neto *et al.*, 1990; Almeida and Cotta, 1996; Gondim *et al.*, 2007), such as the use of filtering solutions to reduce the importance of these nonhomogeneities and/or the employment of a posteriori integral balances so as to analytically improve the eigenfunction expansions convergence rates. This aspect has been observed to be particularly relevant when dealing with nonlinear boundary conditions (Ribeiro and Cotta, 1993; Mikhailov and Cotta, 1998), when the presence of a nonlinear source term, even after filtering, may still promote a slower convergence behavior on the eigenfunction expansions, especially in the vicinity of the nonhomogeneous boundary surface. Nevertheless, the possibility of adopting a nonlinear eigenfunction expansion basis that carries information on the nonlinear coefficients of the originally proposed problem, has not been fully disregarded in previous developments with the GITT. In fact, when dealing with moving boundary problems in which the movement of the boundary is a priori unknown (Diniz *et al.*, 1990; Ruperti *et al.*, 1992; Sias *et al.*, 2009; Monteiro *et al.*, 2011), a nonlinear eigenvalue problem is actually employed, since the domain boundaries are part of the solution and simultaneously coupled to the transformed potentials and to the time-dependent eigenvalues and eigenfunctions.

Thus, the present work provides a novel integral transforms solution for nonlinear convection-diffusion problems, with particular emphasis on the treatment of nonlinear boundary condition coefficients. Instead of collapsing the nonlinearities of the boundary conditions into the corresponding source term, which nevertheless may still exist, the nonlinear coefficients are directly accounted for in the eigenvalue problem formulation, thus yielding a nonlinear eigenfunction expansion basis. As in nonlinear moving boundary problems (Diniz *et al.*, 1990; Ruperti *et al.*, 1992; Sias *et al.*, 2009;

Monteiro *et al.*, 2011), a time-dependent eigenvalue problem then needs to be solved simultaneously with the transformed ODE system. The expected advantage is that convergence will be significantly improved, especially close to the boundaries with nonlinear behavior, and the handling of the nonlinear eigenvalue problem will still lead to mild computational effort, hopefully with some advantage over the traditional GITT approach. Both formal solutions for the linear eigenvalue problem approach and for the introduced nonlinear eigenvalue problem alternative are here described. The general solutions of the related eigenvalue problems by the GITT itself are also briefly provided. Finally, an application dealing with transient heat conduction in a slab with nonlinear convection (Cotta *et al.*, 2015) and/or radiation boundary conditions (Mikhailov and Cotta, 1998) is more closely examined, to illustrate a few relevant computational aspects and demonstrate the improved convergence behavior. Two computational approaches were then tested, either by treating the transcendental equations for the eigenvalues, together with the transformed ODE system, as a differential-algebraic equations (DAEs) system, or by converting the transcendental equations into a system of ODE's for the eigenvalues, by taking the time derivative of the transcendental equations. Different combinations of the governing parameters are then chosen to illustrate the convergence behavior of the nonlinear eigenfunction expansions, and allow for critical comparisons against the classical GITT solution without filtering and a purely numerical solution based on the Method of Lines available in the routine NDSolve of the Mathematica system (Wolfram Research Inc., 2016).

2. Formal solution – linear eigenvalue problem

Before introducing the use of a nonlinear eigenvalue problem for the integral transformation process, the formal GITT solution for nonlinear convection-diffusion problems is revisited. The usual procedure is to first reformulate the problem by collapsing all the nonlinear information into the equation and boundary conditions source terms, while adopting a linear eigenvalue problem for the expansion basis, based on the choice of characteristic linear coefficients in the equation and boundary condition operators. Thus, we consider the nonlinear convection-diffusion problem as follows (Cotta, 1990, 1993):

$$w(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t} = \nabla\cdot[k(\mathbf{x})\nabla T(\mathbf{x},t)] - d(\mathbf{x})T(\mathbf{x},t) + P(\mathbf{x},t,T), \quad \mathbf{x}\in V, t>0 \tag{1a}$$

subjected to the following initial and boundary conditions:

$$T(\mathbf{x},0) = f(\mathbf{x}), \quad \mathbf{x}\in V \tag{1b}$$

$$\alpha(\mathbf{x})T(\mathbf{x},t) + \beta(\mathbf{x})k(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial \mathbf{n}} = \phi(\mathbf{x},t,T), \quad \mathbf{x}\in S \tag{1c}$$

The proposed problem is more general than it seems at first glance, since any sort of nonlinearities in the equation and boundary condition coefficients can be moved to the corresponding source terms, $P(\mathbf{x}, t, T)$ and $\phi(\mathbf{x}, t, T)$, without loss of generality. In this sense, the linear equation (w, k, d) and boundary condition (α, β) coefficients are essentially characteristic expressions that are chosen so as to intrinsically formulate the eigenvalue problem that shall be adopted as a basis for the eigenfunction expansion solution to follow.

However, before applying the integral transforms solution methodology, it is recommended to reduce the importance of the source terms in the original equation and the boundary conditions given by Equation (1a, c), since these terms are essentially responsible for an eventual slower convergence behavior of the eigenfunction expansions (Cotta and Mikhailov, 1997) This can be achieved by applying a filtering scheme, such as in the general form given by:

$$T(\mathbf{x},t) = T^*(\mathbf{x},t) + T_F(\mathbf{x};t) \tag{2}$$

where $T_F(\mathbf{x};\, t)$ is the proposed filter and $T^*(\mathbf{x},\, t)$ is the resulting filtered potential to be determined. After introducing Equation (2) into Equation (1), it results:

$$w(\mathbf{x})\frac{\partial T^*(\mathbf{x},t)}{\partial t} = \nabla\cdot\left[k(\mathbf{x})\nabla T^*(\mathbf{x},t)\right] - d(\mathbf{x})T^*(\mathbf{x},t) + P^*\left(\mathbf{x},t,T^*\right), \quad \mathbf{x}\in V, t>0 \tag{3a}$$

$$T^*(\mathbf{x},0) = f^*(\mathbf{x}), \quad \mathbf{x}\in V \tag{3b}$$

$$\alpha(\mathbf{x})T^*(\mathbf{x},t) + \beta(\mathbf{x})k(\mathbf{x})\frac{\partial T^*(\mathbf{x},t)}{\partial \mathbf{n}} = \phi^*\left(\mathbf{x},t,T^*\right), \quad \mathbf{x}\in S \tag{3c}$$

where:

$$f^*(\mathbf{x}) \equiv f(\mathbf{x}) - T_F(\mathbf{x};0) \tag{4a}$$

$$P^*\left(\mathbf{x},t,T^*\right) = P(\mathbf{x},t,T) - w(\mathbf{x})\frac{\partial T_F(\mathbf{x};t)}{\partial t} + \nabla\cdot[k(\mathbf{x})\nabla T_F(\mathbf{x};t)] - d(\mathbf{x})T_F(\mathbf{x};t) \tag{4b}$$

$$\phi^*\left(\mathbf{x},t,T^*\right) = \phi(\mathbf{x},t,T) - \alpha(\mathbf{x})T_F(\mathbf{x},t) - \beta(\mathbf{x})k(\mathbf{x})\frac{\partial T_F(\mathbf{x};t)}{\partial \mathbf{n}}, \quad \mathbf{x}\in S \tag{4c}$$

As from Equation (2), an explicit filter, $T_F(\mathbf{x};\, t)$, can be chosen, with the inherently filtered expressions for the initial conditions and source terms obtained from Equation (4). In any case, it is in general desirable that the chosen filter at least reduces the importance of the boundary condition source term in Equation (1c), thus leading to a weaker filtered source term, $\phi^*(\mathbf{x},\, t,\, T^*)$. Implicit nonlinear filters have also been proposed for specific cases of nonlinear boundary conditions (Ribeiro and Cotta, 1993; Cotta *et al.*, 2015; Matt, 2013), but may result too cumbersome for more general formulations.

Following the steps in the integral transform approach (Cotta, 1990, 1993, 1994, 1998; Cotta and Mikhailov, 1997, 2006), we define an auxiliary eigenvalue problem, which shall provide the basis for the eigenfunction expansions, in the form:

$$\nabla\cdot\left[k(\mathbf{x})\nabla\psi_i(\mathbf{x})\right] + \left[\mu_i^2 w(\mathbf{x}) - d(\mathbf{x})\right]\psi_i(\mathbf{x}) = 0, \quad \mathbf{x}\in V \tag{5a}$$

$$\alpha(\mathbf{x})\psi_i(\mathbf{x}) + \beta(\mathbf{x})k(\mathbf{x})\frac{\partial\psi_i(\mathbf{x})}{\partial\mathbf{n}} = 0, \quad \mathbf{x}\in S \tag{5b}$$

The eigenvalue problem given by Equation (5) allows for the definition of the integral transform pair as follows:

$$\overline{T}_i(t) = \int_V w(\mathbf{x})\psi_i(\mathbf{x})T^*(\mathbf{x},t)dV, \text{transform} \tag{6a}$$

$$T^{*}(\mathbf{x},t)=\sum_{i=1}^{\infty}\frac{1}{N_i}\psi_i(\mathbf{x})\overline{T}_i(t),\ \text{inverse} \tag{6b}$$

and the normalization integral:

$$N_i=\int_V w(\mathbf{x})\psi_i^2(\mathbf{x})dV \tag{7}$$

After application of the integral transformation procedure, through the operator $\int_V \psi_i(\mathbf{x})(.)dV$ over Equation (3a), and $\int_V w(\mathbf{x})\psi_i(\mathbf{x})(.)dV$ over Equation (3b), the resulting ODE system for the transformed potentials, $\overline{T}_i(t)$, is written as:

$$\frac{d\overline{T}_i(t)}{dt}+\mu_i^2\overline{T}_i(t)=\overline{g}_i\left(t,\overline{\mathbf{T}}\right),\mathrm{t}>0,\mathrm{i},\mathrm{j}=1,2,\ldots \tag{8a}$$

with initial conditions:

$$\overline{T}_i(0)=\overline{f}_i \tag{8b}$$

where:

$$\overline{g}_i\left(t,\overline{\mathbf{T}}\right)=\int_V \psi_i(\mathbf{x})P^{*}\left(\mathbf{x},t,T^{*}\right)dv+\int_S \phi^{*}\left(\mathbf{x},t,T^{*}\right)\left(\frac{\psi_i(\mathbf{x})-k(\mathbf{x})\frac{\partial\psi_i}{\partial\mathbf{n}}}{\alpha(\mathbf{x})+\beta(\mathbf{x})}\right)ds \tag{8c}$$

$$\overline{f}_i=\int_V w(\mathbf{x})\psi_i(\mathbf{x})f^{*}(\mathbf{x})dv \tag{8d}$$

$$\overline{\mathbf{T}}=\left\{\overline{T}_1(t),\overline{T}_2(t),\ldots\right\}^T \tag{8e}$$

System (8), after truncation to a sufficiently large finite order N, is numerically solved through well-established initial value problem solvers, readily available in scientific subroutines libraries, or directly as built-in function in mixed symbolic-numerical platforms, such as the function NDSolve of the Mathematica platform, (Wolfram Research Inc., 2016) which implement automatic relative error control schemes. The desired final solution is then reconstructed as:

$$T(\mathbf{x},t)=\sum_{i=1}^{N}\frac{1}{N_i}\psi_i(\mathbf{x})\overline{T}_i(t)+T_F(\mathbf{x},t) \tag{9}$$

The truncation order N may be adaptively chosen along the numerical integration march, so as to always work with truncation orders that are just enough to satisfy the user prescribed accuracy requirements, at selected positions ($\mathbf{x}$) and time values (t).

The eigenvalue problem that provides the basis for the eigenfunction expansion can be efficiently solved through the GITT itself, as proposed in (Cotta, 1993; Mikhailov and Cotta, 1994) and successfully employed in various applications (Sphaier and Cotta, 2000; Naveira-Cotta *et al.*, 2009; Knupp *et al.*, 2013, 2015a, b). The idea is to employ the GITT formalism to reduce the eigenvalue problem described by partial differential equations into standard algebraic eigenvalue problems, which can be solved by existing routines for matrix eigensystem analysis. Therefore, the eigenfuctions of the original auxiliary problem can be expressed by eigenfunction expansions based on a simpler auxiliary eigenvalue problem, for which exact analytic solutions are available.

The solution of problem (5) is thus itself proposed as an eigenfunction expansion:

$$\psi_i(\mathbf{x}) = \sum_{n=1}^{\infty} \tilde{\Omega}_n(\mathbf{x})\overline{\psi}_{i,n}, \quad \text{inverse} \tag{10a}$$

$$\overline{\psi}_{i,n} = \int_V \hat{w}(\mathbf{x})\, \psi_i(\mathbf{x})\tilde{\Omega}_n(\mathbf{x})dv, \quad \text{transform} \tag{10b}$$

where:

$$\tilde{\Omega}_n(\mathbf{x}) = \frac{\Omega_n(\mathbf{x})}{\sqrt{N_{\Omega n}}}, \text{with } N_{\Omega_n} = \int_V \hat{w}(\mathbf{x})\Omega_n^2(\mathbf{x})dv \tag{10c, d}$$

in terms of a simpler auxiliary eigenvalue problem, given as:

$$\nabla.\hat{k}(\mathbf{x})\nabla\Omega_n(\mathbf{x}) + \left(\lambda_n^2\hat{w}(\mathbf{x}) - \hat{d}(\mathbf{x})\right)\Omega_n(\mathbf{x}) = 0, \mathbf{x} \in V \tag{11a}$$

with boundary conditions:

$$\alpha(\mathbf{x})\Omega_n(\mathbf{x}) + \beta(\mathbf{x})\hat{k}(\mathbf{x})\frac{\partial\Omega_n(\mathbf{x})}{\partial\mathbf{n}} = 0, \quad \mathbf{x} \in S \tag{11b}$$

where the coefficients, $\hat{w}(\mathbf{x})$, $\hat{k}(\mathbf{x})$, and $\hat{d}(\mathbf{x})$, are simpler forms of the equation coefficients chosen so as to allow for an analytical solution of the auxiliary problem. Thus, the solution of problem (5), which needs to be known in terms of the eigenfunctions $\Omega_n(\mathbf{x})$ and related eigenvalues λ_n, offers a basis itself for the eigenfunction expansion of the original eigenvalue problem (5). Equation (5a) is now operated on with $\int_V \tilde{\Omega}_i(\mathbf{x})(\cdot)\ dV$, to yield the transformed algebraic system:

$$(\mathbf{A} + \mathbf{C})\{\overline{\psi}\} = \mu^2\mathbf{B}\{\overline{\psi}\} \tag{12a}$$

with the elements of the $M \times M$ matrices given by:

$$\begin{aligned} A_{ij} = {} & \int_S \frac{\tilde{\Omega}_i(\mathbf{x}) - \hat{k}(\mathbf{x})\frac{\partial\tilde{\Omega}_i(\mathbf{x})}{\partial\mathbf{n}}}{\alpha(\mathbf{x}) + \beta(\mathbf{x})} ds - \int_S \left(k(\mathbf{x}) - \hat{k}(\mathbf{x})\right)\tilde{\Omega}_i(\mathbf{x})\frac{\partial\tilde{\Omega}_j(\mathbf{x})}{\partial\mathbf{n}} ds \\ & + \int_V \left(k(\mathbf{x}) - \hat{k}(\mathbf{x})\right)\nabla\tilde{\Omega}_i(\mathbf{x})\cdot\nabla\tilde{\Omega}_j(\mathbf{x})dv + \int_V \left(d(\mathbf{x}) - \hat{d}(\mathbf{x})\right)\tilde{\Omega}_i(\mathbf{x})\tilde{\Omega}_j(\mathbf{x})dv \end{aligned} \tag{12b}$$

$$C_{ij} = \lambda_i^2\delta_{ij}; B_{ij} = \int_V w(\mathbf{x})\tilde{\Omega}_i(\mathbf{x})\tilde{\Omega}_j(\mathbf{x})dv \tag{12c,d}$$

where δ_{ij} is the Kronecker operator.

Therefore, the eigenvalue problem given by Equation (5) is reduced to the standard algebraic eigenvalue problem given by Equation (12), which can be solved with existing software for matrix eigensystem analysis, yielding the eigenvalues μ, whereas the corresponding calculated eigenvectors from this numerical solution, $\overline{\psi}_i$, are to be used in the inversion formula, given by Equation (10a), to find the desired eigenfunction, while increasing the number of terms in the truncated expansion, M, to meet the user prescribed accuracy.

The convergence behavior of the eigenfunction expansion in Equation (9) is inherently dependent on the importance of the filtered source terms that compose the transformed source term $\bar{g}_i(t, \overline{\mathbf{T}})$. The most frequently employed procedure is the adoption of a single analytical filter (Cotta, 1993; Almeida and Cotta, 1996) which in general reproduces the steady-state solution of the original problem or a quasi-steady behavior upon linearization. In some cases, the single filter strategy might not be able to offer an effective and uniform filtering over the whole time domain, and multiple successive filters may eventually be required for further improvement of the final convergence rates (Cotta and Mikhailov, 1997; Almeida and Cotta, 1996). One possibility is the use of available or easy to obtain approximate transient solutions for the proposed problem, either in discrete or continuous forms, as obtained from numerical or analytical solution methodologies. Besides, further convergence improvement can be achieved through a posteriori enhancement techniques, such as the integral balance approach (Scofano Neto *et al.*, 1990; Almeida and Cotta, 1996), or as more involved filtering approaches, such as local-instantaneous filtering (Gondim *et al.*, 2007) and implicit nonlinear filtering (Ribeiro and Cotta, 1993; Cotta *et al.*, 2015; Matt 2013).

3. Formal solution – nonlinear eigenvalue problem

Next, it is here investigated the possibility of employing the original nonlinear formulation in the boundary conditions, as represented by the corresponding nonlinear eigenvalue problem, in the construction of the eigenfunction expansions. The aim is to achieve improved convergence behavior, in comparison to the traditional approach with a linear eigenvalue problem, especially for regions close to the nonlinear boundary conditions. Thus, the nonlinear diffusion or convection-diffusion problem below is considered, with no need of collapsing the nonlinear boundary condition coefficients information into the nonlinear source terms, as previously presented:

$$w(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t} = \nabla.k(\mathbf{x})\nabla T - d(\mathbf{x})T + P(\mathbf{x},t,T), \quad \text{in } \mathbf{x}\in V, \quad t>0 \tag{13a}$$

with initial and boundary conditions:

$$T(\mathbf{x},0) = f(\mathbf{x}), \quad \mathbf{x}\in \mathrm{V} \tag{13b}$$

$$\alpha(\mathbf{x},t,T)T + \beta(\mathbf{x},t,T)k(\mathbf{x})\frac{\partial T}{\partial \mathbf{n}} = \phi(\mathbf{x},t,T), \quad \mathbf{x}\in S, \quad t>0 \tag{13c}$$

where α and β are the nonlinear boundary condition coefficients and $\mathbf{n}$ is the outward drawn normal vector to surface S. All the boundary condition coefficients and source terms are allowed to be nonlinear, besides being explicitly dependent also on the space and time variables for the sake of generality.

The first step in application of the GITT is the proposition of a filtering solution, which reduces the effects on convergence rates due to the equation and boundary source terms. As before, this solution is here denoted by $T_F(\mathbf{x}; t)$ and is considered a filtering solution in the form of Equation (2). The resulting formulation for the filtered potential, T^*, then becomes:

$$w(\mathbf{x})\frac{\partial T^*(\mathbf{x},t)}{\partial t} = \nabla.k(\mathbf{x})\nabla T^* - d(\mathbf{x})T^* + P^*\left(\mathbf{x},t,T^*\right), \quad \mathbf{x}\in V, \quad t>0 \tag{14a}$$

with initial and boundary conditions:

$$T^*(\mathbf{x},0) \;=\; f^*(\mathbf{x}), \mathbf{x}\in \mathrm{V} \tag{14b}$$

$$\alpha(\mathbf{x},t,T)T^* + \beta(\mathbf{x},t,T)k(\mathbf{x})\frac{\partial T^*}{\partial \mathbf{n}} = \phi^*\left(\mathbf{x},t,T^*\right), \quad \mathbf{x}\in \mathrm{S} \tag{14c}$$

where the filtered source terms and initial condition are given by:

$$P^*(\mathbf{x},t,T) = P(\mathbf{x},t,T) - \left[w(\mathbf{x})\frac{\partial T_F(\mathbf{x};t)}{\partial t} - \nabla.k(\mathbf{x})\nabla T_F + d(\mathbf{x})T_F\right] \tag{15a}$$

$$\phi^*\left(\mathbf{x},t,T^*\right) = \phi(\mathbf{x},t,T) - \left[\alpha(\mathbf{x},t,T)T_F + \beta(\mathbf{x},t,T)k(\mathbf{x})\frac{\partial T_F}{\partial \mathbf{n}}\right] \tag{15b}$$

$$f^*(\mathbf{x}) = f(\mathbf{x}) - T_F(\mathbf{x};0) \tag{15c}$$

In case the filtering solution satisfies identically the boundary conditions, the boundary source term becomes zero, and only the filtered equation source term remains as defined in Equation (15a).

At this point, it suffices to proceed with the integral transform solution for the filtered potential, T^*. Now, taking a different path from the usual formalism in the GITT (Cotta, 1990, 1993, 1994, 1998; Cotta and Mikhailov, 1997, 2006), a nonlinear eigenvalue problem that preserves the original boundary condition coefficients is preferred instead of the one with linear characteristic coefficients, in the form:

$$\nabla.k(\mathbf{x})\nabla\psi_i(\mathbf{x};t) + \left(\mu_i^2(t)w(\mathbf{x}) - d(\mathbf{x})\right)\psi_i(\mathbf{x};t) = 0, \mathbf{x}\in \mathrm{V} \tag{16a}$$

with boundary conditions:

$$\alpha(\mathbf{x},t,T)\psi_i(\mathbf{x};t) + \beta(\mathbf{x},t,T)\mathrm{k}(\mathbf{x})\frac{\partial\psi_i(\mathbf{x};t)}{\partial \mathbf{n}} = 0, \mathbf{x}\in \mathrm{S} \tag{16b}$$

and the solution for the associated time-dependent eigenfunctions, $\psi_i(\mathbf{x};t)$, and eigenvalues, $\mu_i(t)$, is at this point assumed to be known.

Problem (16) allows for the definition of the following integral transform pair:

$$\overline{T}_i(t) = \int_V w(\mathbf{x})\psi_i(\mathbf{x};t)T^*(\mathbf{x},t)dv, \quad \text{transform} \tag{17a}$$

$$T^*(\mathbf{x},t) = \sum_{i=1}^{\infty}\frac{1}{N_i(t)}\psi_i(\mathbf{x};t)\overline{T}_i(t), \quad \text{inverse} \tag{17b}$$

and the normalization integrals:

$$N_i(t) = \int_V w(\mathbf{x})\psi_i^2(\mathbf{x};t)dv \tag{18}$$

After application of the integral transformation procedure, the resulting ODE system for the transformed potentials, $\overline{T_i}(t)$, is written as:

$$\frac{d\overline{T}_i(t)}{dt} + \sum_{j=1}^{\infty}A_{i,j}\left(t,\overline{\mathbf{T}}\right)\overline{T}_j(t) = \overline{g}_i\left(t,\overline{\mathbf{T}}\right), \mathrm{t}>0, \mathrm{i},\mathrm{j} = 1,2,\ldots \tag{19a}$$

with initial conditions:

$$\overline{T}_i(0) = \overline{f}_i \tag{19b}$$

where,

$$A_{i,j}\left(t, \overline{\mathbf{T}}\right) = \delta_{ij}\mu_i^2(t) + A_{i,j}^*\left(t, \overline{\mathbf{T}}\right) \tag{19c}$$

and:

$$A_{i,j}^*\left(t, \overline{\mathbf{T}}\right) = -\frac{1}{N_j(t)}\int_V w(\mathbf{x})\frac{\partial}{\partial t}\left[\psi_i(\mathbf{x};t)\right]\psi_j(\mathbf{x};t)dv \tag{19d}$$

$$\overline{g}_i\left(t, \overline{\mathbf{T}}\right) = \int_V \psi_i(\mathbf{x};t)P^*(\mathbf{x},t,T)dv + \int_S \phi^*(\mathbf{x},t,T)\left(\frac{\psi_i(\mathbf{x};t) - k(\mathbf{x})\frac{\partial \psi_i}{\partial \mathbf{n}}}{\alpha(\mathbf{x},t,T) + \beta(\mathbf{x},t,T)}\right)ds \tag{19e}$$

$$\overline{f}_i = \int_V w(\mathbf{x})\tilde{\psi}_i(\mathbf{x};0)f^*(\mathbf{x})dv \tag{19f}$$

System (19) is again numerically solved through well-established initial value problem solvers, such as the function NDSolve of the Mathematica platform (Wolfram Research Inc., 2016). It should be recalled that the eigenvalue problem in Equation (16) has now to be solved simultaneously with the transformed system given by Equation (19), yielding the time variable eigenfunctions, eigenvalues and norms, as will be further discussed in the next section. The desired final solution is then reconstructed by:

$$T(\mathbf{x},t) = \sum_{i=1}^{N}\frac{1}{N_i(t)}\psi_i(\mathbf{x};t)\overline{T}_i(t) + T_F(\mathbf{x};t) \tag{20}$$

The GITT itself is now employed in the solution of the nonlinear eigenvalue problem, Equation (16). The basic idea is to reduce the eigenvalue problem described by the partial differential equation into a nonlinear algebraic eigenvalue problem, which can be solved by known approaches for matrix nonlinear eigensystem analysis. Therefore, the eigenfuctions of the original auxiliary problem can be expressed by eigenfunction expansions based on a simpler auxiliary eigenvalue problem, with linear coefficients, for which exact analytic solutions exist.

Consider the following nonlinear eigenvalue problem defined in region V and boundary surface S:

$$L\psi(\mathbf{x};t) = \mu^2(t)w(\mathbf{x})\psi(\mathbf{x};t), \quad \mathbf{x} \in V \tag{21a}$$

$$B\psi(\mathbf{x};t) = 0, \quad \mathbf{x} \in S \tag{21b}$$

where the operators L and B are given by:

$$L = -\nabla\cdot(k(\mathbf{x})\nabla) + d(\mathbf{x}) \tag{21c}$$

$$B = \alpha(\mathbf{x},t,T) + \beta(\mathbf{x},t,T)k(\mathbf{x})\frac{\partial}{\partial\mathbf{n}} \tag{21d}$$

and $w(\boldsymbol{x})$, $k(\boldsymbol{x})$, and $d(\boldsymbol{x})$ are known linear functions in region V, while $\alpha(\mathbf{x}, t, T)$, $\beta(\mathbf{x}, t, T)$ are nonlinear functions on the boundary surface S.

The problem given by Equation (21a-d) can be rewritten as:

$$\hat{L}\psi(\mathbf{x};t) = \left(\hat{L}-L\right)\psi(\mathbf{x};t) + \mu^2(t)w(\mathbf{x})\psi(\mathbf{x};t), \quad \mathbf{x}\in V \tag{22a}$$

$$\hat{B}\psi(\mathbf{x};t) = \left(\hat{B}-B\right)\psi(\mathbf{x};t), \quad \mathbf{x}\in S \tag{22b}$$

where $\hat{L}$ and $\hat{B}$ are the simpler operators with linear coefficients, given by:

$$\hat{L} = -\nabla\cdot\left(\hat{k}(\mathbf{x})\nabla\right) + \hat{d}(\mathbf{x}) \tag{22c}$$

$$\hat{B} = \hat{\alpha}(\mathbf{x}) + \hat{\beta}(\mathbf{x})\hat{k}(\mathbf{x})\frac{\partial}{\partial\mathbf{n}} \tag{22d}$$

which are employed to select an auxiliary problem:

$$\hat{L}\Omega(\mathbf{x}) = \lambda^2\hat{w}(\mathbf{x})\Omega(\mathbf{x}), \quad \mathbf{x}\in V \tag{23a}$$

$$\hat{B}\Omega(\mathbf{x}) = 0, \quad \mathbf{x}\in S \tag{23b}$$

where $\hat{w}(\mathbf{x})$, $\hat{k}(\mathbf{x})$, $\hat{d}(\mathbf{x})$, $\hat{\alpha}(\mathbf{x})$, and $\hat{\beta}(\mathbf{x})$ are known coefficients in V and S, properly chosen so that the eigenvalue problem given by Equation (23.a, b) allows for a straightforward solution for the eigenvalues, λ, and corresponding eigenfunctions, $\Omega(\mathbf{x})$.

Therefore, making use of the eigenfunctions orthogonality property, problem (23) allows the definition of the following integral transform pair:

$$\text{transform: } \overline{\psi}_i(t) = \int_V \hat{w}(\mathbf{x})\tilde{\Omega}_i(\mathbf{x})\psi(\mathbf{x};t)dv \tag{24a}$$

$$\text{inverse: } \psi(\mathbf{x};t) = \sum_{i=1}^{\infty}\tilde{\Omega}_i(\mathbf{x})\overline{\psi}_i(t) \tag{24b}$$

where:

$$\tilde{\Omega}_i(\mathbf{x}) = \frac{\Omega_i(\mathbf{x})}{\sqrt{N_{\Omega_i}}}, \text{with } N_{\Omega_i} = \int_V \hat{w}(\mathbf{x})\Omega_i^2(\mathbf{x})dv \tag{24c, d}$$

Equation (21a) is now operated on with $\int_V \tilde{\Omega}_i(\mathbf{x})(\cdot)dv$, to yield the transformed nonlinear algebraic system:

$$\lambda_i^2\overline{\psi}_i(t) = \int_S \gamma_i\left(\hat{B}-B\right)\psi(\mathbf{x};t)ds + \int_V \tilde{\Omega}_i(\mathbf{x})\left(\hat{L}-L\right)\psi(\mathbf{x};t)dv +$$

$$+\mu^2(t)\int_V \tilde{\Omega}_i(\mathbf{x})w(\mathbf{x})\psi(\mathbf{x};t)dv, i = 1,2,... \tag{25a}$$

$$\gamma_i = \frac{\tilde{\Omega}_i(\mathbf{x}) - \hat{k}(\mathbf{x})\frac{\partial \tilde{\Omega}_i(\mathbf{x})}{\partial \mathbf{n}}}{\hat{\alpha}(\mathbf{x}) + \hat{\beta}(\mathbf{x})} \tag{25b}$$

After introducing the truncated to the Mth term inversion formula, Equation (24b), into Equation (25a), the resulting system is written in matrix form as:

$$(\mathbf{A}(t) + \mathbf{C})\{\overline{\psi}(t)\} = \boldsymbol{\mu}^2(t)\mathbf{B}(t)\{\overline{\psi}(t)\} \tag{26a}$$

with the elements of the $M \times M$ matrices and vector μ^2given by:

$$a_{ij}(t) = -\int_S \gamma_i\left(\hat{B} - B\right)\tilde{\Omega}_j(\mathbf{x})ds - \int_V \tilde{\Omega}_i(\mathbf{x})\left(\hat{L} - L\right)\tilde{\Omega}_j(\mathbf{x})dv \tag{26b}$$

$$c_{ij} = \lambda_i^2 \delta_{ij} \tag{26c}$$

$$b_{ij}(t) = \int_V w(\mathbf{x})\tilde{\Omega}_i(\mathbf{x})\tilde{\Omega}_j(\mathbf{x})dv \tag{26d}$$

$$\boldsymbol{\mu}^2(t) = \left\{\mu_1^2(t), \mu_2^2(t), \ldots, \mu_M^2(t)\right\}^T \tag{26e}$$

By choosing to use the relation:

$$\int_V \tilde{\Omega}_i(\mathbf{x})\nabla\cdot\left(\hat{k}(\mathbf{x})\nabla\tilde{\Omega}_j(\mathbf{x})\right)dv = \int_S \hat{k}(\mathbf{x})\tilde{\Omega}_i(\mathbf{x})\frac{\partial \tilde{\Omega}_j(\mathbf{x})}{\partial \mathbf{n}}ds - \int_V \hat{k}(\mathbf{x})\nabla\tilde{\Omega}_i(\mathbf{x})\cdot\nabla\tilde{\Omega}_j(\mathbf{x})dv \tag{27}$$

the elements of **A** can be calculated through the following working formula:

$$\begin{aligned}&\int_S \frac{\tilde{\Omega}_i(\mathbf{x}) - \hat{k}(\mathbf{x})\frac{\partial \tilde{\Omega}_i(\mathbf{x})}{\partial \mathbf{n}}}{\hat{\alpha}(\mathbf{x}) + \hat{\beta}(\mathbf{x})}\Bigg[(\alpha(\mathbf{x}, t, T) - \hat{\alpha}(\mathbf{x}))\tilde{\Omega}_j(\mathbf{x}) \\ &+ \left(\beta(\mathbf{x}, t, T)k(\mathbf{x}) - \hat{\beta}(\mathbf{x})\hat{k}(\mathbf{x})\right)\frac{\partial \tilde{\Omega}_j(\mathbf{x})}{\partial \mathbf{n}}\Bigg]ds - \int_S \left(k(\mathbf{x}) - \hat{k}(\mathbf{x})\right)\tilde{\Omega}_i(\mathbf{x})\frac{\partial \tilde{\Omega}_j(\mathbf{x})}{\partial \mathbf{n}}ds \\ &+ \int_V \left(k(\mathbf{x}) - \hat{k}(\mathbf{x})\right)\nabla\tilde{\Omega}_i(\mathbf{x})\cdot\nabla\tilde{\Omega}_j(\mathbf{x})dv + + \int_V \left(d(\mathbf{x}) - \hat{d}(\mathbf{x})\right)\tilde{\Omega}_i(\mathbf{x})\tilde{\Omega}_j(\mathbf{x})dv\end{aligned} \tag{28}$$

Therefore, the eigenvalue problem given by Equation (21a, b) is reduced to the nonlinear algebraic eigenvalue problem given by Equation (26a), which can be solved with existing methodologies for matrix eigensystem analysis, numerically yielding the eigenvalues $\mu(t)$, whereas the corresponding calculated eigenvectors from this numerical solution, $\overline{\psi}_i(t)$, are to be used in the inversion formula, given by Equation (24b), to find the desired eigenfunction. By increasing the number of terms in the truncated expansion, one can obtain the results with prescribed accuracy.

4. Application

The proposed nonlinear eigenfunction expansion procedure is now considered in more details in an application of transient heat conduction across a slab with nonlinear boundary condition coefficients, as typical of natural convection (air in the present application) (Cotta *et al.*, 2015), radiation (Mikhailov and Cotta, 1998) or combined convection-radiation heat exchange at the surface. The mathematical formulation of the problem here considered, in dimensionless form, is given by:

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \tag{29a}$$

with initial and boundary conditions given, respectively, by:

$$T(x,0) = 1, \quad 0 \leqslant x \leqslant 1 \tag{29b}$$

$$\frac{\partial T(0,t)}{\partial x} = 0; \quad \frac{\partial T(1,t)}{\partial x} + Bi(T(1,t))T(1,t) = 0, \quad t > 0 \tag{29c, d}$$

and for the present application the non-linear function $Bi(T)$ is taken as:

$$Bi(T(1,t)) = Bi_c T^{1/3}(1,t) + Bi_r\left[\left(1+\gamma T(1,t)+\frac{\gamma^2}{2}T^2(1,t)\right)\left(1+\frac{\gamma}{2}T(1,t)\right)\right] \tag{29e}$$

where:

$$T = \frac{\hat{T}-\hat{T}_\infty}{\hat{T}_0-\hat{T}_\infty}, x = \frac{\hat{x}}{L}, t = \frac{\alpha_s \hat{t}}{L^2}, Bi_c = \frac{h_c L}{k_s}, Bi_r = \frac{h_r L}{k_s}, h_r = 4\varepsilon\sigma\hat{T}_\infty^3, \gamma = \frac{\hat{T}_0-\hat{T}_\infty}{\hat{T}_\infty} \tag{29f-l}$$

where ^ denotes the dimensional variables (temperature, time and position) in the original formulation. The correspondence between the above formulation and the general one provided in Equation (13) is given by the following relations:

$$w(\mathbf{x}) = 1, k(\mathbf{x}) = 1, d(\mathbf{x}) = 0, P(\mathbf{x},t,T) = 0, f(\mathbf{x}) = 1 \tag{30a-e}$$

and for the boundary conditions:

$$\text{for } x = 0: \alpha(0,t,T) = 0, \beta(0,t,T) = 1, \phi(0,t,T) = 0$$

$$\text{for } x = 1: \alpha(1,t,T) = Bi(T(1,t)), \beta(1,t,T) = 1, \phi(1,t,T) = 0 \tag{31a-f}$$

The nonlinear problem (29) was first directly solved through the traditional GITT without considering the nonlinear coefficient in the eigenvalue problem, according to Section 2, without filtering and adopting a linearized boundary coefficient:

$$\alpha(1) = Bi_{ef} = Bi_c + Bi_r\left[\left(1+\gamma+\frac{\gamma^2}{2}\right)\left(1+\frac{\gamma}{2}\right)\right] \tag{32a}$$

The above characteristic linear boundary condition coefficient choice then yields the following nonlinear source term, in the nonlinear boundary condition (1c):

$$\phi(1,t,T) = \left[Bi_{ef} - Bi(T(1,t))\right]T(1,t) \tag{32b}$$

Then, the novel approach with nonlinear eigenvalue problem has been considered through the appropriate choice of the auxiliary eigenvalue problem for the integral transformation of the original one. Except for $\alpha(1, t, T)$, all of the other coefficients are linear and remain the same in both eigenvalue problems. The only differing choice is in fact for the boundary condition coefficient, $\widehat{\alpha}(1) = Bi_0$. The nonlinear eigenvalue problem is written as:

$$\frac{\partial^2 \psi(x;t)}{\partial x^2} + \mu^2(t)\psi(x;t) = 0, \quad 0 < x < 1 \tag{33a}$$

$$\frac{\partial \psi(0;t)}{\partial x} = 0; \quad \frac{\partial \psi(1;t)}{\partial x} + Bi(T(1,t))\psi(1;t) = 0 \tag{33b, c}$$

which is readily solved as:

$$\psi(x;t) = \cos[\mu(t)x] \tag{34}$$

yielding the following transcendental equation for the eigenvalues:

$$-\mu(t)\sin[\mu(t)] + Bi(T(1,t))\cos[\mu(t)] = 0 \tag{35a}$$

where:

$$T(1,t) = \sum_{i=1}^{\infty} \frac{1}{N_i(t)} \cos \mu_i(t) \overline{T}_i(t) \tag{35b}$$

$$N_i(t) = \int_0^1 \left(\cos\left[\mu_i(t)x\right]\right)^2 dx = \frac{1}{4}\left(2 + \frac{\sin\left[2\mu_i(t)\right]}{\mu_i(t)}\right) \tag{35c}$$

The transformed system is then obtained from Equation (19a), where the related coefficients are given by:

$$A^*_{i,j}\left(t, \overline{\mathbf{T}}\right) = \frac{1}{N_j(t)} \frac{d\mu_i(t)}{dt} \int_0^1 x \cos\left[\mu_j(t)x\right] \sin\left[\mu_i(t)x\right] dx \tag{36a}$$

$$\overline{g}_i\left(t, \overline{\mathbf{T}}\right) = 0 \tag{36b}$$

$$\overline{f}_i = \frac{1}{\mu_i(0)} \sin\left[\mu_i(0)\right] \tag{36c}$$

Equations (19a) and (36a-c), together with the transcendental Equation (35a), form a nonlinear system of DAEs, upon truncation to a sufficiently large finite order N, which can be numerically solved to provide the transformed potentials and the time-variable eigenvalues, under user prescribed accuracy control (Wolfram Research Inc., 2016). Generally, a system of DAEs can be converted to a system of ODEs by differentiating it with respect to the independent variable t. The index of a DAE is the number of times needed to differentiate the DAEs to get a system of ODEs. The DAE solver methods built into NDSolve work with index-1 systems, so for higher-index systems an index reduction may be necessary to get a solution. There are a variety of solution methods built into NDSolve for solving DAEs. Two methods work with the general residual form of index-1 DAEs, F(t, y, y′) = 0. In particular, the IDA method (Implicit Differential-Algebraic solver from the SUNDIALS package), (Wolfram Research Inc., 2016) is based on backward differentiation formulas which are more appropriate for dealing with potentially stiff systems, which are common when working with eigenfunction expansion approaches (Cotta, 1993).

Alternatively, the transcendental Equation (35a) can be differentiated with respect to time, and thus *a priori* generate ODEs for the eigenvalues, that can be simultaneously solved with the transformed potentials, as previously implemented in the context of moving boundary problems (Ruperti *et al.*, 1992). Therefore:

$$-\frac{d\mu(t)}{dt}\sin[\mu(t)]-\mu(t)\cos[\mu(t)]\frac{d\mu(t)}{dt}$$
$$+\frac{dBi(T(1,t))}{dt}\cos[\mu(t)]-Bi(T(1,t))\sin[\mu(t)]\frac{d\mu(t)}{dt}=0 \tag{37a}$$

and since:

$$\frac{dBi(T(1,t))}{dt}=\frac{dBi(T(1,t))}{dT(1,t)}.\frac{\partial T(1,t)}{\partial t} \tag{37b}$$

then:

$$\frac{d\mu_i(t)}{dt}\{-\sin[\mu_i(t)]-\mu_i(t)\cos[\mu_i(t)]-Bi(T(1,t))\sin[\mu_i(t)]\}+$$
$$+\frac{dBi(T(1,t))}{dT(1,t)}\cos[\mu_i(t)]\frac{\partial T(1,t)}{\partial t}=0 \tag{37c}$$

with initial conditions obtained from the transcendental equations system as follows:

$$-\mu_i(0)\sin[\mu_i(0)]+Bi(T(1,0))\cos[\mu_i(0)]=0, i=1,2,3,\ldots \tag{37d}$$

where:

$$T(1,0)=\sum_{j=1}^{\infty}\frac{1}{N_j(0)}\cos\mu_j(0)\overline{T}_j(0)=\sum_{j=1}^{\infty}\frac{1}{N_j(0)}\cos\mu_j(0)\overline{f}_j \tag{37e}$$

Equation (37c) are then written for the first N eigenvalues and simultaneously solved with Equation (19), forming a system of $2\times N$ ODE's, to yield the transformed potentials and eigenvalues as a function of time. Then, the inverse formula (20) can be directly employed to provide the desired original potential.

5. Results and discussion

The application in the previous section was implemented in the *Mathematica* platform version 10. As the ordinary differential system solver, we have employed the NDSolve function in the *Mathematica* system, in the DAE mode (DAEs). The software responds with an interpolating function that provides a continuous representation of the potentials and the eigenvalues along the time domain. The problem was thus solved through integral transforms, first with the classical approach with a linear eigenvalue problem, collapsing the nonlinearities in the boundary condition source term. Then, the solution through the nonlinear eigenvalue problem of Equation (21) was implemented, when the nonlinearities are directly accounted for in the eigenvalue problem, with different number of terms in the eigenfunction expansion so as to inspect for the convergence behavior.

Tables I and II provide a brief convergence analysis of both the traditional GITT with linear eigenvalue problem of Section 2, and the presently introduced approach

Table I. Potential convergence behavior as obtained from the nonlinear eigenvalue problem and the traditional GITT approaches

(a)[a]

N	$T(x=0.2,t=0.1)$ Nonlinear eig.probl.	$T(x=0.2,t=0.1)$ Linear eig.probl.	$T(x=0.8,t=0.1)$ Nonlinear eig.probl.	$T(x=0.8,t=0.1)$ Linear eig.probl.	$T(x=1.0,t=0.1)$ Nonlinear eig.probl.	$T(x=1.0,t=0.1)$ Linear eig.probl.
1	1.01966	1.03587	0.670233	0.640969	0.480618	0.429363
5	0.976123	0.974836	0.727676	0.730316	0.531293	0.519378
10	0.976108	0.975809	0.727709	0.727208	0.531246	0.525529
15	0.976107	0.975981	0.727707	0.727967	0.531242	0.527504
20	0.976107	0.97604	0.727708	0.727645	0.531241	0.528473
25	0.976107	0.976067	0.727708	0.727819	0.531240	0.529048
30	0.976107	0.976081	0.727708	0.727711	0.531240	0.529429
Num.[c]	0.976107		0.727710		0.531245	

(b)[b]

N	$T(x=0.2,t=0.5)$ Nonlinear eig.probl.	$T(x=0.2,t=0.5)$ Linear eig.probl.	$T(x=0.8,t=0.5)$ Nonlinear eig.probl.	$T(x=0.8,t=0.5)$ Linear eig.probl.	$T(x=1.0,t=0.5)$ Nonlinear eig.probl.	$T(x=1.0,t=0.5)$ Linear eig.probl.
1	0.638206	0.655539	0.433928	0.405630	0.322272	0.271717
5	0.639165	0.637994	0.432577	0.434801	0.320907	0.310043
10	0.639162	0.638872	0.432583	0.432001	0.320899	0.315579
15	0.639162	0.639026	0.432582	0.432684	0.320898	0.317353
20	0.639162	0.639079	0.432582	0.432395	0.320898	0.318224
25	0.639162	0.639103	0.432582	0.432551	0.320898	0.318740
30	0.639162	0.639116	0.432582	0.432454	0.320898	0.319082
Num.[c]	0.639166		0.43259		0.320906	

Notes: [a]$Bi_c = 1, Bi_r = 1, \gamma = 1/3, t = 0.1$; [b]$Bi_c = 1, Bi_r = 1, \gamma = 1/3, t = 0.1$; [c]NDSolve routine – *Mathematica* system (Wolfram Research Inc., 2016)

Table II. Potential convergence behavior as obtained from the nonlinear eigenvalue problem and the traditional GITT approaches

(a)[a]

N	$T(x=0.2,t=0.1)$ Nonlinear eig.probl.	$T(x=0.2,t=0.1)$ Linear eig.probl.	$T(x=0.8,t=0.1)$ Nonlinear eig.probl.	$T(x=0.8,t=0.1)$ Linear eig.probl.	$T(x=1.0,t=0.1)$ Nonlinear eig.probl.	$T(x=1.0,t=0.1)$ Linear eig.probl.
1	0.967554	0.972346	0.367690	0.351195	0.072600	0.0478759
5	0.934230	0.932317	0.416487	0.418872	0.0841374	0.0617875
10	0.934227	0.933313	0.416496	0.412839	0.0841284	0.0673294
15	0.934226	0.933626	0.416495	0.415385	0.084127	0.0708148
20	0.934226	0.933763	0.416496	0.414085	0.0841266	0.0730425
25	0.934226	0.933834	0.416496	0.414837	0.0841265	0.0745414
30	0.934226	0.933874	0.416496	0.414361	0.0841264	0.0756033
Num.[c]	0.934223		0.416494		0.0841299	

(b)[b]

N	$T(x=0.2,t=0.5)$ Nonlinear eig.probl.	$T(x=0.2,t=0.5)$ Linear eig.probl.	$T(x=0.8,t=0.5)$ Nonlinear eig.probl.	$T(x=0.8,t=0.5)$ Linear eig.probl.	$T(x=1.0,t=0.5)$ Nonlinear eig.probl.	$T(x=1.0,t=0.5)$ Linear eig.probl.
1	0.39419	0.395303	0.150352	0.142777	0.0303365	0.0194637
5	0.394198	0.392250	0.150334	0.150577	0.0303307	0.0217978
10	0.394198	0.392614	0.150335	0.148351	0.0303302	0.0238440
15	0.394198	0.392730	0.150335	0.149295	0.0303301	0.0251356
20	0.394198	0.392781	0.150335	0.148813	0.0303301	0.0259619
25	0.394198	0.392807	0.150335	0.149091	0.0303301	0.0265180
30	0.394198	0.392822	0.150335	0.148915	0.0303301	0.0269121
Num.[c]	0.394199		0.150339		0.0303349	

Notes: [a]$Bi_c = 1, Bi_r = 1, \gamma = 1/3, t = 0.1$; [b]$Bi_c = 1, Bi_r = 1, \gamma = 1/3, t = 0.1$; [c]NDSolve routine – *Mathematica* system (Wolfram Research Inc., 2016) with a nonlinear eigenvalue problem, Section 3. A total truncation order of $N=30$ terms in the eigenfunction expansions has been considered in both cases, and results for dimensionless temperatures at specific positions ($x=0.1$, 0.8 and 1.0) and times ($t=0.1$ and 0.5) are presented for increasing truncation orders, $N=1,5,10,15,20,25$ and 30. In addition, the last row provides the numerical results achieved by the routine NDSolve with the Method of Lines for numerically solving the original partial differential problem. In the first set (Table I) a combined convection-radiation situation is considered with $Bi_c=1$; $Bi_r=1$; $\gamma=1/3$. From the columns for increasing truncation orders, for the two values of time considered ($t=0.1$ and 0.5), one can already observe the excellent convergence behavior of the novel nonlinear eigenvalue problem approach, with six fully converged significant digits in all positions, and at least a five digits agreement with the numerical solution of NDSolve with a refined mesh. It can also be noticed that the proposed solution is not sensitive to the positions closer to the nonlinear boundaries, with a fairly uniform convergence behavior throughout the domain. As for the traditional GITT, the solution is fully converged to four digits at the innermost positions, and only to three significant digits at the boundary position, within this range of truncation orders. Thus, deviations at the third significant digit can already be observed, in comparison to the proposed nonlinear GITT and the NDSolve results, at this same boundary position. Once a purely radiative boundary condition with a higher characteristic Biot number is considered, as shown in Table II for the case $Bi_c=0$; $Bi_r=20$; $\gamma=1/3$, the slower convergence behavior of the traditional GITT approach at the boundary position, propagating toward the interior of the slab, is more clearly observable. Again, the nonlinear eigenvalue problem approach is fully converged to all six significant digits presented, and even already converged at truncation orders as low as $N=10$. The agreement with the NDSolve numerical results is also within four to five significant digits throughout the range of space coordinate and time variable. On the other hand, the traditional solution is still not yet converged even at the second significant digit in the worst situation, which occurs at the boundary position. However, at the innermost position, convergence to four significant digits can still be observed, with an agreement to at most three digits with the NDSolve and the new GITT solutions. Nevertheless, to the graph scale, very little deviation is observable among the three sets of numerical results, as shown in Figure 1(a, b) for the temperature distribution across the slab at $t=0.3$, respectively, for the cases $Bi_c=1$; $Bi_r=1$; $\gamma=1/3$ and $Bi_c=0$; $Bi_r=20$; $\gamma=1/3$. Figures 2(a, b) illustrate the transient behavior of the first nonlinear eigenvalue and the corresponding nonlinear eigenfunction evaluated at the boundary $x=1$, for the case $Bi_c=1$; $Bi_r=1$; $\gamma=1/3$, which clearly demonstrates that the variations due to the time variable surface temperature are in fact of relevance to the nonlinear eigenvalue problem behavior. Figures 3(a, b) provide again the first eigenvalue and its corresponding eigenfunction at the boundary, but now for the purely radiative case $Bi_c=0$; $Bi_r=20$; $\gamma=1/3$. Clearly in this case, the transient is approaching a steady-state behavior within the elapsed dimensionless time of $t=0.5$. Finally, Table III illustrates the excellent adherence between the two GITT solutions with nonlinear eigenvalue problem, either by considering the DAEs system (DAE solution) or the derivation of an ODE system for the eigenvalues, as obtained from the transcendental equations. Also shown is the numerical solution obtained from the Method of Lines with the NDSolve solution. It can be seen that the two GITT solutions are equivalent, as expected, and agree in all six signficant digits provided in the table, and have at least four digits agreement with the numerical solution, for the worst situation at the nonlinear boundary $x=1$.

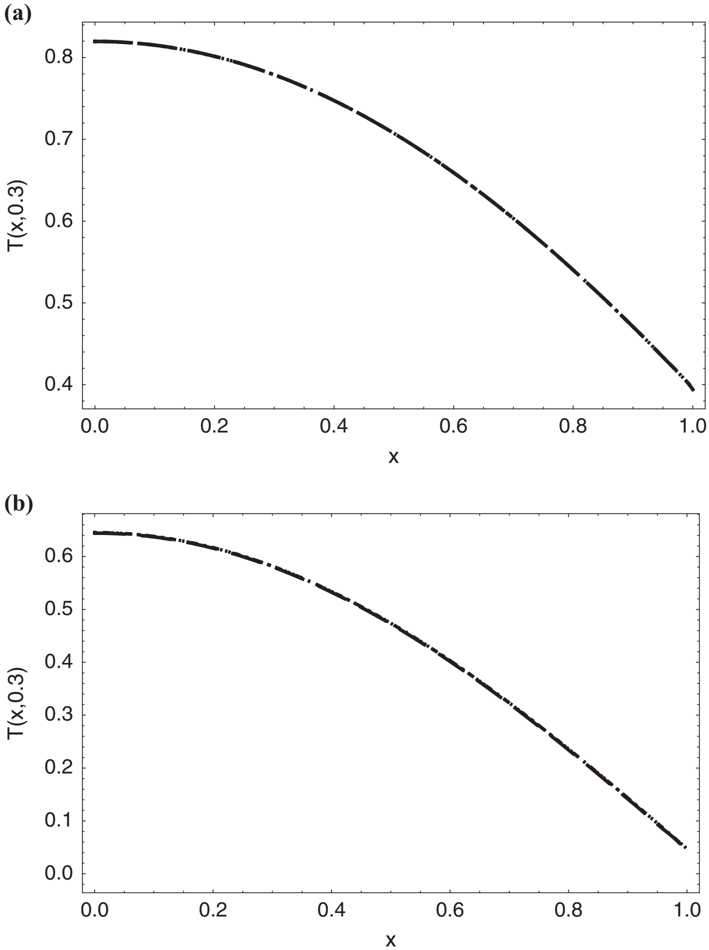
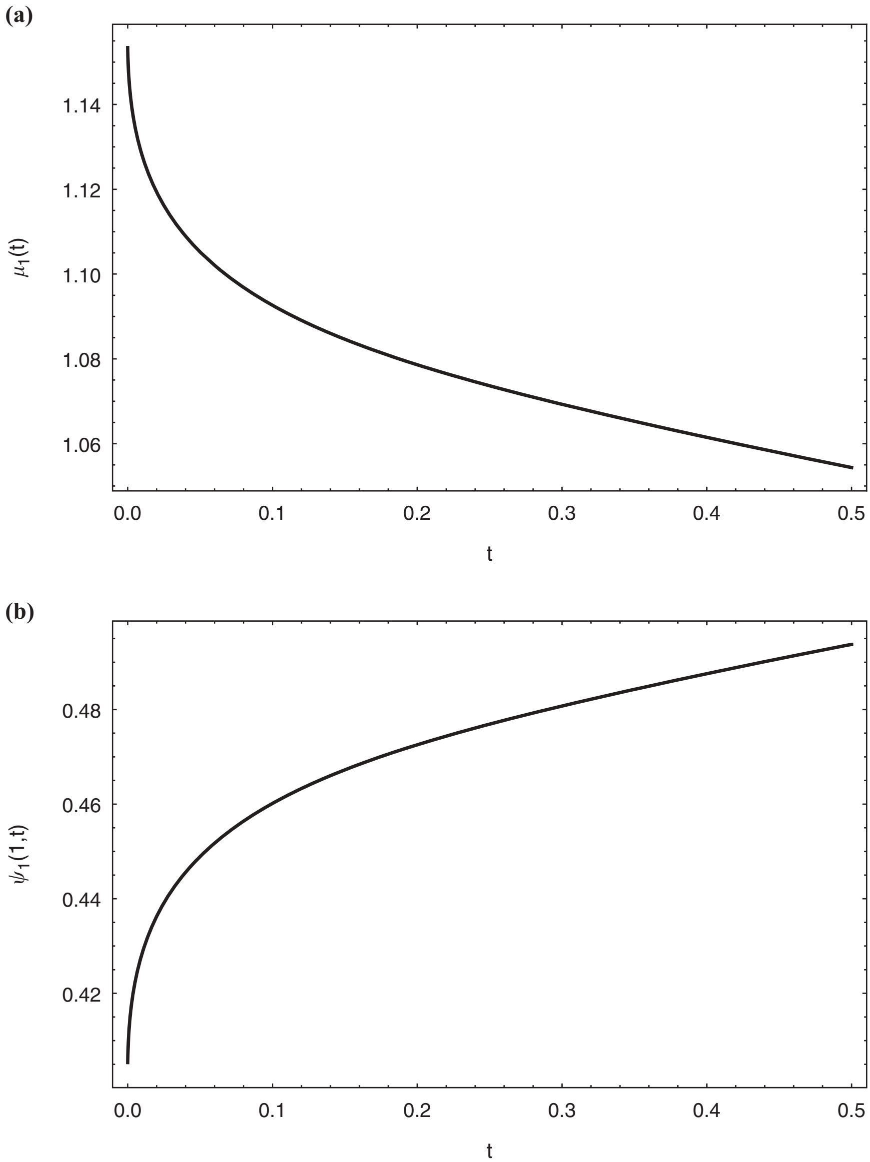


Figure 1.
Results for the potential distribution with: (a) $Bi_c = 1$, $Bi_c = 1$ and $\gamma = 1/3$; (b) $Bi_c = 0$, $Bi_r = 20$ and $\gamma = 1/3$

Notes: Dotted: NDSolve/*Mathematica*; dotted-dashed: GITT with nonlinear eigenvalue problem; dashed: GITT with linear eigenvalue problem

6. Conclusions

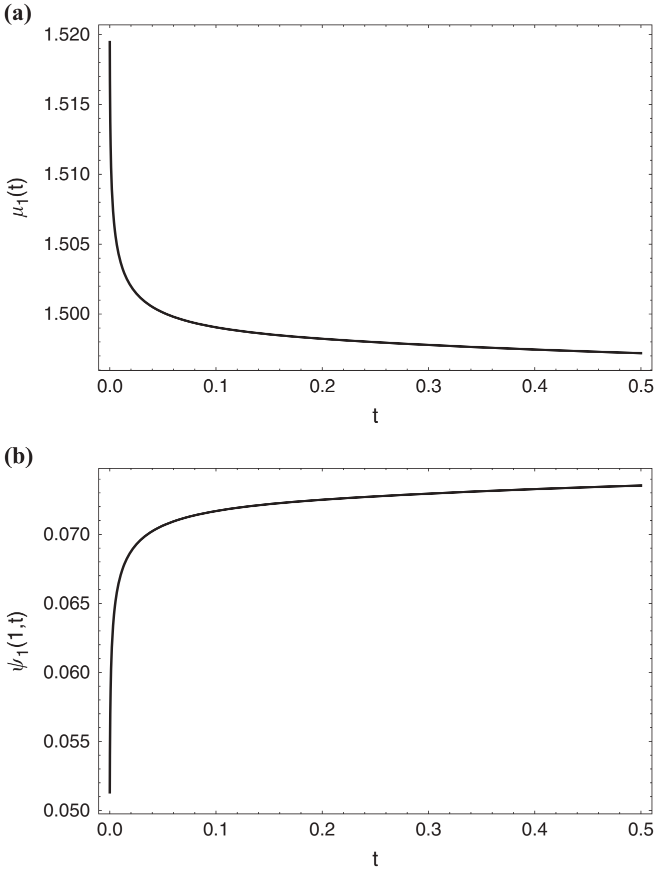
A nonlinear eigenvalue problem approach for the integral transform solution of convection-diffusion problems with nonlinear boundary condition coefficients has been here proposed. The approach is based on the integral transformation of the original eigenvalue problem itself, yielding a nonlinear algebraic eigenvalue problem coupled to the ordinary differential system for the transformed potentials. An example of diffusion with nonlinear boundary conditions is examined more closely, and the approach is demonstrated by handling both the general nonlinear eigenvalue problem approach and the traditional GITT solution with a linear eigenvalue problem. The algorithm can be implemented either as a DAEs system formed by the transformed system and the transcendental equations for the nonlinear eigenvalues or, alternatively, as an ODE system which involves the differentiation in time of the eigenvalues



Note: Case: $Bi_c = 1$, $Bi_r = 1$ and $\gamma = 1/3$

Figure 2.
Transient behavior of: (a) first eigenvalue; (b) first eigenfunction at $x = 1$

transcendental equations, yielding a simultaneous system of ODEs for both the time-dependent eigenvalues and the transformed potentials. The excellent convergence behavior of the proposed nonlinear eigenfunction expansions was then illustrated and the approach was numerically verified, by comparing against a numerical solution through the Method of Lines with refined mesh, as available in the NDSolve routine (Wolfram Research Inc., 2016), and the traditional GITT solution with a linear



Note: Case: $Bi_c = 0$, $Bi_r = 20$ and $\gamma = 1/3$

Figure 3. Transient behavior of: (a) first eigenvalue; (b) first eigenfunction at $x = 1$

Table III. Comparison of the numerical solution (NDSolve), against the nonlinear GITT solutions with ODEs or DAEs for the eigenvalues

	$T(x,t=0.3)$				
x	Numerical[a] NDSolve	ODEs eig.probl.	Dev. (%)	DAEs eig.probl.	Dev. (%)
0	0.645277	0.645279	0.0001834	0.645279	0.0001651
0.1	0.638131	0.638132	0.0001755	0.638132	0.0001580
0.2	0.616835	0.616836	0.0001516	0.616836	0.0001364
0.3	0.581824	0.581824	0.0001072	0.581824	0.0000954
0.4	0.533816	0.533816	0.0000359	0.533816	0.0000284
0.5	0.473817	0.473816	0.0000769	0.473816	0.0000798
0.6	0.403108	0.403107	0.0002560	0.403107	0.0002548
0.7	0.323232	0.323230	0.0005594	0.323230	0.0005557
0.8	0.235965	0.235963	0.0011315	0.235963	0.0011290
0.9	0.143284	0.143281	0.0025180	0.143281	0.0025260
1.0	0.0473142	0.0473096	0.0096640	0.0473096	0.0097402

Notes: $Bi_c = 0$; $Bi_r = 20$; $\gamma = 1/3$; $t = 0.3$. [a]NDSolve routine – *Mathematica* system (Wolfram Research Inc., 2016) eigenvalue problem. This novel approach can be further extended to various classes of nonlinear convection-diffusion problems, either already solved by the GITT with a linear coefficients basis, or new challenging applications with more involved nonlinearities.

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