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"PORTFOLIO SELECTION IN AN ECONOMY WITH MARKETABILITY AND SHORT SALES RESTRICTIONS"

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I. INTRODUCTION

The equilibrium pricing of marketable and risky assets in an economy with non-marketable assets was examined by Mayers[7], [8] and Brito [4]. Assuming that

(i) short-sales of all marketable assets are feasible and that

(ii) the distribution of terminal returns on non-marketable assets is exogenously given and does not depend upon the set of marketable investment opportunities available to investors,

they derive equilibrium relationships for the pricing of marketable assets.

This paper will relax assumption (i), examining the portfolio selection problem of an investor who holds non-marketable assets and cannot short-sell marketable and risky assets. He faces both marketability and short-sales restrictions in the economic setting of this work. The most obvious and probably the most significant example of non-marketable assets are human capital or occupational assets. Non-marketable assets and occupational assets will thus be treated as equivalent expressions throughout this paper.

Our investor is assumed to be an expected utility of terminal wealth maximizer in the context of a one-period model and his utility function is assumed to be a monotonically increasing and concave function of wealth. The investor's expectations are assumed to be represented by a joint distribution of all assets (marketable and non-marketable) that is multivariate normal with a positive-definite variance-covariance matrix of marketable and risky assets and this distribution is assumed to induce a convergent and well defined expected utility integral. Under these assumptions the portfolio decisions of our investor can be examined in the mean-standard deviation space with a strictly convex indifference map. Finally it is assumed that a riskless asset exists in the economy and that our investor can freely borrow and lend at the riskless rate.

It will be shown that, in an economy with marketability restrictions and without short-sales, the risky portfolio selection problem of an investor can be considered in the Reward-to-Variability ratio (RV)-correlation with the occupational asset of the investor (ρ) space. The indifference map induced in this space is convex, RV is a "good" and ρ is a "bad" and thus only portfolios of marketable and risky assets that are efficient in the RV-ρ space are relevant for the investor. It is also shown that the relevant segment of the RV-ρ efficient frontier is concave and if it is optimal for the investor to hold marketable and risky assets his optimal marketable and risky portfolio will be uniquely determined by the tangency between the indifference map and the relevant segment of the efficiency frontier.

II. THE INTERDEPENDENCE BETWEEN RETURNS TO OCCUPATIONAL ASSETS AND THE STRUCTURE OF THE SET OF MARKETABLE ASSETS

The demand for marketable assets of an investor in an economy with marketability restrictions but without short-sales restrictions will follow a Three Fund Separation Theorem. As shown by Brito [4], an investor will acquire a portfolio associated with his occupation that does not depend upon prices of marketable assets and upon risk preferences and will divide the remainder of his wealth between the portfolio that offers the maximum reward-to-variability ratio (given his expectations) and the riskless asset. If Θ denotes the marketable portfolio associated with his occupational asset, which will be denoted O, then Θ is the unique marketable and risky portfolio such that, given the investor's expectations, $\text{Cov}(\tilde{\Theta}, \tilde{X}) = -\text{Cov}(\tilde{O}, \tilde{X})$, for any marketable portfolio X[1]. The portfolio Θ thus hedges away the covariance risk of the occupation and it seems appropriate to call it a corrective portfolio.

Brito [4] and Mayers [7], [8] assume that the distribution of returns on occupational assets is exogenously given and thus independent of the set of marketable investment opportunities available to investors. To evaluate the reasonableness of the assumption consider the portfolio choices of an investor in two extreme cases. First assume that the set of marketable assets is such that returns to his occupational asset are independent of the returns to all marketable and risky assets. Here it follows from Brito [4] that the corrective portfolio of the investor is null. Now assume that the set of marketable assets is reorganized in such a way that there exists a marketable and risky portfolio perfectly correlated with the investor's occupational asset. He can market his "non-marketable" occupational asset by holding an adequate position in the perfectly correlated marketable portfolio. Such a position will also satisfy the covariance condition stated above and is thus the corrective portfolio of the investor. In the first case the investor seeks to market portions of this occupational asset getting around the marketability restrictions although the structure of marketable assets does not allow him to do so. Consequently his corrective portfolio is null. In the second case the structure of marketable assets allows him to fully market his occupational asset and he does so through his corrective portfolio holdings. Corrective portfolios are thus the way investors market portions of their "non-marketable" occupational assets and the portions that are marketed depend upon the structure of marketable assets in the economy and are the portions that covary with such marketable assets. The human capital literature has implicitly recognized[2] the "moral hazard" problem, i.e., if individuals can market portions of their human capital assets they change the intensity with which they work, they consume more leisure. These changes will modify the distribution of returns to the individual's human capital asset which should be different in the two extreme cases discussed. More generally, this distribution will be determined endogenously, depending upon the structure of marketable assets in the economy and the marketable holdings of the investor. To assume that the distribution of returns on occupational assets is exogenously given and independent of the structure of the set of marketable assets is to assume away moral hazard problems.

To assume away moral hazard problems is not critical in Brito [4] and Mayers [7]. Considering the set of trading rules and the structure of marketable assets in the economy, investors decide on the amount of leisure to consume and on the intensity with which they should work. This determines the distribution of returns on their occupational assets which is then used by the author's to derive equilibrium conditions across marketable assets. Their results[3] could (and should) be interpreted in such a partial equilibrium context. However, it seems strong to assume away moral hazard problems if one is considering the impact of changes in the structure of marketable assets in the economy. These changes will induce changes in the distribution of returns to the occupational assets of investors and even a partial equilibrium interpretation is not possible. The results of Mayers [8] should be interpreted with caution.

It can be argued that moral hazard reduces the efficiency of the economic system[4]. The efficiency criterion is strictly a commodity dominance criterion, i.e., more goods would be produced if the portions of occupational assets that are marketed decrease. The criterion suggests that trading rules that restrict individuals' ability to indirectly market portions of their occupational assets are desirable. Since such a marketing takes place through corrective portfolios that, in general, will involve short-sales, it could be argued that short-sales restrictions are desirable. If short-sales restrictions are imposed upon investors, the Three Fund Separation Theorem does not hold and it is unclear how brokerage houses should advise their clients. They should advise differently clients with different occupations but it is even unclear how they should advise clients in the same occupation. They could determine for a given client his efficient frontier of total combinations (occupational + marketable holdings) in the mean-standard deviation space. This frontier has two undesirable properties,

(i) it is not an occupation frontier and even within an identical occupation group it depends upon the individual wealth levels,

(ii) it mixes decisions with respect to marketable and risky portfolio holdings and decisions with respect to riskless asset holdings.

The brokerage community may prefer to focus on the marketable and risky portfolio decisions of the investor avoiding any explicit dependence upon wealth.

III. THE RELEVANCE OF THE RV-ρ SPACE

It is not clear whether correlation between marketable assets and the occupational asset of an investor is a "good" or a "bad" if short-sales of risky assets are allowed in an economy with marketability restrictions[5]. However, if short-sales of risky assets are restricted, an investor cannot diversify his occupation risk by short-selling assets highly correlated with his occupational asset and it would appear that the correlation of a marketable and risky asset is a "bad" which should be associated with the asset.

Defining

W = marketable wealth of the investor,

O = occupational asset of the investor offering random terminal value $\tilde{O}$,

X = a well defined and feasible portfolio of marketable assets offering random terminal value $\tilde{X}$ and having market value W,

E_x, σ_x = mean and standard deviation of the random terminal value of X,

RV_x = reward to variability ratio of X,

ρ_x = correlation between the random terminal value of X and of the occupational asset,

E_o, σ_o = mean and standard deviation of $\tilde{O}$,

r = one plus the riskless rate of interest = $1 + r_f$,

$\tilde{T}$ = total random terminal value available to the investor determined by his marketable holdings and his occupational asset,

E_t, σ_t = mean and standard deviation of $\tilde{T}$,

one obtains

$$E_t = E_o + E_x \tag{1}$$

$$\sigma_t^2 = \sigma_o^2 + \sigma_x^2 + 2\rho_x \sigma_x \sigma_o \quad . \tag{2}$$

Three variables associated with the marketable holdings of the investor, E_x, σ_x and ρ_x, completely determine his total mean-variance combination, given the endowed occupational asset and E_o, σ_o. The mean-variance indifference map of the investor, defined on total random returns (aggregate of returns to marketable and non-marketable assets), will induce an indifference map in the mean-standard deviation-correlation space. The properties of the mean-variance indifference map determine that E_x is a "good" and that σ_x and ρ_x are "bads" in the induced $E_x - \sigma_x - \rho_x$ space[6].

The feasible region of marketable and risky assets in such a space does not appear to be regular[7] in the general case. However, if one explicity reçognizes the existence of a riskless asset in the economy, the portfolio decisions of the investor within a constant correlation group[8] can be examined, as shown in figure 1. From traditional mean-variance analysis it is known that, if a risky marketable asset is combined with the riskless asset, the locus obtained is linear in the constant ρ plane. This implies that the portfolio offering the highest reward-to-variability (RV) ratio will be the dominant one within a constant ρ group and the overall feasible region of marketable assets (risky and riskless) must have linear intersections with any constant ρ plane. These properties are illustrated in figure 1[9].

The result suggests that only RV_x and ρ_x are relevant when considering the set of all feasible combinations of any risky portfolio X and the riskless asset. If this is the case RV_x and ρ_x should completely determine the locus in the mean-standard deviation space of the total combinations associated with leveraged positions in X. The $E_t - \sigma_t$ locus of total combinations associated with the risky feasible portfolio X is the hyperbola determined by combinations of $\tilde{O} + \tilde{X}$ and $\tilde{O} + Wr$[10]. Such a locus is shown in figure 2. It follows from Merton [9] and Gonzalez [6] that the slope of the locus associated with X at the mean-standard deviation combination of $\tilde{O} + Wr$ is RV_x / ρ_x and thus if the $RV_x - \rho_x$ space is

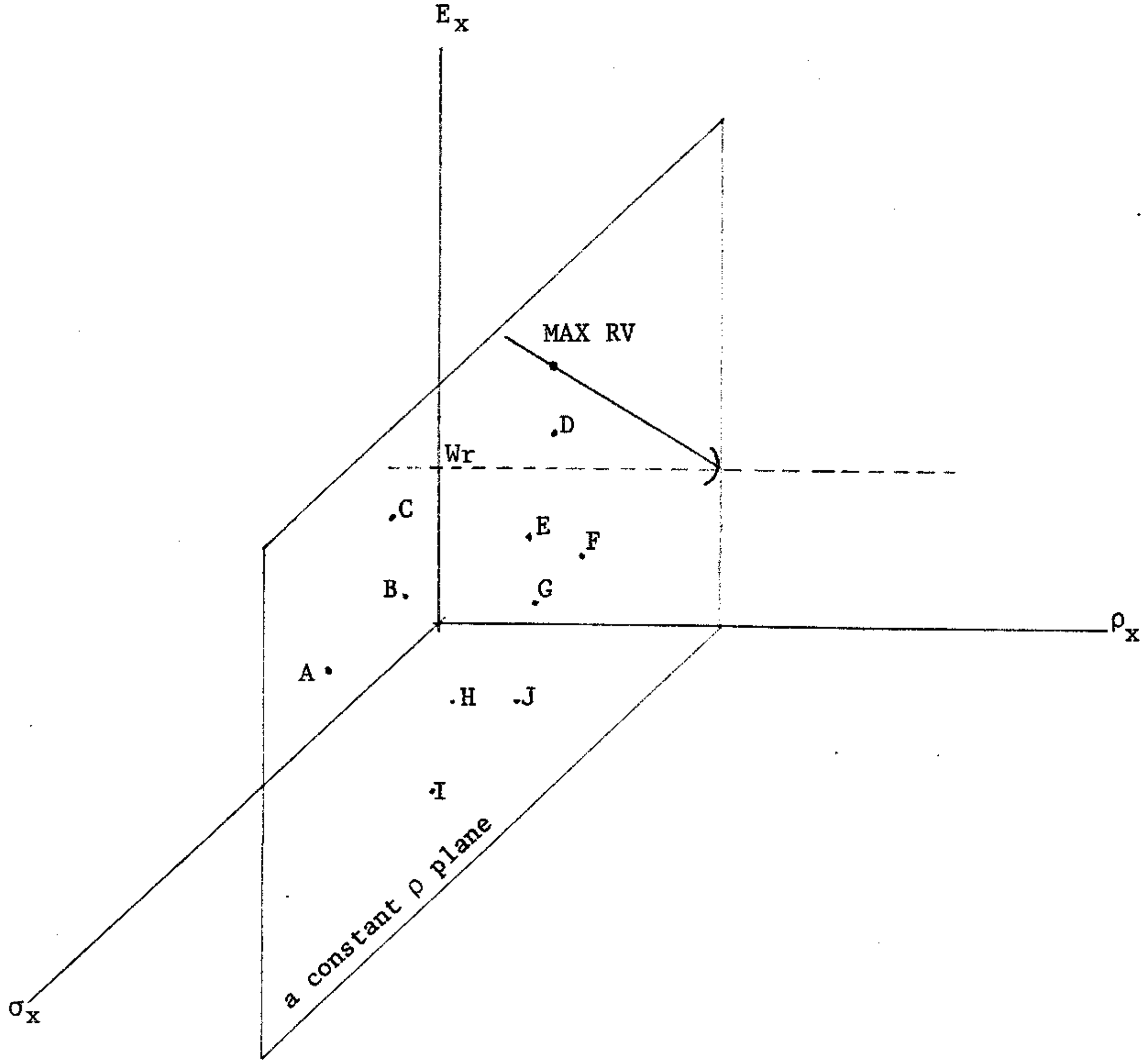

FIGURE 1

DOMINANCE WITHIN A CONSTANT ρ PLANE

embedded into $E_t - \sigma_t$ at $\tilde{O} + Wr$ it will determine how the investor can locally change his $E_t - \sigma_t$ combination through feasible holdings of X. The dashed segment in figure 2 involves short-sales and is not feasible. If one defines X_m as the minimum variance total combination on the locus associated with the risky portfolio X then it also follows from Merton [9] and Gonzalez [6] that X_m completely determines the locus[11] and

$$E(\tilde{X}_m) = (E_o + Wr) - \rho_x \sigma_o RV_x \quad (3)$$

$$\sigma(\tilde{X}_m) = \sigma_o (1 - \rho_x^2) \quad . \quad (4)$$

Given the endowed wealth and occupational asset (and E_o, σ_o) of the investor, RV_x and ρ_x completely determine $E(\tilde{X}_m)$ and $\sigma(\tilde{X}_m)$ and the whole hyperbolic locus. It follows that only RV_x and ρ_x are relevant when considering the set of all feasible combinations of any risky portfolio X and the riskless asset. This suggests that the risky marketable portfolio problem of the investor may be solved in the $RV_x - \rho_x$ space.

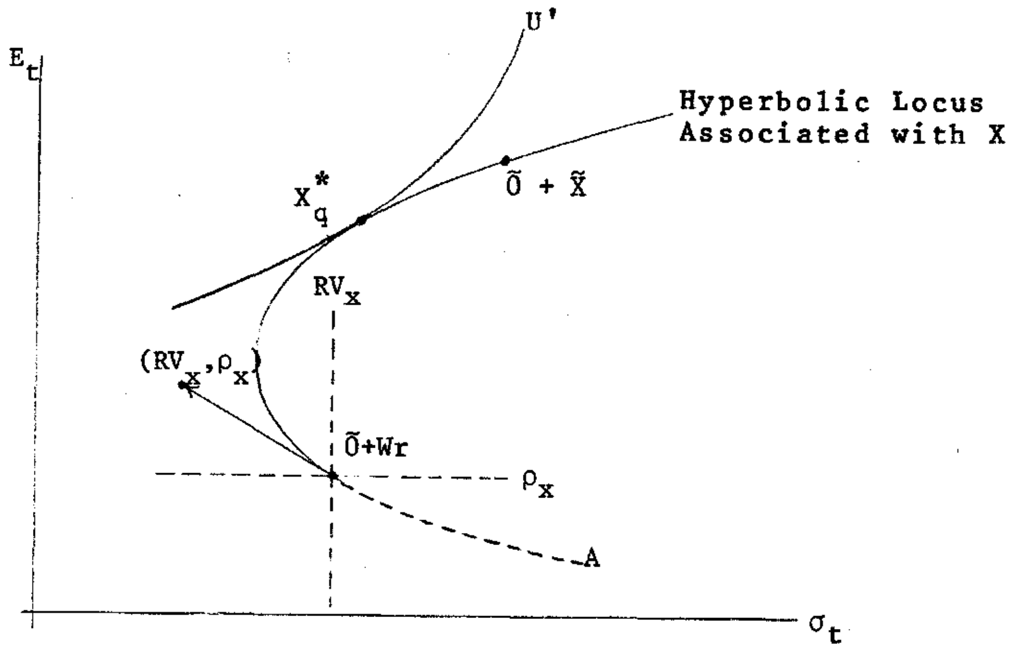

FIGURE - 2

THE LOCUS OF TOTAL COMBINATIONS ASSOCIATED

WITH LEVERAGED POSITIONS ON X

IV. THE INDIFFERENCE MAP INDUCED IN RV-ρ

The expected utility of a total combination offering mean E and standard deviation σ can be defined as $U(E,\sigma)$ and under the assumptions of this paper this function will induce a convex indifference map in E_t-σ_t. As discussed in the preceding section, the E_t- σ_t locus associated with combinations of the riskless asset and any risky portfolio X (plus the occupational asset) is concave and uniquely determined by RV_x and ρ_x. If one defines a feasible combination of X and the riskless asset as $X_q = qX + (1-q)Wr$, $q \geq 0$ then the solution of

$$\underset{\{q \geq 0\}}{\text{Max}}\ U\left[E(\tilde{O} + X_q),\ \sigma(\tilde{O} + \tilde{X}_q)\right]$$

is unique as indicated in figure 2, where X_q^* denotes the combination associated with the unique optimal q. The indifference map in $E_t - \sigma_t$ will induce an indifference map in RV-ρ and one can define $U^I(RV_x, \rho_x) = \underset{\{q \geq 0\}}{\text{Max}}\ U\left[E(\tilde{O}+X_q), \sigma(\tilde{O}+X_q)\right]$. Uniqueness of X_q^* implies that $U^I(.)$ meets the comparability and transitivity axioms, although its convexity properties are unclear at this stage.

Let X and Y be any two marketable portfolios offering the same induced utility in the RV-ρ space. From the definition of $U^I(.)$ there will exist portfolio weights a and b associated with the optimal combinations of X and Y with the riskless asset[12], such that $U^I(RV_x, \rho_x) = U^I(RV_y, \rho_y) = U\left[E(\tilde{O}+\tilde{X}_a), \sigma(\tilde{O}+\tilde{X}_a)\right] = U\left[E(\tilde{O}+\tilde{Y}_b), \sigma(\tilde{O}+\tilde{Y}_b)\right]$. Convexity of the indifference map in E_t- σ_t and the hyperbolic nature of the loci of combinations in E_t- σ_t require that either

(i) $a = b = 0$ or

(ii) $a > 0$ and $b > 0$,[13]

A possible situation is shown in figure 3. Consider an s convex combination of X_a and Y_b which will be denoted $(s: X_a, Y_b)$. It is trivial to show that such a combination can be obtained by leveraging the portfolio[14]

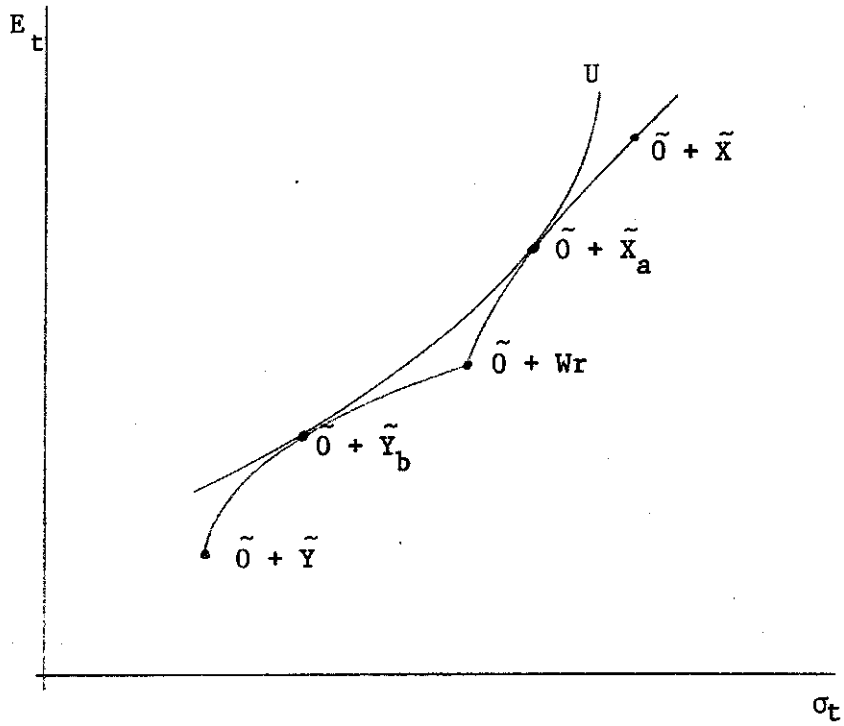

FIGURE 3

INDUCED INDIFFERENCE BETWEEN X AND Y

$$\frac{sa}{sa + (1 - s)b} X + \frac{(1 - s)b}{sa + (1 - s)b} Y . \tag{5}$$

One can now proceed to examine the utility induced by a k convex combination of X and Y. If (i) occurs then full investment of marketable wealth in the riskless asset is preferred to any long position in a portfolio of X and Y:

$$U^I\left[RV_{(k:x,y)}, \rho_{(k:x,y)}\right] = U^I (RV_x, \rho_x) = U^I (RV_y, \rho_y) = \\ = U\left[E(\tilde{O} + Wr), \sigma(\tilde{O} + Wr)\right] \tag{6}$$

If (ii) occurs, from relation (5) there will exist $s > 0$ such that

$$\frac{sa}{sa + (1 - s)b} = k \iff s = \frac{b}{a(1 - k) + b}$$

and from relation (5) there exists a leveraged combination of (k:X,Y) on the E_t-σ_t locus of convex combinations of X_a and Y_b. Since this locus is hyperbolic in E_t- σ_t any point on the locus will be on an indifference curve above the one containing the mean-standard deviation combinations of X_a and Y_b. It follows that if (ii) occurs

$$U^I\left[RV_{(k:x,y)}, \rho_{(k:x,y)}\right] > kU^I (RV_x, \rho_x) + (1-k) U^I (RV_y, \rho_y). \tag{7}$$

Relations (6) and (7) imply convexity of the indifference map induced in RV-ρ.

The utility function induced in RV-ρ not only meets the comparability and transitivity axioms but it also induces a convex indifference map. Moreover, $U^I(.)$ not only allows ranking of feasible and risky marketable portfolios but it implies a special structure upon such a ranking. It can be shown that the marketable and risky portfolio offering the maximum RV ratio is dominant within a constant ρ class and the marketable and risky portfolio offering the minimum ρ is dominant within a constant RV class, i.e., only portfolios that are "efficient" in RV-ρ are relevant for the portfolio selection problem of the investor[15]. It follows that any induced indifference curve is convex and $\partial U^I/\partial RV > 0$ and $\partial U^I/\partial \rho < 0$, i.e., an indifference curve is also an increasing function in the RV-ρ space.

V. THE FEASIBLE REGION AND PORTFOLIO SELECTION IN THE RV-ρ SPACE

Before examining the feasible region let us consider the locus of feasible combinations of two feasible risky portfolios in the RV-ρ space. Let X and Y be feasible risky portfolios with market value W and to simplify the notation define

$\underline{X} = (RV_x, \rho_x)$ = vector associated with X in RV-ρ, for any X,

$K = (k: X,Y)$, $k \in [0, 1]$ = a feasible combination of X and Y[16].

From the linearity of the covariance and expected value operators it follows that[17]

$$\underline{K} = (RV_k, \rho_k) = \frac{k\sigma_x}{\sigma_k}\,\underline{X} + \frac{k\sigma_y}{\sigma_k}\,\underline{Y} \quad . \tag{8}$$

By varying k within [0, 1] this equation will determine the locus. A few properties of the locus can be easily derived. From the definition of k one obtains

$$\frac{k\sigma_x}{\sigma_k}, \ \frac{(1-k)\sigma_y}{\sigma_k} \geqslant 0$$

$$\frac{k\sigma_x}{\sigma_k} + \frac{(1-k)\sigma_y}{\sigma_k} \geqslant 1 .$$

The locus is thus fully contained in the angular segment determined by $\underline{X}$ and $\underline{Y}$ and is always beyond the linear segment $\underline{X}\,\underline{Y}$ for $k \in (0,1)$[18], if $\underline{X}$ and $\underline{Y}$ are linearly independent. The derivation of other properties of the locus require the use of implicit differentiation methods. They are discussed in the Appendix. The slope and the concavity of the locus are continuous and well defined for all k different from the k where the unconstrained maximum or minimum ρ occurs. The locus must be as shown in figure 4 for the various pairs ($\underline{X}$, $\underline{Y}$).

Only portfolios that are efficient in the RV-ρ space are relevant for portfolio selection. Recall that one could obtain the efficient frontier of the investor in $E_t - \sigma_t$ by considering the mean and covariance properties of his occupational asset 0. Since the RV-ρ space is induced by the $E_t - \sigma_t$ space the

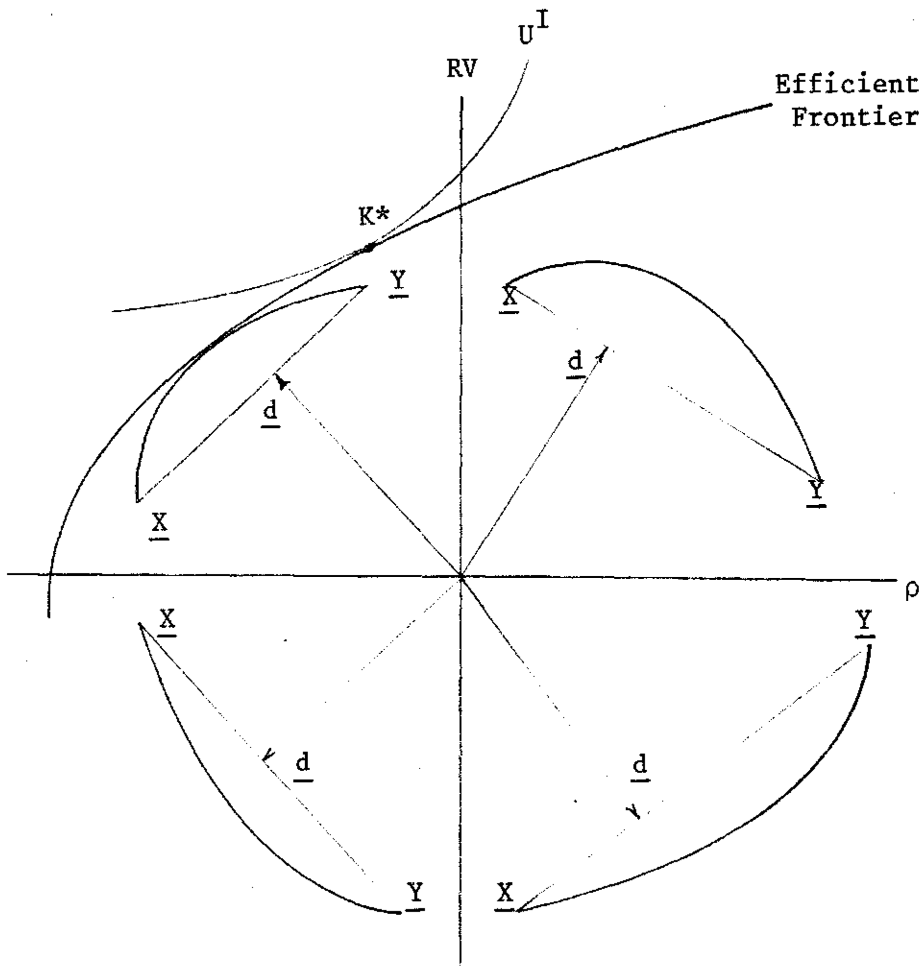

FIGURE 4

THE LOCUS OF CONVEX COMBINATIONS AND PORTFOLIO CHOICE IN RV-ρ efficient frontiers in the two spaces must be closely related. In particular, if the marketable and risky portfolios X and Y define adjacent efficient corner solutions in $E_t - \sigma_t$[19] they must be adjacent efficient corner solutions in RV-ρ and any convex combination of X and Y must be efficient in either space. Moreover, it is known that in $E_t - \sigma_t$ for any convex combination of X and Y there could exist an indifference map such that the convex combination would be the optimal combination. Thus any convex combination of X and Y could also be optimal in RV-ρ and the convexity of the indifference map induced in RV-ρ implies that the locus of convex combinations of X and Y in RV-ρ must be concave[20].

Adjacent efficient corner solutions in $E_t - \sigma_t$ are adjacent efficient corner solutions in RV-ρ but the reverse is not true, in general. It may be efficient for our investor to fully invest his marketable wealth in the riskless asset which has an undefined RV ratio and cannot be represented in the RV-ρ space. In this event full investment in the riskless asset is associated with a corner solution in $E_t - \sigma_t$. The two corner solutions adjacent to the corner solution of the riskless asset are not adjacent in $E_t - \sigma_t$ but are adjacent in RV-ρ. It can be shown that full investment in the riskless asset is inefficient if and only if there exists a portfolio Z such that either $RV_Z \geq 0$ and $\rho_z < 0$ or $RV_z > 0$ and $\rho_z = 0$[21]. If such a portfolio exists, the indeterminacy of the RV ratio of the riskless asset does not need to be of concern. The efficient RV-ρ frontier is always concave and the optimal marketable and risky portfolio of our investor will be uniquely determined by the convex indifference map. If such a Z portoflio does not exist the RV-ρ efficient frontier will be convex in a neighborhood of $\rho = 0$. However, the investor would never hold long positions in marketable and risky portfolios associated with this convex segment. If R and S are the adjacent efficient corner solutions in RV-ρ that determine the convex segment then R and S are not adjacent corner solutions in $E_t - \sigma_t$. It follows that, in general, only marketable and risky portfolios represented on concave segments of the efficient RV-ρ frontier are relevant for portfolio selection in an economy with marketability and short sales restrictions. Convexity of the indifference map implies that the optimal marketable and risky portfolio of the investor is always unique. A possible situation is shown in figure 4. There K^* indicates the optimal risky marketable portfolio.

The risky portfolio selection of an investor can be examined in the RV-ρ space. Implicitly assuming that the investor always takes optimal decisions with respect to his holdings of the riskless asset, the space allows investment advisors to focus on his risky portfolio decision. Moreover, the RV-ρ efficient frontier does not depend upon individual wealth levels and is identical for all investors within the same occupation group.

VI. CONCLUSIONS

In this paper the risky marketable portfolio selection problem of an investor constrained to hold fixed amounts of non-marketable occupational assets in an economy without short-sales of marketable and risky assets was examined under the usual mean-variance assumptions. It was shown that investor's problem can be reduced to the two-parameter Reward-to variability ratio (RV) - correlation with his occupational asset (ρ) space. In this space the investor's induced indifference map is convex, RV is a "good" and ρ is a "bad" and only risky marketable portfolios that are efficient in RV-ρ are relevant for the investor. Moreover, the relevant segment of the RV-ρ efficient frontier was shown to be concave and thus if it is optimal for the investor to hold marketable and risky assets his optimal marketable and risky holdings will be uniquely determined by the tangency between the induced RV-ρ indifference map and the RV-ρ efficient frontier. The optimal risky marketable portfolio of the investor is a function of his preferences, his wealth and his occupation.

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FOOTNOTES

(1) $\tilde{X}$ denotes the random terminal value offered by the non-random portfolio X.

(2) See Becker [3].

(3) As well as the results of this paper and the RV-ρ space.

(4) See Arrow [1], [2].

(5) One can diversify occupational risk either by holding long positions on marketable assets negatively correlated with the occupational asset or by selling marketable assets posivitely correlated with the occupational asset.

(6) Under the assumptions of this paper $\partial E(U)/\partial E_t > 0$ and $\partial E(U)/\partial \sigma_t < 0$ (see Fama and Miller [5]). From (1) and (2) one obtains $\frac{\partial E(U)}{\partial E_x} = \frac{\partial E(U)}{\partial E_t}\frac{\partial E_t}{\partial E_x} > 0$ $\frac{E(U)}{\partial \rho_x} = \frac{\partial E(U)}{\partial \sigma_t}\frac{\partial \sigma_t}{\partial \rho_x} < 0$ and, similarly, $\frac{\partial E(U)}{\partial \sigma_x} < 0$.

(7) To be more precise, the region does not appear to be connected. Convex combinations of portfolios identically correlated with the occupational asset in general will not be identically correlated with the occupational asset.

(8) I.e., a group of marketable portfolios identically correlated with the investor's occupational asset.

(9) Since full investment of W into the riskless asset is a feasible alternative to the investor the intersection of the overall feasible region in $E_x - \sigma_x - \sigma_x$ with the E_x axis is (Wr, 0,0).

(10) Recall that the occupational asset must be fully held by the investor. If he invests all of his marketable wealth in the risky portfolio X his terminal wealth is $\tilde{O} + \tilde{X}$ and it will be $\tilde{O}$ + Wr if he fully invests in the riskless asset. Dividing his marketable wealth between X and the riskless asset will produce total combinations along the hyperbolic locus of combinations of $\tilde{O} + \tilde{X}$ and $\tilde{O}$ + Wr.

(11) Recall that the locus is a hyperbola with center on the mean axis and with a horizontal principal axis.

(12) The scalars a and b are portfolio weights analogous to q above. They represent the proportions of X and Y in the optimal feasible combinations of these portfolios and the riskless asset.

(13) Just recall that the same indifference curve must be tangent to the feasible segments of the hyperbolic loci associated with X and Y at X_a and Y_b, respectively. This double tangency condition is violated if the two conditions are not meet.

(14) Just observe that $(s:X_a, Y_b) = sX_a + (1-s)Y_b = s\{aX + (1-a)Wr\} + (1-s)$ $\{bY + (1-b)Wr\} = saX + (1-s)bY + \{s\ (1-a) + (1-s)\ (1-b)\}\ Wr$.

(15) The proofs of these statements will be sent to the reader, upon request. They follow directly from the properties of the feasible segment of the hyperbolic locus of total combinations associated with leveraged positions in risky portfolios.

(16) It would seem more adequate to define it as K_{xy}. However, there will never exist any doubt about the portfolios X and Y involved and the simpler notation was chosen.

(17) Recalling that

$$RV_k = \frac{E_k - Wr}{\sigma_k} = k\,\frac{E_x - Wr}{\sigma_k} + (1-k)\,\frac{E_y - Wr}{\sigma_k} = \frac{k\sigma_x}{\sigma_k} RV_x + \frac{(1-k)\sigma_y}{\sigma_k} RV_y$$

$$\rho_k = \frac{Cov(\tilde{K},\tilde{0})}{\sigma_o\,\sigma_k} = \frac{k\ Cov\ (\tilde{X},\tilde{0})}{\sigma_k\,\sigma_o} + \frac{(1-k)\ Cov(\tilde{Y},\tilde{0})}{\sigma_k\,\sigma_o} = \frac{k\sigma_x}{\sigma_k}\,\rho_x + \frac{(1-k)\sigma_y}{\sigma_k}\,\rho_y$$

the result follows.

(18) Since the variance-covariance matrix of marketable and risky assets is assumed to be positive definite, if $\underline{X}\ \underline{Y}$ are linearly independent $\underline{K}$ will be on the linear segment $\underline{X}\ \underline{Y}$ only if k=0 or k=1. For $k \in (0,1)$ relations (9) will hold as an strict inequality and thus $\underline{K}$ will be fully contained on the side of $\underline{X}\ \underline{Y}$ opposite to the origin, i.e., it will be beyond the linear segment.

(19) Notice that 0 is held throughout the feasible region in $E_t - \sigma_t$ and thus the corner solutions defined by X and Y are 0 + X and 0 + Y.

(20) If the locus were convex it would be necessary to impose special conditions upon the convex indifference map in RV-ρ to assure that every convex combination of X and Y could be chosen. Since no special conditions are needed in $E_t - \sigma_t$ they are not needed in RV-ρ and the locus of convex combinations of X and Y must then be concave in RV-ρ, i.e., $d_{RV}(X,Y) \geq 0$ and the critical k^*_ρ must be outside the relevant (0,1) range. This result can also be derived considering only the properties of the loci of feasible combinations in the RV-ρ space.

(21) Just recall that if $RV_x - \rho_x$ is embedded in $E_t - \sigma_t$ at $\tilde{0}$ + Wr then $\underline{Z}$ gives the local rate of change of expected value vs. standard deviation that can be attained by holding the risky marketable portfolio Z.

BIBLIOGRAPHY

(1) Arrow, K.; "Economic Welfare and the Allocation of Resources to Invention", In *Essays in the Theory of Risk Bearing* (K. Arrow, ed.), Chicago: Markham Publishers, 1971

(2) - ; "The Economics of Moral Hazard: A Further Comment", In *Essays in the Theory of Risk Bearing* (K. Arrow, ed.), Chicago: Markham Publishers, 1971.

(3) Becker, G.; *Human Capital Theory*, New York: National Bureau of Economic Research, 1964.

(4) Brito, N.; "Marketability Restrictions and the Valuation of Capital Assets Under Uncertainty", Research Paper Nº 269, Graduate School of Business, Stanford University (June 1975), forthcoming in *The Journal of Finance*.

(5) Fama, E. and Miller, M.; *The Theory of Finance*, New York: Holt, Reinhart and Winston, 1972.

(6) Gonzales, N.; "The Capital Asset Pricing Model Under Conditions of Uncertain Inflation", Unpublished Ph. D. Dissertation, Stanford University, August, 1973.

(7) Mayers, D.; "Non-Marketable Assets and Capital Market Equilibrium Under Uncertainty", in *Studies in the Theory of Capital Markets*, M.C. Jensen ed., Praeger Publishers, 1972.

(8) Mayers, D.; "Non-Marketable Assets, Market Segmentation and the Level of Asset Prices", *Journal of Finance and Quantitative Analysis*, March 1976.

(9) Merton, R.; "An Analytical Derivation of the Efficient Portfolio Frontier", *Journal of Finance and Quantitative Analysis*, September, 1972.

APPENDIX

PROPERTIES OF THE LOCUS OF FEASIBLE COMBINATIONS OF RISKY PORTFOLIOS IN THE RV-ρ SPACE

Using implicit differentiation methods and relation (8) one can obtain other properties of the locus of feasible combinations of the feasible and risky portfolios X and Y in the RV-ρ space. Denoting the critical k at which the minimum or maximum ρ combination occurs by k_ρ^* one can show that

(i) the slope and the concavity of the locus are continuous and well defined for all $k \neq k_\rho^*$,

(ii) the slope and the concavity of the locus change of sign at $k=k_\rho^*$; moreover, the concavity changes of sign only at k_ρ^*,

(iii) at $k = k_\rho^*$ the vertical line parallel to the RV axis is tangent to the locus.

These properties and relation (9) imply that the analytical properties of $\underline{X}$, $\underline{Y}$ and of the linear segment $\underline{X}\,\underline{Y}$ completely determine the concavity properties of the locus. The distance vector of $\underline{X}\,\underline{Y}$ from the origin, which is defined as $\underline{d}$, is of particular importance. From relation (9) the projection of $\underline{K}$ along $\underline{d}$, which is defined as $k_{\underline{d}}$, will always be positive and $|k_d| > |d|$ for $k \varepsilon (0,1)$. The concavity of the locus is thus determined by the projection of $\underline{d}$ along the RV axis, which is defined as d_{RV}. Without any loss of generality let X be the portfolio such that $\rho_x < \rho_y$ or such that $RV_x > RV_y$ if $\rho_x = \rho_y \geqslant 0$ or such that $RV_x < RV_y$ if $\rho_x = \rho_y < 0$. If $d_{RV} \geqslant 0$ and $k_\rho^* \varepsilon (0,1)$ represents the minimum [maximum] ρ combination then the locus is concave [convex] for $k \varepsilon [0,k_\rho^*)$ and is convex [concave] for $k \varepsilon (k_\rho^*, 1]$. The opposite would hold if $d_{RV} < 0$. If $k_\rho^* \notin (0,1)$ the locus of convex combinations will have constant concavity, it will be concave if $d_{RV} \geqslant 0$ and convex if $d_{RV} < 0$. These properties imply that the locus must be as shown in figure (4) for the various pairs ($\underline{X}$, $\underline{Y}$). Proofs of these results will be sent to the reader upon request.