On clique graphs with linear size

Fábio Protti[1]
Jayme Luiz Szwarcfiter[2]

Abstract. The *clique graph* $H = K(G)$ of G is the intersection graph of the collection of maximal cliques of G. In this case, G is a *clique-inverse graph* of H. By examining $K(G)$, we describe some sufficient conditions for the number of maximal cliques of G to be bounded by $O(|V(G)|)$. These conditions are then applied to analyze the complexity of recognizing clique-inverse graphs of various classes of graphs.

Keywords: clique graphs, clique-inverse graphs, intersection graphs

1 Introduction

Special graph classes have been investigated with growing interest since the publication of the book by Golumbic in 1980 [10]. Several computer science and other real word applications have required the introduction of new classes and related algorithmic problems. It is worth remarking that the recent book by Brandstädt, Le, and Spinrad [3] compiles results on almost two hundred classes, whereas an information system on graph class inclusions, which can be acessed at `http://www.informatik.uni-rostock.de/ gdb/isgci/Isgci.html`, deals with more than three hundred classes.

Intersection graph classes play an important role in this universe, since they have real applications to areas as biology, computing, matrix analysis, and statistics. A complete guide on intersection graph theory is the also recent book by McKee and McMorris [16]. Let $\mathcal{F} = \{S_1, \ldots, S_n\}$ be any family of sets. The *intersection graph* $\Omega(\mathcal{F})$ of $\mathcal{F}$ is the graph having $\mathcal{F}$ as vertex set, with S_i adjacent to S_j if and only if $i \neq j$ and $S_i \cap S_j \neq \emptyset$.

When the sets of $\mathcal{F}$ are entities which refer to a given graph G, we may regard the intersection graph of $\mathcal{F}$ as the result of the application of some function or *operator* on G. For instance, if $\mathcal{F}$ is the collection of maximal cliques of G, then $\Omega(\mathcal{F})$ is the *clique graph* $K(G)$ of G. The notation K refers to the *clique graph operator*. Another well-known example is the *line graph* of G, the intersection graph of the edges of G. A source on graph operators is the book by Prisner [18], which considers most of the operators ever cited in the literature.

[1] Universidade Federal do Rio de Janeiro, NCE, Caixa Postal 2324, 20001-970, Rio de Janeiro, RJ, Brasil. Partially supported by the Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro, FAPERJ, Brasil. E-mail: fabiop@nce.ufrj.br

[2] Universidade Federal do Rio de Janeiro, Instituto de Matemática, NCE and COPPE, Caixa Postal 2324, 20001-970, Rio de Janeiro, RJ, Brasil. Partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, and Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro - FAPERJ, Brasil. E-mail: jayme@nce.ufrj.br

One important problem on graph operators is characterizing images. In other words: given a class $\mathcal{F}$ of graphs and an operator op, describe the image $op(\mathcal{F})$, that is, the class consisting of the graphs H such that $H = op(G)$ for some $G \in \mathcal{F}$. With respect to the clique graph operator, this problem has been solved for many classes, for instance [2, 4, 11, 12, 22]. Images of other operators have also been studied, see [18, 16]. However, much less is known about the related problem of characterizing *inverse images*: given $\mathcal{F}$ and op, which are the graphs G such that $op(G) \in \mathcal{F}$? This work is a study on the inverse image of the clique graph operator. The inverse image of a class $\mathcal{F}$ by K is called the class of *clique-inverse graphs* of $\mathcal{F}$, and is denoted by $K^{-1}(\mathcal{F})$. Formally, $G \in K^{-1}(\mathcal{F})$ if and only if $K(G) \in \mathcal{F}$. In what follows, we present an overview of the results of this work.

Let G be a connected graph such that $|V(G)| = n$, and let $\tau(G)$ be the number of maximal cliques of G. It is known that $\tau(G)$ may reach the value $3^{n/3}$ [17]. Therefore, the cardinality of the vertex set of the clique graph $K(G)$ may be exponential on n, since by definition it is equal to $\tau(G)$. In the present work, it is shown (Section 3) that if $K(G)$ has either bounded clique number, bounded maximum degree, or bounded chromatic number, then $|V(K(G))|$ is $O(n)$. Moreover, if $K(G)$ contains an induced subgraph H which is either a chordless cycle, a chordless path, or an independent set, then $|V(H)|$ is $O(n)$.

The above results are used to analyze the complexity of recognizing some classes of clique-inverse graphs. Let r and d be positive constants. Recognizing $K^{-1}(\mathcal{F})$ can be done in polynomial time when $\mathcal{F}$ is one of the following classes: K_r-free graphs, graphs with maximum degree d, and planar graphs (Section 3). On the other hand (Section 4), it is NP-hard for clique-inverse graphs of (co-)chordal graphs, split graphs, co-bipartite graphs, co-chordal-bipartite graphs, (co-)interval graphs, (co-)comparability graphs, permutation graphs, block-cutpoint graphs, AT-free graphs, k-colorable graphs ($k \geq 3$), and graphs with independence number s ($s \geq 1$).

Graphs with a polynomially bounded number of maximal cliques have been previously considered in [1] and [19]. In the latter work, it has been proved that if a graph contains no induced subgraph isomorphic to the complement of p disjoint edges, then G contains at most $n^{2(p-1)}$ maximal cliques. Clique-inverse graphs were the subjects of [15] and [20]. They are also called *roots* (relative to the clique operator), see e.g. [18]. Clique-inverse graphs of complete graphs are called *clique-complete*. A characterization of the minimal clique-complete graphs with no universal vertex (a vertex which is adjacent to every other vertex of the graph) has been formulated in [15]. It corresponds to a description of minimal graphs whose maximal cliques do not satisfy the Helly property. In the same work, it is shown that recognizing clique-complete graphs is a Co-NP-complete problem. In [21], characterizations for clique-inverse graphs of K_3-free and K_4-free graphs are presented in terms of forbidden subgraphs. A general study of classes of clique-inverse graphs, which originated the present work, can be found in [20], where characterizations of clique-inverse graphs of bipartite graphs, chordal bipartite graphs, and trees, are also presented.

2 Notation and definitions

Let G be a finite undirected graph with no loops nor multiple edges. Denote the vertex set of G by $V(G)$, and the edge set by $E(G)$. Assume that G is connected and $|V(G)| = n$. A subgraph H of G is a graph where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a set X of vertices of G, denote by $G[X]$ the subgraph of G induced by X. Let $\overline{G}$ represent the complement of G. Denote by $\Delta(G)$ the maximum degree of a vertex of G.

Denote by C_k and P_k, respectively, the chordless cycle and the chordless path with k vertices. If G contains no cycles, then G is a *tree*. If G contains no induced C_k for $k \geq 4$, then G is a *chordal graph*. If $\overline{G}$ is chordal, then G is a *co-chordal graph*. If G and $\overline{G}$ are chordal, then G is a *split graph*.

A *clique* is a subset of vertices of G such that there is an edge between any two of these vertices. A *maximal clique* is one not properly contained in any other. Denote by $\tau(G)$ the number of maximal cliques of G. The *clique number* of G, denoted by $\omega(G)$, is the size of a maximum clique of G. Denote by K_r the clique with r vertices. If G is chordal and every edge of it belongs to exactly one maximal clique, then G is a *block-cutpoint* graph.

An *independent set* is a subset of vertices of G inducing a subgraph with no edges. Denote by I_k an independent set with k vertices. A *maximal independent set* is one not properly contained in any other. The *independence number* of G, denoted by $\alpha(G)$, is the size of a maximum independent set of G. If $\alpha(G) \geq k$, G is said to be k*-independent*.

A *color* is an independent set of G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors needed to cover $|V(G)|$. If $\chi(G) \leq k$, G is said to be k*-colorable*. If $\chi(G) \leq 2$, G is *bipartite*. If G is bipartite and contains no induced C_{2k} for $k \geq 3$, then G is *chordal bipartite*. If $\overline{G}$ is bipartite (chordal bipartite), then G is *co-bipartite* (*co-chordal-bipartite*).

If G can be drawn on the plane with no crossing edges, then G is a *planar graph*. An *interval graph* is the intersection graph of a family of intervals on the real line. If $\overline{G}$ is an interval graph, then G is a *co-interval graph*. If the edges of G can be oriented so that the existence of the oriented edges (x, y) and (y, z) implies the existence of the oriented edge (x, z), then G is said to be a *comparability graph*. If $\overline{G}$ is a comparability graph, then G is a *co-comparability* graph. If both G and $\overline{G}$ are comparability graphs, then G is said to be a *permutation graph*. An *asteroidal triple* in a graph G is an independent set $\{x_1, x_2, x_3\} \subseteq V(G)$ such that there is a path from x_i to x_j avoiding the neighborhood of x_k. A graph G is *AT-free* if it contains no asteroidal triples.

The *clique graph* $K(G)$ of G is the intersection graph of the collection of maximal cliques of G. If $H = K(G)$ for some graph G, say that H is a *clique graph*, and G a *clique-inverse graph* of H. Given a class $\mathcal{F}$ of graphs, $K^{-1}(\mathcal{F})$ is the class of graphs whose clique graphs are members of $\mathcal{F}$, and it is called the class of *clique-inverse graphs* of $\mathcal{F}$. Clearly, if H is not a clique graph, then $K^{-1}(\{H\}) = \emptyset$. Denote by $CLIQUE$ the class of clique graphs. Then, for any class $\mathcal{F}$ of graphs, $K^{-1}(\mathcal{F}) = K^{-1}(\mathcal{F} \cap CLIQUE)$.

3 Clique graphs with linear size

We start this section by analyzing some cases in which $K(G)$ has bounded clique number or bounded maximum degree.

Lemma 1. *Let G be a graph. If $\omega(K(G))$ is bounded by a constant, then $|V(K(G))|$ is $O(n)$.*

Proof. Assume that $\omega(K(G)) \leq r$, for some positive constant r. Observe that any vertex v of G may belong to at most r maximal cliques, otherwise the cliques containing v would correspond to a clique of size at least $r+1$ in $K(G)$, a contradiction. Therefore, $\tau(G) \leq rn$, that is, $|V(K(G))|$ is $O(n)$. □

Corollary 2. *Let G be a graph. If $\Delta(K(G))$ is bounded by a constant, then $|V(K(G))|$ is $O(n)$.*

Proof. Assume that $\Delta(K(G)) \leq d$, for some positive constant d. Clearly, $\omega(K(G)) \leq \Delta(K(G)) + 1 = d + 1$. Therefore, by Lemma 1, $|V(K(G))|$ is $O(n)$. More precisely, $|V(K(G))| \leq (d+1)n$. □

Next lemma states that if $K(G)$ contains a certain type of induced subgraph S, then the size of S is $O(n)$.

Lemma 3. *Let G be a connected graph. Let H be an induced subgraph of $K(G)$ such that $|V(H)| = k$. If H is isomorphic either to P_k, C_k, or I_k, then $k = O(n)$.*

Proof. Write $V(H) = \{M_1, M_2, \ldots, M_k\}$ (each vertex of H corresponds to a maximal clique of G). Let us divide the proof in the three possible cases. Assume without loss of generality that $k \geq 3$.

Case 1: H is isomorphic to P_k. Assume w.l.o.g. that H is the chordless path $M_1 M_2 \ldots M_k$. In this case, $M_i \cap M_j \neq \emptyset$ for $j - i = 1$ and $M_i \cap M_j = \emptyset$ for $j - i > 1$ $(1 \leq i < j \leq k)$. Let $v_i \in M_i \cap M_{i+1} (1 \leq i \leq k-1)$. Then, the v_i's are pairwise distinct, otherwise two non-consecutive cliques would intersect, a contradiction. Therefore, $k - 1 \leq n$, that is, $k = O(n)$.

Case 2: H is isomorphic to C_k. Assume w.l.o.g. that H is the chordless cycle $M_1 M_2 \ldots M_k M_1$. Then the subgraph of $K(G)$ induced by the vertices $M_1, \ldots, M_{k-1}$ is isomorphic to P_{k-1}. Therefore, by Case 1, $k-1$ is $O(n)$, that is, $k = O(n)$.

Case 3: H is isomorphic to I_k. In this case, observe that each vertex $v \in V(G)$ may belong to at most one maximal clique from $M_1, M_2, \ldots, M_k$, since $M_i \cap M_j = \emptyset$ for $1 \leq i < j \leq k$. Each M_i must contain at least two vertices, since G is connected. Thus, k may reach at most the value $n/2$, that is, $k = O(n)$. □

Corollary 4. *Let G be a graph. If $K(G)$ is isomorphic to a chordless path or cycle, then $|V(K(G))|$ is $O(n)$.* □

Corollary 5. *Let G be a graph. If $\chi(K(G))$ is bounded by a constant, then $|V(K(G))|$ is $O(n)$.*

Proof. Assume that $\chi(K(G)) = c \leq k$, for some positive constant k. Each color of $K(G)$ is an edgeless subgraph H_i of $K(G)$. By Lemma 3, $|V(H_i)| = O(n)$. Thus, $|V(K(G))| = \sum_{i=1}^{c} |V(H_i)| = O(n)$. □

4 Families of clique-inverse graphs with polynomial time recognition algorithms

Let G be a connected graph such that $\omega(K(G)) \leq d$, for some positive constant d. Then, it is clear that $G \in K^{-1}(K_r - FREE)$, where $r = d+1$. The next result says that this class can be recognized in polynomial time.

Theorem 6. *Let r be a positive constant. Then there is a polynomial-time recognition algorithm for the class $K^{-1}(K_r - FREE)$.*

Proof. If $G \in K^{-1}(K_r - FREE)$, then $\omega(K(G)) \leq r - 1$. By Lemma 1, $|V(K(G))|$ is $O(n)$. More precisely, $|V(K(G))| \leq (r-1)n$, that is, G has at most $(r-1)n$ maximal cliques. Given a graph G, one can test whether G has at most $(r-1)n$ maximal cliques in $O(n^4)$ time by applying to $\overline{G}$ the algorithm in [13], which generates all the maximal independent sets of a graph with delay $O(n^3)$. If G has more than $(r-1)n$ maximal cliques, then the answer to the question "$G \in K^{-1}(K_r - FREE)$?" is clearly "no". Otherwise, construct $K(G)$ by taking the maximal cliques generated by the algorithm. This task takes $O(n^3)$ time, since G has at most $(r-1)n$ maximal cliques, and each intersection test between two cliques takes $O(n^2)$ time. Finally, test whether $\omega(K(G)) \leq r-1$ in $O(n^r)$ time. Therefore, the entire procedure answers the question "$G \in K^{-1}(K_r - FREE)$?" in $O(n^k)$ time, where $k = max\{r, 4\}$. □

Corollary 7. *The classes $K^{-1}(TRIANGLE - FREE)$, $K^{-1}(BIPARTITE)$, $K^{-1}(CHORDAL\ BIPARTITE)$, and $K^{-1}(TREE)$ admit polynomial-time recognition algorithms.*

Proof. If G belongs to some of these classes, then $\omega(K(G)) \leq 3$. Therefore, apply Theorem 6 for $r = 3$. □

Corollary 8. *Let d be a positive constant. Then there is a polynomial-time recognition algorithm for the class $K^{-1}(MAXIMUM\ DEGREE\ d)$.*

Proof. If $\Delta(K(G)) = d$, then $\omega(K(G)) \leq d + 2$. Therefore, apply Theorem 6 for $r = d + 2$. □

Corollary 9. *There is a polynomial-time recognition algorithm for the class $K^{-1}(PLANAR)$.*

Proof. If $G \in K^{-1}(PLANAR)$, then $\chi(K(G)) \leq 4$. Therefore, by Lemma 5, $|V(K(G))|$ is $O(n)$. More precisely, $|V(K(G))| \leq 2n$, that is, G has at most $2n$ maximal cliques. Given a graph G, apply the same technique described in Theorem 6. Test first whether G has at most $2n$ maximal cliques in $O(n^4)$ time. If the answer is "yes", construct $K(G)$ in $O(n^3)$ time and test whether it is planar in $O(n)$ time. The entire procedure decides whether $G \in K^{-1}(PLANAR)$ in $O(n^4)$ time. □

5 Families of clique-inverse graphs whose recognition is NP-hard

In this section, we describe NP-hard cases of recognizing clique-inverse graphs of various classes. We start by analyzing the recognition of $K^{-1}(CHORDAL)$. In [4], this class is referred as the class of *clique-chordal graphs*.

In what follows, let Π be the problem of recognizing $K^{-1}(COMPLETE)$, which is known to be Co-NP-complete [15].

Theorem 10. *Recognizing $K^{-1}(CHORDAL)$ is a Co-NP-complete problem.*

Proof. Let G be a graph. A certificate showing that $G \notin K^{-1}(CHORDAL)$, that is, $K(G)$ does not belong to $CHORDAL$, is a set $M_1, \ldots, M_k$ of maximal cliques of G inducing C_k ($k \geq 4$) in $K(G)$ (recall that maximal cliques of G are vertices of $K(G)$, and vice-versa). By Lemma 3, $k = O(n)$. Therefore, one can verify this certificate in polynomial time, that is, recognizing $K^{-1}(CHORDAL)$ is in Co-NP. In the sequel, construct a reduction from Π. Given a graph H, instance for Π, construct in polynomial time on the size of H a new graph G, instance for the recognition of $K^{-1}(CHORDAL)$, as follows: G is formed by two copies H_1 and H_2 of H where every vertex of H_1 is adjacent to all the vertices of H_2. Let us show that $K(H)$ is not a clique if and only if $K(G)$ is not chordal. If $K(H)$ is not a clique, there exist M' and M'' maximal cliques of H such that $M' \cap M'' = \emptyset$. Let $M_1', M_1'' \in H_1$ and $M_2', M_2'' \in H_2$ such that M_1', M_2' correspond to M' and M_1'', M_2'' correspond to M''. Observe that $M_1' \cup M_2'$, $M_1' \cup M_2''$, $M_1'' \cup M_2'$, and $M_1'' \cup M_2''$ are maximal cliques of G inducing C_4 in $K(G)$. Therefore, $K(G)$ is not chordal. On the other hand, if $K(H)$ is a clique, then $K(G)$ is also a clique, that is, $K(G)$ is chordal. □

Corollary 11. *Recognizing $K^{-1}(SPLIT)$ is a Co-NP-complete problem.*

Proof. Let G be a graph. A certificate showing that $K(G)$ is not a split graph is a set of maximal cliques of G inducing in $K(G)$ one of the graphs C_4, C_5, and $2K_2$, which are the forbidden subgraphs for split graphs [5]. In any case, the number of maximal cliques in the certificate is fixed, and thus one can verify the certificate in polynomial time, that is, recognizing $K^{-1}(SPLIT)$ is in Co-NP. The detailed reduction of the proof is similar to that of Theorem 10. □

Corollary 12. *Recognizing K^{-1}(INTERVAL) is a Co-NP-complete problem.*

Proof. Let G be a graph. A certificate showing that $K(G)$ is not an interval graph is a set $\{M_1, M_2, \ldots, M_k\}$ of maximal cliques of G inducing in $K(G)$ some of the subgraphs depicted in Figure 1, which are the forbidden subgraphs for interval graphs [14]. Observe that I and II have fixed size, III is the graph C_k, and IV,V contain P_{k-2} as an induced subgraph (if $k > 2$). In any case, by Lemma 3, k is $O(n)$. Therefore, one can verify the certificate in polynomial time, that is, recognizing K^{-1}(*INTERVAL*) is in Co-NP. The reduction is again similar to Theorem 10. □

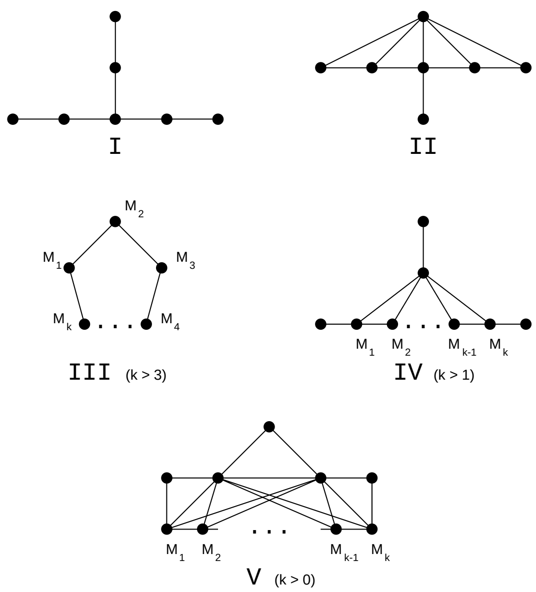

Fig. 1. Forbidden subgraphs for interval graphs.

As another corollary of Theorem 10, one can prove that recognizing the class $K^{-1}(BLOCK - CUTPOINT)$ is a Co-NP-complete problem. Details can be found in [20].

Theorem 13. *Recognizing $K^{-1}(CO - INTERVAL)$ is a NP-hard problem.*

Proof. Let H be an instance for Π, and construct an instance G for the recognition of $K^{-1}(CO - INTERVAL)$ as follows: G is formed by two copies H_1 and H_2 of H and an additional vertex u such that every vertex of H_1 is adjacent to u and to all the vertices of H_2. Let us show that $K(H)$ is not a clique if and only if $\overline{K(G)}$ is not an interval graph. If $K(H)$ is not a clique, then there are M' and M'' maximal cliques of H such that $M' \cap M'' = \emptyset$. Let $M_1', M_1'' \in H_1$

and $M_2', M_2'' \in H_2$ such that M_1', M_2' correspond to M', and M_1'', M_2'' correspond to M''. Observe that $M_1' \cup M_2'$, $M_1' \cup M_2''$, $M_1'' \cup M_2'$, $M_1'' \cup M_2''$, $\{u\} \cup M_1'$, and $\{u\} \cup M_1''$ are maximal cliques of G inducing in $K(G)$ the graph $\overline{C_6}$. Since $\overline{C_6}$ belongs to Gallai's family [6], $K(G)$ is not a comparability graph. Therefore, $\overline{K(G)}$ is not a co-comparability graph. Since every interval graph is a co-comparability graph [9], $\overline{K(G)}$ is not an interval graph. Conversely, if $K(H)$ is a clique, then $K(G)$ is also a clique. Therefore, $\overline{K(G)}$ is a graph with no edges, and thus an interval graph. □

Corollary 14. *Recognizing $K^{-1}(COMPARABILITY)$ is a NP-hard problem.*

Proof. Let H be an instance for Π, and construct G, instance for the recognition of $K^{-1}(COMPARABILITY)$, in the same way as in Theorem 13. We already know that if $K(H)$ is not a clique, then $K(G)$ is not a comparability graph. On the other hand, if $K(H)$ is a clique, then $K(G)$ is also a clique, and thus a comparability graph. □

Theorem 15. *Recognizing $K^{-1}(CO - COMPARABILITY)$ is a NP-hard problem.*

Proof. Let H be an instance for Π, and construct G, instance for the recognition of $K^{-1}(CO - COMPARABILITY)$, as follows:

$V(G) = V(H) \cup \{w, r, s\} \cup \{x_{uv} \mid u, v \text{ are not neighbors in } H\}$,

$E(G) = E(H) \;\cup\; \{w, r\} \;\cup\; \{r, s\}$

$\cup \{ \{u, x_{uv}\}, \{v, x_{uv}\}, \{w, x_{uv}\}, \{u, w\}, \{v, w\} \mid u, v \text{ are not neighbors in } H\}$.

Let us show now that $K(H)$ is not a clique if and only if $\overline{K(G)}$ is not a comparability graph. If $K(H)$ is not a clique, then there are two disjoint maximal cliques M_1 and M_2 in H. Therefore, there are $u \in M_1$ and $v \in M_2$ such that u is not adjacent to v. Observe that $M_1, M_2, \{u, w, x_{uv}\}, \{v, w, x_{uv}\}, \{w, r\}$, and $\{r, s\}$ are maximal cliques of G inducing in $K(G)$ the following graph T: a clique $\{v_1, v_2, v_3\}$ and an independent set $\{w_1, w_2, w_3\}$ where w_i is adjacent to v_j if $j = i$. Therefore, $\overline{K(G)}$ contains $\overline{T}$ as an induced subgraph. Since $\overline{T}$ belongs to Gallai's family [6], $\overline{K(G)}$ is not a comparability graph. Conversely, assume that $K(H)$ is a clique. Observe that the maximal cliques of G can be partitioned in four collections: (i) $\mathcal{M}_H$, consisting of the maximal cliques of G which are identical to those of H; (ii) $\mathcal{M}_w$, consisting of the sets of the form $\{ \{u, w, x_{uv}\} \mid u, v \text{ are not neighbors in } H\}$; (iii) the clique $M_{wr} = \{w, r\}$; (iv) the clique $M_{rs} = \{r, s\}$. Cliques in $\mathcal{M}_H$ induce in $\overline{K(G)}$ an independent set, since $K(H)$ is a clique. Cliques in $\mathcal{M}_w \cup \{M_{wr}\}$ share vertex w, and thus also induce in $\overline{K(G)}$ an independent set. Therefore, the edges of $\overline{K(G)}$ can be partitioned in four sets:

$\mathcal{A}$: edges of the form $\{M, M'\}$ such that $M \in \mathcal{M}_H$, $M' \in \mathcal{M}_w$, $M \cap M' = \emptyset$;

$\mathcal{B}$: edges of the form $\{M, M_{wr}\}$ for $M \in \mathcal{M}_H$;

$\mathcal{C}$: edges of the form $\{M, M_{rs}\}$ for $M \in \mathcal{M}_H$;

$\mathcal{D}$: edges of the form $\{M', M_{rs}\}$ for $M' \in \mathcal{M}_w$.

Observe that there exist the edges $\{M, M_{wr}\} \in \mathcal{B}$ and $\{M, M_{rs}\} \in \mathcal{C}$ for every clique $M \in \mathcal{M}_H$. There also exists the edge $\{M', M_{rs}\}$ for every clique $M' \in \mathcal{M}_w$. It is necessary to show that $\overline{K(G)}$ is a comparability graph. Set an orientation to its edges as follows: for edges in $\mathcal{A}$, from $M \in \mathcal{M}_H$ to $M' \in \mathcal{M}_w$; for edges in $\mathcal{B}$, from $M \in \mathcal{M}_H$ to M_{wr}; for edges in $\mathcal{C}$ and $\mathcal{D}$, to M_{rs}. In order to verify that this is indeed a transitive orientation, just observe that if there exist oriented edges (M, M') of $\mathcal{A}$ and (M', M_{rs}) of $\mathcal{D}$, then there also exists the oriented edge (M, M_{rs}) of $\mathcal{C}$. Therefore, $\overline{K(G)}$ is a comparability graph. □

By using Theorem 15, one can prove that recognizing $K^{-1}(AT-FREE)$ is a Co-NP-complete problem, and recognizing $K^{-1}(PERMUTATION)$ is a NP-hard problem. Details of the proofs can be found in [20]. In the same work, it is also shown that recognizing $K^{-1}(CO-CHORDAL)$, $K^{-1}(CO-BIPARTITE)$, and $K^{-1}(CO-CHORDAL-BIPARTITE)$ are NP-hard problems.

The next theorem deals with the chromatic number of $K(G)$. As expected, to decide whether $\chi(K(G)) \leq 3$ is a NP-complete problem.

Theorem 16. *Recognizing $K^{-1}(3-COLORABLE)$ is a NP-complete problem.*

Proof. Let G be a graph. A certificate showing that $\chi(K(G)) \leq 3$ consists of the set of maximal cliques of G together with a mapping that associates each maximal clique to a color in such a way that: (i) the mapping uses at most three colors; (ii) intercepting cliques receive distinct colors. By Lemma 5, $|V(K(G))| = \tau(G) = O(n)$. More precisely, $\tau(G) \leq \frac{3}{2}n$. Therefore, this certificate can be verified in polynomial time, that is, recognizing $K^{-1}(3-COLORABLE)$ is in NP. In order to prove the NP-hardness, let us extend the transformation from 3-SAT to 3-COLOR, described in [8]. Let $C = \{C_1, C_2, \ldots, C_p\}$ be a set of clauses on the variables $x_1, x_2, \ldots, x_n$. Assume that every clause contains exactly three distinct literals. Write $C_i = (a_i \vee b_i \vee c_i)$, where $\{a_i, b_i, c_i\}$ is contained in $\{x_1, x_2, \ldots, x_n, \overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}\}$. Construct a graph H such that $\chi(H) \leq 3$ if and only if C is satisfiable, as follows:

$V(H) = \{v_1, v_2, v_3\} \cup \{x_i, \overline{x_i} | 1 \leq i \leq n\} \cup \{y_{ij} | 1 \leq i \leq p, 1 \leq j \leq 6\}$,
$E(H) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\} \cup \{\{x_i, \overline{x_i}\} | 1 \leq i \leq n\}$
$\cup \{\{v_3, x_i\}, \{v_3, \overline{x_i}\} | 1 \leq i \leq n\}$
$\cup \{\{a_i, y_{i1}\}, \{b_i, y_{i2}\}, \{c_i, y_{i3}\} | 1 \leq i \leq p\}$
$\cup \{\{v_2, y_{i6}\}, \{v_3, y_{i6}\} | 1 \leq i \leq p\}$
$\cup \{\{y_{i1}, y_{i2}\}, \{y_{i1}, y_{i4}\}, \{y_{i2}, y_{i4}\} | 1 \leq i \leq p\}$
$\cup \{\{y_{i3}, y_{i5}\}, \{y_{i3}, y_{i6}\}, \{y_{i5}, y_{i6}\} | 1 \leq i \leq p\}$
$\cup \{\{y_{i4}, y_{i5}\} | 1 \leq i \leq p\}$.

Now, construct a clique-inverse graph G of H. Let $V(G) = \{u_i | 1 \leq i \leq p+1\} \cup \{w_1, w_2\} \cup \{w_{i3}, q_i, \overline{q_i} | 1 \leq i \leq n\} \cup \{z_{ij} | 1 \leq i \leq p, 1 \leq j \leq 7\}$. In order to define $E(G)$, let R_i $(1 \leq i \leq n)$ be the subset of vertices of G defined as follows: if y_{jk} is adjacent to x_i in H (for k, j, and i such that $1 \leq k \leq p$, $j \in \{1, 2, 3\}$, and $1 \leq i \leq n$), then $z_{jk} \in R_i$. Analogously, let S_i $(1 \leq i \leq n)$ be as follows: if y_{jk} is adjacent to $\overline{x_i}$ in H, then $z_{jk} \in R_i$. Now, define $E(G)$ in the following way:

$E(G) = \{\{u_i, u_j\} | 1 \leq i < j \leq p+1\} \ \cup \ \{w_1, u_{p+1}\}$
$\cup \{\{w_2, u_i\} | 1 \leq i \leq p\}$

$\cup \{\{w_{i3}, q_i\} | 1 \leq i \leq n\}$
$\cup \{\{w_{i3}, \overline{q_i}\} | 1 \leq i \leq n\}$
$\cup \{\{w_{i3}, w_{j3}\} | 1 \leq i < j \leq n\}$
$\cup \{\{w_{i3}, u_j\} | 1 \leq i \leq n, 1 \leq j \leq p\}$
$\cup \{\{z_{i6}, u_i\} | 1 \leq i \leq p\}$
$\cup \{\{z_{i1}, z_{i3}\}, \{z_{i2}, z_{i3}\}, \{z_{i3}, z_{i4}\} | 1 \leq i \leq p\}$
$\cup \{\{z_{i4}, z_{i6}\}, \{z_{i5}, z_{i6}\} | 1 \leq i \leq p\}$
$\cup \{\{w_{i3}, r\}, \{q_i, r\} | 1 \leq i \leq n, r \in R_i\}$
$\cup \{\{w_{i3}, s\}, \{\overline{q_i}, s\} | 1 \leq i \leq n, s \in S_i\}$
$\cup \{\{r_1, r_2\} | r_1, r_2 \in R_i, 1 \leq i \leq n\}$
$\cup \{\{s_1, s_2\} | s_1, s_2 \in S_i, 1 \leq i \leq n\}$.

Observe that there is a direct correspondence between maximal cliques of G and vertices of H, in such a way that two cliques intersect if and only if the corresponding vertices are adjacent:

maximal cliques of G	vertices of H
$\{w_1, u_{p+1}\}$	v_1
$\{w_2, u_1, \ldots, u_{p+1}\}$	v_2
$\{w_{13}, \ldots, w_{n3}, u_1, \ldots, u_{p+1}\}$	v_3
$\{z_{i6}, u_i\}$	y_{i6}
$\{z_{i5}, z_{i6}\}$	y_{i5}
$\{z_{i3}, z_{i6}\}$	y_{i3}
$\{z_{i4}, z_{i5}\}$	y_{i4}
$\{z_{i2}, z_{i4}\}$	y_{i2}
$\{z_{i1}, z_{i4}\}$	y_{i1}
$\{q_i, w_{i3}\} \cup R_i$	x_i
$\{\overline{q_i}, w_{i3}\} \cup S_i$	$\overline{x_i}$

The above table shows that $H = K(G)$. Therefore, C is satisfiable if and only if $\chi(H) \leq 3$ if and only if $\chi(K(G)) \leq 3$. In other words, given a set C of clauses, we have constructed in polynomial time on the size of C a graph G such that C is satisfiable if and only if the clique graph of G has chromatic number at most three. Thus, recognizing $K^{-1}(3 - \mathit{COLORABLE})$ is a NP-complete problem. □

Theorem 17. *Let s be a positive integer. Then the problem of recognizing the class $K^{-1}(s - \mathit{INDEPENDENT})$ is NP-complete.*

Proof. Let G be a graph. A certificate showing that $\alpha(K(G)) \geq s$ consists of a set $\{M_1, \ldots, M_s\}$ of pairwise disjoint maximal cliques of G. By Lemma 3, s reaches at most the value $n/2$. Therefore, this certificate can be verified in polynomial time, that is, recognizing $K^{-1}(s - \mathit{INDEPENDENT})$ is in NP. The proof of the NP-hardness uses the same idea in Theorem 16, by extending the transformation of 3-SAT into STABLE SET described in [7]. The details can be found in [20]. □

References

1. E. Balas and C. S. Yu. On graphs with polynomially solvable maximum-weight clique problem. *Networks* 19 (1989) 247–253.
2. H.-J. Bandelt and E. Prisner. Clique graphs and Helly graphs. *J. Comb. Theory, Series B* 51 (1991) 34–45.
3. A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes: a Survey.* SIAM Monographs on Discrete Mathematics and Applications, Philadelphia (1999).
4. A. Brandstädt, F. F. Dragan, V. D. Chepoi, and V. I. Voloshin. Dually chordal graphs. *SIAM J. Discrete Math.* 11 (1998) 437–455.
5. S. Földes and P. L. Hammer. Split Graphs. In: F. Hoffman *et al.*, eds., *Proc. 8th Southeastern Conf. on Combinatorics, Graph Theory and Computing* 311–315 (1977). Louisiana State University.
6. T. Gallai. Transitive orienterbare graphen. *Acta Math. Sci. Hung.* 18 (1967) 25–66.
7. M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman, New York, 1979.
8. M. R. Garey, D. S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science* 1 (1976) 237–267.
9. P. C. Gilmore and A. J. Hoffman. A characterization of comparability graphs and of interval graphs. *Canad. J. Math.* 16 (1964) 539–548.
10. M. C. Golumbic. Algorithmic Graph Theory and Perfect Graphs. Academic Press, New York (1980).
11. S. T. Hedetniemi and P. J. Slater. Line graphs of triangleless graphs and iterated clique graphs. *Lecture Notes in Mathematics* 303 (1972) 139–147.
12. B. Hedman. Clique graphs of time graphs. *J. Combin. Theory Ser. B* 37 (1984) 270–278.
13. D. S. Johnson, M. Yannakkakkis, and C. H. Papadimitriou. On generating all maximal independent sets. *Information Processing Letters* 27 (1988) 119-123.
14. C. G. Lekkerkerker and J. C. Boland. Representation of a finite graph by a set of intervals on the real line. *Fundamenta Mathematicae* 51 (1962) 45–64.
15. C. L. Lucchesi, C. P. Mello, and J. L. Szwarcfiter. On clique-complete graphs. *Discrete Mathematics* 183 (1998) 247–254.
16. T. A. McKee and F. R. McMorris. Topics in Intersection Graph Theory. SIAM Monographs on Discrete Mathematics and Applications, Philadelphia (1999).
17. J. W. Moon and L. Moser. On cliques in graphs. *Israel J. Math.* 3 (1965) 23–28.
18. E. Prisner. *Graph Dynamics.* Pitman Research Notes in Mathematics 338, Longman, London (1995).
19. E. Prisner. Graphs with few cliques. In: Y. Alavi and A. Schwenk, eds., *Graph Theory, Combinatorics and Applications: Proceedings of the 7th Quadrennial International Conference on the Theory and Applications* 945–956 (1995). John Wiley and Sons.
20. F. Protti. *Classes de grafos clique-inversos.* PhD Thesis, Universidade Federal do Rio de Janeiro (1998). In portuguese.
21. F. Protti and J. L. Szwarcfiter. Clique-inverse graphs of K_3-free and K_4-free graphs. Tech. Rep. 05/99, NCE/UFRJ, Rio de Janeiro, Brazil. Submitted.
22. J. L. Szwarcfiter and C. F. Bornstein. Clique graphs of chordal and path graphs. *SIAM J. Discrete Math.* 7 (1994) 331–336.

This article was processed using the LaTeX macro package with LLNCS style