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Abstract

A mathematical model for the small vibration of an elastic string is considered. The model takes into account the change of tension due to the movement of the end points of the string. Under the assumptions that the velocity of the moving ends be less than the characteristic velocity of the equation, the global existence and the local uniqueness of the solution are proved.

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1 Introduction

The mathematical model of D'ALEMBERT for small vibration of elastic strings is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{\tau}{m}, \tag{1}$$

where $u(x,t)$ is the transverse displacement of the string and m is the mass per unit length. The basic assumption of the model is that the string must be well stretched so that the tension τ can be regarded as constant during the vibration. However, due to the change of length of the string, the tension may have a small variation during the vibration. A model which takes this into consideration for the string with fixed ends is proposed later in [2, 5], known as the equation of KIRCHHOFF-CARRIER:

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{m}\left\{\tau_0 + \frac{\kappa}{2L_0}\int_\alpha^\beta \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

where τ_0 is the tension and $L_0 = \beta - \alpha$ is the length of the string at rest; κ is the Young's modulus. The nonlinear term containing the displacement gradient

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has since attracted much attention in mathematical analysis of the model by various authors [6, 8, 10]. In [9] Medeiros *et al* further generalized the Kirchhoff-Carrier model for a linear elastic string with moving ends. In [7] Liu & Rincon also proposed the model for nonlinear elastic strings in general given by the following equation:

$$\widehat{L}u(x,t) = \frac{\partial^2 u}{\partial t^2} - \left(a(t) + b(t)\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

$$\alpha(t) < x < \beta(t), \qquad t > 0,$$

where $\alpha(t)$ and $\beta(t)$ are the positions of the left and the right ends at the instant t respectively; the horizontal length of the string is given by $\gamma(t) = \beta(t) - \alpha(t) > 0$.

The two coefficient functions $a(t)$ and $b(t)$ depend on the elastic properties of the string. Let s be the length of the string. We define the mean Lagrangian strain ε as

$$\varepsilon = \varepsilon(s) = \frac{s - L_0}{L_0}, \tag{4}$$

where L_0 is the length of the string at some reference state. The stress-strain relation of an elastic string is nonlinear in general and is given by

$$\tau = \sigma(\varepsilon). \tag{5}$$

For the vibration model, the string is assumed to be under tension and the elastic modulus is non-negative,

$$\sigma(\varepsilon) > 0, \qquad \sigma'(\varepsilon) \geq 0. \tag{6}$$

From the model proposed in [7], we have the following relations:

$$a(t) = \frac{1}{m}\sigma(\varepsilon(\gamma(t))), \qquad b(t) = \frac{1}{2mL_0}\sigma'(\varepsilon(\gamma(t))). \tag{7}$$

From (6), it follows that

$$a(t) > 0, \qquad b(t) \geq 0. \tag{8}$$

When the stress-strain relation is linear [9],

$$\tau = \sigma(\epsilon) = \tau_0 + \kappa\,\epsilon,$$

where κ is the Young's modulus and τ_0 is the tension of the string at the reference state with length L_0, then from (7) we have

$$a(t) = \frac{\tau_0}{m} + \frac{\kappa}{m}\frac{\gamma(t) - L_0}{L_0}, \qquad b(t) = \frac{\kappa}{2mL_0}. \tag{9}$$

Let $\widehat{Q} = \{(x,t) \in \mathbb{R}^2;\ \alpha(t) < x < \beta(t),\ t > 0\}$ be the non-cylindrical domain with boundary $\widehat{\Sigma} = \bigcup_{0<t<T} \{\alpha(t), \beta(t)\} \times \{t\}$ and consider the following problem:

$$\text{(I)} \quad \begin{cases} \widehat{L}u(x,t) = f(x,t) & \forall\,(x,t) \in \widehat{Q}, \\ u(x,t) = 0 & \forall\,(x,t) \in \widehat{\Sigma}, \\ u(x,0) = u_0(x), & \dfrac{\partial u}{\partial t}(x,0) = u_1(x); \quad \alpha(0) < x < \beta(0). \end{cases} \tag{10}$$

In [9] the existence and the uniqueness of the local solution of the problem (10) have been proved under the following hypotheses:

h1: $\alpha, \beta \in C^2([0,T);\mathbb{R})$,

$\alpha(t) < \beta(t)$, $\alpha'(t) < 0$, $\beta'(t) > 0$, for $0 \le t < T$,

$|\alpha'(t) + \gamma'(t)y| < (m_0/2)^{1/2}$, for $0 \le t < T$, $0 \le y \le 1$.

h2: $a \in W^{1,\infty}(0,\infty)$, $a(t) \ge m_0 > 0$.

Theorem 1 *Let* $\Omega_t = (\alpha(t), \beta(t))$ *and* $\Omega_0 = (\alpha(0), \beta(0))$. *Then, under the hypotheses* (h1) *and* (h2), *given* $u_0 \in H_0^1(\Omega_0) \bigcap H^2(\Omega_0)$, $u_1 \in H_0^1(\Omega_0)$, $f \in C^0([0,T); H_0^1(\Omega_t))$, *there exists* $0 < T_0 < T$ *and a unique solution* $u : \widehat{Q} \to \mathbb{R}$ *of the problem* (I) *satisfying the following conditions:*

$$u \in L^\infty(0, T_0; H_0^1(\Omega_t) \bigcap H^2(\Omega_t)),$$

$$u' \in L^\infty(0, T_0; H_0^1(\Omega_t)),$$

$$u'' \in L^2(o, T_0; L^2(\Omega_t)).$$

Numerical simulations and stability analysis in [7] seem to indicate that it is unnecessary to restrict the conditions $\alpha'(t) < 0$ and $\beta'(t) > 0$ at the end points as required by the hypothesis (h1) in order that the length of the string can only increase.

In this paper, we consider the case in which the change of length due to transverse displacements is insignificant and consequently the equation is linear by neglecting the nonlinear term, but the tension is not constant in general because of the moving ends.

We remark that although we neglect the term with $b(t)$ as coefficient, it is different from assuming $b(t) = 0$. Because, otherwise, it would require $a(t)$ to be a constant from our formulation. The case with constant tension with moving boundary, has been considered elsewhere [1, 3, 4].

Moreover, we shall consider initial conditions less restrictive than those required in Theorem 1 cited from [9]. The main result is given in Theorem 3, in which we do not require the length of the string be only increasing. However, we do require that the velocities of the end points, $\alpha'(t)$ and $\beta'(t)$, be less than the characteristic velocity of the equation, which seems to be reasonable from physical intuition. Such a condition has also been assumed in [3, 4] in the case with constant tension.

2 Formulation of Linear Problem

The equation (3) without the nonlinear term is given by

$$\widetilde{L}u(x,t) = \frac{\partial^2 u}{\partial t^2} - a(t)\frac{\partial^2 u}{\partial x^2} = 0. \tag{11}$$

We can formulate the following linear problem:

$$\textbf{(II)} \quad \begin{cases} \widetilde{L}u(x,t) = f(x,t) & \forall\,(x,t) \in \widehat{Q}, \\ u(x,t) = 0 & \forall\,(x,t) \in \widehat{\Sigma}, \\ u(x,0) = u_0(x), & \dfrac{\partial u}{\partial t}(x,0) = u_1(x), \qquad \alpha(0) < x < \beta(0). \end{cases} \tag{12}$$

where $\widehat{Q}$ and $\widehat{\Sigma}$ has been defined previously and

$$\alpha(t) < x < \beta(t), \qquad t > 0.$$

The horizontal length of the string is given by $\gamma(t) = \beta(t) - \alpha(t) > 0$.

In order to prove the existence and uniqueness of the solution of problem (II) we shall consider an equivalent problem defined in a fixed domain. Consider the change of variable:

$$(x,t) \in \widehat{Q} \mapsto (y,t) \in Q, \qquad y = \frac{x - \alpha(t)}{\gamma(t).} \tag{13}$$

where $Q = (0,1) \times (0,T)$. Let $u(x,t) = v(y,t)$ and introduce the following operator define in Q,

$$Lv(y,t) = \frac{\partial^2 v}{\partial t^2} + a(y,t)\frac{\partial^2 v}{\partial y^2} + b(y,t)\frac{\partial^2 v}{\partial t \partial y} + c(y,t)\frac{\partial v}{\partial y}. \tag{14}$$

We obtain the following equivalent problem in the rectangular domain $(0,1) \times (0,T)$:

$$\textbf{(III)} \quad \begin{cases} Lv(y,t) = g(y,t) & \forall\,(y,t) \in (0,1) \times (0,T), \\ v(0,t) = v(1,t) = 0, & 0 < t < T, \\ v(y,0) = v_0(y), & \dfrac{\partial v}{\partial t}(y,0) = v_1(y), \quad 0 \le y \le 1, \end{cases} \tag{15}$$

where

$$a(y,t) = \frac{1}{4}b(y,t)^2 - \frac{1}{\gamma(t)^2}a(t), \tag{16}$$

$$b(y,t) = -2\,\frac{\alpha' + \gamma'(t)y}{\gamma(t)}, \tag{17}$$

$$c(y,t) = -\frac{1}{\gamma(t)}\Big(\alpha'' + \gamma''(t)y + \gamma'(t)b(y,t)\Big), \tag{18}$$

with $a(t)$ defined in (8).

Observation: Since the equation is hyperbolic, the coefficients in the equation (14) must satisfy the following inequality:

$$b^2(y,t) - 4a(y,t) > 0,$$

which is equivalent to the condition (8), $a(t) > 0$.

3 Existence and Uniqueness

We shall first establish the existence and uniqueness of problem (III) as an auxiliary theorem and then prove the main result for the original problem (II).

Let $((\ ,\ ))$, $\|\,.\,\|$ and $(\ ,\ )$, $|\,.\,|$ be respectively the scalar product and the norms in $H_0^1(0,1)$ and $L^2(0,1)$, and consider the following hypotheses:

H1: $\alpha, \beta \in C^2([0,T]; \mathbb{R})$.

H2: $a(y,t) < 0$.

The hypothesis (**H2**) implies that

$$a(t) - \max_{0\le t\le T} \left\{ |\alpha'(t)|^2; |\beta'(t)|^2 \right\} > 0,$$

which requires that the velocities of the end points be smaller than the characteristic velocity of the equation. Under these conditions, we have the following result for problem (III):

Theorem 2 *Under the hypotheses* (**H1**) *and* (**H2**) *and given the initial data*

$$v_0 \in H_0^1(0,1), \quad v_1 \in L^2(0,1), \quad g \in L^2([0,T]; L^2(0,1)),$$

there exists $T > 0$ *and a unique weak solution of Problem* (III) $v : Q \to \mathbb{R}$*, satisfying the following conditions:*

1. $v \in L^\infty(0,T; H_0^1(0,1))$,
2. $v' \in L^\infty(0,T; L^2(0,1))$.

Proof. To prove the theorem, we introduce the approximate solutions. Let $T > 0$ and denote by V_m the subspace spanned by $\{u_1, u_2, ..., u_m\}$, where $\{u_\nu;\ \nu = 1, \cdots m\}$ are the first m base vectors of the space $H_0^1(0,1)$. If $v_m(t) \in V_m$ then it can be represented by

$$v_m(t) = \sum_{\nu=1}^{m} g_{\nu m}(t) u_\nu(y),$$

where $g_{\nu m}$ is the solution of the system of ordinary differential equations:

$$\begin{cases} (Lv_m, w) = (g, w) & \forall\, w \in V_m, \\ v_m(0) = v_{0m} \to v_0 & \text{in} \quad H_0^1(0,1), \\ v_m'(0) = v_{1m} \to v_1 & \text{in} \quad L^2(0,1). \end{cases} \tag{19}$$

The system (19) has local solution in the interval $(0, T_m)$. To extend the local solution to the interval $(0, T)$ independent of m the following a priori estimate is needed.

A Priori Estimate:

Taking $w = v'_m(t)$ in the equation (19) we obtain

$$\left(v''_m, v'_m\right) + \left(a(y,t)\frac{\partial^2 v_m}{\partial y^2}, v'_m\right) + \left(b(y,t)\frac{\partial v'_m}{\partial y}, v'_m\right) + \left(c(y,t)\frac{\partial v_m}{\partial y}, v'_m\right) = \left(g, v'_m\right). \tag{20}$$

From the first term of (20) we have

$$\left(v''_m, v'_m\right) = \frac{1}{2}\frac{\partial}{\partial t}|v'_m|^2. \tag{21}$$

Integrating the second term of (20) by parts and using the boundary conditions $v'_m(1,t) = v'_m(0,t) = 0$, we obtain

$$\begin{aligned}\int_0^1 a(y,t)\frac{\partial^2 v_m}{\partial y^2} v'_m\, dy &= -\int_0^1 \frac{\partial a(y,t)}{\partial y}\frac{\partial v_m}{\partial y} v'_m\, dy - \frac{1}{2}\int_0^1 a(y,t)\frac{\partial}{\partial t}(\frac{\partial v_m}{\partial y})^2\, dy \\ &= -\int_0^1 \frac{\partial a(y,t)}{\partial y}\frac{\partial v_m}{\partial y} v'_m\, dy - \frac{1}{2}\int_0^1 \frac{\partial}{\partial t}\left(a(y,t)(\frac{\partial v_m}{\partial y})^2\right) dy \\ &+\frac{1}{2}\int_0^1 \left(\frac{\partial a(y,t)}{\partial t}\right)(\frac{\partial v_m}{\partial y})^2\, dy.\end{aligned} \tag{22}$$

By the generalized mean-value theorem, there exists a $\xi \in [0,1]$ such that

$$\begin{aligned}&-\frac{1}{2}\frac{\partial}{\partial t}\int_0^1 a(y,t)(\frac{\partial v_m}{\partial y})^2\, dy \\ &= -\frac{1}{2}\frac{\partial}{\partial t}\left(a(\xi,t)\int_0^1 (\frac{\partial v_m}{\partial y})^2\right) dy = -\frac{1}{2}\frac{\partial}{\partial t}\left(a(\xi,t)||v_m||^2\right).\end{aligned} \tag{23}$$

Substituting (23) in (22) we obtain

$$\begin{aligned}\int_0^1 a(y,t)\frac{\partial^2 v_m}{\partial y^2} v'_m\, dy &= -\int_0^1 \frac{\partial a(y,t)}{\partial y}\frac{\partial v_m}{\partial y} v'_m\, dy - \frac{1}{2}\frac{\partial}{\partial t}\left(a(\xi,t)||v_m||^2\right) \\ &+\frac{1}{2}\int_0^1 \left(\frac{\partial a(y,t)}{\partial t}\right)(\frac{\partial v_m}{\partial y})^2\, dy.\end{aligned} \tag{24}$$

From the third term of (20), we have

$$\begin{aligned}\int_0^1 b(y,t)\frac{\partial v'_m}{\partial y} v'_m dy &= \frac{1}{2}\int_0^1 b\frac{\partial}{\partial y}(v'_m)^2 dy = \frac{1}{2}(v'_m)^2 b\Big|_0^1 \\ &-\frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}(v'_m)^2 dy = -\frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}(v'_m)^2.\end{aligned} \tag{25}$$

Substituting (21), (24) and (25) in (20) we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}|v'_m(t)|^2 &- \frac{1}{2}\frac{\partial}{\partial t}\Big(a(\xi,t)\|v_m\|^2\Big) = \int_0^1 \frac{\partial a(y,t)}{\partial y}\frac{\partial v_m}{\partial y}v'_m\,dy \\
&- \frac{1}{2}\int_0^1 \Big(\frac{\partial a(y,t)}{\partial t}\Big)(\frac{\partial v_m}{\partial y})^2\,dy + \frac{1}{2}\int_0^1 \frac{\partial b(y,t)}{\partial y}(v'_m)^2 dy \qquad (26)\\
&- \int_0^1 c(y,t)\frac{\partial v_m}{\partial y}v'_m(t)dy + \int_0^1 g(t)v'_m dy,
\end{aligned}
$$

which after integration in $[0,t)$, with $t \in [0,T_m)$, becomes

$$
\begin{aligned}
\frac{1}{2}|v'_m(t)|^2 &- \frac{1}{2}a(\xi,t)\|v_m(t)\|^2 = \frac{1}{2}|v'_m(0)|^2 + \frac{1}{2}a(\xi,0)\|v_m(0)\|^2 \\
&+ \int_0^t \Big\{\int_0^1 \frac{\partial a(y,t)}{\partial y}\frac{\partial v_m}{\partial y}v'_m\,dy - \frac{1}{2}\int_0^1 \Big(\frac{\partial a(y,t)}{\partial t}\Big)(\frac{\partial v_m}{\partial y})^2\,dy \\
&+ \frac{1}{2}\int_0^1 \frac{\partial b(y,t)}{\partial y}(v'_m)^2 dy - \int_0^1 c(y,t)\frac{\partial v_m}{\partial y}v'_m(t)dy \\
&+ \int_0^1 g(t)v'_m\,dy\Big\}\,ds.
\end{aligned}
\qquad (27)
$$

Now, consider the terms on the right hand side of (27). We have

$$\int_0^1 \frac{\partial a(y,t)}{\partial y}\frac{\partial v_m}{\partial y}v'_m\,dy \le c_1\Big(\|v_m(t)\|^2 + |v'_m(t)|^2\Big), \qquad (28)$$

$$\int_0^1 \Big(\frac{\partial a(y,t)}{\partial t}\Big)(\frac{\partial v_m}{\partial y})^2\,dy \le c_2\|v_m\|^2, \qquad (29)$$

$$\int_0^1 \frac{\partial b(y,t)}{\partial y}(v'_m)^2 dy \le c_3|v'_m|^2, \qquad (30)$$

$$\int_0^1 c(y,t)\frac{\partial v_m}{\partial y}v'_m(t)dy \le c_4\|v_m\|^2, \qquad (31)$$

$$\int_0^1 g(t)v'_m\,dy \le \frac{1}{2}\Big(|g|^2 + |v'_m|^2\Big), \qquad (32)$$

where we have used the relations (16), (17), (18) and the hypothesis (H1) to conclude that $a(y,t)$ is a continuous function in Q. Moreover,

$$|\frac{\partial a(y,t)}{\partial y}| = \frac{2}{\gamma^2}|\gamma'|\,|\alpha' + \gamma' y| \le 2c_1,$$

$$|\frac{\partial a(y,t)}{\partial t}| = |\frac{-2(\alpha' + \gamma' y)}{\gamma^3}|\,|\gamma'(\alpha' + \gamma' y) - \gamma(\alpha'' + \gamma'' y)| \le c_2,$$

and in a similar manner for $|\dfrac{\partial b(y,t)}{\partial y}|$ and $|c(y,t)|$.

After substitution of (28), (29), (30), (31), (32) in (27), for $t < T_m < T$, we obtain

$$\frac{1}{2}|v_m'(t)|^2 - \frac{1}{2}a(\xi,t)\|v_m(t)\|^2 \leq c + \hat{c}\int_0^T (\|v_m\|^2 + |v_m'|^2)ds, \tag{33}$$

where $c = \frac{1}{2}(|v_m'(0)|^2 + \|v_m(0)\|^2 + |g|^2)$ is a constant, since by hypothesis the sequences $v_m(0) = v_{0m}$ and $v_m'(0) = v_{1m}$ are bounded in $H_0^1(0,1)$ and $L^2(0,1)$ respectively, and $g \in L^2(0,T;L^2(0,1))$. Moreover, by hypothesis (**H2**), $a(y,t) < 0$ is bounded, hence we can take $k = \min\{\frac{1}{2}, -\frac{1}{2}a(\xi,t)\}$ and obtain

$$|v_m'(t)|^2 + \|v_m\|^2 \leq K_1 + K_2\int_0^T \Big(|v_m'(s)|^2 + \|v_m(s)\|^2\Big)ds, \tag{34}$$

where $K_1 = \dfrac{c}{k}$ and $K_2 = \dfrac{\hat{c}}{k}$.

By the use of Gronwall's Lemma in the inequality (34), we obtain the estimate,

$$|v_m'(t)|^2 + \|v_m(t)\|^2 \leq C, \qquad \forall t \in [0,T], \quad T > 0. \tag{35}$$

Therefore, there exists a subsequence $(v_\mu)_{\mu\in I\!N}$ of the sequence $(v_m)_{m\in I\!N}$ such that

$$\begin{aligned} v_\mu &\rightharpoonup^* v \quad \text{in} \quad L^\infty(0,T;H_0^1(0,1)),\\ v_\mu' &\rightharpoonup^* v' \quad \text{in} \quad L^\infty(0,T;L^2(0,1)). \end{aligned} \tag{36}$$

From (19) we have

$$\frac{\partial}{\partial t}\Big(\frac{\partial v_\nu}{\partial t}, w\Big) - \Big(\frac{\partial a(y,t)}{\partial y}\frac{\partial v_\nu}{\partial y}, w\Big) - \Big(a(y,t)\frac{\partial v_\nu}{\partial y}, \frac{\partial w}{\partial y}\Big) + \frac{\partial}{\partial t}\Big(b(y,t)\frac{\partial v_\nu}{\partial y}, w\Big)$$
$$-\Big(\frac{\partial b(y,t)}{\partial t}\frac{\partial v_\nu}{\partial y}, w\Big) + \Big(c(y,t)\frac{\partial v_\nu}{\partial y}, w\Big) = \Big(g, w\Big), \qquad \forall w \in H_0^1(0,1) \tag{37}$$

in the sense of distribution in [0,T].

From $(36)_2$ and by the definition of weak star convergence, it follows that

$$\Big(v_\mu', w\Big) \longrightarrow \Big(v', w\Big) \qquad \forall w \in H_0^1(0,1)$$

in the sense of distribution in [0,T]. Therefore, we have

$$\frac{\partial}{\partial t}\Big(v_\mu', w\Big) \longrightarrow \frac{\partial}{\partial t}\Big(v', w\Big) \qquad \forall w \in H_0^1(0,1). \tag{38}$$

Similarly, from $(36)_1$ it leads to

$$\Big(\frac{\partial v_\mu}{\partial y}, w\Big) \longrightarrow \Big(\frac{\partial v}{\partial y}, w\Big) \qquad \forall w \in H_0^1(0,1),$$

in the sense of distribution in [0,T] and

$$\left(\frac{\partial v_\mu}{\partial y}, \frac{\partial w}{\partial y}\right) \longrightarrow \left(\frac{\partial v}{\partial y}, \frac{\partial w}{\partial y}\right) \qquad \forall\, w \in H_0^1(0,1)$$

in the sense of distribution in [0,T]. From hypothesis (H1) it follows that

$$\left(\frac{\partial a(y,t)}{\partial y}\frac{\partial v_\mu}{\partial y}, w\right) \longrightarrow \left(\frac{\partial a(y,t)}{\partial y}\frac{\partial v}{\partial y}, w\right) \qquad \forall\, w \in H_0^1(0,1), \tag{39}$$

$$\left(a(y,t)\frac{\partial v_\mu}{\partial y}, \frac{\partial w}{\partial y}\right) \longrightarrow \left(a(y,t)\frac{\partial v}{\partial y}, \frac{\partial w}{\partial y}\right) \qquad \forall\, w \in H_0^1(0,1), \tag{40}$$

$$\frac{\partial}{\partial t}\left(b(y,t)\frac{\partial v_\mu}{\partial y}, w\right) \longrightarrow \frac{\partial}{\partial t}\left(b(y,t)\frac{\partial v}{\partial y}, w\right) \qquad \forall\, w \in H_0^1(0,1), \tag{41}$$

$$\left(\frac{\partial b(y,t)}{\partial t}\frac{\partial v_\mu}{\partial y}, w\right) \longrightarrow \left(\frac{\partial b(y,t)}{\partial t}\frac{\partial v}{\partial y}, w\right) \qquad \forall\, w \in H_0^1(0,1), \tag{42}$$

$$c(y,t)(\frac{\partial v_\mu}{\partial y}, w) \longrightarrow c(y,t)(\frac{\partial v}{\partial y}, w); \qquad \forall\, w \in H_0^1(0,1), \tag{43}$$

in the sense of distribution in [0,T]. In the limit $\mu \to \infty$ and by convergence of the terms (38) through (43) the approximate solutions v_μ converge to the solution v in the sense of distribution in [0,T].

Initial Conditions:

Let $\varphi \in C^1([0,T])$ satisfying $\varphi(0) = 1$ and $\varphi(T) = 0$ and consider $w(x,t) = u(x)\varphi'(t)$. From the convergence $(36)_1$, we have

$$\int_0^T (v_\mu, u)\varphi' dt \longrightarrow \int_0^T (v,u)\varphi' dt \qquad \forall u \in H_0^1(0,1).$$

On the other hand, from $(36)_2$, we obtain

$$\int_0^T (v'_\mu, u)\varphi\, dt \longrightarrow \int_0^T (v', u)\varphi\, dt \qquad \forall u \in H_0^1(0,1).$$

From the last two relations it follows that

$$\int_0^T \frac{\partial}{\partial t}\{(v_\mu, u)\varphi\}\, dt \longrightarrow \int_0^T \frac{\partial}{\partial t}\{(v,u)\varphi\}\, dt,$$

which leads to

$$(v_\mu(0), u) \longrightarrow (v(0), u), \qquad \forall\, u \in H_0^1(0,1).$$

From (19) we have

$$(v_\mu(0), u) \longrightarrow (v_0, u), \qquad \forall\, u \in H_0^1(0,1).$$

Therefore, $(v(0),u)) = (v_0,u)$ or $v(0) = v_0$.

For the other initial condition, we define the following function:

$$\varphi_\delta(t) = \begin{cases} -\dfrac{t}{\delta}+1, & 0 \le t \le \delta, \\ 0, & \delta \le t \le T. \end{cases} \tag{44}$$

Since the equation

$$(v''_\mu, w) + a(y,t)((v_\mu, w)) + \Big(b(y,t)\frac{\partial v'_\mu}{\partial y}, w\Big) + c(y,t)\Big(\frac{\partial v_\mu}{\partial y}, w\Big) = (g,w),$$

is valid for all $w \in V_m$. Then for μ fixed, the equation is valid for all $w \in H_0^1(0,1)$ and hence is valid for $w \in C_0^\infty(0,1)$. Multiplying the equation by φ_δ and integrating, we obtain

$$\begin{aligned} &\int_0^\delta (v''_\mu(t), w)\varphi_\delta\,dt + \int_0^\delta (a(y,t)((v_\mu, w))\varphi_\delta\,dt + \int_0^\delta (b(y,t)\frac{\partial v'_\mu}{\partial y}, w)\varphi_\delta\,dt \\ &\qquad + \int_0^\delta c(y,t)(\frac{\partial v_\mu}{\partial y}, w)\varphi_\delta\,dt = \int_0^\delta (g(t),w)\varphi_\delta\,dt. \end{aligned} \tag{45}$$

Since

$$\begin{aligned} \int_0^\delta (v''_\mu(t), w)\varphi_\delta\,dt &= \int_0^\delta \frac{\partial}{\partial t}(v'_\mu, w)\varphi_\delta\,dt = (v'_\mu, w)\varphi_\delta\Big|_0^\delta + \int_0^\delta (v'_\mu, w)\frac{1}{\delta}\,dt \\ &= -(v'_\mu(0), w) + \int_0^\delta (v'_\mu(t), w)\frac{1}{\delta}\,dt, \end{aligned}$$

by substitution into the equation (45), we obtain

$$\begin{aligned} &-(v'_\mu(0), w) + \frac{1}{\delta}\int_0^\delta (v'_\mu(t), w)dt + \int_0^\delta a(y,t)((v_\mu, w))(\frac{-t}{\delta}+1)dt \\ &+ \int_0^\delta (b(y,t)\frac{\partial v'_\mu}{\partial y}, w)(\frac{-t}{\delta}+1)dt + \int_0^\delta c(y,t)(\frac{\partial v_\mu}{\partial y}, w)(\frac{-t}{\delta}+1)dt \\ &\qquad = \int_0^\delta (g(t),w)(\frac{-t}{\delta}+1)dt. \end{aligned}$$

Taking the limit $\mu \to \infty$, it gives

$$\begin{aligned} &-(v_1, w) + \frac{1}{\delta}\int_0^\delta (v', w)dt + \int_0^\delta (a(y,t)((v, w))(\frac{-t}{\delta}+1)dt \\ &+ \int_0^\delta b(y,t)(\frac{\partial v'}{\partial y}, w)(\frac{-t}{\delta}+1)dt + \int_0^\delta c(y,t)(\frac{\partial v}{\partial y}, w)(\frac{-t}{\delta}+1)dt \\ &\qquad = \int_0^\delta (g(t),w)(\frac{-t}{\delta}+1)dt. \end{aligned}$$

Let $\delta \to 0$ and use the fundamental theorem of calculus, then it follows that $(v'(0), w) = (v_1, w)$, or $v'(0) = v_1$.

Uniqueness of Solution:

Suppose that the problem admits two solutions, v and $\hat{v}$, and consider $w = v - \hat{v}$. Then w is a solution of the problem:

$$\begin{cases} w'' + a(y,t)\dfrac{\partial^2 w}{\partial y^2} + b(y,t)\dfrac{\partial w'}{\partial y} + c(y,t)\dfrac{\partial w}{\partial y} = 0, \\ w(0) = w'(0) = 0. \end{cases} \tag{46}$$

and we want to show that $w = 0$ in $[0,T)$.

Consider the function ψ defined by

$$\psi(t) = \begin{cases} -\displaystyle\int_t^s w(r)dr, & \text{em} \quad 0 < t \le s, \\ 0, & \text{em} \quad s \le t \le T, \end{cases}$$

then $\psi \in L^\infty(0,T;H_0^1(0,1))$. Let $w_1(t) = \displaystyle\int_0^t w(r)\,dr$, then we have

$$\psi(t) = -\int_t^s w(r)dr = \int_0^t w(r)dr - \int_0^s w(r)dr = w_1(t) - w_1(s).$$

Moreover, $\psi'(t) = w_1'(t) = w(t)$ and $\psi(s) = 0$.

Multiplying the equation (46) by ψ and integrating in Ω, we obtain

$$\int_0^1 w''\psi dy + \int_0^1 a(y,t)\frac{\partial^2 w}{\partial y^2}\psi dy + \int_0^1 b(y,t)\frac{\partial w'}{\partial y}\psi dy + \int_0^1 c(y,t)\frac{\partial w}{\partial y}\psi dy = 0. \tag{47}$$

But

$$\frac{\partial}{\partial t}(w',\psi) = (w'',\psi) + (w',w),$$

and the first term of (47) becomes

$$\int_0^1 w''\psi dy = \frac{\partial}{\partial t}(w',\psi) - \frac{1}{2}\frac{\partial}{\partial t}|w(t)|^2. \tag{48}$$

Integrating by parts the second term of (47) and using $\psi' = w$ and $\psi \in H_0^1(0,1)$, we have

$$\begin{aligned} \int_0^1 a(y,t)\frac{\partial^2 w}{\partial y^2}\psi\, dy &= \int_0^1 a(y,t)\frac{\partial^2 \psi'}{\partial y^2}\psi\, dy \\ &= -\int_0^1 \frac{\partial a(y,t)}{\partial y}\psi\frac{\partial \psi'}{\partial y} - \frac{1}{2}\int_0^1 a(y,t)\frac{\partial}{\partial t}(\frac{\partial \psi}{\partial y})^2\, dy \\ &= -\int_0^1 \frac{\partial a(y,t)}{\partial y}\psi\frac{\partial \psi'}{\partial y} - \frac{1}{2}\frac{\partial}{\partial t}(a(\xi,t)||\psi||^2) + \frac{1}{2}\frac{\partial a(\xi,t)}{\partial t}||\psi||^2, \end{aligned} \tag{49}$$

where the generalized mean value theorem has been used and $\xi \in [0,1]$.

For the third term of (47), we have

$$\int_0^1 b(y,t)\frac{\partial w'}{\partial y}\psi dy = \int_0^1 b\frac{\partial}{\partial t}\Big(\frac{\partial w}{\partial y}\psi\Big)dy - \frac{1}{2}\int_0^1 b\frac{\partial}{\partial y}w^2 dy$$

$$= \int_0^1 b\frac{\partial}{\partial t}\Big(\frac{\partial w}{\partial y}\psi\Big)dy - \frac{1}{2}\Big(b(y,t)(w)^2\Big|_0^1 - \int_0^1 \frac{\partial b}{\partial y}w^2 dy\Big)$$

$$= \int_0^1 b\frac{\partial}{\partial t}\Big(\frac{\partial w}{\partial y}\psi\Big)dy + \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}w^2 dy$$

$$= \int_0^1 \frac{\partial}{\partial t}\Big(b\frac{\partial w}{\partial y}\psi\Big)dy - \int_0^1 \frac{\partial b}{\partial t}\frac{\partial w}{\partial y}\psi dy + \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}w^2 dy$$

$$= \int_0^1 \frac{\partial}{\partial t}\Big(b\frac{\partial w}{\partial y}\psi\Big)dy + \int_0^1 \frac{\partial b'}{\partial y}\psi w\, dy + \int_0^1 \frac{\partial b}{\partial t}\frac{\partial \psi}{\partial y}w\, dy + \frac{1}{2}\int_0^1 \frac{\partial b}{\partial y}w^2 dy. \tag{50}$$

Finally for the fourth term of (47), integrating by parts, we have

$$\int_0^1 c(y,t)\frac{\partial w}{\partial y}\psi\, dy = -\int_0^1 \frac{\partial c(y,t)}{\partial y}\psi w\, dy - \int_0^1 c(y,t)\frac{\partial \psi}{\partial y}w\, dy. \tag{51}$$

Substituting (48), (49), (50) and (51) in (47), we obtain

$$\frac{\partial}{\partial t}(w',\psi) - \frac{1}{2}\frac{\partial}{\partial t}|w|^2 - \frac{1}{2}\frac{\partial}{\partial t}(a(\xi,t)\|\psi\|^2) + \int_0^1 \frac{\partial}{\partial t}(b(y,t)\frac{\partial w}{\partial y}\psi)\, dy$$

$$= -\frac{1}{2}\frac{\partial a(\xi,t)}{\partial t}\|\psi\|^2 - \int_0^1 \frac{\partial b'}{\partial y}\psi w\, dy - \int_0^1 \frac{\partial b}{\partial t}\frac{\partial \psi}{\partial y}w\, dy \tag{52}$$

$$-\frac{1}{2}\int_0^1 \frac{\partial b(y,t)}{\partial y}(w)^2 dy + \int_0^1 \frac{\partial c(y,t)}{\partial y}\psi w\, dy + \int_0^1 c(y,t)\frac{\partial \psi}{\partial y}w\, dy.$$

Integrating (52) in $0 \le t \le s$ and multiplying by -1, we have

$$\frac{1}{2}|w(s)|^2 - \frac{1}{2}a(\xi,0)\|\psi(0)\|^2$$

$$= \int_0^s \Big\{\frac{1}{2}\frac{\partial a(\xi,t)}{\partial t}\|\psi\|^2 + \frac{1}{2}\int_0^1 \frac{\partial b(y,t)}{\partial y}(w)^2\, dy + \int_0^1 \frac{\partial b'}{\partial y}\psi w\, dy \tag{53}$$

$$+\int_0^1 \frac{\partial b}{\partial t}\frac{\partial \psi}{\partial y}w\, dy + \int_0^1 \frac{\partial c(y,t)}{\partial y}\psi w\, dy + \int_0^1 c(y,t)\frac{\partial \psi}{\partial y}w\, dy\Big\}dt,$$

where we have used the conditions $w(0) = w'(0) = \psi(s) = 0$, and

$$\int_0^s \frac{\partial}{\partial t}(w',\psi)dt = (w'(s),\psi(s)) - (w'(0),\psi(0)) = 0, \tag{54}$$

$$\frac{1}{2}\int_0^s \frac{\partial}{\partial t}|w(t)|^2 dt = \frac{1}{2}\Big(|w(s)|^2 - |w(0)|^2\Big) = \frac{1}{2}|w(s)|^2, \tag{55}$$

$$\int_0^s \frac{\partial}{\partial t}\Big(b(y,t)\frac{\partial w}{\partial y},\psi\Big) = \Big(b(y,s)\frac{\partial w(s)}{\partial y},\psi(s)\Big) - \Big(b(y,0)\frac{\partial w(0)}{\partial y},\psi(0)\Big) = 0. \quad (56)$$

Using the generalized mean value theorem for the first term on the right hand side of (53) we have

$$\frac{1}{2}\int_0^s \frac{\partial a(\xi,t)}{\partial t}\|\psi\|^2\,dt = \frac{1}{2}\frac{\partial a(\xi,\eta)}{\partial t}\int_0^s \|\psi\|^2\,dt \quad (57)$$

where $\eta \in [0,s]$.

For the remaining terms on the right hand side of (53) we have

$$\frac{1}{2}\int_0^s\int_0^1 \frac{\partial b}{\partial y}(w)^2\,dy\,dt \le c_1\int_0^s |w|^2\,dt, \quad (58)$$

$$\int_0^s\int_0^1 \frac{\partial b'}{\partial y}\psi w\,dy\,dt \le \int_0^s c_2\theta|w|^2 + \frac{1}{\theta}|\psi|^2\,dt, \quad (59)$$

$$\int_0^s\int_0^1 \frac{\partial b}{\partial t}\frac{\partial \psi}{\partial y}w\,dy\,dt \le \int_0^s c_3\theta|w|^2 + \frac{1}{\theta}\|\psi\|^2\,dt, \quad (60)$$

$$\int_0^s\int_0^1 \frac{\partial c(y,t)}{\partial y}\psi w\,dy\,dt, \le \int_0^s c_4\theta|w|^2 + \frac{1}{\theta}|\psi|^2\,dt, \quad (61)$$

$$\int_0^s\int_0^1 c(y,t)\frac{\partial \psi}{\partial y}w\,dy \le \int_0^s c_5\theta|w|^2 + \frac{1}{\theta}\|\psi\|^2\,dt, \quad (62)$$

where $\theta, c_1, c_2, c_3, c_4, c_5$ are positive real numbers. Substituting the above inequalities in (53) and using the continuous imbedding of the space $H_0^1(0,1)$ in $L^2(0,1)$, we obtain

$$|w(s)|^2 - (a(\xi,0)\|\psi(0)\|^2 \le \left(\frac{\partial a(\xi,\eta)}{\partial t} + \frac{\hat{C}}{\theta}\right)\int_0^s \|\psi\|^2\,dt + C\int_0^s |w|^2\Big\}dt, \quad (63)$$

where C and $\hat{C}$ are positive constants. But $\psi(t) = w_1(t) - w_1(s)$, therefore, $\psi(0) = w_1(s)$ and

$$\|\psi(t)\|^2 = \|w_1(t) - w_1(s)\|^2 = 2(\|w_1(t)\|^2 + \|w_1(s)\|^2) - \|w_1(t) + w_1(s)\|^2$$
$$\le 2(\|w_1(t)\|^2 + \|w_1(s)\|^2).$$

Consequently, we have

$$|w(s)|^2 + \delta\|w_1(s)\|^2 \le C\int_0^s |w(t)|^2 + \|w_1(t)\|^2\,dt, \quad (64)$$

where

$$\delta = -a(\xi,0) - 2s\left(\left|\frac{\partial a(\xi,\eta)}{\partial t}\right| + \frac{\hat{C}}{\theta}\right).$$

To prove the uniqueness we must show that $\delta > 0$. Indeed, taking

$$T_0 = \frac{1}{2} \min_y |a(y,0)| \Big/ \max_{y,t} \left| \frac{\partial a(y,t)}{\partial t} \right| \quad \text{and} \quad \theta = \frac{\epsilon}{2T_0 \hat{C}} \tag{65}$$

for ϵ positive and sufficiently small, we obtain for $s < T_0$ that

$$\delta > |a(\xi,0)| - 2T_0 \left| \frac{\partial a(\xi,\eta)}{\partial t} \right| - \epsilon > 0,$$

where the hypothesis (**H2**) has been used. Note that if $\dfrac{\partial a(y,t)}{\partial t} \equiv 0$, then $\delta > 0$ for all $t > 0$. Returning to the equation (64), we have

$$|w(s)|^2 + \|w_1(s)\|^2 \le K \int_0^{T_0} \left(|w(t)|^2 + \|w_1(t)\|^2 \right) dt, \tag{66}$$

where $K = C/\delta > 0$. Using the Gronwall's inequality, we conclude that $w(t) = 0$ for all $t < T_0$. This completes the proof of Theorem 2. □

The Main Result for Problem (II):

Now let us restate the previous results for the original problem (II) and prove the following theorem:

Theorem 3 *Let $\Omega_t = (\alpha(t), \beta(t))$, $\Omega_0 = (\alpha(0), \beta(0))$ and the initial data $u_0 \in H_0^1(\Omega_0)$, $u_1 \in L^2(\Omega_0)$, $f \in L^2([0,T]; L^2(\Omega_t))$. Then there exists $T_0 > 0$ and a unique weak solution of problem* (II), *$u : \widehat{Q} \to I\!R$, satisfying the following conditions:*

1. *$u \in L^\infty(0, T_0; H_0^1(\Omega_t))$,*
2. *$u' \in L^\infty(0, T_0; L^2(\Omega_t))$.*

Proof. If v is a solution of Theorem 2, then consider $u(x,t) = v(y,t)$, where $x = \alpha + \gamma y$. We also have $g(y,t) = f(x,t) = f(\alpha + \gamma y, t)$ and $v_0(y) = u(x,0) = u_0(\alpha(0) + \gamma(0)y)$, $v_1(y) = u'(x,0) = u_1(\alpha(0) + \gamma(0)y) + (\alpha'(0) + \gamma'(0)y)u_0'((\alpha(0) + \gamma(0)y)$.

To verify that $u(x,t)$, under the hypotheses of Theorem 2, is a solution of problem (II), it is sufficient to observe that the mapping: $(x,t) \to \left(\dfrac{x-\alpha}{\gamma}, t \right)$ of the domain $\hat{Q}$ into $Q = (0,1) \times (0,T)$ is of class C^2, where T_0 is given by (65). Consequently we have

1. $\dfrac{\partial u^2}{\partial x^2}(x,t) = \dfrac{1}{\gamma^2} \dfrac{\partial v^2}{\partial y^2}(y,t)$,

2. $u''(x,t) = v''(y,t) + b(y,t)\frac{\partial v'}{\partial y} + \frac{1}{4}b^2(y,t)\frac{\partial v^2}{\partial y^2}(y,t) + c(y,t)\frac{\partial v}{\partial y}$.

Therefore, the operator $Lv(y,t)$ defined in (14), with $y = \dfrac{x-\alpha}{\gamma}$, is transformed into the operator

$$\widetilde{L}u(x,t) = \frac{\partial^2 u}{\partial t^2} - a(t)\frac{\partial^2 u}{\partial x^2} = f(x,t),$$

with initial conditions u_0 and u_1.

The regularity of $v(y,t)$ given by Theorem 2 implies that $u(x,t)$ is a solution of problem (II) and the uniqueness of the solution of problem (II) is a direct consequence of the uniqueness of problem (III). □

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