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Assessing temporal trends in Copula based Value-at-Risk using Local Estimation

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Abstract

In this paper we use local estimation to assess temporal trends in copula based Value-at-Risk (VaR), despite the developed method be able to be applied to any risk measure based in the probability distribution of an asset portfolio. The estimation of any quantile-based risk measure, in particular the VaR, relies on the correct specification of the multivariate probability distribution of the assets composing the portfolio. Temporal changes in the portfolio volatility may follow from the autocorrelations in the squares of each margin, as well as from changes over time in the dependence structure among the components. In order to assess and model temporal trends in copula parameters, we used local likelihood methods. First it is carried on some Monte Carlo simulation experiments in order to illustrate the methodology, then we apply the methods for the VaR valuation of 10 portfolios composed of international stock indexes. Through the use of out-of-sample tests, we found that the local estimation outperforms the global estimation procedure.

Keywords: Copulas; Local maximum likelihood estimation; Value-at-Risk.

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1 Introduction

In this paper we use local estimation to assess temporal trends in copula based risk measures. For this, we chose the well known Value-at-Risk (VaR), despite the developed method be able to be applied to any risk measure based in the probability distribution of an asset portfolio. Loosely speaking, the VaR of a portfolio is the value large enough to cover its losses over a N-day holding period with a probability of $(1-\alpha)$, usually denoted by VaR(α, N). The bottom line in the computation of the VaR, is the estimation of the $(1-\alpha)$-quantile of the distribution F of the portfolio. Since F is unknown, it reduces to the estimation of F. The historical or empirical VaR, for example, is computed using the empirical distribution function as an estimate of F.

For multivariate portfolios, or portfolios with 2 or more assets, the VaR may be computed by estimating the multivariate distribution of the d assets composing the portfolio. Under this approach one explicitly takes into account the dependence structure of the d assets composing the portfolio. However, the most commonly used technique by practioners is based on the simplifying assumption of multivariate normality.

It is well documented in the literature all the pitfalls in the non-conditional multivariate normality assumption. The use of time-varying volatility models and copulas can solve part of these pitfalls. However, to capture all dynamics found in the multivariate distribution of financial returns, the fit of marginal time-varying volatility is not enough, somehow it is also necessary the modeling of the temporal changes in the dependence structure. This can be accomplished using multivariate GARCH models (Engle, 2000 and Tse and Tsui, 2002), or time-varying correlation coefficient (Cherubini et al. (2004)), or time-varying copulas (Dias and Embrechts (2004) and Mendes (2005)). One should note that besides the VaR estimation, the time-varying estimation of the multivariate distribution is the basis for many important financial applications, for example, portfolio selection, option pricing and asset pricing models.

In this paper we use an alternative approach for assessing temporal trends in the parameters of the copula linking the assets composing the portfolio. We propose the use of a local estimation procedure to capture the dynamics of the parameters in a non-parametric way. We provide some illustrations using the Gaussian and Gumbel copulas. Then we apply the methodology to 10 equally weighted portfolios composed of international indexes. Having found temporal trends in the copula parameters, this information may be incorporated in a parametric model which may then be used for predictions. Finally, we perform out-of-sample tests to evaluate the performance of the proposed estimation procedure. The back tests used are the well known Kupiec test and the loss function test. To the best of our knowledge no one has used yet local maximum likelihood estimation for copulas.

Copulas have become standard tools in fields of finance and insurance (see Georges et al. (2001), Embrechts et al. (2003), Cherubini et al. (2004), Fermanian and Scaillet (2004), among others). Applications using dynamic copulas were proposed more recently. Patton (2001) extends the standard definition of copula to the conditional case. Thus, he introduces the copula theory to model time-varying conditional dependence. His interest consists in taking into account the well-known heteroscedastic pattern, widely reported in the financial literature, for the volatility of any financial return time series. Further, there are many situations where the entire conditional joint density is required, such as the pricing of financial options with multiple underlying assets, or in the calculation of portfolio VaR, as previously discussed. Modeling exchange rates, he assumed a bivariate Gaussian conditional copula with the correlation coefficient following a GARCH-type model. He considered also structural breaks and asymmetric copulas. Similarly, Genest et al. (2003) allowed the Kendall's correlation coefficient to evolve through time according current values of the conditional marginal variances.

Cherubini et al. (2004) make another application with conditional copulas. They fit GARCH(1,1) for the marginals and model the dependence structure with a Gaussian copula with time-varying correlation coefficient. The chosen model for the copula parameter is the modified logistic function (also known as hyperbolic tangent function), necessary to keep ρ_t belonging to the interval $(-1, 1)$.

Fermanian and Scaillet (2004) introduced the concept of pseudo-copulas. They showed that the copula models defined in Patton (2001), Rockinger and Jondeau (2001), and also in Genest et al. (2003) are all pseudo-copulas. They proposed a nonparametric estimator of the conditional pseudo-copulas, derived its normal asymptotic distribution, and built up a goodness of fit test statistics.

Time varying dependence structure was also considered by Van Den Goorbergh et al. (2005) for modelling the relation between bivariate option prices and the dependence structure of the underlying financial assets. Patton (2003) found time variation to be significant in a copula model for asymmetric dependence between two exchange rates where the dependence parameter followed an ARMA type process.

In Section 2 we provide a brief review of copula definitions and the classical way of parameter estimation. In Section 3, it is introduced the theory about local maximum likelihood estimation procedure. In Section 4, we provide illustrations of the proposed method using the Gaussian and Gumbel copulas in order to assess models suitability for capturing the evolution of copula parameters. In Section 5 we present an application where we compute the conditional-local-estimated VaR for several pairs of stock indexes. And finally, in Section 6, we provide some concluding remarks.

2 Copulas and classical estimation

To simplify the notation, from now on we set $d = 2$ even though the inference method in the paper is intended and work for dimensions $d \geq 2$. Let (X_1, X_2) be a continuous random variable (rv) in $\Re^2$ with joint distribution function (cdf) F and margins F_i, $i = 1, 2$. Consider the probability integral transformation of X_1 and X_2 into uniformly distributed random variables (rvs) on $[0, 1]$ (denoted $uniform(0, 1)$), that is, $(U_1, U_2) = (F_1(X_1), F_2(X_2))$. The copula C pertaining to F is the joint cdf of (U_1, U_2). As multivariate distributions with $uniform(0, 1)$ margins, copulas provide very convenient models for studying dependence structure with tools that are scale-free. It summarizes the dependence structure independently of the specification of the marginal distribution.

From Sklar's theorem (Sklar, 1959) we know that for continuous random variables there exists a unique 2-dimensional copula C such that for all $(x_1, x_2) \in [-\infty, \infty]^2$,

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)). \tag{1}$$

To measure monotonic dependence, one may use the copula based Kendall's τ correlation coefficient. Kendall's τ does not depend upon the marginal distributions and is given by:

$$\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1 \ . \tag{2}$$

This invariance property is not shared by the Pearson linear correlation coefficient ρ, which is actually the Spearman correlation coefficient between X_1 and X_2. In order to measure (upper) tail dependence one may use the upper tail dependence coefficient defined as:

$$\lambda_U = \lim_{\alpha \to 0^+} \lambda_U(\alpha) = \lim_{\alpha \to 0^+} .Pr\big(X_1 > F_1^{-1}(1-\alpha) \mid X_2 > F_2^{-1}(1-\alpha)\big)$$

If this limit exists, and where F_i^{-1} is the generalized inverse of F_1 or F_2, i.e., $F_i^{-1}(u_i) = sup\{x_i | F_i(x_i) \leq u_i\}$, for $i = 1, 2$. The lower tail dependence coefficient λ_L

is defined in a similar way. Both the upper and the lower tail dependence coefficients may be expressed using the pertaining copula:

$$\lambda_U = \lim_{u \to 1^-} \frac{\overline{C}(u,u)}{1-u}, \;\; where \;\; \overline{C}(u_1,u_2) = Pr\big(U_1 > u_1, U_2 > u_2\big) \;\; and \;\; \lambda_L = \lim_{u \to 0^+} \frac{C(u,u)}{u}$$

If these limits exist. The measures $\lambda_U \in (0,1]$ (or $\lambda_L \in (0,1]$) quantify the amount of extreme dependence within the class of asymptotically dependent distributions. If $\lambda_U = 0 (\lambda_L = 0)$, the two variables X_1 and X_2 are said to be asymptotically independent in the upper (lower) tail.

In the case the true copula belongs to a parametric family $\{C_\theta, \theta \in \Theta\}$, estimates of the parameters may be obtained through the IFM method mentioned in the Introduction, in the context of independent and identically distributed observations. There are mainly two versions: the fully parametric and the semiparametric approaches, detailed in Genest et al. (1993), Shi and Louis (1995), Joe (1999), and Chebrian et al. (2002). The fully parametric approach relies on the assumption of parametric marginal distributions. The $uniform(0,1)$ data, obtained from the estimated marginals, are used to maximize the copula density function with respect to θ. The final results are very sensitive to the right specification of all marginals, but, as shown by Genest, Ghoudi, and Rivest (1995), and Shih and Louis (1995), the resulting estimator is consistent and asymptotically normal distributed. In the semiparametric method, the standardized data in the first step are obtained as the empirical cdfs. In this case, the estimation procedure suffers from loss of efficiency, see Genest and Rivest (1993), even though many authors use it to avoid misidentification of the marginal cdfs (Frahm, Junker, Schmid, 2004).

The behavior of the maximum likelihood estimators of copula parameters were investigated through simulations by Capéraà (1997) in the case of the Gumbel or logistic model, by Genest (1987) in the case of the Frank family, and by Mendes, Melo and Nelsen (2007) in the case of the Joe-Clayton copula. Genest (1987) investigated the performance of four estimators considering samples of size 10 to 50, and found that the method of moments estimator appears to have smaller mean squared error than the maximum likelihood estimator. Goodness of fit tests for copulas and alternative tools for checking the quality of fits are discussed in Fermanian (2003), Chen and Fan (2005), among others.

In what follows we assume the margins have been already properly estimated and concentrate on fitting copulas robustly. In our application to real data, we fit GARCH models to the behavior of the series of marginal returns.

3 Local maximum likelihood estimation

Local likelihood is a kernel-based approach to nonparametric regression. Methods for local likelihood inference were proposed by Tibshirani and Hastie (1987). Consider a model in which the parameter vector is assumed to be locally constant over time. The local likelihood estimator for θ under this model is

$$\hat{\theta}_j = argmax \sum_{J=1}^{n} K(t_J - t_j; h).log\Big(c(u_J, v_J, \theta_j)\Big) \tag{3}$$

where K is a positive symmetric function whose variability is determined solely by the temporal bandwidth $h > 0$ and $c(u, v)$ is the copula density function. The estimator is based upon maximizing a weighted sum of log-likelihood contributions corresponding to different times, and so can be viewed as a generalization of the idea of a weighted average. The weights are determined entirely by the time separation $t_J - t_j$, through the kernel function K. The exact choice of kernel function is not critical. In this paper, we take K to be an appropriately normalized normal density function with zero mean and standard deviation h (a standard choice). The estimator is biased (as are all nonparametric regression estimators), and the choice of the bandwidth h is a classic example of a trade-off between bias and variance. The procedure for the parameter estimation in each time was quite simple. It consisted of several optimizations, one for each time, where the likelihood function was maximized weighted by kernel function.

Uncertainty in local likelihood estimation can be quantified using the semi-parametric bootstrap scheme outlined in Davison & Ramesh (2000). If the number of bootstrap samples is sufficiently large then, under weak assumptions, the empirical variability within the bootstrapped estimates will give a good approximation to the uncertainty within the local likelihood estimator itself (Davison & Hinkley, 1997). The construction of bootstrap confidence intervals about parameter estimates is, however, complicated by the presence of bias in the estimates. Methods for constructing confidence intervals which account for this bias have been proposed (Deciccio & Romano, 1988), but are typically rather complicated and require a large number of bootstrap samples in order to produce reasonable intervals.

Bowman & Azzalini (1997) suggest that it is both straightforward and intuitively reasonable to construct bands which capture variability in the parameter estimates without correcting for bias ("variability bands"), but note that these bands formally amount to pointwise confidence intervals about $E(\hat{\theta}_j)$ rather than about θ_j, and therefore require careful interpretation. In this paper we construct variability bands simply by centering the empirical distribution of bootstrapped estimates about the maximum local likelihood estimate, without imposing any additional assumption of symmetry or normality.

In order to generate the bootstrap samples, we did the following steps: (i) we sampled form the Normal and Gumbel copula with the estimated parameter value for each time, obtaining a bivariate time series; and (ii) we used the algorithm of local maximum likelihood estimation to estimate over time the parameters of the generated bivariate time series. We repeated steps (i) and (ii) five hundred times for the construction of the bootstrap confidence intervals.

3.1 Bandwidth choice and model diagnostics

The bandwidth h determines smoothness in the local regression model, and bandwidth selection is consequently a key element of the local likelihood methodology. In many contexts it is most sensible to choose the bandwidth "subjectively", based upon either:

1) a *priori* information about the scale at which trends might be expected to operate; or

2) a visual assessment of the validity of fitted trends produced by different bandwidths.

We tried different values for h and observed the behavior of the estimates over time and choose the value of 25, because, with this value, we got a good balance between local estimation and smoothness. No matter how the bandwidth is chosen, it is important to verify that the fitted local model provides a reasonable fit to the data.

4 Simulations

We now report the results from four simulation experiments. For the first two experiments, we simulate from a Gaussian copula with constant and time-varying parameter, and estimate using the proposed local method. The other two examples are based in a Gumbel copula. For experiments 2 and 4, we observe the effect of a time-varying parameter in the valuation of the Value at Risk of a hypothetical portfolio.

Experiment 1: True model is Gaussian constant on time (ρ fixed) but the time varying model is assumed and estimated. We generate bivariate data (sample size 500) based on a Gaussian copula with ρ fixed and equal to 0.37. Using the local maximum likelihood method we estimate the model based on a Gaussian copula. For this experiment, the Kendall's tau is constant and equal to 0.24. We assessed the variability of our estimates trough bootstrap confidence intervals over time (figure 1).

Experiment 2: True model is a linear time-varying Gaussian. A global and a local model copula are estimated. In this experiment we mimic a situation where the strength of dependence depends upon time. The true data generating process is Gaussian such that the ρ parameter varies linearly on time t starting as 0.05 till 0.70 in 500 days. For this experiment, the Kendall's tau varies from 0.03 to 0.49 in 500 days. The margins for the VaR valuation are t-Student with 6 and 4 degrees of freedom. We assessed the variability of our estimates trough bootstrap confidence intervals over time (figure 2).

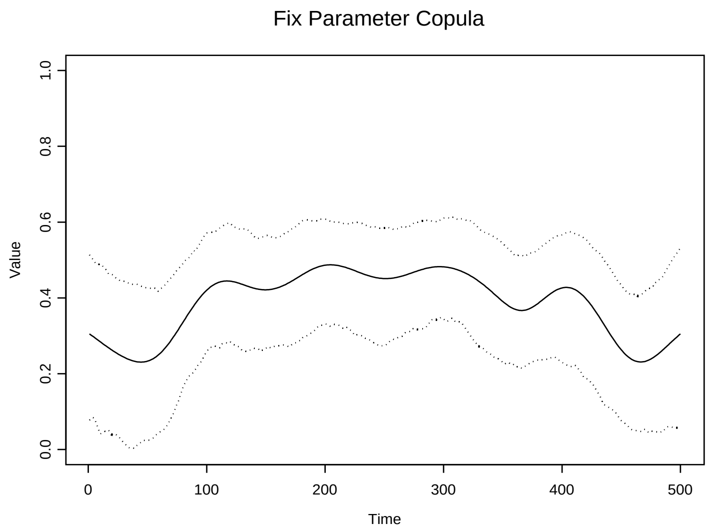

Figure 1 - Plot of the estimates and the 95% confidence interval for the illustrative example 1, which the simulated copula parameter is fix over time.

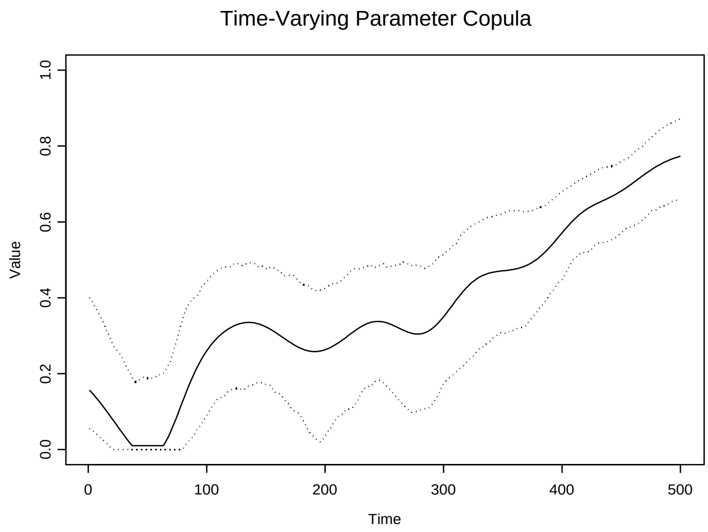

Figure 2 - Plot of the estimates and the 95% confidence interval for the illustrative example 2, which the simulated copula parameter is time-varying.

Considering t-Student marginals with 6 and 4 degrees of freedom for the assets returns, respectively, we also computed the conditional VaR(1%) for the second experiment and plotted it over time (figure 3). This is the VaR of a short position in an equally weighted portfolio of the two assets. The straight line is the VaR(1%) evaluated using a global maximum likelihood estimation procedure.

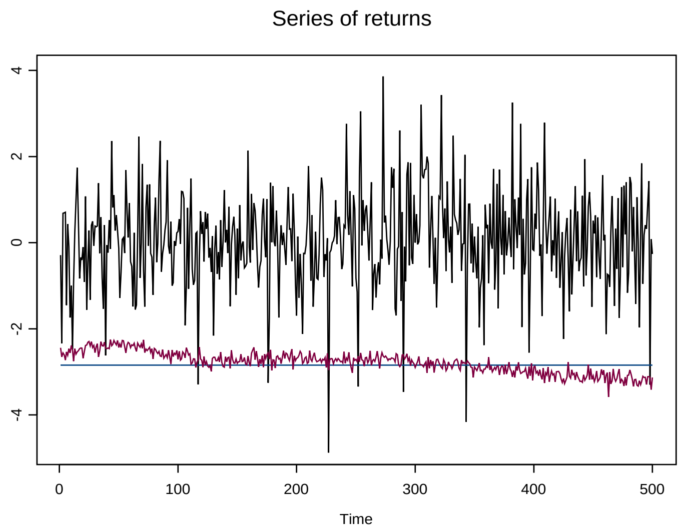

Figure 3 - Plot of the VaR(1%) measures over time for the second example, considering local and global (straight line) maximum likelihood estimation for a short position in an equally weighted portfolio of two assets.

Experiment 3: True model is Gumbel constant on time (δ) but the time varying model is assumed and estimated. We generate bivariate data (sample size 500) based on a Gumbel copula with δ fixed and equal to 1.73. Using the local maximum likelihood method we estimate the model based on a Gumbel copula. For this experiment, the Kendall's tau is constant and equal to 0.42. We assessed the variability of our estimates trough bootstrap confidence intervals over time (figure 4).

Experiment 4: True model is a linear time-varying Gumbel. A global and a local model copula are estimated. In this experiment we mimic a situation where the strength of dependence depends upon time. The true data generating process is Gumbel such that the δ parameter varies linearly on time t starting as 1.00 till 3.00 in 500 days. For this experiment, the Kendall's tau varies from 0.00 to 0.67 in 500 days. The margins for the VaR valuation are t-Student with 6 and 4 degrees of freedom. We assessed the variability of our estimates trough bootstrap confidence intervals over time (figure 5).

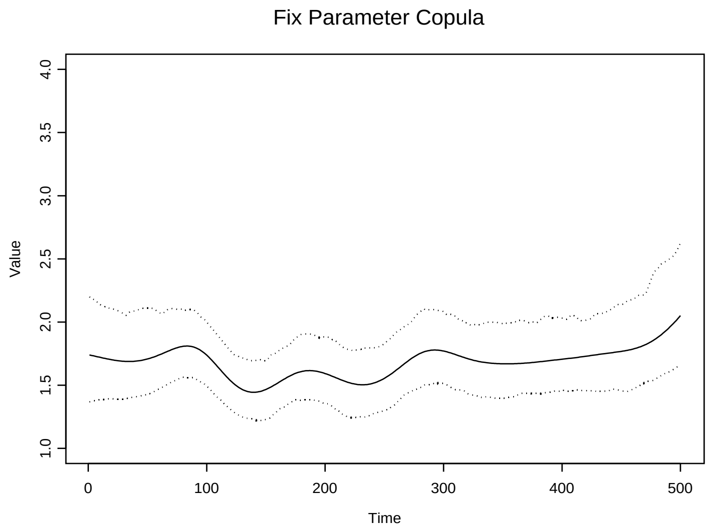

Figure 4 - Plot of the estimates and the 95% confidence interval for the illustrative example 3, which the simulated copula parameter is fix over time.

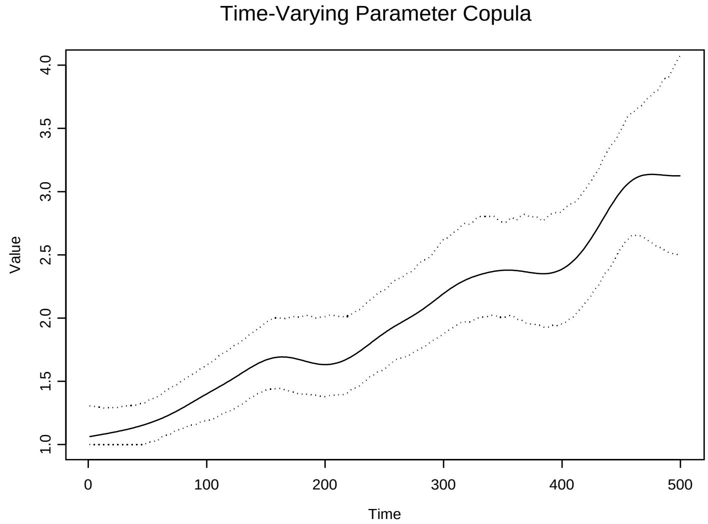

Figure 5 - Plot of the estimates and the 95% confidence interval for the illustrative example 4, which the simulated copula parameter is time-varying

Considering t-Student marginals with 6 and 4 degrees of freedom for the assets returns, respectively, we also computed the conditional VaR(1%) for the fourth experiment and plotted it over time (figure 6). This is the VaR of a short position in an equally weighted portfolio of two assets. The straight line is the VaR(1%) evaluated using a global maximum likelihood estimation procedure.

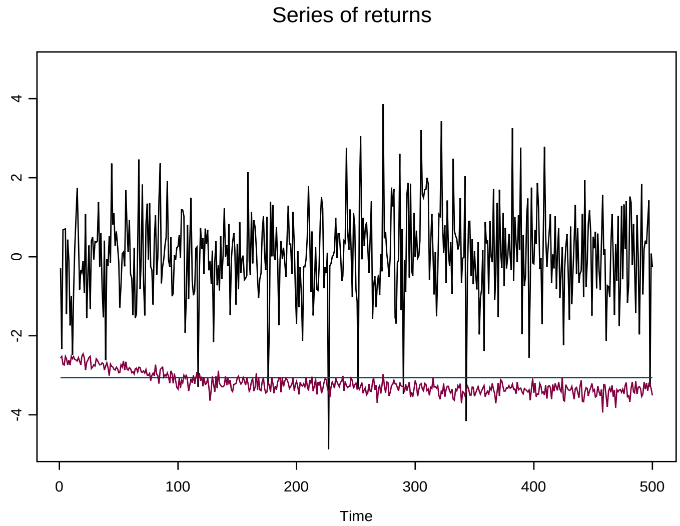

Figure 6 - Plot of the VaR(1%) measures over time for the fourth experiment, considering local and global (straight line) maximum likelihood estimation for a short position in an equally weighted portfolio of two assets.

Through the above simulation experiments, it seems clear the difference between the estimated VaRs when a global maximum likelihood is used in a case where the dependence strength changes on time. In the Gumbel copula example, since the upper tail dependence increases over time, a short position in the portfolio carries on a big risk compared to the Gaussian case, because the probability of extreme negatives returns is high. In such a case, the use of local estimation procedures is more suitable in order to capture the dynamics of the dependence conditioned on time.

5 Applications

In this section we use several pairs of stock markets indexes to empirically show how temporal changes in the copula parameters carry over the estimate of the Value-at-Risk. We test and find that the local estimation of the dependence structure may provide better one-step-ahead VaR forecasts. Moreover, once a temporal trend is detected, it may be incorporated in the k-steps-ahead VaR forecasts, $k \geq 1$.

We explain in detail the application for the pair composed by the Brazil and the Mexico indexes daily log-returns, from January 3rd, 1994 to December 30th, 2005. This period of time provides 3,130 observations for each index. We summarize the results for the other 9 pairs in Table 1, composed by the following indexes: (i) US (S&P500), (ii) UK (FTSE), (iii) Brazil (IBOVESPA), (iv) Argentina (MERVAL) and (v) Mexico (IPC). For the assessment of the performance of our methodology, we separated the last 100 observations from the sample in order to compare, through back-testing, the conditional-local-estimated-copula VaR with the conditional-global-estimated-copula VaR.

For the modeling of the conditional means and variances of the log-returns we use GARCH(1,1) models, which are fitted to each margin by maximum likelihood. The choice of the conditional error distribution for each margin was based on the AIC criterion. The conditional distributions were as follows: (i) S&P500, t-student ($\nu_{SP500} = 8$), (ii) FTSE, t-student ($\nu_{FTSE} = 13$), (iii) IBOVESPA, t-student ($\nu_{IBOVESPA} = 7$), (iv) MERVAL, t-student ($\nu_{MERV} = 5$) and (v) IPC, t-student ($\nu_{IPC} = 5$). Figure 7 shows the bivariate series of log-returns at the left hand side (IBOVESPA x IPC), and the corresponding standardized residuals properly transformed into $Uniform(0;1)$ data at the right hand side.

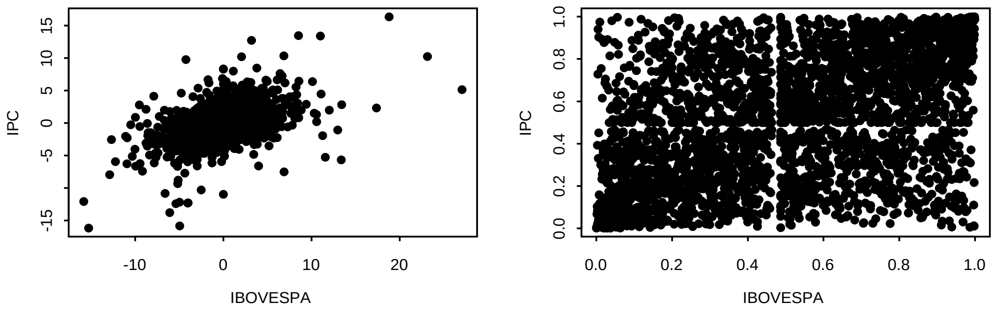

Figure 7 - Bivariate series of log-returns (IBOVESPA and IPC) at the left hand side, and the corresponding residuals standardized into $Uniform(0;1)$ data at the right hand side.

The Gaussian copula is then fitted by global and local maximum likelihood to the standardized data. For this pair, the global fit was provided by a Gaussian copula with the constant parameter estimate of $\widehat{\rho} = 0.32$. Figure 8 shows the realized path of the estimates $\widehat{\rho_t}$ from the local fit for the entire sample period. We can observe a positive trend in the correlation coefficient on time. Specially in the last 500 observations, it is observed an increase in the dependence strength through time. This sort of behavior may be better modeled by local estimation procedures.

For each time t from 3,031st till the 3,130th observation, we performed the following algorithm for the VaR calculation : (i) we fit the GARCH(1,1) for each margin in the sample, (ii) the bivariate residuals are transformed in $Uniform(0;1)$ data, (iii) the parameter of the Gaussian copula is estimated locally, (iv) we estimate a trend in the previous 42 estimates (2 months) of the copula parameter trough ordinary least squares, (v) we incorporate the estimated trend in the last estimate of the parameters locally fitted, (vi) we simulate 1,000 runs from the copula with this parameter forecast, (vii) we apply the inverse cumulative distribution function, according to the t-student conditional distributions used in the GARCH fit and, finally, (viii) using the simulated portfolio returns, the VaR is calculated trough the $\alpha\%$ quantile.

For the globally estimation procedure, the algorithm is the same, except for the steps (iii), (iv) and (v). These steps are substituted by the following step: the parameter of the Gaussian copula is estimated globally. After the last step of the algorithm, one observation is added to the sample and the procedure is repeated until the 1000th observation. For each pair of indexes, we get two time-series of Value at Risk values, which will be compared with the real observations in a out-of-sample back test.

At the right hand side of Figure 9 we see the evolution through time of the one-step-ahead conditional VaR(5%) estimates based on the two fits, using the Brazilian and Mexican indexes in an equally weighted portfolio.

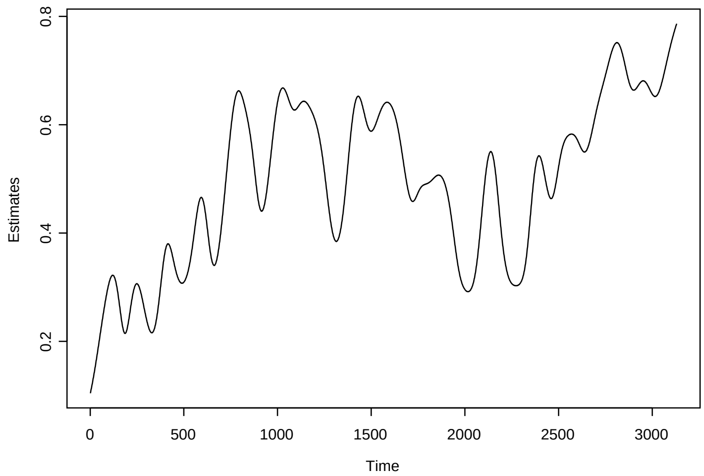

Figure 8 - Evolution through time of the Pearson correlation estimates between IBOVESPA and IPC for the Gaussian fit.

As observed above, one possible application of the proposed models and procedure is the computation of the conditional-local-estimated Value-at-Risk. Supposing a long position in a equally weighted portfolio of these two assets, we show in Figure 9 the series of log-returns of this portfolio with the conditional VaR($\alpha = 5\%$) evaluated with copulas estimated globally and locally (first column). The second column shows the time series of the conditional VaR($\alpha = 5\%$) evaluated with copulas estimated globally (blue color) and locally.

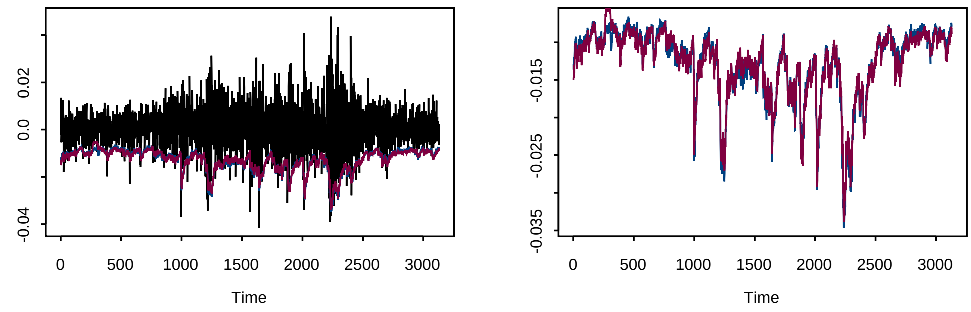

Figure 9 - Series of log-returns with the conditional VaR($\alpha = 5\%$) evaluated with copulas estimated globally and locally (first column) and time series of the conditional VaR (second column)

One should note that the variation in the VaR values comes from the conditional modelling of the variance (GARCH) as well as from the locally estimation of the copula parameters. In order to verify the accuracy of the risk measures, we performed the Kupiec LR test (Kupiec, 1995) on the empirical failure rates. This test is based on binomial theory and tests the difference between the observed and expected number of VaR exceptions of the effective portfolio losses.

The VaR measure is based on a $1-\alpha$ confidence level, so when we observe N losses in excess of VaR out of T observations, it is experienced a N/T proportion of excessive losses. The Kupiecs test answers the question whether N/T is statistically significant different from α. Following binomial theory, the probability of observing N failures out of T observations is $(1-\alpha)^{T-N}.\alpha^N$, so that the test of the null hypothesis $H_0 : \alpha = \alpha^*$ is given by a likelihood ratio test statistic:

$$LR = 2\left(ln\left[\left(1-\frac{N}{T}\right)^{T-N}\left(\frac{N}{T}\right)^{N}\right] - ln\left[(1-\alpha^*)^{T-N}\alpha^{*N}\right]\right) \tag{4}$$

which is distributed as χ^2_1 under H_0. It is well known that the power of this test, that is the ability to reject a bad model, rises with T. In this paper, we are working with 100 observations. Assuming a Gaussian copula, the real VaR exceptions and the Kupiec test p-values are reported in Table 1.

Table 1: *VaR back-testing analysis (globally and locally estimation of Gaussian Copulas). RE - Real exceptions in % and (p-value) of the Kupiec test.*

	Global Estimation		**Local Estimation**	
	α= 5%	α= 1%	α= 5%	α=1%
Portfolio	RE%(p-value)	RE%(p-value)	RE%(p-value)	RE%(p-value)
BR and MX	2.0(0.12)	0.0(0.16)	2.0(0.12)	0.0(0.16)
BR and AR	6.0(0.64)	0.0(0.16)	5.0(0.98)	0.0(0.16)
BR and US	5.0(0.98)	1.0(0.99)	5.0(0.98)	1.0(0.99)
BR and UK	4.0(0.65)	0.0(0.16)	6.0(0.64)	0.0(0.16)
MX and US	4.0(0.65)	0.0(0.16)	5.0(0.98)	0.0(0.16)
MX and UK	2.0(0.12)	0.0(0.16)	2.0(0.12)	0.0(0.16)
MX and AR	5.0(0.98)	1.0(0.99)	5.0(0.98)	1.0(0.99)
AR and US	6.0(0.64)	1.0(0.99)	4.0(0.65)	1.0(0.99)
AR and UK	2.0(0.12)	1.0(0.99)	2.0(0.12)	1.0(0.99)
US and UK	1.0(0.03)	0.0(0.16)	1.0(0.03)	0.0(0.16)

The results form the Kupiec's test of the local-estimation-VaR are quite similar to the ones presented by the global-estimation-VaR. However, when they are not equal, the local-estimation-VaR has a better performance. Since the number of exceptions are low as well as the results of the Kupiec's test are almost equal, we also compared the risk measures based in different estimation methods by means of a loss function.

This was first introduced by Lopez (1998). The general form of these loss functions is:

$$C_t = \begin{cases} f(L_t, VaR_t) & \text{if } L_t < VaR_t \\ g(L_t, VaR_t) & \text{if } L_t \geq VaR_t \end{cases}$$

where $f(x, y)$ and $g(x, y)$ are arbitrary functions such that $f(x, y) \geq g(x, y)$ for a given y. The score of the complete sample is:

$$C = \sum_{i=1}^{N} C_i$$

where N is the size of the sample for the test. Accurate VaR estimates will generate the lowest possible score C. For this test, we considered the suggestion of Blanco and Ihle (1999) for the loss function:

$$C_t = \begin{cases} \frac{L_t - VaR_t}{VaR_t} & \text{if } L_t < VaR_t \\ 0 & \text{if } L_t \geq VaR_t \end{cases}$$

where L_t is the return of the portfolio for the day t. The results for this test are reported in Table 2.

Table 2: *VaR back-testing analysis (globally and locally estimation of Gaussian Copulas). Score of the loss function.*

	α= 5%		α=1%	
	Global	**Local**	**Global**	**Local**
BR and MX	0.97	0.69	0.00	0.00
BR and AR	1.45	0.83	0.00	0.00
BR and US	0.90	0.84	0.01	0.00
BR and UK	0.66	0.66	0.00	0.00
MX and US	1.22	1.12	0.00	0.00
MX and UK	0.73	0.50	0.00	0.00
MX and AR	1.85	1.48	0.10	0.11
AR and US	2.10	1.69	0.37	0.29
AR and UK	1.23	1.08	0.50	0.18
US and UK	0.38	0.32	0.00	0.00

Based in the loss function considered above, it can be observed in Table 2 the better performance of the local-estimation VaRs compared to the global-estimation ones. For $\alpha = 1\%$, the number of exceptions is low so that the losses functions are not differentiated for most of the portfolios though the local estimation also presents better loss scores.

6 Conclusions

In this paper we proposed a model for locally estimate the dependence structure of a set of financial returns. Our modeling strategy allows for the margins to follow some GARCH type model while the copula dependence structure is estimated locally and changes over time. Our exposition was restricted to the bivariate case, but models can be easily implemented and run relatively fast in higher dimensions.

Research on time varying copulas is still in its infancy. There are many open questions, and there is room for both theoretical and computational developments. Many applications will naturally follow. In this paper we provided empirical evidence that the dependence structure (given by its copula) among asset returns may be best represented by a time varying copula or by a local estimation.

Through simulation experiments and out-of-sample back tests, like the Kupiec test and the use of a loss function, based in real portfolios we showed that the Value-at-Risk estimation may be improved by using a local estimation procedure. These findings can be used by investors selecting portfolio components or by fund managers improving the accuracy of their risk measures.

Appendices

Gaussian Copula. It is an Elliptical Copula, which are simply the copulas pertaining to elliptical distributions. The Gaussian or Normal copula is the copula pertaining to the multivariate normal distribution. It is given by

$$C_{\rho}^{Ga}(u,v) = \int_{-\infty}^{\phi^{-1}(u)} \int_{-\infty}^{\phi^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\Big(- \frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)}\Big) dsdt \qquad (5)$$

where ρ is simply the linear correlation coefficient between the two random variables.

Gumbel copula: The well known Gumbel copula (Gumbel, 1960) is an Extreme Value copula as well as an Archimedean copula. It is the dependence structure corresponding to the symmetric logistic model (see Ghoudi, Khoudraji and Rivest, 1998), and has the following form:

$$C_{Gu}^{\delta}(u,v) = \exp\left\{ - \left(\tilde{u}^{\delta} + \tilde{v}^{\delta}\right)^{1/\delta} \right\} \ . \qquad (6)$$

The coefficient of tail dependence is given by $\lambda_U = 2 - 2^{1/\delta}$.

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