Optimum Grid Representations[1]

M. H. C. Fampa[2] S. Klein[3] F. Protti[4] D. C. A. Rêgo[5]

Abstract. A graph G is a *grid intersection graph* if G is the intersection graph of $\mathscr{H} \cup \mathscr{I}$, where $\mathscr{H}$ and $\mathscr{I}$ are, respectively, finite families of horizontal and vertical linear segments in the plane such that no two parallel segments intersect. (This definition implies that every grid intersection graph is bipartite.) Any family of segments realizing G is a *representation* of G. As a consequence of a characterization of grid intersection graphs by Kratochvíl [7], we observe that when a bipartite graph $G = (U \cup W, E)$ with minimum degree at least two is a grid intersection graph, then there exists a normalized representation of G on the $(r \times s)$-grid, where $r = |U|$ and $s = |W|$. A natural problem, with potential applications to circuit layout, is the following: among all the possible representations of G on the $(r \times s)$-grid, find a representation $\mathcal{R}$ such that the sum of the lengths of the segments in $\mathcal{R}$ is minimum. In this work we introduce this problem and present a mixed integer programming formulation to solve it.

Keywords: intersection graph of segments, grid intersection graph, grid representation, integer programming

1 Introduction

Let $\mathscr{F}$ be a finite family of sets. The *intersection graph* $\Omega(\mathscr{F})$ of $\mathscr{F}$ is the graph G whose vertices are in a one-to-one correpondence with the sets in $\mathscr{F}$, and (u, w) is an edge of G, for $u \neq w$, if and only if the sets corresponding to u and w intersect. When $\mathscr{F}$ is a family of linear segments arranged in at most k directions in the plane, G is said to be a k-DIR graph [8]. In addition, G is a PURE-k-DIR graph if G is a k-DIR graph and the segments can be arranged in such a way that no two parallel segments intersect. (According to this definition, every PURE-k-DIR graph is k-partite.)

Grid intersection graphs [6, 8] are exactly the PURE-2-DIR graphs. In other words, a graph G is a grid intersection graph (*gig*, for short) if $G = \Omega(\mathscr{H} \cup \mathscr{I})$, where $\mathscr{H}$ and $\mathscr{I}$ are, respectively, finite families of horizontal and vertical linear segments in the plane such that no two parallel segments intersect. If G is a gig, we will assume that G has bipartition $V = U \cup W$, where vertices in U correspond to segments in $\mathscr{H}$ and vertices in W to segments in $\mathscr{I}$.

Gigs form an important subclass of intersection graphs of linear segments in the plane. The latter class have applications in areas such as air traffic optimization [4], circuit design [9], and computational geometry [3]. The paper [8] by Kratochvíl and Matoušek is a deep theoretic study on intersection graphs of segments.

[1]This work is partially supported by CNPq and FAPERJ, Brazilian Research Agencies.
[2]IM and COPPE/Sistemas, UFRJ, Rio de Janeiro, Brazil, CEP 21945-970. E-mail: fampa@cos.ufrj.br
[3]IM and COPPE/Sistemas, UFRJ, Rio de Janeiro, Brazil, CEP 21945-970. E-mail: sula@cos.ufrj.br
[4]IM and NCE, UFRJ, C.Postal 2134, Rio de Janeiro, Brazil, CEP 20001-970. E-mail: fabiop@nce.ufrj.br
[5]COPPE/Sistemas, UFRJ, Rio de Janeiro, Brazil, CEP 21945-970 . E-mail: deborac@cos.ufrj.br

Denote by $(r \times s)$-*grid* the finite subset of points of the cartesian plane corresponding to the crossings defined by the lines with equations $y = i$, $1 \leq i \leq r$, and $x = j$, $1 \leq j \leq s$. Given positive integers r, s and a graph G, Gavril [5] showed that the problem of recognizing whether G is an intersection graph for segments on the $(r \times s)$-grid is NP-complete (here, parallel segments are allowed to intersect.) Kratochvíl [7] showed that the problem of deciding whether a given bipartite graph is a gig is also NP-complete. Hartman, Newman, and Ziv [6] proved that every planar bipartite graph is a gig. Moreover, they show infinite families of bipartite graphs which are not gigs. In a latter work, Bellantoni, Hartman, Przytycka and Whitesides [1] proved that every bipartite graph of boxicity two is a gig. (The *boxicity* of a graph G is the smallest b such that G is the intersection graph of parallelepipeds in the Euclidean b-space with edges parallel to the coordinate axes.)

In the paper [7], it is presented a characterization of gigs in terms of vertex orderings admitting no forbidden structures called *volkswagens*. (This characterization shall be discussed in the next section.) As a direct consequence of this characterization, we observe that when a bipartite graph $G = (U \cup W, E)$ with minimum degree at least two is a gig, then there exists a normalized representation of G on the $(r \times s)$-grid, where $r = |U|$ and $s = |W|$. Thus, a natural question is to obtain, among all the possible representations of G on the $(r \times s)$-grid, a representation $\mathcal{R}$ such that the sum of the lengths of the segments in $\mathcal{R}$ is minimum. In this work we introduce this problem (Sections 2 and 3) and present a mixed integer programming formulation to solve it (Section 4.)

The problem of finding optimum representations of gigs has potential applications to fields such as circuit layout. One of the most known problems in VLSI layout consists of finding a mapping from the vertices of a graph G to points of a grid, together with an incidence-preserving assignment of the edges of G to paths on the grid, where the paths must follow grid guidelines and cannot overlap (although they may cross at a point.) Finding a VLSI layout with minimum length for the longest wire is NP-hard [2]. Other intersection graphs in connection with VLSI layout and computational geometry are retangle intersection graphs [10].

2 Background

The graphs considered in this work are undirected and without loops or multiple edges. Let G denote a gig. The vertex set and the edge set of G are denoted by V and E, respectively, where $|V| = n$ and $|E| = m$. The degree of a vertex v is denoted by $\delta(v)$, and the minimum degree of a vertex in G by $\delta(G)$. Assume that G has bipartition $V = U \cup W$, where the color classes satisfy $|U| = r$ and $|W| = s$.

Any family of segments realizing G is called a *representation* of G. We denote by $\mathcal{R}_G$ a representation of G, and by $\mathcal{R}_G(v)$ the segment in $\mathcal{R}_G$ corresponding to the vertex $v \in V$. For simplicity, we omit the subscript G when there is no ambiguity. We write $\mathcal{R}(v) = [(x(v), y(v)), (x'(v), y'(v))]$, where $(x(v), y(v))$ and $(x'(v), y'(v))$ are the extreme points of $\mathcal{R}(v)$ on the cartesian plane. The length of $\mathcal{R}(v)$ is denoted by $l(v)$, where $l(v) > 0$.

We assume that $\mathcal{R}$ has the following properties:

P1: If $v \in U$ then $\mathcal{R}(v) = [(x(v), y(v)), (x(v) + l(v), y(v))]$, and if $v \in W$ then $\mathcal{R}(v) = [(x(v), y(v)), (x(v), y(v) + l(v))]$.

P2: Any two horizontal (vertical) segments in $\mathcal{R}$ do not lie on the same line. That is, if $v, z \in U$ then $y(v) \neq y(z)$, and if $v, z \in W$ then $x(v) \neq x(z)$.

2.1 A characterization of grid intersection graphs by Kratochvíl

In [7], Kratochvíl presented a characterization of grid intersection graphs, which is described in what follows. Let $u_1 < u_2 < \ldots < u_r$ and $w_1 < w_2 < \ldots < w_s$ be orderings of the color classes U and W of a bipartite graph G. Relatively to these orderings, a *volkswagen* is a subset of six vertices $u_a, u_b, u_c, w_i, w_j, w_k$ such that $u_a < u_b < u_c$, $w_i < w_j < w_k$, $(u_b, w_j) \notin E$ and $(u_a, w_j), (u_c, w_j), (u_b, w_i), (u_b, w_k) \in E$. See Figure 1.

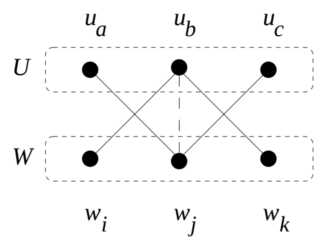

Figure 1: A volkswagen.

Theorem 1 [7] *A bipartite graph G is a gig if and only if its color classes admit orderings without volkswagens.*

Let us reconsider the above theorem in an equivalent form. Define $max(v)$ (resp. $min(v)$) as the maximum (resp. minimum) neighbor of v relatively to the orderings of U and W.

Corollary 2 *A bipartite G is a gig if and only if its color classes admit orderings satisfying the following condition:*

(*) *for every pair v, z of non-adjacent vertices belonging to distinct color classes, if $min(z) < v < max(z)$ then either $max(v) < z$ or $z < min(v)$.*

Let $\mathcal{R}$ be a representation of a gig G and let v, z be non-adjacent vertices in V. Assume for instance that $v \in U$ and $z \in W$, that is, $\mathcal{R}(v)$ is horizontal and $\mathcal{R}(z)$ is vertical. In this case, Condition (*) has the following geometric meaning: if $\mathcal{R}(v)$ lies between the bottommost and the topmost horizontal segments intersecting $\mathcal{R}(z)$, then $\mathcal{R}(z)$ either lies to the right of the rightmost vertical segment intercepting $\mathcal{R}(v)$, or it lies to the left of the leftmost vertical segment intercepting $\mathcal{R}(v)$.

Let us briefly check Corollary 2. Assume that G is a gig and $\mathcal{R}$ is a representation of G satisfying properties P1 and P2. Define now the orderings $u_1 < u_2 < \ldots < u_r$ and $w_1 < w_2 < \ldots < w_s$ in such a way that $y(u_1) < y(u_2) < \ldots < y(u_r)$ and $x(w_1) < x(w_2) < \ldots < x(w_s)$. Observe that if Condition (*) is not true, then there clearly exist non-adjacent vertices v, z such that $min(z), v, max(z), min(v), z, max(v)$ form a volkswagen, which is impossible.

Conversely, from the orderings in U and W satisfying Condition (*), define a representation $\mathcal{R}$ as follows. For each u_i, $1 \leq i \leq r$, let $f_1(u_i)$ and $f_2(u_i)$ be, respectively, the indices of $min(u_i)$ and $max(u_i)$ in the corresponding ordering and define $\mathcal{R}(u_i) = [(f_1(u_i) - \varepsilon, i), (f_2(u_i) + \varepsilon, i)]$, for some $0 < \varepsilon < 1$. Define analogously $\mathcal{R}(w_j) = [(j, f_1(w_j) - \varepsilon), (j, f_2(w_j) + \varepsilon)]$, for $1 \leq j \leq s$. It is not difficult to see that this representation realizes all necessary intersections and avoids spurious crossings.

2.2 Graphs with minimum degree at least two

In the proof of sufficiency of Condition (*) in Corollary 2, we observe that when $\delta(G) > 1$, $f_1(v)$ and $f_2(v)$ are distinct for all $v \in U \cup W$. In this case, if the representation $\mathcal{R}$ is constructed by taking $\varepsilon = 0$, the extreme points of every segment in $\mathcal{R}$ have integer coordinates and are placed on the $(r \times s)$-grid. Let us call such a representation a $(r \times s)$*-grid representation.*

Corollary 3 *Let G be a bipartite graph with $\delta(G) > 1$. Then G is a gig if and only if there exists a $(r \times s)$-grid representation of G.*

The corollary above is interesting in the sense that every gig G satisfying $\delta(G) > 1$ can be represented in a normalized, natural form by using $(r-1) \times (s-1)$ area units. (Observe an example in Figure 2.) In fact, it turns out that the case $\delta(G) > 1$ is the only important one to be considered. We discuss this matter in what follows.

Note that if v is a vertex of degree one in a bipartite graph G, then G is a gig if and only if $G - v$ is a gig. Thus, the construction of a representation of G can be done in three steps: first, we repeatedly delete vertices of degree one until we obtain a graph G' with no such vertices; second, we try to obtain a representation $\mathcal{R}'$ of G'; finally, we simply add to $\mathcal{R}'$ segments corresponding to the vertices deleted in the first step, in order to obtain a representation $\mathcal{R}$ of G.

This shows that vertices of degree one are not an obstacle to finding representations. In the next sections, we will therefore focus our attention on graphs with minimum degree at least two.

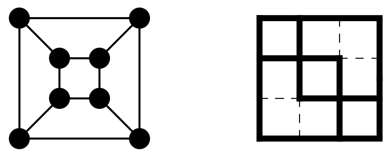

Figure 2: A (4×4)-grid representation of a gig.

3 An optimization problem on grid representations

Let G be a gig with $\delta(G) > 1$. A natural question that arises is the following: among all the possible $(r \times s)$-grid representations of G, how to obtain a representation $\mathcal{R}$ such that the sum of the lengths of the segments in $\mathcal{R}$ is minimum? In this section, we introduce formally the GRID combinatorial optimization problem to deal with these question.

Let $\mathcal{R}$ be a $(r \times s)$-grid representation of G. We define

$$len(\mathcal{R}) = \sum_{v \in V} l(v)$$

.

Next, we define $len(G)$ as the minimum value $len(\mathcal{R})$ amongst all possible $(r \times s)$-grid representations $\mathcal{R}$ of G:

$$len(G) = min\{\ len(\mathcal{R}) \mid \mathcal{R} \text{ is a } (r \times s)-\text{grid representation of } G\ \}. \qquad (1)$$

The following lemma gives a lower and an upper bound for $len(G)$:

Lemma 4 *Let G be a gig with $\delta(G) > 1$. Then:*

$$4 \leq 2m - n \leq len(G) \leq \lfloor n^2/2 \rfloor - n. \qquad (2)$$

Proof:
The lower bound comes from the observation that, in any $(r \times s)$-grid representation of G,

$$l(v) \geq \delta(v) - 1, \quad \text{for all } v \in V. \tag{3}$$

Moreover, the assumption $\delta(G) > 1$ implies $2m - n \geq 4$. With respect to the upper bound, note that the complete bipartite graph $K_{r,s}$ has maximum value of len amongst all gigs with $n = r+s$ vertices. Therefore $len(G) \leq len(K_{r,s}) = r(s-1) + s(r-1)$. Notice that the expression for $len(K_{r,s})$, when subject to the restriction $r + s = n$, attains the maximum value for $r = \lfloor n/2 \rfloor$ and $s = \lceil n/2 \rceil$. Thus, the lemma follows. □

We now formulate the GRID problem (decision version) as follows:

Instance: A bipartite graph G with $\delta(G) > 1$, an integer ℓ such that $\ell \geq 4$.
Question: Is there a $(r \times s)$-grid representation $\mathcal{R}$ of G such that $len(\mathcal{R}) \leq \ell$?

Theorem 5 GRID *is NP-complete.*

Proof:
We can easily check in polynomial time whether a given $(r \times s)$-grid representation $\mathcal{R}$ of G satisfies $len(\mathcal{R}) \leq \ell$. Therefore, GRID is in NP. On the other hand, recall that the problem Π of recognizing whether a given bipartite graph H is a gig is NP-complete [7]. Let us show a reduction from Π to GRID. Given a bipartite graph H, repeatedly delete vertices of degree one until we obtain a graph H_1 containing no such vertices; then remove isolated vertices from H_1, obtaining a graph H_2 which either is empty or satisfies $\delta(H_2) > 1$. Define $G = H_2 \cup C_4$ and $\ell = \lfloor n_2^2/2 \rfloor - n_2 + 4$, where n_2 is the number of vertices of H_2. By Lemma 4, if H_2 is a gig then $len(H_2) \leq \lfloor n_2^2/2 \rfloor - n_2$. Thus, H is a gig if and only if there is a $(r \times s)$-grid representation $\mathcal{R}$ of G satisfying $len(\mathcal{R}) \leq \ell$. □

4 A mixed integer programming formulation for GRID

In this section we consider a bipartite graph G with $\delta(G) > 1$ and present a mixed integer programming formulation MIPF for GRID, such that G is a gig if and only if there is a feasible solution for MIPF. Furthermore, if MIPF has a feasible solution, then it has an optimal solution which defines an $(r \times s)$-grid representation $\mathcal{R}$ of G such that $len(\mathcal{R})$ is minimum, i.e. $len(\mathcal{R}) = len(G)$.

We define, for each vertex $v \in U \cup W$, the variables $x(v)$, $y(v)$ and $l(v)$. As mentioned in Section 2, $(x(v), y(v))$ represents an extreme point of $\mathcal{R}(v)$ on the cartesian plane and $l(v)$

represents the length of $\mathcal{R}(v)$. Since our goal is to obtain a representation of G on the $(r \times s)$-grid, we impose the following constraints on $x(v)$ and $y(v)$, for each $v \in U \cup W$:

$$1 \leq x(v) \leq s \tag{4}$$

$$1 \leq y(v) \leq r \tag{5}$$

Consider now an edge $(u, w) \in E$ such that $u \in U$ and $w \in W$. Then the following constraints must be satisfied:

$$x(u) \leq x(w) \leq x(u) + l(u) \tag{6}$$

and

$$y(w) \leq y(u) \leq y(w) + l(w). \tag{7}$$

In order to present the next constraints of MIPF, let us partition the set of non-edges of G as follows:

$$E_1 = \{(u, w) \in \overline{E} \mid u \in U \text{ and } w \in W\}$$
$$E_2 = \{(u, u') \in \overline{E} \mid u \in U \text{ and } u' \in U\}$$
$$E_3 = \{(w, w') \in \overline{E} \mid w \in W \text{ and } w' \in W\}$$

We analyze each set above separately. The constant K considered in the following analysis is defined by $K = \max\{r, s\}$.

If $(u, w) \in E_1$, it is straightforward to see that one of the following relations should be satisfied at $\mathcal{R}$:

either

$$x(w) - x(u) - l(u) - 1 \geq 0 \tag{8}$$

or

$$x(u) - x(w) - 1 \geq 0. \tag{9}$$

In addition, either

$$y(u) - y(w) - l(w) - 1 \geq 0 \tag{10}$$

or

$$y(w) - y(u) - 1 \geq 0. \tag{11}$$

The four possibilities above are represented in Figure 3.

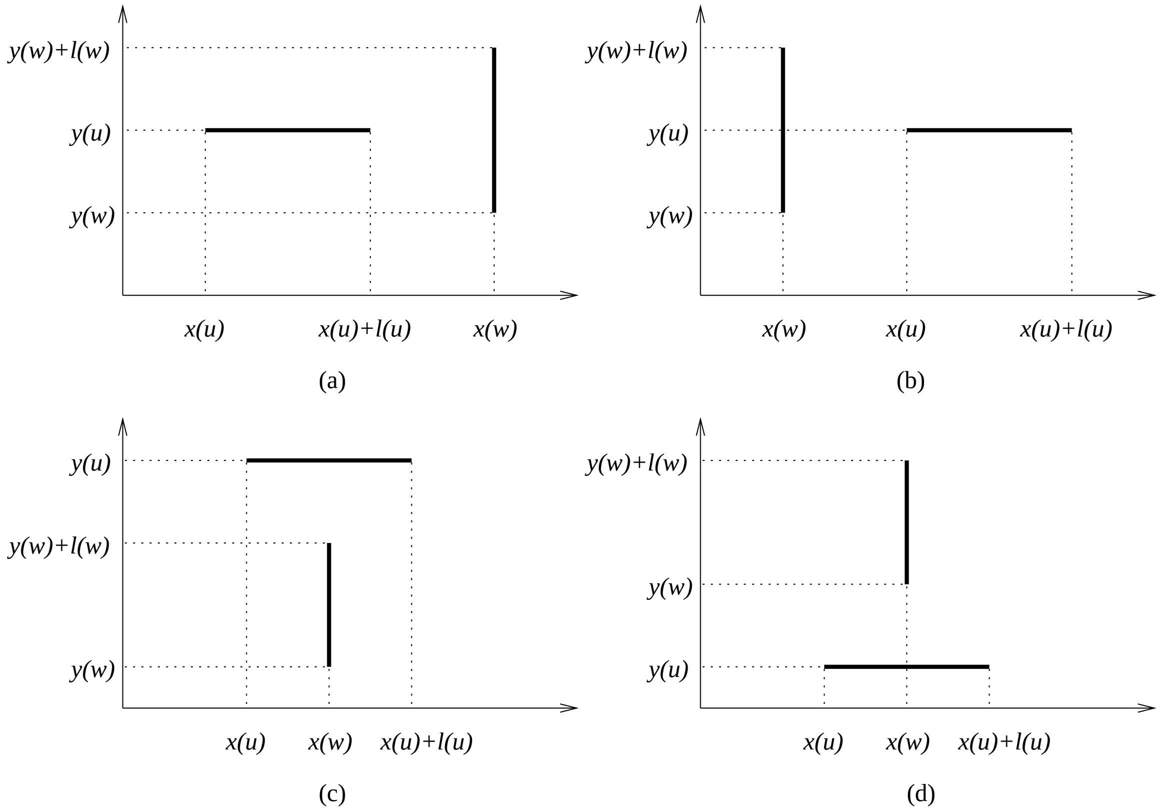

Figure 3: Possibilities for $(u, w) \in E_1$: (a) Relation (8); (b) Relation (9); (c) Relation (10); (d) Relation (11)

Now, in order to guarantee that at least one of the four situations modeled by relations (8)-(11) is satisfied at $\mathcal{R}$, we first define the binary variables $\alpha(u, w)$, $\alpha(w, u)$, $\beta(u, w)$ and $\beta(w, u)$ and impose the following constraints over them:

$$\alpha(u,w) \quad - \quad \frac{x(w) - x(u) - l(u) - 1}{K} \quad \leq \quad 1 \tag{12}$$

$$\alpha(w,u) \quad - \quad \frac{x(u) - x(w) - 1}{K} \quad \leq \quad 1 \tag{13}$$

$$\beta(u,w) \quad - \quad \frac{y(u) - y(w) - l(w) - 1}{K} \quad \leq \quad 1 \tag{14}$$

$$\beta(w,u) \quad - \quad \frac{y(w) - y(u) - 1}{K} \quad \leq \quad 1 \tag{15}$$

Since K was defined large enough, note that the absolute value of each fraction on the expressions (12)-(15) lies between 0 and 1. Therefore, these expressions guarantee that:

- If $x(w) - x(u) - l(u) - 1 < 0$ then $\alpha(u, w) = 0$

- If $x(u) - x(w) - 1 < 0$ then $\alpha(w,u) = 0$
- If $y(u) - y(w) - l(w) - 1 < 0$ then $\beta(u,w) = 0$
- If $y(w) - y(u) - 1 < 0$ then $\beta(w,u) = 0$

Finally, the following constraint guarantees that at least one of the four relations (8)-(11) is satisfied at $\mathcal{R}$:

$$\alpha(u,w) + \alpha(w,u) + \beta(u,w) + \beta(w,u) \geq 1. \tag{16}$$

Observe that $\alpha(u,w)$ and $\alpha(w,u)$ are not simultaneously equal to one, since these variables model mutually exclusive situations. Similarly, $\beta(u,w)$ and $\beta(w,u)$ are not simultaneously equal to one. However, it is possible to have $\alpha(u,w)$ (or $\alpha(w,u)$) and $\beta(u,w)$ (or $\beta(w,u)$) equal to one at the same solution.

We consider now the non-edges $(u,u') \in E_2$. Recall from Section 2 that $\mathcal{R}$ must satisfy P2. Thus, one of the two following relations must be satisfied:

either

$$y(u') - y(u) - 1 \geq 0 \tag{17}$$

or

$$y(u) - y(u') - 1 \geq 0. \tag{18}$$

In order to guarantee that either (17) or (18) is satisfied, we introduce now the binary variables $\gamma(u,u')$ and $\gamma(u',u)$ and the following constraints:

$$\gamma(u,u') \quad - \quad \frac{y(u') - y(u) - 1}{K} \quad \leq \quad 1 \tag{19}$$

$$\gamma(u',u) \quad - \quad \frac{y(u) - y(u') - 1}{K} \quad \leq \quad 1 \tag{20}$$

Therefore:

- If $y(u') - y(u) - 1 < 0$ then $\gamma(u,u') = 0$
- If $y(u) - y(u') - 1 < 0$ then $\gamma(u',u) = 0$

In addition, to guarantee that either (17) or (18) is satisfied, we impose the constraint:

$$\gamma(u,u') + \gamma(u',u) = 1 \tag{21}$$

It remains to analyze the non-edges $(w, w') \in E_3$. This analysis is analogous to the previous one we did for the non-edges $(u, u') \in E_2$. We define the binary variables $\tau(w, w')$ and $\tau(w', w)$ for each $(w, w') \in E_3$, and impose the constraints:

$$\tau(w, w') - \frac{x(w') - x(w) - 1}{K} \leq 1 \tag{22}$$

$$\tau(w', w) - \frac{x(w) - x(w') - 1}{K} \leq 1 \tag{23}$$

$$\tau(w, w') + \tau(w', w) = 1 \tag{24}$$

Now, we are able to present the mixed integer programming formulation MIPF for the GRID problem, considering the objective function given by (1) and the constraints previously presented.

(MIPF) Minimize $\sum_{v \in V} l(v)$

Subject to:

For each $v \in V$:

$$\begin{array}{ll} 1 \leq x(v) \leq s & (4) \\ 1 \leq y(v) \leq r & (5) \end{array}$$

For each $(u, w) \in E$ such that $u \in U,\ w \in W$:

$$\begin{array}{ll} x(w) - x(u) \geq 0 & (6) \\ x(u) + l(u) - x(w) \geq 0 & \\ y(u) - y(w) \geq 0 & (7) \\ y(w) + l(w) - y(u) \geq 0 & \end{array}$$

For each $(u, w) \in \overline{E}$ such that $u \in U,\ w \in W$:

$$\begin{array}{ll} (x(w) - x(u) - l(u) - 1) - K\alpha(u, w) + K \geq 0 & (12) \\ (x(u) - x(w) - 1) - K\alpha(w, u) + K \geq 0 & (13) \\ (y(u) - y(w) - l(w) - 1) - K\beta(u, w) + K \geq 0 & (14) \\ (y(w) - y(u) - 1) - K\beta(w, u) + K \geq 0 & (15) \\ \alpha(u, w) + \alpha(w, u) + \beta(u, w) + \beta(w, u) - 1 \geq 0 & (16) \\ \alpha(u, w), \alpha(w, u), \beta(u, w), \beta(w, u) \in \{0, 1\} & \end{array}$$

For each $(u, u') \in \overline{E}$ such that $u \in U,\ u' \in U$:

$$\begin{array}{ll} (y(u') - y(u) - 1) - K\gamma(u, u') + K \geq 0 & (19) \\ (y(u) - y(u') - 1) - K\gamma(u', u) + K \geq 0 & (20) \\ \gamma(u, u') + \gamma(u', u) = 1 & (21) \\ \gamma(u, u'), \gamma(u', u) \in \{0, 1\} & \end{array}$$

For each $(w, w') \in \overline{E}$ such that $w \in W,\ w' \in W$:

$$\begin{aligned}
&(x(w') - x(w) - 1) - K\tau(w, w') + K \geq 0 && (22)\\
&(x(w) - x(w') - 1) + K\tau(w', w) + K \geq 0 && (23)\\
&\tau(w, w') + \tau(w', w) = 1 && (24)\\
&\tau(w, w'), \tau(w', w) \in \{0, 1\}
\end{aligned}$$

The previous discussions in this section allow us to conclude the following result:

Theorem 6 *Let $G = (U \cup W, E)$ be a bipartite graph with $\delta(G) > 1$, $|U| = r$ and $|W| = s$. Then G is a gig if and only if there is a feasible solution for* MIPF. *Furthermore, if* MIPF *has a feasible solution, then it has an optimal solution which defines an $(r \times s)$-grid representation $\mathcal{R}$ of G such that $len(\mathcal{R})$ is minimum, i.e. $len(\mathcal{R}) = len(G)$.* □

It is worth remarking that, given a bipartite graph G, one can use MIPF simply to test whether G is a gig, by solely verifying the existence of a feasible solution. In this case, computational efforts for minimizing the objective function can be avoided.

The corollary below is an important property of MIPF:

Corollary 7 *In any optimum solution of* MIPF, *the variables $x(v)$, $y(v)$ and $l(v)$, for all $v \in V$, have integer values.*

Proof:

For every pair of distinct vertices $u, u' \in U$, relation (17) or relation (18) must be satisfied, and this implies $|y(u) - y(u')| \geq 1$. Since $|U| = r$ and constraint (5) is satisfied, we conclude that, in any feasible solution of MIPF, $y(u) \in \{1, 2, \ldots, r\}$, for every $u \in U$. Similarly, $x(w) \in \{1, 2, \ldots, s\}$, for every $w \in W$.

Now, let $v \in U$ and consider the horizontal segment $\mathcal{R}(v)$ in an optimum solution s^* of MIPF. We already know that $y(v)$ has an integer value. We need to show that $x(v)$ and $l(v)$ also have integer values. Since $\delta(v) > 1$, $\mathcal{R}(v)$ must intersect at least two distinct vertical segments $\mathcal{R}(w)$ and $\mathcal{R}(w')$, for $w, w' \in W$. Assume that $\mathcal{R}(w)$ and $\mathcal{R}(w')$ are, respectively, the leftmost and the rightmost vertical segments intercepting $\mathcal{R}(v)$ in s^*. The minimization of the objetive function naturally sets $x(v) = x(w)$ and $l(v) = x(w') - x(w)$. To complete the proof, recall from the previous paragraph that $x(w)$ and $x(w')$ have integer values.

The proof for the case $v \in W$ is analogous. □

Finally, observe that relations (2) and (3) are valid inequalities for the optimal solution of MIPF and can be used in a computational implementation.

References

[1] S. Bellantoni, I. B.-A. Hartman, T. Przytycka, and S. Whitesides. Grid Intersection graphs and boxicity. *Discrete Mathematics* 114 (1993) 41–49.

[2] S. N. Bhatt and S. S. Cosmadakis. The complexity of minimizing wire lengths in VLSI layouts. *Information Processing Letters* 25 (1987) 263–267.

[3] H. Edelsbrunner. *Algorithms in Combinatorial Geometry.* Springer-Verlag, Berlin/New York, 1987.

[4] R. Fondacci, O. Goldschmidt, and V. Letrouit. Combinatorial issues in air traffic optimization. *Transportation Science* 32(3) (1998) 256–267.

[5] F. Gavril. Some NP-complete problems on graphs. *Proc. 11th Conf. on Information Sciences and Systems*, Johns Hopkins University, Baltimore, MD, 91–95 (1977).

[6] I. B.-A. Hartman, I. Newman, and R. Ziv. On grid intersection graphs. *Discrete Mathematics* 87 (1991) 41–52.

[7] J. Kratochvíl. A special planar satisfiability problem and a consequence of its NP-completeness. *Discrete Applied Mathematics* 52 (1994) 233–252.

[8] J. Kratochvíl and J. Matoušek. Intersection graphs of segments. *Journal of Combinatorial Theory, Series B* 62 (1994) 289–315.

[9] Y. S. Kuo, T. C. Chern, and W. Shih. Fast Algorithms for Optimal Layer Assignment. *Proceedings of the* 25*th ACM/IEEE Design Automation Conference*, IEEE Computer Society DATC, Washington, D.C., 554–559 (1988).

[10] C. S. Rim and K. Nakajima. On Retangle Intersection and Overlap Graphs. *IEEE Transactions on Circuits and Systems - I: Fund. Theory and Applications* 42, no.9 (1995) 549–553.