A Numerical Application of the Residue Theorem for the Exact Control

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Abstract

In this paper we implement the results obtained by Vasilyev et al [11] on the numerical approximation of the exact control for the string equation. The computational part and the respective graphs are made for a particular case. For that we have applied the Residues Theorem of holomorphic functions, which, as far as we know, is the first time that this theorem is applied in the computational study of exact control problems.

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1 Introduction

Let us consider a flexible elastic string of length L. Then its small transversal vibration can be studied by the following problem:

$$\left\{\begin{array}{lll} y_{tt}(x,t) - y_{xx}(x,t) = 0, & & \text{in } Q = (0,L)\times(0,T) \\ y(0,t) = v(t), & y(L,t) = 0, & 0 \le t \le T \\ y(x,0) = y_0(x), & y_t(x,0) = y_1(x), & 0 \le x \le L \end{array}\right. \tag{1}$$

where $y(x,t)$ denote the displacement of the point x of the string $(0 \le x \le L)$, at the instant t, $(0 \le t \le T)$. The functions $y_0(x)$ and $y_1(x)$ represent, respectively, the initial position and the initial velocity of the string . In (1), $v(t)$ denotes the control variable, which acts on the system through the extreme $x = 0$. The other extreme is held fixed.

The exact control problem for (1) consists of finding, for each initial data $y_0(x)$ and $y_1(x)$ that belong to some class, a control $v(t)$ that lies in some class such that the solution $y(x,t)$ of Problem (1) with these data satisfies the final condition

$$y(x,T) = 0, \qquad y_t(x,T) = 0, \qquad 0 \le x \le L. \tag{2}$$

Let us consider $z(x,t) = y(x, T-t)$. Then Problem (1) and final condition (2) are, respectively, equivalent to

$$\begin{cases} z_{tt}(x,t) - z_{xx}(x,t) = 0, & \text{in } Q = (0,L)\times(0,T) \\ z(0,t) = v(T-t), \quad z(L,T-t) = 0, & 0 \le t \le T \\ z(x,0) = 0, \quad z_t(x,0) = 0, & 0 \le x \le L \end{cases} \tag{3}$$

and

$$z(x,T) = y_0(x), \qquad z_t(x,T) = -y_1(x), \qquad 0 \le x \le L. \tag{4}$$

The HUM (Hilbert Uniqueness Method) was introduced by Lions [5] in order to study the exact control problem for distributed systems, in particular for (1) (or equivalently, (3) with the final condition (4)). A number of authors have studied this method, among them, we can mention, Zuazua [13], Medeiros [6], Komornik [4] and Milla Miranda [8]. In these works the exact control problem for (1) was studied when Ω is an open bounded set of $\mathbb{R}^n$, which generalizes the case $\Omega = (0, L)$. A study of the numerical analysis of the exact control problem for the wave equation was done by Glowinski et al [2]. They used finite elements and conjugate gradient methods. Similar results are obtained in Rincon [9], by using the difference finite method.

F.P.Vasilyev, M.A. Kurzhanskii and A. Razgulin made a numerical study which allows us to obtain an approximation $v_N(t)$ for the exact control $u(t)$ of Problem (3), where $u(t)$ is given by HUM. They used the Fourier series method. In this paper we implement these results for the particular case $y_0 \equiv 1$, $y_1 = \delta_{L/2}$ (δ = Dirac delta), $T = 3L$ and $L = \pi^2$.

We present the graphics of the approximations $u_N(t)$ of $u(t)$ and $y_N(x,t)$ of the solution $y(x,t)$ of (3). For that we use non standard results on numerics series, which are obtained by applying the Residues Theorem of holomorphic functions (see, the Appendix). The novelty in this paper is the use of these results in the computational study of an exact control problem. As far as we know this is the first time that it is done.

2 Preliminaries

In this part we describe the principal results obtained by Vasilyev et al [11], in order to obtain an approximation $u_N(t)$ for the exact control $u(t)$ of problem (3), where $v(t)$ is given by HUM.

Let us consider the problem

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0, & \text{in } Q = (0,L)\times(0,T) \\ y(0,t) = v(t), \quad y(L,t) = 0, & 0 \le t \le T \\ y(x,0) = 0, \qquad y_t(x,0) = 0, & 0 \le x \le L \end{cases} \tag{5}$$

and the final condition

$$y(x,T) = y_0(x), \qquad y_t(x,T) = y_1(x), \qquad 0 \le x \le L. \tag{6}$$

Let us consider $v(t) \in L^2(0,T)$. Then the ultra weak solution $y(x,t;v)$ of (5) has the following form:

$$y(x,t;v) = \frac{2}{L}\sum_{m=1}^{\infty} \sin\frac{m\pi x}{L}\int_0^t v(\tau)\sin\frac{m\pi(t-\tau)}{L}d\tau. \tag{7}$$

Note that

$$y \in C^0([0,T]; L^2(0,L)) \qquad \text{and} \qquad y_t \in C^0([0,T]; H^{-1}(0,L)).$$

Let us define the application

$$A : L^2(0,T) \mapsto H^{-1}(0,L)\times L^2(0,L)$$

$$v \mapsto Av = \{y_t(\cdot,T;v), -y(\cdot,T;v)\},$$

where $y(x,t;v)$ is the ultra weak solution of (5). The adjoint A^* of A is the operador

$$A^* : H_0^1(0,L) \times L^2(0,L) \mapsto L^2(0,T)$$

$$\{\varphi_0,\ \varphi_1\} \mapsto A^*\{\varphi_0,\ \varphi_1\} = \frac{\partial}{\partial x}\varphi(0,\cdot;\{\varphi_0,\ \varphi_1\})$$

where φ is the ultra weak solution of the backward problem:

$$\begin{cases} \varphi_{tt}(x,t) - \varphi_{xx}(x,t) = 0, & \text{in} \quad Q \\ \varphi(0,t) = 0, \quad \varphi(L,t) = 0, & 0 \le t \le T \\ \varphi(x,T) = \varphi_0(x), \quad \varphi_t(x,T) = \varphi_1(x), & 0 \le x \le L \end{cases}$$

Note that $\varphi \in C^1([0,L];L^2(0,T))$.

We define $\Lambda = AA^*$. By the above expressions, we have

$$\Lambda\{\varphi_0,\ \varphi_1\} = \left\{ y_t\Big(\cdot, T; \frac{\partial}{\partial x}\ \varphi(0,\cdot;\{\varphi_0,\ \varphi_1\})\Big),\ -y\Big(\cdot, T; \frac{\partial}{\partial x}\ \varphi(0,\cdot;\{\varphi_0,\ \varphi_1\})\Big) \right\}; \tag{8}$$

Thus

$$\begin{array}{ccc} L^2(0,T) & \xmapsto{A} & H^{-1}(0,L) \times L^2(0,L) = F' \\ & A^* \nwarrow & \uparrow \Lambda \\ & & H_0^1(0,L) \times L^2(0,L) = F \end{array}$$

By Lions [5], we have that Λ is an isomorphism from F to F'. We observe that the system (5) is exactly controllable when $T > 2L$, see Lions [5]. Thus for each $f = \{y_1, -y_0\} \in F'$, there exists a unique $c = \{\varphi_0,\ \varphi_1\} \in F$ such that

$$\Lambda\{\varphi_0,\ \varphi_1\} = \{y_1, -y_0\}. \tag{9}$$

From (8) and (9) it follows that

$$\begin{cases} y_0 = y\Big(\cdot, T; \dfrac{\partial}{\partial x}\ \varphi\ (0, . \,, \{\varphi_0,\ \varphi_1\})\Big) \\ \\ y_1 = y_t\Big(., T; \dfrac{\partial}{\partial x}\ \varphi\ (0, . \,, \{\varphi_0,\ \varphi_1\})\Big) \end{cases}$$

which is the final condition (6) that we want to obtain.

Thus

$$u = \frac{\partial}{\partial x}\ \varphi\ (0, . \,; \{\varphi_0,\ \varphi_1\}). \tag{10}$$

is one control which permits that the ultra weak solution $y(x,t;u)$ of (5) verifies the final condition (6).

The method described above to determine $u(t)$ defined by (10) is the HUM, which determine a unique control of Problem (3) for each $\{y_1, -\, y_0\} \in F'$

Note that

$$u = A^*\{\varphi_0,\ \varphi_1\}$$

$$\Lambda\{\varphi_0,\ \varphi_1\} = \{y_1, -\, y_0\} = f$$

In what follows one determines the approximations $u_N(t)$ of the exact control $u(t)$, given by (10). For that, we project the functions obtained before on finite-dimensional space and use the properties of the isomorphism Λ. In fact,

$$\varphi_0(x) = \sum_{k=1}^{\infty} \varphi_{0k} \sin\frac{k\pi x}{L}; \quad \varphi_1(x) = \sum_{k=1}^{\infty} \varphi_{1k} \sin\frac{k\pi x}{L}$$

where the Fourier coefficients are given by

$$\varphi_{0k} = \frac{2}{L}\int_0^L \varphi_0(x)\sin\frac{k\pi x}{L}dx, \quad \varphi_{1k} = \frac{2}{L}\int_0^L \varphi_1(x)\sin\frac{k\pi x}{L}dx$$

We are interested in a finite-dimensional subspace that contains the sum of the first N terms of φ_0 and φ_1. Then we define the subspace F_N which is composed by elements $c_N = \{\varphi_{0N},\ \varphi_{1N}\}$, where

$$\varphi_{0N}(x) = \sum_{k=1}^{N} c_{0Nk}\sin\frac{k\pi x}{L}, \quad \varphi_{1N}(x) = \sum_{k=1}^{N} c_{1Nk}\sin\frac{k\pi x}{L}$$

with c_{0Nk} and c_{1Nk} belonging to $\mathbb{R}$. One uses the notation $F_N^{'} = \Lambda F_N$. Then $\dim F_N^{'} = \dim F_N = 2N$.

We fix $f = \{y_1, -y_0\} \in F^{'}$ in order to determine the unique exact control $u(t)$ of Problem (3) given by (10). We denote by f_N the orthogonal projection of f in $F_N^{'}$, that is,

$$\|f_N - f\|_{F'} = \min_{g_N \in F_N^{'}} \|g_N - f\|_{F'}$$

Let us denote $\Lambda c_N = f_N$ (therefore $c_N = \Lambda^{-1} f_N$) and $u_N = A^* c_N$. Then

$$u_N \to u \qquad in \qquad L^2(0,T).$$

Thus u_N is the required approximation of $u(t)$. Note that, if f_N is known then so is u_N.

In what follows, we construct an algorithm to determine u_N. Let us introduce

$$e_{2k} = \{\sin\frac{k\pi x}{L}, 0\}, \qquad e_{2k-1} = \{0, \sin\frac{k\pi x}{L}\}, \quad k = 1, 2, ...$$

Then $\{e_1, e_2, ..., e_{2N}\}$ and $\{\Lambda e_1, \Lambda e_2, ..., \Lambda e_{2N}\}$ are basis, respectively, of the subspace F_N and $F_N^{'}$. The elements Λe_i, see Vasilyev et al [11], are give explicitly by

$$\Lambda e_{2k} = \left(\sum_{m=1}^{\infty}\left(\frac{2km\pi^2}{L^3}\right)\alpha_{mk}\ \sin\left(\frac{m\pi x}{L}\right), -\sum_{m=1}^{\infty}\left(\frac{2k\pi}{L^2}\right)\gamma_{mk}\ \sin\left(\frac{m\pi x}{L}\right)\right)$$

$$\Lambda e_{2k-1} = \left(\sum_{m=1}^{\infty}\left(\frac{2m\pi}{L^2}\right)\beta_{mk}\ \sin\left(\frac{m\pi x}{L}\right), -\sum_{m=1}^{\infty}\left(\frac{2}{L}\right)\delta_{mk}\ \sin\left(\frac{m\pi x}{L}\right)\right)$$

where

$$\begin{cases} \alpha_{mk} &= \displaystyle\int_0^T \cos\left(\frac{m\pi(T-\tau)}{L}\right)\cos\left(\frac{k\pi(\tau-T)}{L}\right)d\tau \\ \gamma_{mk} &= \displaystyle\int_0^T \sin\left(\frac{m\pi(T-\tau)}{L}\right)\cos\left(\frac{k\pi(\tau-T)}{L}\right)d\tau = -\beta_{km} \\ \delta_{mk} &= \displaystyle\int_0^T \sin\left(\frac{m\pi(T-\tau)}{L}\right)\sin\left(\frac{k\pi(\tau-T)}{L}\right)d\tau. \end{cases}$$

The function f_N has the form

$$f_N = \sum_{k=1}^{2N} f_{Nk}\Lambda e_k$$

because $f_N \in F_N'$. Since we know Λe_k, in order to determine f_N, it is sufficient to know the coefficients f_{Nk}. This is obtained by solving the following system of linear equations:

$$\sum_{k=1}^{2N} f_{Nk}(\Lambda e_k, \Lambda e_m)_{F'} = (f, \Lambda e_m)_{F'}, \qquad m = 1, 2, \cdots, 2N \tag{11}$$

This system is obtained from the relations

$$(f - f_N, \Lambda e_m)_{F'} = 0, \qquad m = 1, 2, \cdots, 2N$$

which hold because $f - f_N$ is orthogonal to the subspace F'_N.

Using the coefficients f_{Nk} we obtain

$$c_N = \Lambda^{-1} f_N = \sum_{k=1}^{2N} f_{Nk} \Lambda^{-1} \Lambda e_k = \sum_{k=1}^{2N} f_{Nk} e_k$$

$$u_N = A^* c_N = \sum_{k=1}^{2N} f_{Nk} A^* e_k \tag{12}$$

where $u_N(t)$ is the approximate control that we seek.

From the above algorithm, $u_N(t)$ can be determined after solving the linear system (11).

Let $M = (m_{ij})$, $\hat{f} = (f_{N1}, \cdots, f_{N,2N})^T$ and $G = (g_1, \cdots, g_{2N})^T$ be the matrix of order $2N \times 2N$, $2N \times 1$ and $2N \times 1$, respectively. Then the system (11) can be written in the matrix form,

$$M\hat{f} = G$$

defined by

$$m_{ij} = (\Lambda e_i, \Lambda e_j)_{F'}, \quad g_j = (f, \Lambda e_j)_{F'}, \qquad i, j = 1, 2, \cdots 2N. \tag{13}$$

In the particular case $T = 3L$ the entries m_{ij} of the matrix M have the following form:

$$(\Lambda e_{2k_1}, \Lambda e_{2k_2-1})_{F'} = \begin{cases} 0 & \text{if } (k_1 + k_2) \text{ is even} \\ \dfrac{12 k_1 k_2}{k_1^2 - k_2^2} & \text{if } (k_1 + k_2) \text{ is odd.} \end{cases}$$

$$(\Lambda e_{2k_1-1}, \Lambda e_{2k_2-1})_{F'} = \begin{cases} 0 & \text{if } k_1,\ k_2 \text{ are even and } k_1 \neq k_2 \\ 5L & \text{if } k_1 \text{ is even and } k_1 = k_2 \\ -\dfrac{4L}{\pi^2 k_1 k_2} & \text{if } k_1,\ k_2 \text{ are odd and } k_1 \neq k_2 \\ 5L - \dfrac{4L}{\pi^2 k_1^2} & \text{if } k_1 \text{ is odd and } k_1 = k_2 \\ 0 & \text{if } k_1 \text{ is even and } k_2 \text{ is odd} \end{cases}$$

$$(\Lambda e_{2k_1}, \Lambda e_{2k_2})_{F'} = \begin{cases} 0 & \text{if } k_1,\ k_2 \text{ are even and } k_1 \neq k_2 \\ \dfrac{5k_1^2\pi^2}{L} & \text{if } k_1 \text{ is even and } k_1 = k_2 \\ 0 & \text{if } k_1,\ k_2 \text{ are odd and } k_1 \neq k_2 \\ \dfrac{5k_1^2\pi^2}{L} & \text{if } k_1 \text{ is odd and } k_1 = k_2 \\ 0 & \text{if } k_1 \text{ is odd and } k_2 \text{ is even} \end{cases}$$

In the obtention of this result we used the equalities $(26), \cdots, (33)$ of the Appendix.

In addition, for $f = \{\delta_{L/2}, -1\}$, by applying Proposition A.2 and result (25) of the Appendix, the entries g_j of the matrix G defined by (13) are given by

$$(f, \Lambda e_{2k-1})_{F'} \quad = \quad \begin{cases} \dfrac{L}{k\pi}((-1)^{k/2} - 1) & \text{if } k \text{ is even} \\ -\dfrac{6L}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

$$(f, \Lambda e_{2k})_{F'} \quad = \quad \begin{cases} 0 & \text{if } k \text{ is even} \\ 3(-1)^{(k-1)/2} & \text{if } k \text{ is odd.} \end{cases}$$

3 Computational Results

In this section we determine the graphs of the approximate control $u_N(t)$ and of the approximate ultra weak solution $y_N(x,t)$ at the instant $t = T$. Here $y(x,t)$ denotes the solution of Problem (5) with $T = 3L$, exact control $u(t)$ and initial data $y_0 \equiv 1$, $y_1 = \delta_{L/2}$.

By the characteristic of the matrix M defined by (13), the Crout form or $(L^T DL)$ (see Golub et al [3]) is an appropriate method for solving the system, that is, for obtaining the coefficients f_{Nk}.

Note that if we know the f_{Nk}'s, then we will know $u_N(t)$ defined by (12), because

$$A^* e_k = \frac{k\pi}{L} \cos \frac{k\pi(t-T)}{L} \qquad\qquad k = 1, 2 \cdots .$$

$$A^* e_{k-1} = \sin \frac{k\pi(t-T)}{L} \qquad\qquad k = 1, 2 \cdots .$$

In order to obtain $y_N(x, T; u)$, we substitute the exact control $u(t)$ in (7) by $u_N(t)$, and compute the respective integral,

$$y(x, t; u_N) = \frac{2}{L} \sum_{m=1}^{\infty} \sin \frac{m\pi x}{L} \int_0^T \; u_N(\tau) \sin \frac{m\pi(T-\tau)}{L} d\tau.$$

We have the following graphs with $L = \pi^2$.

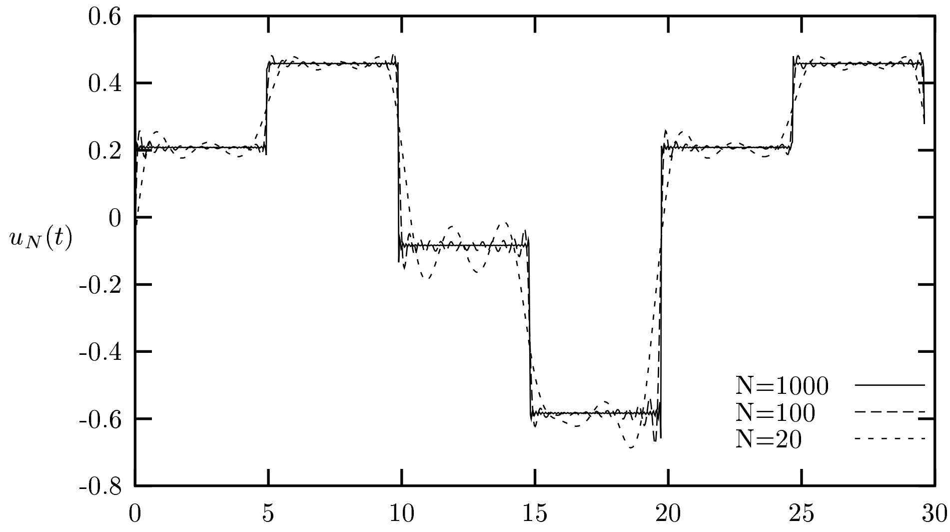

Figure 1: Approximation Control $u_N(t)$

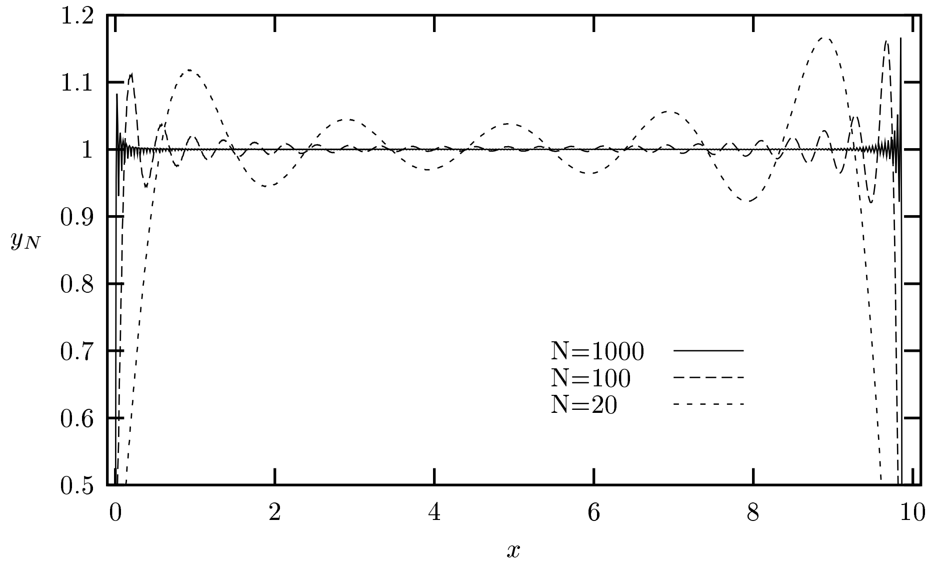

Figure 2: Approximation Solution $y_N(x, T, u)$

Figure 1 describes the variation of the approximate control $u_N(t)$, when $N = 20,\ 100$ and 1000 and we note that $u_N(t)$ approximates a continuous function almost everywhere when N increases. Since we use the Fourier series, the Gibbs phenomenon appears near the points of discontinuity.

Figure 2 shows the variation of the approximate solution $y_N(x, T)$ at the instant $t = T = 3L$. We note that $y_N(x, T)$, for all $x \in (0, L)$, approximates the function $y_0 \equiv 1$, except near the extremes $x = 0$ and $x = L$. This is due to the Gibbs phenomenon.

4 Error Estimates

In this part we analyze the error of the results obtained when N increase. In order to reduce the influence of the other numerical errors, we have considered sufficiently large values of $N_x = L/\Delta x = 1000$ and $N_t = T/\Delta t = 1000$.

The error estimate in the $L^2(0,\ L)$ norm for the solution $y(x, t)$ at the instant $t = T$ is given by

$$\begin{aligned} \|E_y\| &= \left\{ \left((1 - y_N(0, T))^2 + (1 - y_N(L, T))^2 \right) \frac{\Delta x}{2} \right. \\ &+ \left. \sum_{j=1}^{N_x - 1} ((1 - y_N(j\Delta x,\ T))^2\, \Delta x \right\}^{1/2} \end{aligned}$$

Since we do not know the exact control $u(t)$, for the computation of the error, the approximation $u_N(t)$ with $N = 1600$ is used instead of $u(t)$. The formula that approximates the error of the control $u_N(t)$, in the $L^2(0, T)$ norm, is given by

$$\begin{aligned} \|E_u\| &= \left\{ \left((u_N(0) - u_{1600}(0))^2 + (u_N(T) - u_{1600}(T))^2 \right) \frac{\Delta t}{2} \right. \\ &+ \left. \sum_{j=1}^{N_t - 1} (u_N(j\Delta t) - u_{1600}(j\Delta t))^2 \Delta t \right\}^{1/2} \end{aligned}$$

We confirm that each of the above errors depend algebraically on N, that is, $\|E\| \approx a(N)^{-\alpha}$ (for some $\alpha > 0$) for N large enough. This conclusion is based on the analysis of the following data:

N	$\|E_y\|$	$\|E_v\|$
1600	0,06342	0
800	0,088386	0,030861
400	0,124107	0,077274
200	0,17467	0,1217316
100	0,24622	0,179067
50	0,350747	0,256549
25	0,506850	0,358164

In fact, we note that if at N, we have the numerical error E_y and E_u, then at $4N$, we will have, approximately, the numerical errors $E_y/2$ and $E_u/2$. This suggests us to formulate the following hypotheses:

$$\|E_y\| \approx a\, N^{-1/2}; \qquad a = \tan\theta \tag{14}$$

$$\|E_u\| \approx b\, N^{-1/2}; \qquad b = \tan\theta \tag{15}$$

In the first case, making the graph of the error for the exact solution, $\|E_y\|$ vs $N^{-1/2}$ (see Figure 3) we observe that the error can be approximated by a straight line passing through the origin of coordinates. This confirms our hypothesis (14). In the second case, for the graph of the control error, $\|E_u\|$ vs $N^{-1/2}$ (see Figure 4), we consider only the values $N = 25,\ 50,\ 100,\ 200,\ 400,\ 800$, since we have taken $u_N(t)$ with $N = 1600$ instead of the exact control $u(t)$ in the computation of the errors, and it is natural that we can not obtain good approximations for $u(t)$ when N is near of the value 1600. Based on the hypothesis (14) and (15) we affirm that the order of convergence of the applied method is 1/2. In what follows we present the graphs of the estimate errors.

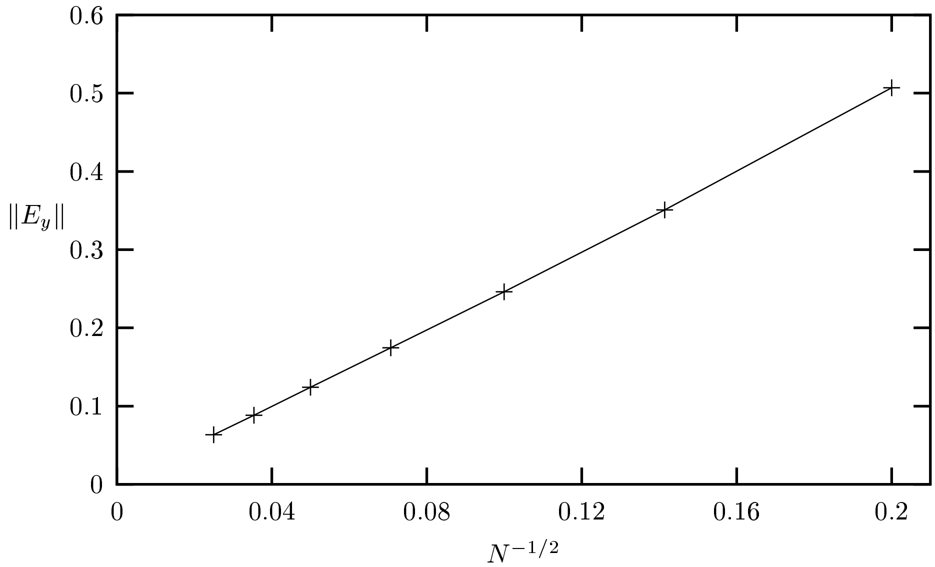

Figure 3: Error for Approximation Solution $y_N(x, T, u)$

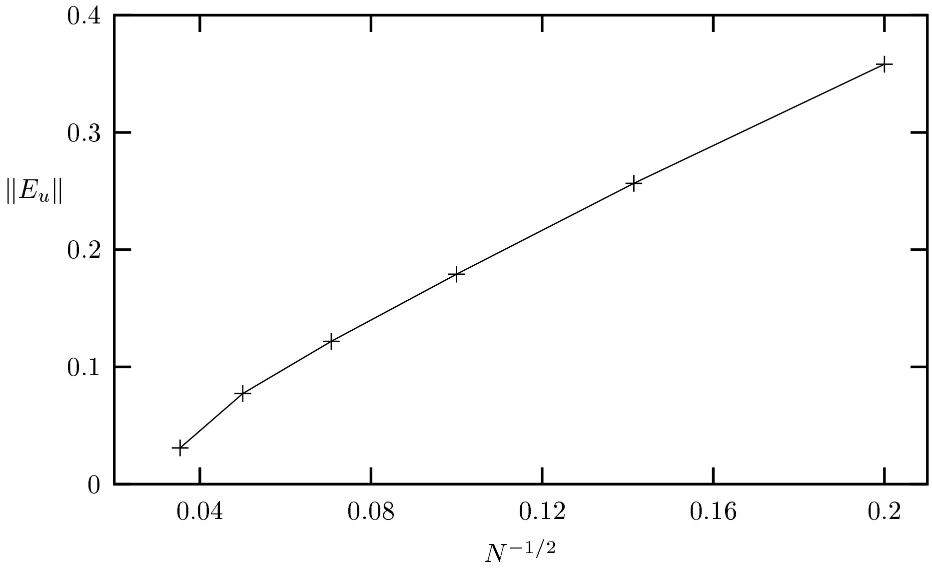

Figure 4: Error for the exact control $u_N(T)$

Appendix

In this part by using the Residue Theorem of holomorphic functions we obtain some results on numerical series. For other similar results, see Weinberger [12].

In what follows we fix some notations and write some results. Let $\mathcal{O}$ be an open set of $\mathbb{C}$ and $f : \mathcal{O} \to \mathbb{C}$ a holomorphic function. Let z_0 be a pole of order k of $f(z)$. The residue of $f(z)$ at z_0, which is denoted by $\Re_{z_0}$, is by definition

$$\Re_{z_0}[f] = \frac{1}{(k-1)!} \lim_{z \to z_0} \left\{ \frac{d^{k-1}}{dz^{k-1}} (z - z_0)^k \ f(z) \right\}.$$

Note that if $f(z) = g(z)/h(z)$, where $g(z)$ and $h(z)$ are holomorphic functions in $\mathcal{O}$ and z_0 is a pole of order one of $f(z)$, then

$$\Re_{z_0}[f] = \frac{g(z_0)}{h'(z_0)} \tag{16}$$

In this case $g(z_0) \neq 0$, $h(z_0) = 0$ and $h'(z_0) \neq 0$.

Proposition A.1 *Let $f(z)$ be a holomorphic function in $\mathbb{C}$, except in the poles $z_1, z_2 \ \cdots$, such that $|z_j| \to \infty, \quad j \to \infty$. Suppose that there exists a sequence (R_q) of real positive numbers with $R_q \to \infty \quad q \to \infty$, such that*

$$\lim_{q \to \infty} \left(R_q \max_{|z| = R_q} |f(z)| \right) = 0 \tag{17}$$

Then

$$\sum_{j=1}^{\infty} \Re_{z_j}[f] = 0$$

Proof We denote by γ_q the curve $|z| = R_q$. By hypothesis (17) we have that there exists q_0 such that the curves γ_q, $q > q_0$, do not intersect the poles z_j since each time that γ_q intersect some pole, one has $\max\limits_{|z| = R_q} |f(z)| = \infty$.

Let us consider γ_q with $q \geq q_0$. By Residue Theorem we have

$$\int_{\gamma_q} f(z)\, dz = 2i\pi \sum_j \Re_{z_j}[f] \tag{18}$$

where the summation is made on all poles that are inside $q \in \gamma_q$. We have

$$|\int_{\gamma_q} f(z) dz| \leq \left(\max_{|z|=R_q} |f(z)\right) 2\pi R_q \to 0, \quad q \to \infty$$

which implies

$$\int_{\gamma_q} f(z) dz \to 0, \quad q \to \infty \tag{19}$$

Expressions (18) and (19) give the proposition. ■

Let $z = x + iy$ be a complex number. Then

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \quad \text{and} \quad |\sin z|^2 = \sin^2 x + \sinh^2 y \tag{20}$$

In the sequel we obtain results on numerical series which have been used in the paper.

Proposition A.2 *Let $k_e > 0$ be an even number. Then*

$$\sum_{p=1}^{\infty} \frac{1}{(2p-1)^2 - k_e^2} = 0$$

Proof Let us consider the function

$$f(z) = \frac{1}{z^2 - k_e^2} \, \frac{\cos \frac{\pi}{2}(z+1)}{\sin \frac{\pi}{2}(z+1)}.$$

As $\cos \dfrac{\pi}{2}(\pm k_e + 1) = 0$ it follows that $z = \pm k_e$ are removable singularities. The poles of $f(z)$ are $z_n = n, \quad n = \pm 1, \pm 3, \pm 5, \cdots$, and their order is one.

By (16) and noting that $\dfrac{d}{dz}\left(\sin \dfrac{\pi}{2}(z+1)\right) = \dfrac{\pi}{2} \cos \dfrac{\pi}{2} (z+1)$, we have that

$$\Re_n[f] = \frac{2}{\pi\, (n^2 - k_e^2)}; \quad n = \pm 1 \ \pm 3, \cdots.$$

Thus, the sum of all residues is

$$\sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{2}{\pi (n^2 - k_e^2)} \tag{21}$$

On the other hand, we have

$$|f(z)|^2 = \frac{1}{|z^2 - k_e^2|^2} \, \frac{|\cos \frac{\pi}{2}(z+1)|^2}{|\sin \frac{\pi}{2}(z+1)|^2}$$

Noting that, for $z = x + iy$,

$$|z^2 - k_e^2| \geq \left(|z| - k_e \right)^2 \quad \textit{and} \quad \frac{\pi}{2}(z+1) = \frac{\pi}{2}(x+1) + i\frac{\pi}{2}y$$

and using (20), we obtain from the last equality

$$|f(z)|^2 \leq \frac{1}{(|z| - k_e)^4} \, \frac{\cos^2 \frac{\pi}{2}(x+1) + \sinh^2 \frac{\pi}{2}y}{\sin^2 \frac{\pi}{2}(x+1) + \sinh^2 \frac{\pi}{2}y} \tag{22}$$

Let us consider, $R_q = 2q, \quad q = 1, 2, \cdots, \quad$ and $\quad |z| = 2q$; $z = x + iy$. We have:

1. If $|x| \geq 2q - \frac{1}{2}$, then

$$\begin{cases} q\pi + \dfrac{\pi}{4} \;\leq\; \dfrac{\pi}{2}\,(x+1) \;\leq\; q\pi + \dfrac{\pi}{2}, & x > 0 \\ -q\pi + \dfrac{\pi}{2} \;\leq\; \dfrac{\pi}{2}\,(x+1) \;\leq\; -q\pi + \dfrac{3\pi}{4}, & x < 0 \end{cases}$$

For these values of x, one has

$$\cos^2 \frac{\pi}{2}\,(x+1) \;\leq\; \sin^2 \frac{\pi}{2}\,(x+1).$$

This inequality and (22) imply that

$$|f(z)|^2 \;\leq\; \frac{1}{(|z| - k_e)^4} \;=\; \frac{1}{(2q - k_e)^4}. \tag{23}$$

2. If $|x| \;<\; 2q - \frac{1}{2}$, then

$$y^2 \;>\; 4q^2 - \left(2q - \frac{1}{2}\right)^2 = 2q - \frac{1}{4} \;\geq\; \frac{7}{4}$$

that is, $|y| \;>\; \dfrac{\sqrt{7}}{2}$. From (22) it follows that

$$|f(z)|^2 \;\leq\; \frac{1}{(|z| - k_e)^4}\,\theta(y) \tag{24}$$

where $\theta(y)$ is the function defined by

$$\theta(y) \;=\; \frac{1 + \sinh^2 \frac{\pi}{2} y}{\sinh^2 \frac{\pi}{2} y}, \quad \text{whenever} \quad y \neq 0$$

Noting that

$$\sinh^2 \frac{\pi}{2} y \;=\; \frac{1}{4}(e^{\pi y} - 2 + e^{-\pi y}) \quad \text{and} \quad \frac{d}{dy}\left(\sinh^2 \frac{\pi}{2} y\right) \;=\; \frac{\pi}{4}(e^{\pi y} - e^{-\pi y}),$$

we obtain

$$\frac{d}{dy}\theta(y) \;=\; -\frac{1}{(\sinh^2 \frac{\pi}{2} y)^2}\,\frac{\pi}{4}\,(e^{\pi y} - e^{-\pi y}), \quad \text{whenever} \quad y \neq 0.$$

From that we conclude that $\theta(y)$ is decreasing in $]0,\,\infty[$ and increasing in $]-\infty,\,0[$. Therefore

$$\theta(y) \;<\; \begin{cases} \theta(\frac{\sqrt{7}}{2}) & \text{for} \quad y > \frac{\sqrt{7}}{2} \\ \theta(\frac{-\sqrt{7}}{2}) & \text{for} \quad y < \frac{-\sqrt{7}}{2} \end{cases}$$

Nothing that $\theta(y) \;=\; \theta(-y)$, for $y \in \mathbb{R}, \quad y \neq 0$, we derive then

$$\theta(y) \;<\; \theta\left(\frac{\sqrt{7}}{2}\right), \qquad |y| > \frac{\sqrt{7}}{2}.$$

Thus by (24) we obtain

$$|f(z)|^2 \;\leq\; \frac{1}{(2q - k_e)^4}\,\theta\left(\frac{\sqrt{7}}{2}\right)$$

The above inequality and (23) give that

$$|f(z)| \le \frac{1}{(2q-k_e)^2} M, \qquad |z| = 2q, \quad 2q > k_e$$

where $M^2 = \theta\left(\frac{\sqrt{7}}{2}\right)$. The last inequality implies

$$\lim_{q\to\infty}\left(2q \max_{|z|=2q} |f(z)|\right) = 0.$$

Proposition $A.1$, (21) and this limit give us that

$$\sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{2}{\pi(n^2-k_e^2)} = 0,$$

which implies the proposition. ∎

By applying similar arguments as in the proof of Proposition A.2, we derive the following results on numerical series. We will present results and the respective function $f(z)$ that allow us to obtain each one of them.

Let $k_e > 0$ be an even number. Then

$$\begin{aligned} &\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{(2p-1)((2p-1)^2-k_e^2)} = \frac{\pi}{4}\left(\frac{(-1)^{k/2}}{k_e^2} - \frac{1}{k_e^2}\right) \\ &f(z) = \frac{1}{z(z^2-k_e^2)} \, \frac{1}{\sin\frac{\pi}{2}(z+1)}. \end{aligned} \tag{25}$$

Let $k_{1e} > 0$ and $k_{2e} > 0$ be even numbers with $k_{1e} \neq k_{2e}$. Then

$$\begin{aligned} &\sum_{p=1}^{\infty} \frac{1}{((2p-1)^2-k_{1e}^2)} \frac{1}{((2p-1)^2-k_{2e}^2)} = 0 \\ &f(z) = \frac{1}{(z^2-k_{1e}^2)(z^2-k_{2e}^2)} \, \frac{\cos\frac{\pi}{2}(z+1)}{\sin\frac{\pi}{2}(z+1)}. \end{aligned} \tag{26}$$

Let k_e be an even number. Then

$$\begin{aligned} &\sum_{p=1}^{\infty} \frac{1}{((2p-1)^2-k_e^2)} = \frac{\pi^2}{16k_e^2} \\ &f(z) = \frac{1}{(z^2-k_e)^2} \, \frac{\cos\frac{\pi}{2}(z+1)}{\sin\frac{\pi}{2}(z+1)}. \end{aligned} \tag{27}$$

Let $k_{1o} > 0$ and $k_{2o} > 0$ be odd numbers with $k_{1o} \neq k_{2o}$. Then

$$\begin{aligned} &\sum_{p=1}^{\infty} \frac{1}{((2p)^2-k_{1o}^2)} \, \frac{1}{((2p)^2-k_{2o}^2)} = -\frac{1}{2\,k_{1o}^2\,k_{2o}^2} \\ &f(z) = \frac{1}{(z^2-k_{1o}^2)(z^2-k_{2o}^2)} \, \frac{\cos\frac{\pi}{2}z}{\sin\frac{\pi}{2}z}. \end{aligned} \tag{28}$$

Let $k_o > 0$ be an odd number. Then

$$\begin{aligned} &\sum_{p=1}^{\infty} \frac{1}{((2p)^2-k_o^2)^2} = \frac{\pi^2}{16\,k_o^2} - \frac{1}{2\,k_o^4} \\ &f(z) = \frac{1}{(z^2-k_o^2)} \, \frac{\cos\frac{\pi}{2}z}{\sin\frac{\pi}{2}z}. \end{aligned} \tag{29}$$

Let $k_{1e} > 0$ and $k_{2e} > 0$ be even numbers with $k_{1e} \neq k_{2e}$. Then

$$\begin{aligned} &\sum_{p=1}^{\infty} \frac{(2p-1)^2}{((2p-1)^2 - k_{1e}^2)^2 \, ((2p-1)^2 - k_{2e}^2)} = 0 \\ &f(z) = \frac{z^2}{(z^2 - k_{1e}^2)(z^2 - k_{2e}^2)} \, \frac{\cos \frac{\pi}{2}(z+1)}{\sin \frac{\pi}{2}(z+1)}. \end{aligned} \tag{30}$$

Let $k_e > 0$ be an even number. Then

$$\begin{aligned} &\sum_{p=1}^{\infty} \frac{(2p-1)^2}{((2p-1)^2 - k_e^2)^2} = \frac{\pi^2}{16} \\ &f(z) = \frac{z^2}{(z^2 - k_e^2)} \, \frac{\cos \frac{\pi}{2}(z+1)}{\sin \frac{\pi}{2}(z+1)}. \end{aligned} \tag{31}$$

Let $k_{1o} > 0$ and $k_{2o} > 0$ be odd numbers with $k_{1o} \neq k_{2o}$. Then

$$\begin{aligned} &\sum_{p=1}^{\infty} \frac{(2p)^2}{((2p)^2 - k_{1o}^2) \, ((2p)^2 - k_{2o}^2)} = 0 \\ &f(z) = \frac{z^2}{(z^2 - k_{1o}^2)(z^2 - k_{2o}^2)} \, \frac{\cos \frac{\pi}{2} z}{\sin \frac{\pi}{2} z}. \end{aligned} \tag{32}$$

Let $k_o > 0$ be an odd number. Then

$$\begin{aligned} &\sum_{p=1}^{\infty} \frac{(2p)^2}{((2p)^2 - k_o^2)^2} = \frac{\pi^2}{16} \\ &f(z) = \frac{z^2}{(z^2 - k_o^2)} \, \frac{\cos \frac{\pi}{2} z}{\sin \frac{\pi}{2} z}. \end{aligned} \tag{33}$$

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