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Abstract

Starlike graphs are the intersection graphs of substars of a star. We describe characterizations by forbidden subgraphs for starlike graphs and for a special subclass of it.

Key Words: graph classes, forbidden subgraphs, intersection graphs, starlike graphs, starlike-threshold graphs.

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1 Introduction

Graph classes and intersection graphs are traditional topics in graph theory. In fact, theses studies have been receiving much attention, recently. For example, we mention the two new books by Brandstädt, Le and Spinrad [1] and by McKee and McMorris [11].

Chordal graphs form one of the most well studied classes of graphs. In special, they can be considered as special intersection graphs (Buneman [2], Gavril [7], Walter [18]). There exist characterizations by forbidden subgraphs for various subclasses of chordal graphs. In fact, there are characterizations of this type for classes as interval graphs (Lekkerkerker and Boland [10]), proper interval graphs (Roberts [17]), strongly chordal graphs (Farber [5]), chordal bipartite graphs (Golumbic and Goss [8]), split graphs (Földes and Hammer [6]), threshold graphs (Chvátal and Hammer [4]). Recently, such a characterization has been also described for directed path graphs (Panda [14]). Nevertheless, remain open the problems of finding forbidden subgraph characterizations for both undirected and rooted directed path graphs.

In the present paper, we describe characterizations by forbidden subgraphs for two other classes of chordal graphs, namely starlike graphs and starlike-threshold graphs. Starlike graphs were introduced by Gustedt [9], in the study of the pathwidth problem for chordal graphs. Further, this class has been considered by Peng et al. [15], Moscarini et al. [13], Cerioli and Szwarcfiter [3]. See also McMorris and Shier [12] and Prisner [16]. Starlike-threshold graphs were implicitly introduced by Chvátal and Hammer [4]. They arise naturally when studying edge clique graphs of threshold graphs (Cerioli and Szwarcfiter [3]).

All graphs considered are connected, finite and simple. The vertex and edge sets of an undirected graph G are represented by $V(G)$ and $E(G)$, respectively. A vertex is *universal* if it is adjacent to all the other vertices. Two adjacent vertices which are adjacent exactly to the same vertices of G are called *twins*. For $v, w \in V(G)$, the *distance* between v and w, denoted

$d(v, w)$, is the number of edges in a shortest $v - w$ path of G. For $M \subseteq V(G)$, say that M is a *clique* when M induces a complete subgraph in G. A *maximal clique* is one not properly contained in any other. The set M is *dominating* when every vertex outside M is adjacent to some vertex of M. A maximal clique which is dominating is called a *central clique* of G.

A *chordal graph* is the intersection graph of a set $\mathcal{S}$ of subtrees of some tree. Among all trees T which would give rise to the same chordal graph G, one of minimum size is called a *clique-tree* of G. In this situation there exists an one-to-one correspondence between maximal cliques of G and vertices of T. Moreover, if M_j is the maximal clique of G corresponding to $w_j \in V(T)$ and $v_i \in V(G)$ is the vertex corresponding to the subtree S_i of $\mathcal{S}$, then $v_i \in M_j$ if and only if $w_j \in S_i$.

A *star* is a tree having one universal vertex, called *center*. A *starlike graph* is the intersection graph of substars of a star. The following characterization is useful.

Theorem 1 ([9]) *A graph G is starlike if and only if it admits a central clique M, such that for $v_i, v_j \in V(G) \setminus M$, v_i and v_j are either twins or non adjacent.*

Let G be a starlike graph, $M_1, \ldots, M_q$ its maximal cliques, where M_1 is a central clique satisfying the above theorem. Denote $M_i' = M_i \setminus M_1$, for $i > 1$. Then $M_1, M_2', \ldots, M_q'$ is a partition of $V(G)$, called *starlike partition.* It follows that G is a starlike graph if and only if it admits a starlike partition. A *starlike-threshold* graph is a starlike graph admitting a starlike partition $M_1, M_2', \ldots, M_q'$ satisfying $M_i \cap M_1 \supseteq M_{i+1} \cap M_1$, $1 \leq i < q$. In this case, call $M_1, M_2', \ldots, M_q'$, a *starlike-threshold partition* of G.

Characterizations by forbidden subgraphs for the classes of starlike and starlike-threshold graphs are described in Sections 2 and 3, respectively.

2 Starlike Graphs

The following is a characterization for starlike graphs.

Theorem 2 *A graph G is starlike if and only if G does not contain any of the six graphs of Figure 1 as an induced subgraph.*

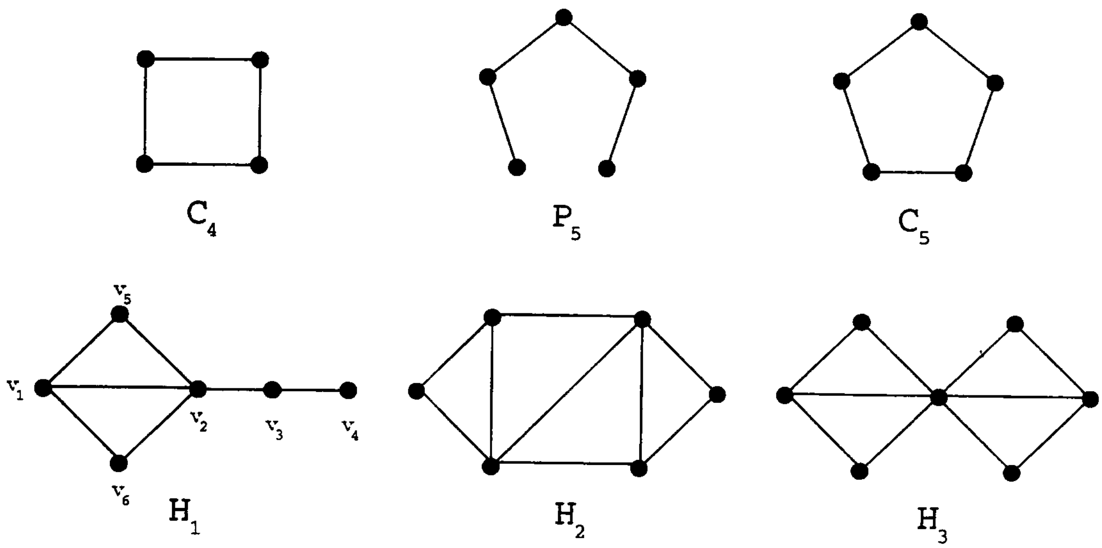

Figure 1: The forbidden subgraphs for starlike graphs.

Proof: Let G be a starlike graph, M_1 a central clique and $M_1, M_2', \ldots, M_q'$ a starlike partition of it. Clearly, G is chordal and consequently it does not contain C_4 nor C_5 as induced subgraphs. In addition, the diameter of G is at most three, that is, it does not contain a P_5 too. Examine the graph H_1 of Figure 1. Referring to its vertices as in the figure, it follows that $v_1 \in M_a'$, $v_2, v_3 \in M_1$ and $v_4 \in M_b'$, with $a, b > 1$. Since v_5, v_6 are not adjacent to v_3, we conclude that $v_5, v_6 \notin M_1$. Because v_5, v_6 are adjacent to v_1, it follows $v_5, v_6 \in M_a'$. However, the latter situation can not occur, because v_5, v_6 are not adjacent. Hence G can not contain H_1 as an induced subgraph. The proofs for H_2 and H_3 are similar to that for H_1.

Conversely, let G be a graph which does not contain any of the graphs of Figure 1, as an induced subgraph. We show that G is starlike. Examine the alternatives, regarding the subset C of the non simplicial vertices of G.

If C is a clique, then choose a maximal clique $M_1 \supseteq C$. Denote by $M_2, \dots, M_q$ the remaining maximal cliques of G. Construct a star T, with vertex set $\{w_1, \dots, w_q\}$ and center w_1. Define a family $\mathcal{S}$ of subsets of $V(T)$, as follows. There is one subset $S_i \in \mathcal{S}$, for each vertex $v_i \in V(G)$. Let $M(v_i) \subseteq \{M_1, \dots, M_q\}$ be the subset of maximal cliques of G, containing v_i. Then define $S_i = \{w_a \in V(T) : M_a \in M(v_i)\}$.

The following argument leads to the conclusion that each subset S_i induces a (connected) substar of T. If v_i is a simplicial vertex, then S_i consists of a single vertex of T and the conclusion is trivial. When v_i is not a simplicial vertex, it is contained in the maximal clique M_1. Consequently, S_i contains the center w_1 of T, meaning that S_i indeed induces a substar of T.

Further, we describe a second property, regarding T and $\mathcal{S}$. We assert that two vertices v_i, v_j of G are adjacent precisely when the corresponding subsets S_i, S_j intersect. For, divide its argument into the following two cases. If one of these two vertices, say v_i, is simplicial, then $v_i v_j \in E(G)$ implies that v_j also belongs to the maximal clique M_a containing v_i. Consequently, S_i and S_j both contain the vertex $w_a \in V(T)$, therefore $S_i \cap S_j \neq \emptyset$. On the other hand, when both v_i, v_j are not simplicial vertices, then S_i and S_j contain $w_1 \in V(T)$, and again $S_i \cap S_j \neq \emptyset$. Conversely, suppose that $S_i \cap S_j \neq \emptyset$. Examine the center w_1 of T. If $w_1 \notin S_i \cap S_j$, then one of the subtrees, say S_i, consists of a sole vertex $w_a \in V(T)$, $a \neq 1$. Since S_j must also contain w_a, it follows that v_i and v_j belong to the same maximal clique M_a of G. Consequently, $v_i v_j \in E(G)$, as required. The last alternative is $w_1 \in S_i \cap S_j$. In this situation, both vertices v_i, v_j are contained in M_1, also implying that $v_i v_j \in E(G)$.

From the above two properties it follows that G is the intersection graph of the family of substars $\mathcal{S}$ of T. That is, G is starlike and the theorem is true.

It remains to examine the second alternative, concerning the subset $C \subseteq V(G)$, that is when C is not a clique. In this case, employ the hypothesis that G does not contain the forbidden subgraphs of Figure 1. In special, G

not containing C_4, C_5 nor P_5 means that G is chordal. That is, G is the intersection graph of a family $\mathcal{S}$ of subtrees of a tree T. Without loss of generality choose T as to be a clique-tree of G. Denote by $S_i \in \mathcal{S}$ the subtree corresponding to $v_i \in V(G)$. We know that each $w_a \in V(T)$ corresponds to a maximal clique M_a of G, and also $v_i \in M_a$ if and only if $w_a \in S_i$.

Employing the assumption that C is not a clique, choose non adjacent vertices $v_i, v_j \in C$. Then S_i and S_j are disjoint subtrees of T. Denote by $w_a \in S_i$ the vertex of S_i closest to S_j in T. Similarly, $w_b \in S_j$ is the closest to S_i in T. Because v_i, v_j are both non simplicial vertices it follows that S_i contains some vertex $w_c \neq w_a$ and S_j contains $w_d \neq w_b$, such that $w_c w_a \in E(T)$ and $w_d w_b \in E(T)$. Consequently, the four distinct vertices w_c, w_a, w_b, w_d, in this order, belong to a same path of T. Because M_c, M_d are maximal cliques, G contains vertices $v_p \in M_c$ and $v_q \in M_d$ satisfying $v_p \notin M_a$ and $v_q \notin M_b$.

Examine the possible alternatives with respect to the distance between v_i and v_j in G. Clearly, $d(v_i, v_j) > 1$.

Case 1: $d(v_i, v_j) = 2$.

There exists some vertex $v_k \in V(G)$, simultaneously adjacent to v_i and v_j. Consider the following alternatives of containments of v_k in M_c and M_d.

Case 1.1: $v_k \notin M_c, M_d$.

Then the subgraph induced in G by $\{v_i, v_j, v_k, v_p, v_q\}$ is a P_5.

Case 1.2: $v_k \in M_c$ and $v_k \notin M_d$.

Because M_a is a maximal clique of G, there exists $v_l \in M_a$ satisfying $v_l \notin M_c$. The following situations should further be considered.

Case 1.2.1: $v_l \notin M_b$.

Then the subgraph induced by $\{v_i, v_j, v_k, v_p, v_q, v_l\}$ is a graph H_1.

Case 1.2.2: $v_l \in M_b$.

The following situations still apply.

Case 1.2.2.1: $v_l \notin M_d$.

The vertices $\{v_i, v_j, v_p, v_q, v_l\}$ induce a P_5 in G.

Case 1.2.2.2: $v_l \in M_d$.

In this situation, the subgraph induced in G by $\{v_i, v_j, v_k, v_p, v_q, v_l\}$ is a graph H_2.

Case 1.3: $v_k \notin M_c$ and $v_k \in M_d$.

Similar to Case 1.2.

Case 1.4: $v_k \in M_c, M_d$.

Because M_a and M_b are maximal cliques, there exist vertices $v_l \in M_a$ and $v_r \in M_b$, satisfying $v_l \notin M_c$ and $v_r \notin M_d$.

Case 1.4.1: $v_l \notin M_b$ and $v_r \notin M_a$.

It follows that the subset of vertices $\{v_i, v_j, v_k, v_p, v_q, v_l, v_r\}$ induces a graph H_3 in G.

Case 1.4.2: $v_l \in M_b$ and $v_r \notin M_a$.

This case is further subdivided, as follows.

Case 1.4.2.1: $v_l \notin M_d$.

Then $\{v_i, v_j, v_p, v_q, v_l\}$ induces a P_5.

Case 1.4.2.2: $v_l \in M_d$.

Then $\{v_i, v_j, v_p, v_q, v_l, v_r\}$ induces a graph H_1.

Case 1.4.3: $v_l \notin M_b$ and $v_r \in M_a$.

Similar to Case 1.4.2.

Case 1.4.4: $v_l \in M_b$ and $v_r \in M_a$.

If $v_l \notin M_d$ or $v_r \notin M_c$, then a P_5 arises. Otherwise, $v_l \in M_d$ and $v_r \in M_c$, implying that $\{v_i, v_j, v_p, v_q, v_l, v_r\}$ induces a graph H_2.

Examine the remaining alternatives, for the distance between v_i and v_j.

Case 2: $d(v_i, v_j) = 3$.

Let v_i, v_l, v_r, v_j be a shortest $v_i - v_j$ path of G. Then $v_l \notin M_b$ and $v_r \notin M_a$. The analysis is subdivided as follows.

Case 2.1: $v_l \notin M_c$.

Then $\{v_i, v_j, v_p, v_l, v_r\}$ induces a P_5.

Case 2.2: $v_l \in M_c$ and $v_r \notin M_d$.

Then $\{v_i, v_j, v_q, v_l, v_r\}$ induces a P_5.

Case 2.3: $v_l \in M_c$ and $v_r \in M_d$.

Denote by w_e the closest vertex to w_a in the $w_a - w_b$ path of T, such that $v_r \in M_e$. Because M_a is a maximal clique, G has a vertex $v_t \in M_a$ and $v_t \notin M_c$. In addition, $d(v_i, v_j) = 3$ implies that $w_e \neq w_a, w_b$ and $v_t \notin M_b$. The following alternatives occur.

Case 2.3.1: $v_t \notin M_e$.

Then $\{v_i, v_p, v_q, v_l, v_r, v_t\}$ induces a graph H_1.

Case 2.3.2: $v_t \in M_e$.

In this situation, $\{v_i, v_j, v_p, v_t, v_r\}$ induces a P_5.

Case 3: $d(v_i, v_j) > 3$.

Then any chordless $v_i - v_j$ path contains a P_5.

Cases 1 through 3 exhaust all the possibilities for the distance between a pair of non adjacent non simplicial vertices of G. All the possibilities lead to forbidden graphs of Figure 1. Consequently, any two non simplicial vertices of G are adjacent. This completes the proof of the theorem. ∎

Corollary 1 *A graph is starlike if and only if the set of its non simplicial vertices is a clique.*

3 Starlike-threshold Graphs

The following characterizes starlike-threshold graphs.

Theorem 3 *A graph is starlike-threshold if and only if it is starlike and does not contain a P_4, as an induced subgraph.*

Proof: By hypothesis, G is a starlike-threshold graph. Let $M_1, M_2', \ldots, M_q'$ be a starlike-threshold partition of it with maximal cliques $M_1, M_2, \ldots, M_q$ where $M_i' = M_i \setminus M_1$, $i > 1$, and $M_1 \supseteq M_2 \cap M_1 \supseteq \ldots \supseteq M_q \cap M_1$. Clearly, G is starlike and we show that it has no induced P_4. Suppose the theorem false and let (v_1, v_2, v_3, v_4) be an induced P_4 of it. Examine the cliques of $M_1, M_2', \ldots, M_q'$ where the vertices v_1, v_2, v_3, v_4 are included, and consider the following alternatives:

Case 1: $v_1 \in M_1$ and $v_2 \notin M_1$

Then $v_3 \notin M_1$. Since $v_2v_3 \in E(G)$, v_2 and v_3 must belong to a same clique M'_i. However $v_1v_2 \in E(G)$ and $v_1v_3 \notin E(G)$ imply that $M_1, M'_2, \ldots, M'_q$ is not a starlike partition of G, a contradiction.

Case 2: $v_1, v_2 \in M_1$

Then $v_3, v_4 \notin M_1$. Because $v_3v_4 \in E(G)$, v_3 and v_4 belong to a same clique M'_i. However, $v_2v_3 \in E(G)$ and $v_2v_4 \notin E(G)$ imply that $M_1, M'_2, \ldots, M'_q$ is not a starlike partition of G, a contradiction.

Case 3: $v_1 \notin M_1$ and $v_2 \in M_1$

Then $v_4 \notin M_1$ and there are two alternatives for v_3. Suppose $v_3 \in M_1$. Because $v_1v_4 \notin E(G)$, $v_1 \in M'_i$ implies $v_4 \in M'_j$ and $i \neq j$. Moreover, $v_1v_2 \in E(G)$ and $v_2v_4 \notin E(G)$ imply $i < j$. On the other hand, $v_1v_3 \notin E(G)$ and $v_3v_4 \in E(G)$ mean $i > j$. Consequently, $M_1, M'_2, \ldots, M'_q$ is not a starlike-threshold partition of G, a contradiction. Examine the remaining alternative $v_3 \notin M_1$. Similarly as in Cases 1 and 2, $v_3v_4 \in E(G)$ implies that $v_3, v_4 \in M'_i$, for some i. In this case, again, $M_1, M'_2, \ldots, M'_q$ is not a starlike partition of G, since $v_2v_3 \in E(G)$ and $v_2v_4 \notin E(G)$.

Case 4: $v_1, v_2 \notin M_1$

Then $v_1, v_2 \in M'_i$, for some i. Because $v_2v_3 \in E(G)$ and $v_1v_3 \notin E(G)$ it follows that $v_3 \notin M_1$. However, $v_2v_3 \in E(G)$ implies $v_3 \in M'_i$ and $v_1v_3 \notin E(G)$ means $v_3 \notin M'_i$, an impossibility.

Since none of the above cases may occur we conclude that G can not contain any induced P_4, as required.

Conversely, suppose that G is a starlike graph with no induced P_4. Let $M_1, M'_2, \ldots, M'_q$ be a starlike partition of G, and $M_1, M_2, \ldots, M_q$ its maximal cliques with $M'_i = M_i \setminus M_1$, $i > 1$. We prove below that $M_1, M'_2, \ldots, M'_q$ is a starlike-threshold partition of G.

Suppose it is not. Hence there exist maximal cliques M_i and M_j, $i, j \neq 1$, such that $M_1 \cap M_i$ neither contains nor is contained in $M_1 \cap M_j$. Let $v_i \in (M_1 \cap M_i) \setminus M_j$ and $v_j \in (M_1 \cap M_j) \setminus M_i$. Since v_i and v_j are vertices in

M_1, $v_iv_j \in E(G)$. Since M'_i and M'_j are non empty sets, there exist vertices $u_i \in M'_i$ and $u_j \in M'_j$. Because $M_1, M'_2, \ldots, M'_q$ is a starlike partition, $u_iv_i \in E(G)$ and $u_jv_j \in E(G)$ while $u_iv_j \notin E(G)$, $u_jv_i \notin E(G)$ and $u_iu_j \notin E(G)$. The conclusion is that the subgraph induced by $\{u_i, v_i, v_j, u_j\}$ is a P_4 of G, a contradiction. ∎

Corollary 2 *A graph is starlike-threshold if and only if it does not contain any of the graphs of Figure 2, as an induced subgraph.*

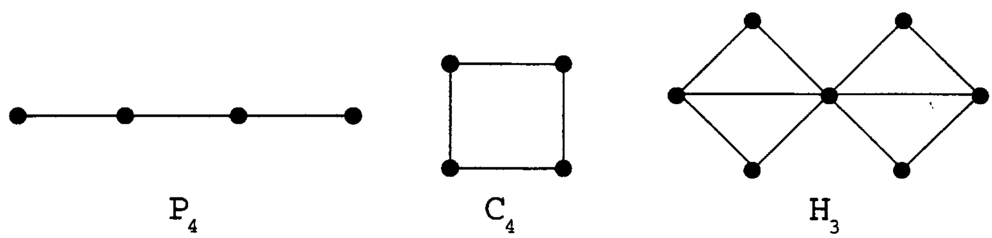

Figure 2: The forbidden subgraphs for starlike-threshold graphs.

It is easy to conclude that starlike-threshold graphs are interval graphs. In fact, they admit an intersection model by intervals of a line, using intervals of at most two distinct lengths.

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