# Generalized many-dimensional excited random walk in Bernoulli environment

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# Resumo

Estudamos uma extensão do passeios aleatórios excitados generalizados (GERW, do inglês Generalized Excited Random Walk) em $\mathbb{Z}^d$ introduzido em [*Ann. Probab. 40 (5), 2012*] por Menshikov, Popov, Ramírez and Vachkovskaia. Nossa extensão consiste no estudo de uma versão do GERW onde a excitação pode ou não acontecer com uma probabilidade que depende do tempo. Em particular, dada uma sequência de parâmetros $\{p_n\}_{n\geq 1}$ tal que $p_n \in (0,1]$ para todo $n \geq 1$, sempre que o processo visitar um sítio no tempo $n$ pela primeira vez, com probabilidade $p_n$, este ganha um drift em uma direção dada (podendo ser qualquer uma dentro da esfera unitária). Caso contrário, com probabilidade $1-p_n$, comporta-se com um $d$-martingal com vetor de média zero. Sempre que o processo visita um sitio já anteriormente visitado, este agirá como um $d$-martingal com vetor de media zero. Chamamos este modelo de GERW em um ambiente de Bernoulli, ou de forma reduzida de $p_n$-GERW.

Sob as mesmas hipóteses de [*Ann. Probab. 40 (5), 2012*] (saltos limitados, elipsidade uniforme) e com uma sequência $\{p_n\}_{n\geq 1}$ que decai polinomialmente, denominamos $p_n = \mathcal{C}n^{-\beta} \wedge 1$ com $\beta > 0$ e $\mathcal{C}$ uma constante positiva, demonstramos uma série de resultados para o $p_n$-GERW, os quais dependerão da dimensão e do valor de $\beta$. Especificamente, para $\beta < 1/6$ e $d \geq 2$, demonstramos que o $p_n$-GERW possui probabilidade positiva de nunca retornar a origem na direção do drift, para $\beta > 1/2$, $d \geq 2$ e $\beta = 1/2$, $d = 2$, obtemos sob certas condições um Teorema Central do Limite Funcional. Por último, para $\beta = 1/2$ e $d \geq 4$ obtemos, também sob condições adequadas, que o $p_n$-GERW é um processo rígido, além disso todo ponto limite $Y$ satifaz $W_t \cdot \ell + c_1\sqrt{t} \preceq Y_t \cdot \ell \preceq W_t \cdot \ell + c_2\sqrt{t}$ onde $c_1$ e $c_2$ são constantes positivas, $W$ é um Movimento Browniano e $\ell$ a direção do drift.

*Palavras-chave:Passeios Aleatórios Excitados, Teorema Central do Limite Funcional, Passeios Aleatórios Não-Markovianos, Balisticidade*

# Abstract

We study an extension of the generalized excited random walk (GERW) on $\mathbb{Z}^d$ introduced in [*Ann. Probab. 40 (5), 2012*] by Menshikov, Popov, Ramírez and Vachkovskaia. Our extension consists in studying a version of the GERW where excitation may/may not occur according to a time-dependent probability. Specifically, given a sequence of parameters $\{p_n\}_{n\geq 1}$, with $p_n \in (0,1]$ for all $n \geq 1$, whenever the process visits a site at time $n$ for the first time, with probability $p_n$ it gains a drift in a given direction (could be any direction of the unit sphere). Otherwise, with probability $1-p_n$, it behaves as a $d$-martingale with zero-mean vector. Whenever the process visits an already-visited site, the process acts again as a $d$-martingale with zero-mean vector. We refer to the model as a GERW in Bernoulli environment, in short $p_n$-GERW.

Under the same hypothesis of [*Ann. Probab. 40 (5), 2012*] (bounded jumps, uniform ellipticity) and with a sequence $\{p_n\}_{n\geq 1}$ which decays polynomially, namelly $p_n = \mathcal{C}n^{-\beta} \wedge 1$ with $\beta > 0$ and $\mathcal{C}$ is a positive constant, we show a series of results for the $p_n$-GERW depending on the value of $\beta$ and on the dimension. Specifically, for $\beta < 1/6$ and $d \geq 2$, we show that the $p_n$-GERW has a positive probability of never returning to the origin in the drift direction, for $\beta > 1/2$, $d \geq 2$ and $\beta = 1/2$ and $d = 2$ we obtain, under certain conditions, a Functional Central Limit Theorem. Finally, for $\beta = 1/2$ and $d \geq 4$ we obtain, under suitable conditions, that a sequence of $p_n$-GERW is a tight process, and every limit point $Y$ satisfies $W_t \cdot \ell + c_1\sqrt{t} \preceq Y_t \cdot \ell \preceq W_t \cdot \ell + c_2\sqrt{t}$ where $c_1$ and $c_2$ are positive constants, $W$ is a Brownian motion and $\ell$ is the direction of the drift.

*Keywords: Excited Random Walk, Functional Central Limit Theorem, Non-Markovian Random Walks, Ballisticity.*

# Contents

## 1. Introduction

The many-dimensional excited random walk (ERW) is a model introduced in 2003 by Benjamini and Wilson [3]. It is a discrete time non Markovian random walk in $\mathbb{Z}^d$, with $d \geq 2$. It jumps as a simple random walk biased in direction $e_1$ (with bias $\delta$) every time it visits a site for the first time, where $\{e_i : 1 \leq i \leq d\}$ denotes the canonical base of $\mathbb{Z}^d$, otherwise it jumps as a simple symmetric random walk.

In [3], Benjamini and Wilson proved that ERW is transient in direction $e_1$, i.e., $\lim_{n\to\infty} X_n \cdot e_1 = \infty$ almost surely. Furthermore, they also show that, if $d \geq 4$, ERW is ballistic to the right, i.e.,

$$\liminf_{n\to\infty} \frac{X_n \cdot e_1}{n} > 0 \,, \qquad a.s..$$

Later on, Kozma extended the proof of ballisticity to $d = 3$ in [12], and $d = 2$ in [13]. In 2007, Bernard and Ramirez [4] proved a Law of Large Numbers (LLN) and a Central Limit Theorem (CLT) for $d \geq 2$. Specifically, they prove that

$$\lim_{n\to\infty} \frac{X_n \cdot e_1}{n} = v \,, \qquad a.s.,$$

for some $v = v(\delta, d) \in \mathbb{R}^+$, and that

$$\left\{ \frac{X_{\lfloor nt \rfloor} \cdot e_1 - \lfloor nt \rfloor v}{\sqrt{n}} \right\}_{t\geq 0} ,$$

converges in distribution as $n \to \infty$ (with respect to the Skorohod topology on the space of càdlàg functions) to a Brownian Motion with a finite variance depending on $\delta$ and $d$. Their proof relies on the introduction of an appropriate regeneration structure that was first used in the context of random walks in random environments, see for instance [19].

The proofs of directional transience in [3], the LLG and the CLT in [4], rest upon two important ingredients. A coupling between the ERW and the simple symmetric random walk (SSRW) which implies that the distance between the ERW and the SSRW at time $n$, in the direction $e_1$, is non decreasing in $n$, while for the others directions it is zero. Using this coupling, the authors provide a lower bound on the cardinality of the set of visited sites by the ERW up to time $n$ (the range of ERW) in terms of *tan points* for the SSRW, i.e., those sites $x \in \mathbb{Z}^d$ such that $x$ is the first site visited in the set $\{x + ke_1 : k \geq 0\}$. A direct consequence of the coupling is that when the SSRW reaches a tan point, the ERW visits a new site and thus it is pushed in direction $e_1$ by a positive drift. Then in [4], using this coupling, the authors proved that the range of the ERW up to time $n$ in dimension $d \geq 2$ is of order at least $n^{3/4}$ with large probability. This fact alone is not enough to provide a direct proof for a linear speed of the process, however it is instrumental to guarantee the existence of a renewal structure for the process which leads to the limit theorems.

A drawback of the technique based on tan points is that it is tailored to the basic model of ERW and it is not robust, i.e., the coupling with the SSRW would not work if for example we consider a random walk with bounded jumps, rather than nearest neighbor jumps, or even if we suppose a drift not parallel to any canonical direction. A more robust technique was developed by Menshikov, Popov, Ramirez and Vachkovskaia in [15]. The model they considered is a generalization of the ERW and is as follows: on already visited sites the process behaves like a $d$-dimensional martingale with bounded jumps and zero mean vector (rather than a SSRW) and whenever the process visits a site for the first time it behaves as follows: it has bounded jumps, satisfies an uniformly elliptic condition and a drift condition in an arbitrary direction $\ell$ of the unit sphere in $\mathbb{R}^d$. They call this model *generalized excited random walk* (GERW). They show that the GERW with a drift condition in direction $\ell$ is ballistic in that direction. Besides that, they proved a LLG and a CLT (both for dimensions $d \geq 2$) for a special case of the GERW, which they called *excited random walk in random environment*. This special model consists in an excited random walk in an i.i.d. random environment, which means that the process still has a mean drift in direction $\ell$ when it visits a site for the first time and whenever it hits an already visited site it has a zero mean drift (for more details see page 2110 in [15]). Along with that, the probability transitions for nearest neighbors of the process are explicit. Similarly to what was done for ERW, the first step in their proof consists in controlling the range of the process. Proposition 4.1 in [15] states that the range of the GERW is smaller than $n^{1/2+\alpha}$ with probability that decays as a stretched exponential, where $\alpha > 0$ does not depend on the parameters of the model. A similar result can be found in [14] (see, Theorem 1.3). However, the range considered therein is for, what the authors call, a directed submartigale in direction $\ell$. The proof of Proposition 4.1 completely avoids the use of the coupling and the tan points. Again, this control on the range of the process allows the construction of a regeneration structure for the GERW. If $\{X_n\}_{n\geq 0}$ denotes the GERW, the regeneration structure consists in a properly defined sequence of finite regeneration times $\tau_k$, $k \geq 1$, that correspond to those times when the process $\{X_n \cdot \ell\}_{n\geq 0}$ reaches for the first time the level $X_{\tau_k} \cdot \ell$ and never comes back below $X_{\tau_k} \cdot \ell$ after time $\tau_k$. The renewal structure considered in [15] follows the standard approach and notation presented in [4] and [19].

What makes the GERW (and the ERW) an interesting model is the self-interaction encoded in the different behavior the process has on sites visited for the first time as compared to sites already visited. Similar works worth mentioning along these lines are [2], [17], and [1]. It is customary to think that initially all sites have a *cookie*. Whenever the process visits a site for the first time, it eats the cookie and gains a drift in a given direction. On subsequent visits to a site, since there is no cookie left, the process has no drift (for this reason ERW are also referred to as cookie random walks). A natural question is what happens to GERW when on the first visit to a site the random walk may or may not find/eat a cookie, with a probability that may possibly depend on the time of the first visit. Would the process still be ballistic in the direction of the drift? What about LLN and CLT?

In order to address this question, we introduce and study a model which is a variation of the GERW. Specifically, given a sequence of parameters $\{p_n\}_{n\geq 1}$ with $p_n \in (0,1]$ for all $n \geq 1$, if at time $n$ the process visits a site for the first time, it finds a cookie with probability $p_n$ (thus gaining a drift). Otherwise, with probability $1-p_n$, it finds no cookie (no drift) and behaves as a $d$-martingale with zero-mean vector. If instead the process has already visited the site, there is no cookie and the process acts again as a $d$-martingale with zero-mean vector. We call this model $p_n$-GERW. In this Thesis, we will focus on two specific cases for the sequence $\{p_n\}_{n\geq 1}$:

1) HOMOGENEOUS: the sequence $\{p_n\}_{n\geq 1}$ will be constant, i.e., for a $p \in (0,1]$ we have $p_n = p$ for all $n \geq 1$. This case will be called $p$-GERW. As it turns out, the homogeneous case bears no novelty as it can be reduced to the GERW with certain adjustments (see discussion at the beginning of Section 1.2). However, the analysis of this simple case, in particular the understanding of how $p$ will affect the results, turns out to be important for the time dependent case.
2) POLYNOMIAL DECAY: $p_n = \mathcal{C}n^{-\beta} \wedge 1$ with $\beta > 0$ and $\mathcal{C}$ is a positive constant.

Our models are well motivated since for the many-dimensional excited random walk a relevant question is if it is still possible to guarantee properties such as directional transience and ballisticity by reducing the number of cookies in the system. This goes in the same direction of well-known results in dimension one. Specifically, in dimension one for the nearest neighbor ERW with an independent cookie environment, the mean number of cookies per site should be greater than one for the system to be ballistic, see [11].

As regards to the $p$-GERW, under the same hypothesis of [15][1], we show that the $p$-GERW is ballistic for all $p \in (0,1]$ (see, Theorem 1.1) and under the same stronger assumptions, that is, the increments of the regeneration times be i.i.d., we obtain a Law of Large Numbers and a Central Limit Theorem (see, Theorem 1.2 and Theorem 1.3, respectively).

As regards to the $p_n$-GERW (with $p_n = \mathcal{C}n^{-\beta} \wedge 1$), the results we obtain depend on the value of $\beta$ and on the dimension $d$:

- For $\beta < 1/6$ and $d \geq 2$, we show that the $p_n$-GERW has a positive probability of never returning to the origin in the direction $\ell$ (see, Theorem 1.7).
- For $\beta > 1/2$ and $d \geq 2$ we obtain that, under certain conditions, the $p_n$-GERW suitably rescaled converges in distribution to a Gaussian Process (see, Theorem 1.4). Note that, for $\beta > 1$ the $p_n$-GERW only

[1]Bounded jumps, an uniformly elliptic condition and a drift condition in an arbitrary direction of the unit sphere.

eats finitely many cookies almost surely and therefore the process eventually behaves as a $d$-dimensional martingale; in this case the convergence in distribution follows by Theorem 7.1.4 in [7].

- For $\beta = 1/2$, the dimension $d$ makes a difference:
  - For $d = 2$ the $p_n$-ERW, which is a specific type of $p_n$-GERW (see, Section 1.1.2), converges in distribution (under a suitable rescale) to a Brownian Motion (see, Theorem 1.5).
  - For $d \geq 4$, we obtain that the $p_n$-ERW suitably rescaled is a tight process and every limit point is stochastically dominated from above and below in direction $\ell$ by a Brownian Motion plus a continuous function in $[0, \infty)$ (see, Theorem 1.6).

Let us stress that there are two cases which are not listed above, ($d = 3, \beta = 1/2$) and ($d \geq 2, \beta \in [1/6, 1/2)$). For the first case we believe the behaviour of the $p_n$-ERW would be equivalent to what we have for $d \geq 4$ and $\beta = 1/2$. However, due to some technical difficulties, we were not able to prove it (see discussion at the end of Section 3.3). For the case $d \geq 2$ and $\beta \in [1/6, 1/2)$ we believe the same result for the $p_n$-GERW in $d \geq 2$ and $\beta < 1/6$ holds, i.e., the $p_n$-GERW has a positive probability to never return to the origin in the $\ell$ direction. However we were not able to prove it (see discussion at the Section 3.5 in Remark 3.3).

1.1. **The model.** We now formally introduce the $p_n$-GERW. Recall that $d \geq 2$ is the fixed dimension and let $\{p_n\}_{n\geq 1}$ be a sequence of parameters with $p_n \in (0, 1]$ for all $n \geq 1$. In a broader sense our process is a random element $(X, \pi)$ of $(\mathbb{Z}^d)^{\mathbb{Z}_+} \times [0, 1]^{\mathbb{Z}^d}$ endowed with the product Borel $\sigma$-algebra. The second coordinate $\pi = \{\pi(x)\}_{x\in\mathbb{Z}^d} \in [0, 1]^{\mathbb{Z}^d}$ is a random element whose marginals have uniform distribution in $[0, 1]$ and independents. We denote by $Q$ the probability law of $\pi$. The first coordinate $X = \{X_n\}_{n\geq 0}$ is a $\mathbb{Z}^d$ valued process with $X_0 = 0$ which is adapted to a filtration $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$, where $\mathcal{F}_n = \sigma(X_1, \ldots, X_n, \pi(X_1), \ldots, \pi(X_n))$ and $\sigma(Y)$ represents the smallest $\sigma$-algebra generated by a random vector $Y$. We denote the law of $(X, \pi)$ by $\mathbb{P}$ and by $\mathbb{E}$ its expectation, we can think of $\mathbb{P}$ as the semi-direct measure $Q \otimes P_{\hat{\pi}}$, where $P_{\hat{\pi}}$ is the quenched measure for $X$, i.e., the conditional probability law of $X$ given a realization $\hat{\pi}$ of $\pi$. Now fix $\ell \in \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ is the unit sphere of $\mathbb{R}^d$, and let $||\cdot||$ be the euclidean norm in $\mathbb{R}^d$. The process $X$ is called a $p_n$-GERW in direction $\ell$, if it satisfies the following conditions:

**Condition I** (Bounded increments)**.** *There exists a constant $K > 0$ such that $\sup_{n\geq 0} ||X_{n+1} - X_n|| < K$ on every realization.*

**Condition II.** *There exists $\lambda > 0$ such that:*

- *almost surely on the event $\{X_k \neq X_n$ for all $k < n\}$, either*

$$\mathbb{E}[X_{n+1} - X_n|\mathcal{F}_n] \cdot \ell \geq \lambda\ , \ \text{if } \pi(X_n) \leq p_n\ ,$$

*or*

$$\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = 0 \ , \ \text{if } \pi(X_n) > p_n \, .$$

- *almost surely on the event* $\{\exists\, k < n \text{ such that } X_k = X_n\}$,

$$\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = 0 \, .$$

**Condition III.** *There exist* $h, r > 0$ *such that*

- Uniformly elliptic in direction $\ell$: *for all* $n$

$$\mathbb{P}\left[(X_{n+1} - X_n) \cdot \ell > r | \mathcal{F}_n\right] \geq h \ , \ a.s.. \tag{UE1}$$

- Uniformly elliptic on the event $\{\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = 0\}$: *on the event* $\{\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = 0\}$, *for all* $\ell' \in \mathbb{S}^{d-1}$, *with* $||\ell'|| = 1$

$$\mathbb{P}\left[(X_{n+1} - X_n) \cdot \ell' > r | \mathcal{F}_n\right] \geq h \ , \ a.s.. \tag{UE2}$$

We now present a helpful way to write the $p_n$-GERW, especially in the proofs. Let $\{X_n\}_{n \geq 0}$ be a $p_n$-GERW in direction $\ell$. Increasing the probability space the following representation for the process holds: Set $\{U_i\}_{i \geq 1}$ as a sequence of i.i.d. random variables with uniform distribution in $[0, 1]$ [2]. For $i \geq 1$ we define $E_i$ as the event that the $p_n$-GERW is, at time $i$, in an already visited site, i.e., $E_i := \{\exists \ k < i \ \text{ such that } \ X_k = X_i\}$ and $E_0 := \emptyset$. We can write $\{X_n\}_{n \geq 0}$ as

$$\begin{aligned} X_n &= \sum_{i=1}^{n} (X_i - X_{i-1}) \\ &= \sum_{i=1}^{n} \left( 1_{\{E_{i-1}\}} \xi_i + 1_{\{E^c_{i-1}\}} 1_{\{U_i > p_i\}} \xi_i + 1_{\{E^c_{i-1}\}} 1_{\{U_i \leq p_i\}} \gamma_i \right) , \end{aligned} \tag{1}$$

where $\{\xi_i, \mathcal{F}_i\}_{i \geq 1}$ is an increment of a $d$-martingale with zero mean and $\{\gamma_i, \mathcal{F}_i\}_{i \geq 1}$ is a random vector such that $\mathbb{E}[\gamma_i \cdot \ell | \mathcal{F}_{i-1}] \geq \lambda$ for all $i \geq 1$. Note that due to Condition I (bounded jumps), we also have that $||\xi_i|| < K$ as well as $||\gamma_i|| < K$.

1.1.1. *A slightly different version of the* $p_n$-GERW *(*$p_n$-GERW**).* Benefiting from (1), we now introduce a new condition, which allows us to relax the bounded jump assumption of Condition I. However, in the proof of Theorem 1.4 we will also need to assume an asymptotic condition on the increments $\{\xi_i\}_{i \geq 1}$ to assure the convergence to a non-degenerate continuous covariance matrix (see, (2)). For that let us denote $C = ((c_{i,j}))$ a continuous, $d \times d$ matrix-valued function, defined in $[0, \infty)$, satisfying $C(0) = 0$ and

$$\sum_{i,j=1}^{d} (c_{i,j}(t) - c_{i,j}(s)) \alpha_i \alpha_j \geq 0 \quad \text{for any } \alpha \in \mathbb{R}^d, \quad t > s \geq 0 \, .$$

[2]Note that this sequence is part of the marginals of $\pi$. To avoid clutter, we set this i.i.d. uniform distribution as $\{U_i\}_{i \geq 1}$ from now on.

The new condition is:

**Condition I*.**

i) *For all $k \geq 1$ and $\theta < \beta - 1/2$, where $\beta > 1/2$, we have*

$$\sup_{k\geq 1} \frac{\mathbb{E}[\|\gamma_k\|]}{k^\theta} < \infty \quad \textit{and} \quad \sup_{k\geq 1} \frac{\mathbb{E}[\|\xi_k\|]}{k^\theta} < \infty \,.$$

ii) *It holds that*

$$\frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} \xi_i \xi_i^T \to C(t) \quad \textit{as } n \to \infty \,, \tag{2}$$

*in probability and*

$$\lim_{k\to\infty} k^{-1/2}\mathbb{E}\left[\sup_{1\leq i\leq k} \|\xi_i\|\right] = 0\,.$$

If a process $X$ satisfies Condition I*, II and III, and the sequence $\{U_i\}_{i\geq 1}$ (in the corresponding representation as (1)) is uncorrelated with both sequences $\{\gamma_i\}_{i\geq 1}$ and $\{\xi_i\}_{i\geq 1}$, we call $X$ as the *$p_n$-generalized excited random walk** ($p_n$-GERW*).

1.1.2. *A specific case of* $p_n$-GERW *(*$p_n$-ERW*).* Considering the representation in (1) for a $p_n$-GERW, if we further assume that the sequence $\{\xi_i\}_{i\geq 1}$ is i.i.d. with zero mean vector and finite variance and the sequence $\{\gamma_i\}_{i\geq 1}$ is also i.i.d. with finite variance (recall that $\gamma_i$ satisfies $\mathbb{E}[\gamma_i \cdot \ell | \mathcal{F}_{i-1}] \geq \lambda$), then we obtain a specific type of the $p_n$-GERW, which we call $p_n$-ERW in the direction $\ell$. In the Example below, we provide a concrete example of a $p_n$-ERW.

**Example:** Here we provide a concrete example of a $p_n$-GERW which may be thought of as a generalization of the classical ERW. Specifically it evolves as the classical ERW but when a site is visited for the first time, it finds a cookie (thus gaining a drift) with probability $p_n$. This generalization reduces to the classical ERW when $p_n = 1$ for all $n \geq 1$. Fix $\delta \in (1/2, 1]$ and let $q^{(0)}(x, e_i)$, $x \in \mathbb{Z}^d$, $i = 1, ..., d$, be defined as

$$q^{(0)}(x, e_1) = \delta/d, \;\; q^{(0)}(x, -e_1) = (1-\delta)/d \,,$$
$$q^{(0)}(x, \pm e_i) = 1/2d \quad \text{for all } i = 2, \dots, d \,,$$

and $q^{(1)}(x, e_i)$, $x \in \mathbb{Z}^d$, $i = 1, ..., d$, be the transition probabilities of a SRW in $\mathbb{Z}^d$. Let $\{X_n\}_{n\geq 0}$ be a process in $\mathbb{Z}^d$ with transition probabilities

$$P\Big[X_{n+1} = x + e_i \Big| X_n = x, \sum_{j=0}^{n-1} 1_{\{X_j = x\}} = 0\Big] =$$
$$= 1_{\{\pi(x)\leq p_n\}} q^{(0)}(x, e_i) + 1_{\{\pi(x) > p_n\}} q^{(1)}(x, e_i) \,,$$

and for every $m \in \{1, 2, \dots, n-1\}$ we have

$$P\Big[X_{n+1} = x + e_i \Big| X_n = x, \sum_{j=0}^{n-1} 1_{\{X_j = x\}} = m\Big] = q^{(1)}(x, e_i) \ .$$

The $\{X_n\}_{n\geq 0}$ is clearly a $p_n$-ERW when $p_n = p$ for all $n \geq 1$. See Figure 1.1 for a simulation of this $p$-ERW in $\mathbb{Z}^2$. The simulation suggests that the $p$-ERW is ballistic in direction $e_1$. Indeed, we will prove that the $p$-ERW satisfies a ballistic Law of Large Numbers and a Central Limit Theorem.

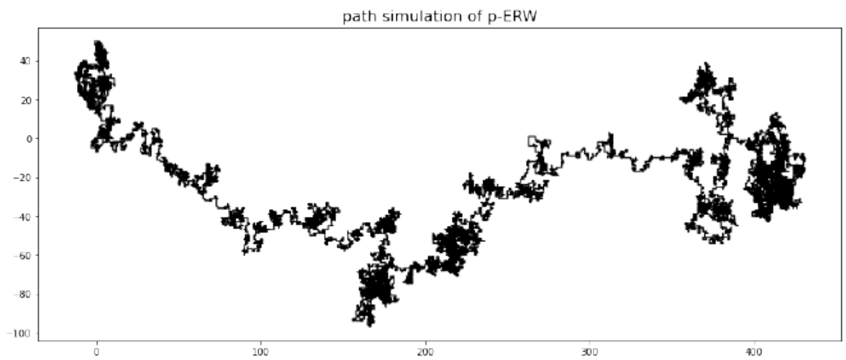


FIGURE 1.1. 20000 steps simulation of $p$-ERW for $d = 2$, $p = 0.25$, $q^{(0)}(x, e_1) = 0.375$, $q^{(0)}(x, -e_1) = 0.125$ and $q^{(0)}(x, \pm e_2) = 0.25$. The initial position of the random walk is $X_0 = (0, 0)$ and the final $X_{20000} = (397, -20)$.

1.2. **Main results for the $p$-GERW.** As mentioned in the Introduction the homogeneous case can be reduced to the GERW. Specifically, the $p$-GERW with a given $\lambda$ in Condition II reduces to a GERW with $p\lambda$ in Condition $C^+$ in [15]. As a matter of fact, if we denote by $\widetilde{\mathcal{F}}_n = \sigma(X_1, \dots, X_n)$, by integrating $\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] \cdot \ell$ with respect to $\pi(X_1), \dots, \pi(X_n)$ we obtain $\mathbb{E}[X_{n+1} - X_n | \widetilde{\mathcal{F}}_n] \cdot \ell \geq p\lambda$, which is Condition $C^+$ in [15] with a $\lambda' = p\lambda$. Despite the close connection with GERW, some of the techniques developed to prove results for the $p$-GERW will be useful for the time dependent case.

Our first result is that for every $p \in (0, 1]$, the $p$-GERW is ballistic in $\ell$ direction.

**Theorem 1.1** (Ballisticity of $p$-GERW)**.** *Let $X$ be a $p$-GERW in direction $\ell \in \mathbb{S}^{d-1}$. Then*

$$\liminf_{n\to\infty} \frac{X_n \cdot \ell}{n} > 0 \ , \quad a.s..$$

The next two results are the Law of Large Numbers and the Central Limit Theorem which hold for a special case of $p$-GERW. Specifically, we need to introduce a fourth condition (see, Condition IV in Section 2) which is related to the distribution of the increments of the regeneration times associated to the $p$-GERW. A $p$-GERW satisfying this fourth condition will be called *$p$-Strong General Excited Random Walk* ($p$-SGERW). It can be shown (see, Corollary 2.3) that the $p$-ERW introduced in the Example in Section 1.1 is an example of $p$-SGERW.

**Theorem 1.2** (Law of Large Numbers)**.** *Assume the process $X$ is a $p$-*SGERW *in direction $\ell$ (i.e., satisfies Conditions I,II, III and IV), then there exists $v \in \mathbb{R}^d$ such that $v \cdot \ell > 0$ and*

$$\lim_{n\to\infty} \frac{X_n}{n} = v \;, \quad a.s.. \tag{3}$$

Let $X$ be a $p$-SGERW in direction $\ell$ and $v \in \mathbb{R}^d$ from (3). Let us define the process

$$B_t^n = \frac{X_{\lfloor nt \rfloor} - \lfloor nt \rfloor v}{n^{1/2}} \;, \; t \geq 0 \;. \tag{4}$$

**Theorem 1.3** (Central Limit Theorem)**.** *The process $B_\cdot^n$ converges in distribution, as $n \to \infty$, to a $d$-dimensional Brownian Motion with a non-degenerate covariance matrix.*

1.3. **Main results for the $p_n$-GERW.** Let us recall that we shall focus on the case in which the sequence $\{p_n\}_{n\geq 1}$ is of the form $p_n = \mathcal{C}n^{-\beta} \wedge 1$, and $\mathcal{C}$ is a positive constant.

Before we state the main results for the $p_n$-GERW, let us introduce some notation. We define the following process

$$\hat{B}_t^n = \frac{X_{\lfloor nt \rfloor}}{n^{1/2}} + (nt - \lfloor nt \rfloor)\frac{(X_{\lfloor nt \rfloor+1} - X_{\lfloor nt \rfloor})}{n^{1/2}} \;, \tag{5}$$

where the process $X$ will make reference to a specified process in each Theorem (e.g., $p_n$-GERW*).

We set $C_{\mathbb{R}^d}[0,T]$ as the space of continuous functions $f : [0,T] \to \mathbb{R}^d$, with $d \geq 1$, for all $T > 0$ and the metric we use is the uniform. The space of continuous functions on $f : [0,\infty) \to \mathbb{R}^d$, it will be denoted by $C_{\mathbb{R}^d}[0,\infty)$ and the metric we use here is

$$\rho(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \sup_{0\leq t\leq k} (||f(t) - g(t)|| \wedge 1) \;,$$

where $f$ and $g$ are continuous functions with domain $[0,\infty)$ and range $\mathbb{R}^d$.

The first result we obtain is a Central Limit Theorem for the $p_n$-GERW* (see, Section 1.1.1) when $\beta > 1/2$ and $d \geq 2$.

**Theorem 1.4.** *Let the process $X$ be a $p_n$-*GERW* *in direction $\ell$, in $\mathbb{Z}^d$, with $d \geq 2$, $p_n = \mathcal{C}n^{-\beta} \wedge 1$, with $\beta > 1/2$. Then $\hat{B}_\cdot^n$ converges in distribution to a process with independent Gaussian increments with sample paths in $C_{\mathbb{R}^d}[0,\infty)$.*

**Remark 1.1.** *Let us point out that the* $p_n$-GERW* *when* $\beta > 1$ *eventually behaves as a d-dimensional martingale and in this latter case our theorem (essentially) reduces to Theorem 7.1.4 in [7].*

**Remark 1.2.** *Let us point out that Theorem 1.4 holds true under slightly weaker conditions. Specifically item i) of Condition I* and the sequence* $\{p_n\}_{n\geq 1}$ *can be more general. As it emerges from the proof, for the statement of Theorem 1.4 be true we only need that* $\sum_{i=1}^{\lfloor nt \rfloor} p_i \mathbb{E}[||\gamma_i||] = o(\sqrt{n})$ *(for more details, see Remark 3.1 ).*

From Theorem 1.4 we obtain the following Corollary holding in the specific case of $p_n$-ERW.

**Corollary 1.1.** *Let the process* $X$ *be a* $p_n$-ERW *in direction* $\ell$ *in* $\mathbb{Z}^d$, *with* $d \geq 2$, $p_n = Cn^{-\beta} \wedge 1$, *with* $\beta > 1/2$. *Then* $\hat{B}^n_\cdot$ *converges in distribution to a d-dimensional Brownian Motion in* $C_{\mathbb{R}^d}[0, \infty)$.

We set now $\beta = 1/2$. Here we will have different results depending on the dimension of the process. First we will present the Central Limit Theorem in $d = 2$ for the $p_n$-ERW.

**Theorem 1.5.** *Let the process* $X$ *be a* $p_n$-ERW *in direction* $\ell$, *in* $\mathbb{Z}^d$ *with* $d = 2$, $p_n = Cn^{-1/2} \wedge 1$. *Then* $\hat{B}^n_\cdot$ *converges in distribution to a* 2*-dimensional Brownian Motion in* $C_{\mathbb{R}^2}[0, \infty)$.

**Remark 1.3.** *Note that in Corollary 1.1 and Theorem 1.5 the* $p_n$-ERW *under a suitable rescaling converges in distribution to a Brownian Motion and has no ballisticity, differently from what happens in the* ERW *(see [3], [12] and [13]).*

We state now our result for the $p_n$-ERW in higher dimensions. Here we obtain that the $p_n$-ERW suitably rescaled is a tight process and every limit point is stochastically dominated (henceforth denoted by $\preceq$) in the direction $\ell$ from above and below by a Brownian Motion plus a continuous function in $[0, \infty)$.

Let us define the set $D_k \subset \{e_1, \ldots, e_d\}$, where $d \geq 4$ and $k$ is the cardinality of $D_k$ with $1 \leq k \leq d - 3$. Now set $\ell_{D_k}$ as a direction in the unit sphere in dimension $d$, that is, $\ell_{D_k} \in \mathbb{S}^{d-1}$, such that $\ell_{D_k} = \sum_{i=1}^{k} \alpha_i x_i$, where $\alpha_i \in [0, 1]$ and $x_i \in D_k$, both for all $1 \leq i \leq k$. In essence, $\ell_{D_k}$ is a direction in the unit sphere in dimension $d$ that can be determined by the canonical directions of the set $D_k$.

**Theorem 1.6.** *Let the process* $X$ *be a* $p_n$-ERW *in direction* $\ell_{D_k}$, *in* $\mathbb{Z}^d$ *with* $d \geq 4$, $p_n = Cn^{-1/2} \wedge 1$. *Then the sequence of processes* $\hat{B}^n_\cdot$ *is tight in* $C_{\mathbb{R}^d}[0, \infty)$ *and there exists a Brownian Motion* $W_\cdot$ *such that for every limit point* $Y_\cdot$ *from the process* $\hat{B}^n_\cdot$ *it holds that*

$$W_t \cdot \ell_{D_k} + f(t) \preceq Y_t \cdot \ell_{D_k} \preceq W_t \cdot \ell_{D_k} + g(t)\,,$$

*where $f$ and $g$ are continuous function on $[0,\infty)$, such that $f(t) = c_1\sqrt{t}$ and $g(t) = c_2\sqrt{t}$ with $c_2 > c_1 > 0$.*

**Remark 1.4.** *The constants $c_1$ and $c_2$ will be described at the end of the proof of Theorem 1.6.*

Now we propose a conjecture for the $p_n$-ERW in direction $\ell \in \mathbb{S}^{d-1}$, in $\mathbb{Z}^d$, with $d \geq 3$ and $p_n = Cn^{-1/2} \wedge 1$.

**Conjecture 1.1.** *Let the process $X$ be a $p_n$-ERW in direction $\ell \in \mathbb{S}^{d-1}$, in $\mathbb{Z}^d$ with $d \geq 3$, $p_n = Cn^{-1/2} \wedge 1$. Then $\hat{B}^n_\cdot$ converges in distribution as the following*

$$\hat{B}^n_t \cdot \ell \to W_t \cdot \ell + c\sqrt{t} \quad \text{as } n \to \infty\,,$$

*where $W_\cdot$ is Brownian Motion and $c$ is a positive constant.*

The proof of Theorem 1.5 and Theorem 1.6 rely on a control on the range of a $p_n$-ERW, which is stated in the following proposition (whose proof is given in Section 3.4).

Considering the representation in (1) for $X$ a $p_n$-ERW (see, Section 1.1.2), let us denote by $\pi_d$ the probability of a random walk with i.i.d. increments (with zero mean and finite variance) given by the corresponding $\{\xi_i\}_{i\geq 0}$ never returning to the origin.

**Proposition 1.1.** *Let the process $X$ be a $p_n$-ERW in direction $\ell$, in $\mathbb{Z}^d$ with $d \geq 2$, $p_n = Cn^{-1/2} \wedge 1$. Let $\mathcal{R}^X_n$ be the range of the process up to time $n$ (i.e., the number of different sites visited up to time $n$). Then, for all sufficiently large $n$ that we have that*

$$\mathbb{P}\left[|\mathcal{R}^X_n| \leq \delta n\right] = 1\,,$$

*for every $\delta > \pi_d$.*

Below, we propose a conjecture about the range of the $p_n$-ERW in $\mathbb{Z}^d$, in direction $\ell \in \mathbb{S}^{d-1}$, with $p_n = \mathcal{C}n^{-\beta} \wedge 1$, with $\beta \geq 1/2$ and $d \geq 2$.

**Conjecture 1.2.** *Let the process $X$ be a $p_n$-ERW in direction $\ell \in \mathbb{S}^{d-1}$, in $\mathbb{Z}^d$ with $d \geq 2$, $p_n = \mathcal{C}n^{-\beta} \wedge 1$, with $\beta \geq 1/2$. Let $\mathcal{R}^X_n$ be the range of the process up to time $n$. Then we have*

$$\frac{|\mathcal{R}^X_n|}{n} \to \pi_d \quad \text{as } n \to \infty \text{ a.s.}.$$

*Note that for $d = 2$, we have that $\pi_d = 0$, whereas for $d \geq 3$, $\pi_d \in (0, 1]$.*

**Remark 1.5.** *If Conjecture 1.2 holds true, we would be able to extend the result in Theorem 1.6 to $d = 3$ and to any direction in the unit sphere (see discussion in Section 3.3.1).*

We now provide our last result, which states that, for $\beta < 1/6$ and $d \geq 2$ the $p_n$-GERW in direction $\ell$, in $\mathbb{Z}^d$ has a positive probability of never returning to the origin in the $\ell$ direction.

Before stating the theorem, we need some notation. For every $\ell \in \mathbb{S}^{d-1}$, let $\mathbb{M}_\ell$ denote the positive half-space in direction $\ell$, that is, $\mathbb{M}_\ell = \{x \in \mathbb{Z}^d : x \cdot \ell > 0\}$. We define $A$ as the *excitation-allowing set*, which means the set of sites where there is the possibility of having cookies (see Condition II*). We set the event $\{\eta(X_0) = \infty\}$ as the event in which the process $X$ never returns to the origin in the drift direction.

**Theorem 1.7.** *Let $X$ be a $p_n$-GERW in direction $\ell$, in $\mathbb{Z}^d$ with $d \geq 2$, where $p_n = (q_0 + n)^{-\beta}$, with $\beta < 1/6$, $q_0$ is a non negative integer and excitation-allowing set $A \subset \mathbb{Z}^d$ such that $\mathbb{M}_\ell \subset A$. There exists $\psi > 0$ depending on the parameters of the model such that*

$$\mathbb{P}\left[\eta(X_0) = \infty\right] \geq \mathbb{P}\left[X_n \cdot \ell > 0 \text{ for all } n \geq 1\right] \geq \psi\,.$$

Figure 1.2 provides a summary of the main results concerning $p$-GERW and the $p_n$-GERW, with $p_n = \mathcal{C}n^{-\beta} \wedge 1$ for different values of $\beta$ and dimension $d$.

| | |
|---|---|
| $p$-GERW $(d \geq 2,\, p \in (0,1])$ | ballisticity in the drift direction for every $p > 0$. |
| $p$-SGERW $(d \geq 2,\, p \in (0,1])$ | besides ballisticity, LLN and CLT. |
| $p_n$-GERW $(\beta < 1/6,\, d \geq 2)$ | positive probability of never returning to the origin (in the direction $\ell$) |
| $p_n$-GERW* $(\beta > 1/2,\, d \geq 2)$ | convergence in distribution to a Guassian Process |
| $p_n$-ERW $(\beta = 1/2,\, d = 2)$ | convergence in distribution to a Brownian Motion |
| $p_n$-ERW $(\beta = 1/2,\, d \geq 4)$ | all sub-sequences converge, in distribution, to a process which is stochastically dominated in the drift direction below and above by a Brownian Motion plus a continuous function. |

FIGURE 1.2. Summary of the results for GERW.

This text is organized as follows: The renewal structure and the proof of the main results for the $p$-GERW are presented in Section 2. Our main contributions are given in Section 3 where we study the $p_n$-GERW, with $p_n = n^{-\beta}$. We prove several asymptotic results depending on the value of $\beta$ and on the dimension $d$ (see Section 3.1, 3.2, 3.3 and 3.5). In Section 3.4 we provide the proof of Proposition 1.1, in which we analyze the asymptotic behavior of the range of the $p_n$-ERW in $d \geq 2$ and $\beta = 1/2$. Finally, Appendix A, B contain some proofs which were omitted in the main text and Appendix C some auxiliary results.

## 2. Proofs of the main theorems for the $p$-GERW

2.1. **Regeneration Structure.** We start this section defining the regeneration structure for the $p$-GERW which is a key element in the proofs of the theorems stated in Section 1.2. We will follow closely the regeneration structure constructed in [15]. We make small adjustments in the definition of the regeneration times that will not affect the main properties of the structure. Afterwards we prove the Theorems 1.1, 1.2 and 1.3. Our main contribution is to establish the necessary properties on the regeneration times which is a challenging task for the $p$-GERW.

Consider $\{X_n\}_{n\geq 0}$ a $p$-GERW in direction $\ell \in \mathbb{S}^{d-1}$. Fix $a>0$ and define

$$\begin{aligned} \rho(X_m) &:= \inf\{n \geq m : X_n \cdot \ell \geq X_m \cdot \ell + a\} \, , \\ \eta(X_m) &:= \inf\{n \geq m : X_n \cdot \ell < X_m \cdot \ell\} \, . \end{aligned} \tag{6}$$

We now define the sequence of regeneration times $\{\tau_k\}_{k\geq 0}$. First, we set $\tau_0 \equiv 0$ and then we define $\tau_{k+1}$ from $\tau_k$ recursively for $k \geq 0$. If $\tau_k = \infty$, then $\tau_{k+1} = \infty$. Assuming $\tau_k < \infty$, we define

$$\rho_1^{(k)} := \rho(X_{\tau_k}) \, , \qquad \eta_1^{(k)} := \begin{cases} \eta(X_{\rho_1^{(k)}}) \, , & \rho_1^{(k)} < \infty \, , \\ \infty \, , & \rho_1^{(k)} = \infty \, , \end{cases}$$

where $\rho$ and $\eta$ are given in (6). Moreover, for every $i \geq 2$, we recursively define

$$\rho_i^{(k)} := \begin{cases} \inf\{n \geq \eta_{i-1}^{(k)} : X_n \cdot \ell \geq \max_{k \leq \eta_{i-1}^{(k)}} X_k \cdot \ell + a\} \, , & \eta_{i-1}^{(k)} < \infty \, , \\ \infty \, , & \eta_{i-1}^{(k)} = \infty \, , \end{cases}$$

$$\eta_i^{(k)} := \begin{cases} \eta(X_{\rho_i^{(k)}}) \, , & \rho_i^{(k)} < \infty \, , \\ \infty \, , & \rho_i^{(k)} = \infty \, . \end{cases}$$

Setting $q_k := \inf\{n \geq 1 : \rho_n^{(k)} < \infty, \ \eta_n^{(k)} = \infty\}$, we define $\tau_{k+1} := \rho_{q_k}^{(k)}$. The time $\tau_k$, with $k \geq 1$, represents the $k$-th regeneration time. Clearly $\tau_k$ is not a $\mathcal{F}_n$-stopping time for every $k \geq 1$ and it depends on the whole future of $\{(X_n, \pi(X_n))\}_{n\geq 1}$. For a better understanding of the sequence of the regeneration times see Figure 2.1.

**Remark 2.1.** *The particular choice of $a$ is irrelevant in this manuscript. Without harm to the proofs presented here we could have chosen $a=1$. The choice of $a$ is only used to show the non-degeneracy of the covariance matrix in Theorem 1.3, the proof is not presented here since it follows from the same arguments as in Theorem 4.1 of [18].*

To deal with the information produced by $\{(X_n, \pi(X_n))\}_{n\geq 1}$ until time $\rho_i^{(k)}$ for each $i \geq 0$ and $k \geq 1$, we also define the $\sigma$-algebras: $\mathcal{G}_0^{(0)} := \mathcal{F}_0$

$$\mathcal{G}_0^{(k)} = \sigma\big(\tau_1, \dots, \tau_k, (X_{n\wedge\tau_k})_{n\geq 0}, (\pi(X_{n\wedge\tau_k}))_{n\geq 0}\big) \text{ for } k \geq 1 \, ,$$

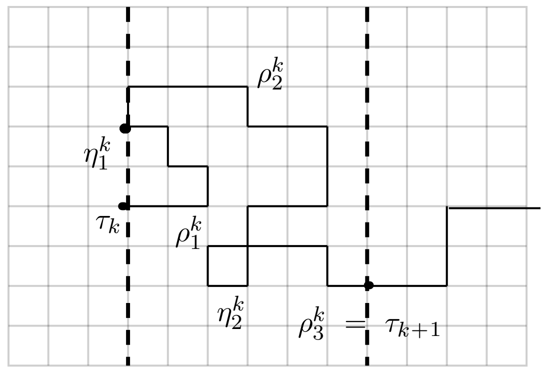


FIGURE 2.1. A representation of the regeneration times for a $p$-ERW in $\mathbb{Z}^2$ with drift direction $e_1$ and $a = 1$. The dotted horizontal lines represents that the RW never goes below that position in direction $e_1$.

and for all $k \geq 1$ and $i \geq 1$

$$\mathcal{G}_i^{(k)} = \sigma\big(\tau_1, \dots, \tau_k, (X_{n\wedge \rho_i^k})_{n\geq 0}, (\pi(X_{n\wedge\rho_i^k}))_{n\geq 0}\big) \text{ for } k \geq 1,\ i \geq 1\ .$$

The next result is an estimation on the tail probabilities for the increments of the regeneration times which will guarantee existence of moments needed in the proof of Theorem 1.1.

**Proposition 2.1.** *Consider a $p$-GERW and $\{\tau_k\}_{k\geq 0}$ its sequence of associated regeneration times. Then there exist positive constants $C'$ and $\phi$ such that for every $n \geq 1$*

$$\sup_{k\geq 0} \mathbb{P}\big[\tau_{k+1} - \tau_k > n \big| \mathcal{G}_0^{(k)}\big] \leq C' e^{-n^{\phi}}\ , \quad a.s..$$

Proposition 2.1 will be proved in Section 2.3 (with some details provided in Appendix A). As a consequence we have the two next results:

**Corollary 2.1.** *For every $k \geq 1$ we have that $\tau_k < \infty$ a.s..*

**Corollary 2.2.** *For every $k \geq 0$ and $m \geq 1$, we have that,*

$$\mathbb{E}[\tau_{k+1}^m | \mathcal{G}_0^k] < \infty\,.$$

Below, we present a helpful result, which is a version of Proposition 2.2 in [15]. It is worth mentioning that we work on larger spaces, since in our model we have the cookie environment. However the proof follows the same steps as that in [15] and it is deferred to Appendix A. Recall that $(\mathbb{Z}^d)^{\mathbb{Z}_+}$ is the space of trajectories for $X$.

**Proposition 2.2.** *Suppose $X = \{X_n\}_{n\geq 0}$ is a $p$-GERW in direction $\ell$ and $\{\tau_k\}_{k\geq 1}$ its sequence of associated regeneration times. Let $A$ be a Borel subset of $(\mathbb{Z}^d)^{\mathbb{Z}_+}$, then we have:*

*(i) For every* $k \geq 1$*,*

$$\mathbb{P}\left[X_{\tau_k+\cdot} \in A | \mathcal{G}_0^{(k)}\right] = \sum_{n=1}^{\infty} 1_{\{\tau_k=n\}} \mathbb{P}\left[X_{n+\cdot} \in A | \eta(X_n) = \infty, \mathcal{F}_n\right] \text{, a.s..}$$

*(ii) For every* $k, j \geq 1$*,*

$$\mathbb{P}\left[X_{\rho_j^{(k)}+\cdot} \in A | \mathcal{G}_j^{(k)}\right] =$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 1_{\{\tau_k=n\}} 1_{\{\rho_j^{(k)}=n+m\}} \mathbb{P}\left[X_{n+m+\cdot} \in A | \eta(X_n) = \infty, \mathcal{F}_{n+m}\right] \text{, a.s..}$$

*Where* $X_{k+\cdot}$ *represents the trajectory of the process from* $k$ *onwards.*

Finally we have all the elements to formally introduce Condition IV and define the $p$-SGERW.

**Condition IV.** *Let* $\{\tau_k\}_{k\geq 0}$ *be the associated sequence of regeneration times of the* $p$-GERW*. If under* $\mathbb{P}$ *we have:*

*(i) the increments* $\tau_{k+1} - \tau_k$*,* $k \geq 0$*, are independent, and for* $k \geq 1$ *they are also identically distributed as* $\tau_1 | \eta(X_0) = \infty$*.*

*(ii) the random variables* $X_{\tau_1}$*,* $X_{\tau_{k+1}} - X_{\tau_k}$*,* $k \geq 1$*, are independent, and* $X_{\tau_{k+1}} - X_{\tau_k}$*,* $k \geq 1$*, are identically distributed as* $X_{\tau_1} | \eta(X_0) = \infty$*.*

As mentioned in the Introduction, a $p$-GERW is called a $p$-SGERW if it also satisfies Condition IV.

In the next corollary we show that the $p$-ERW, introduced in the Example in Section 1.1, is an example of $p$-SGERW, that is, it satisfies Condition IV.

**Corollary 2.3.** *The* $p$-ESRW *satisfies Condition IV; hence, it is a* $p$-SGERW*.*

*Proof.* Let $\{X_n\}_{n\geq 0}$ be a $p$-ESRW and $\{\tau_k\}_{k\geq 1}$ be the associated sequence of regeneration times. Recall that $\tau_k < \infty$ a.s. and let $A$ be a Borel subset and $\theta$ the canonical shift on $(\mathbb{Z}^d)^{\mathbb{Z}_+}$, then for all $k \geq 1$

$$\begin{aligned}
\mathbb{P}[X_{\tau_k+\cdot} - X_{\tau_k} \in A | \mathcal{G}_0^{(k)}] &= \mathbb{P}[X_{\tau_k+\cdot} \in A \circ \theta_{\tau_k} | \mathcal{G}_0^{(k)}] \\
&= \sum_{n=1}^{\infty} 1_{\{\tau_k=n\}} \mathbb{P}[X_{n+\cdot} \in A \circ \theta_n | \eta(X_n) = \infty, \mathcal{F}_n] \qquad (7) \\
&= \mathbb{P}[X_\cdot \in A | \eta(X_0) = \infty] \,,
\end{aligned}$$

where we used Proposition 2.2 part $(i)$ in the second equality. The last equality instead is due to the fact that the process has a totally new area of sites with independent cookie configurations to explore, since we have $\{\eta(X_n) = \infty\}$, then the history of it does not matter. Besides that the $p$-ESRW has homogeneous transition probabilities that together with independence implies that $X_{n+\cdot} | \{\tau_k = n\}$ has the same distribution as $X_\cdot | \{\eta(X_0) = \infty\}$, since from direction $e_1$ both processes evolve on identically distributed environments. From the previous equality (7), we have that the $p$-ERW satisfies Condition IV. □

As a consequence of Proposition 2.1 and Condition IV we have the important result for the $p$-SGERW and its associated sequence of regeneration times.

**Corollary 2.4.** *Let $X$ be a $p$-SGERW with associated sequence of regeneration times $\{\tau_k\}_{k\geq 0}$. Then we have that*

*(i)* $\mathbb{E}[\tau_k^m] < \infty$ *and* $\mathbb{E}[\tau_k^m | \eta(X_0) = \infty] < \infty$, *for all* $m \geq 1$;
*(ii)* $\mathbb{E}[X_{\tau_k}^m] < \infty$ *and* $\mathbb{E}[X_{\tau_k}^m | \eta(X_0) = \infty] < \infty$, *for all* $m \geq 1$.

2.2. **On the proofs of the main theorems for the $p$-GERW.** Here we just outline the proofs of our main theorems. As mentioned above, the proofs are analogous to those presented in [15] and they follow from the results on the regeneration times presented in Section 2.1. We point out that the main contribution of this section is the proof of Proposition 2.1, which makes the regeneration structure work for the $p$-GERW.

The first step in the proof of Theorem 1.1 is to show the following result.

**Lemma 2.1.** *Consider $X$ a $p$-GERW in direction $\ell$ in $\mathbb{Z}^d$, where $d \geq 2$ and with an associated sequence of regeneration times $\{\tau_n\}_{n\geq 1}$. Then there exists a constant $C > 0$ such that,*

$$\limsup_{n\to\infty} \frac{\tau_n}{n} < C\,, \quad a.s..$$

The proof is the same as that of Lemma 3.1 in [15] and it will be deferred to the end of this section. All we need is finite fourth moment of $\tau_k$ and $X_{\tau_k}$, for $k \geq 1$, which we have by Corollary 2.2. Afterwards the proof of Theorem 1.1 is also the same as that of Theorem 1.1 in [15].

Concerning the proof of Theorem 1.2 for the $p$-SGERW, Condition IV allows us to follow closely the proof of Proposition 2.1 in [19], which is a Law of Large Numbers for random walks in random environment. There the proof is for nearest-neighbor jumps, but it is simple to adjust it for the case of uniformly bounded jumps. From that proof we obtain

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{\mathbb{E}[X_{\tau_1} | \eta(X_0) = \infty]}{\mathbb{E}[\tau_1 | \eta(X_0) = \infty]} = v\,, \quad a.s..$$

At last, to show Theorem 1.3 we can follow closely the proof of Theorem 4.1 in [18] defining the covariance matrix $A$ as

$$A = \frac{\mathbb{E}[(X_{\tau_1} - \tau_1 v)^t (X_{\tau_1} - \tau_1 v) | \eta(X_0) = \infty]}{\mathbb{E}[\tau_1 | \eta(X_0) = \infty]}\,.$$

Here, as in [18], we have to choose $a > 2\sqrt{d}$ to show that the matrix $A$ is non-degenerate.

2.2.1. *Proof of ballisticity of p-GERW.* We start showing Lemma 2.1 which will be an important tool to prove Theorem 1.1.

*Proof of Lemma 2.1.* Let $Z = \sup_{k\geq 1} \mathbb{E}[\tau_{k+1} - \tau_k | \mathcal{G}_0^{(k)}]$ and by Proposition 2.1 we have $Z < \infty$. Consider the process

$$M_n = \sum_{k=0}^{n-1} \left( \tau_{k+1} - \tau_k - \mathbb{E}[\tau_{k+1} - \tau_k | \mathcal{G}_0^{(k)}] \right) ,$$

for $n \geq 1$. Since $M_n$ is integrable and adapted to $\mathcal{G}_0^{(n)}$, $\{M_n, \mathcal{G}_0^{(n)}\}_{n\geq 1}$ is a martingale. For a positive constant $C$ such that $C > Z$, we have that

$$\begin{aligned}
\mathbb{P}[\tau_n > nC] &= \mathbb{P}\left[ \tau_n - \sum_{k=0}^{n-1} \mathbb{E}[\tau_{k+1} - \tau_k | \mathcal{G}_0^{(k)}] > nC - \sum_{k=0}^{n-1} \mathbb{E}[\tau_{k+1} - \tau_k | \mathcal{G}_0^{(k)}] \right] \\
&\leq \mathbb{P}[M_n > n(C - Z)] \\
&\leq \frac{\mathbb{E}[M_n^4]}{n^4 (C-Z)^4} \, .
\end{aligned} \tag{8}$$

By Corollary 2.4 we have $\mathbb{E}[M_n^4] < \infty$ for all $n$. Let $Y_k = \tau_{k+1} - \tau_k - \mathbb{E}[\tau_{k+1} - \tau_k | \mathcal{G}_0^{(k)}]$ then we obtain $\mathbb{E}[Y_k] = 0$. Now we will find a upper bound for $\mathbb{E}[M_n^4]$.

$$\begin{aligned}
\mathbb{E}[M_n^4] = \mathbb{E}\left[ \left( \sum_{k=0}^{n-1} Y_k \right)^4 \right] &= \mathbb{E}\left[ \sum_{0\leq i,j,k,l \leq n-1} Y_i Y_j Y_k Y_l \right] \\
&\leq n \left( \sup_{0\leq i\leq n-1} \mathbb{E}[Y_i^4] \right) + 3(n^2 - n) \left( \sup_{0\leq i \leq n-1} \mathbb{E}[Y_i^2] \times \sup_{0\leq j\leq n-1} \mathbb{E}[Y_j^2] \right) \\
&\leq C_1 n^2 \, .
\end{aligned} \tag{9}$$

The first inequality in (9) holds because the only two terms which do not vanish are those of the form $\mathbb{E}[Y_i^4]$ and $\mathbb{E}[Y_i^2 Y_j^2]$. The last inequality follows from Proposition 2.1 together with the fact that $C_1$ is a positive constant. Thus by (8) and (9) we have

$$\mathbb{P}[\tau_n > nC] \leq \frac{C_2}{n^2}$$

$$\sum_{n=1}^{\infty} \mathbb{P}[\tau_n > nC] \leq \sum_{n=1}^{\infty} \frac{C_2}{n^2} < \infty \, .$$

Then by Borell-Cantelli

$$\limsup_{n\to\infty} \frac{\tau_n}{n} < C \text{ a.s.}.$$

Hence we finish the proof.

□

Now we are ready to prove the ballisticity for the $p$-GERW.

*Proof of Theorem 1.1.* By definition of the regeneration times

$$\liminf_{n\to\infty} \frac{X_{\tau_n}\cdot \ell}{n} \geq 1 \quad \text{a.s..} \tag{10}$$

Indeed, it is enough to write

$$\begin{aligned} X_{\tau_k}\cdot\ell &= \sum_{j=0}^{k-1}(X_{\tau_{j+1}}\cdot\ell - X_{\tau_j}\cdot\ell) \\ &= \underbrace{(X_{\tau_1}\cdot\ell - X_{\tau_0}\cdot\ell)}_{\geq 1} + \cdots + \underbrace{(X_{\tau_k}\cdot\ell - X_{\tau_{k-1}}\cdot\ell)}_{\geq 1} . \end{aligned}$$

Then we have for all $k\geq 1$

$$\frac{X_{\tau_k}\cdot\ell}{k} \geq 1 .$$

Now suppose we have a sequence $\{k_n\}_{n\geq 0}$ nondecreasing and tending to infinity such that $\tau_{k_n} \leq n < \tau_{k_n+1}$. By the definition of the regeneration times we have $X_n\cdot\ell \geq X_{\tau_{k_n}}\cdot\ell$. Therefore,

$$\begin{aligned} \liminf_{n\to\infty}\frac{X_n\cdot\ell}{n} &\geq \liminf_{n\to\infty}\frac{X_{\tau_{k_n}}\cdot\ell}{n} \geq \liminf_{n\to\infty}\frac{k_n}{n}\frac{X_{\tau_{k_n}}\cdot\ell}{k_n} \\ &\geq \liminf_{n\to\infty}\frac{k_n}{\tau_{k_n+1}}\frac{X_{\tau_{k_n}}\cdot\ell}{k_n} \geq \frac{1}{C} . \end{aligned} \tag{11}$$

To obtain the last inequality in (11) we used Lemma 2.1 and (10). Thus we finish the proof. ☐

2.2.2. *Proof of LLN for the* $p$-SGERW. As a consequence of Condition IV and the Strong Law of Large Numbers we obtain

$$\begin{aligned} \lim_{k\to\infty}\frac{\tau_k}{k} &= \mathbb{E}[\tau_1|\eta(X_0)=\infty] \quad \text{a.s. and} \\ \lim_{k\to\infty}\frac{X_{\tau_k}}{k} &= \mathbb{E}[X_{\tau_1}|\eta(X_0)=\infty] \quad \text{a.s..} \end{aligned} \tag{12}$$

Now suppose we have a nondecreasing sequence $\{k_n\}_{n\geq 0}$ and tending to infinity such that $\tau_{k_n}\leq n<\tau_{k_n+1}$. Thus we can do

$$\frac{\tau_{k_n}}{k_n} \leq \frac{n}{k_n} < \frac{\tau_{k_n+1}}{k_n} ,$$

and by (12) we have

$$\lim_{n\to\infty}\frac{k_n}{n} = \frac{1}{\mathbb{E}[\tau_1|\eta(X_0)=\infty]} . \tag{13}$$

One can see that

$$\frac{X_n}{n} = \frac{X_{\tau_{k_n}}}{n} + \frac{X_n - X_{\tau_{k_n}}}{n} . \tag{14}$$

Thus we will analyze the two sum portions limits in (14) and then we will have the desired result. Then, first we have by (12) and (13)

$$\frac{X_{\tau_{k_n}}}{n} = \frac{X_{\tau_{k_n}}}{k_n}\frac{k_n}{n} \to \frac{\mathbb{E}[X_{\tau_1}|\eta(X_0)=\infty]}{\mathbb{E}[\tau_1|\eta(X_0)=\infty]} \quad \text{as } n\to\infty\,. \tag{15}$$

For the second sum portion in (14) by (12) and Condition I we have,

$$\frac{|X_n - X_{\tau_{k_n}}|}{n} \le \frac{K(\tau_{k_n+1}-\tau_{k_n})}{n} = K\left(\frac{\tau_{k_n+1}}{k_n+1}\frac{k_n+1}{n} - \frac{\tau_{k_n}}{k_n}\frac{k_n}{n}\right) \to 0\ , \tag{16}$$

as $n\to\infty$.

Thus we obtain (3) by (15) and (16). By the ballisticity of the $p$-GERW (Theorem 1.1) it is clear that $\mathbb{E}[X_{\tau_1}\cdot\ell|\eta(X_0)=\infty]>0$ then $v\cdot\ell>0$, hence we finish the proof.

□

2.2.3. *Proof of CLT for the* $p$-SGERW. We first define a nondecreasing sequence $\{k_n\}_{n\ge0}$ tending to infinity such that $\tau_{k_n}\le n<\tau_{k_n+1}$ and a random variable

$$Z_j = X_{\tau_{j+1}} - X_{\tau_j} - (\tau_{j+1}-\tau_j)v\ ,\ \text{for } j\ge1\ .$$

The sequence of random variables $\{Z_j\}_{j\ge1}$ by Condition IV and from the definition of $v$ is i.i.d. and centered under $\mathbb{P}[\cdot|\eta(X_0)=\infty]$. Besides, they are square integrable by Corollary 2.4.

Next we have for a $T>0$

$$\sup_{t\le T}\left|B_t^n - \frac{1}{n^{\frac12}}\sum_{j=1}^{k_{\lfloor tn\rfloor}} Z_j\right| \le 2(K+|v|)\sup_{0\le j\le k_{\lfloor nT\rfloor}}\frac{\tau_{j+1}-\tau_j}{n^{\frac12}} \quad \text{a.s..} \tag{17}$$

We obtain (17) by the fact that for all $t\ge0$,

$$\begin{aligned}
&\left|B_t^n - \frac{1}{n^{\frac12}}\sum_{j=1}^{k_{\lfloor tn\rfloor}} Z_j\right| = \left|\frac{X_{\lfloor nt\rfloor}-\lfloor nt\rfloor v}{n^{\frac12}} - \frac{1}{n^{\frac12}}\sum_{j=1}^{k_{\lfloor tn\rfloor}} Z_j\right| \\
&= \left|\frac{X_{\lfloor nt\rfloor}-\lfloor nt\rfloor v - X_{\tau_{k_{\lfloor nt\rfloor}+1}} + X_{\tau_1} + \tau_{k_{\lfloor nt\rfloor}+1}v - \tau_1 v}{n^{\frac12}}\right| \\
&\le \left|\frac{X_{\tau_{k_{\lfloor nt\rfloor}+1}} - X_{\lfloor nt\rfloor}}{n^{\frac12}}\right| + \left|\frac{(\tau_{k_{\lfloor nt\rfloor}+1}-\lfloor nt\rfloor)v}{n^{\frac12}}\right| + \left|\frac{X_{\tau_1}-\tau_1 v}{n^{\frac12}}\right| \\
&\le 2(K+|v|)\sup_{0\le j\le k_{\lfloor nt\rfloor}}\frac{\tau_{j+1}-\tau_j}{n^{\frac12}}\ .
\end{aligned}$$

Now we will show that (17) goes to 0 in $\mathbb{P}$-probability as $n \to \infty$. Then, for $\varepsilon > 0$, we obtain

$$\mathbb{P}\left[\sup_{0\le j\le k_{\lfloor nT\rfloor}} \frac{\tau_{j+1}-\tau_j}{n^{\frac{1}{2}}} > \varepsilon\right] = \mathbb{P}\left[\bigcup_{j=0}^{k_{\lfloor nT\rfloor}} \left\{\tau_{j+1}-\tau_j > \varepsilon n^{\frac{1}{2}}\right\}\right]$$
$$\le \mathbb{P}[\tau_1 > \varepsilon n^{\frac{1}{2}}] + \sum_{j=1}^{k_{\lfloor nT\rfloor}} \mathbb{P}[\tau_{j+1}-\tau_j > \varepsilon n^{\frac{1}{2}}] \tag{18}$$
$$\le \mathbb{P}[\tau_1 > \varepsilon n^{\frac{1}{2}}] + k_{\lfloor nT\rfloor}\mathbb{P}[\tau_1 > \varepsilon n^{\frac{1}{2}} | \eta(X_0)=\infty]$$
$$\le \mathbb{P}[\tau_1 > \varepsilon n^{\frac{1}{2}}] + \frac{nT+1}{\varepsilon^2 n}\mathbb{E}\left[\tau_1^2 1_{\{\tau_1 > \varepsilon n^{\frac{1}{2}}\}} | \eta(X_0)=\infty\right].$$

In the first inequality in (18) we used union bound, the second we obtain by Condition IV, the third by Chebyshev's inequality and the fact that $k_n \le n$ for all $n$.

Now we have

$$\mathbb{P}[\tau_1 > \varepsilon n^{\frac{1}{2}}] + \frac{nT+1}{\varepsilon^2 n}\mathbb{E}\left[\tau_1^2 1_{\{\tau_1 > \varepsilon n^{\frac{1}{2}}\}} | \eta(X_0)=\infty\right] \to 0 \quad \text{as } n \to \infty\,,$$

by Proposition 2.1 for the first portion sum and by Corollary 2.4 for the second one. Thus we conclude that

$$\sup_{0\le j\le k_{\lfloor nT\rfloor}} \frac{\tau_{j+1}-\tau_j}{n^{\frac{1}{2}}} \to 0 \quad \text{as } n\to\infty \text{ in } \mathbb{P}-\text{probability}\,.$$

Thus the Skorohod distance between (see [5], page 124) $B^n_{\cdot}$ and $\sum_{j\le k_{\lfloor n\cdot\rfloor}} Z_j/n^{1/2}$ tends to 0 in $\mathbb{P}$-probability as $n$ goes to infinity. Now, notices that, by the Donsker's invariance principle (see for example [5], page 146)

$$\frac{1}{n^{\frac{1}{2}}}\sum_{j=1}^{\lfloor \cdot n\rfloor} Z_j \xrightarrow{\mathcal{D}} W_{\cdot}\,, \tag{19}$$

where $\xrightarrow{\mathcal{L}}$ means, converges in distribution, and $W$ is a Brownian Motion with covariance matrix $\mathbb{E}[\tau_1|\eta(X_0)=\infty]A$.

Next, using (13) and Dini's theorem, we obtain that for all $T > 0$

$$\sup_{0\le t\le T}\left|\frac{k_{\lfloor tn\rfloor}}{n} - \frac{t}{\mathbb{E}[\tau_1|\eta(X_0)=\infty]}\right| \to 0 \quad \text{as } n\to\infty \text{ a.s..} \tag{20}$$

Now we denote two continuous functions

$$f(t) = \frac{1}{n^{\frac{1}{2}}}\sum_{j=1}^{\lfloor tn\rfloor} Z_j \quad \text{for } t \ge 0\,,$$
$$g(t) = \frac{k_{\lfloor tn\rfloor}}{n} \quad \text{for } t\ge 0\,.$$

Thus the composition of two continuous function is a continuous function and we have

$$(f \circ g)(t) = \frac{1}{n^{\frac{1}{2}}} \sum_{j=1}^{k_{\lfloor tn \rfloor}} Z_j \ .$$

Hence by Lemma in page 151 in [5], (19) and (20) we have

$$\frac{1}{n^{\frac{1}{2}}} \sum_{j=1}^{k_{\lfloor \cdot n \rfloor}} Z_j \xrightarrow{\mathcal{L}} W_A^{(d)} \ ,$$

where $W_A^{(d)}$ is a $d$-dimensional Brownian Motion with covariance matrix $A$. To obtain tightness of $n^{-1/2} \sum_{j \leq k_{\lfloor \cdot n \rfloor}} Z_j$ we use (19), (20) and Theorem 13.2 of [5]. Thus we finish the proof that $B^n_{\cdot}$ converges in law to a Brownian Motion with covariance matrix A, since the Skorohod distance between $B^n_{\cdot}$ and $\sum_{j \leq k_{\lfloor n \cdot \rfloor}} Z_j / n^{1/2}$ tends to 0 in $\mathbb{P}$-probability as $n$ goes to infinity (see for instance [5] page 148).

Remains to us to prove that the covariance matrix $A$ is non-degenerate. We remember that we have to choose $a > 2\sqrt{d}$.

First we consider $w \in \mathbb{R}^d$ with $w^t A w = 0$, then we have

$$\mathbb{P}[w \cdot (X_{\tau_1} - \tau_1 v) = 0 | \eta(X_0) = \infty] = 1 \ . \tag{21}$$

One can see that, the set $\{x \in \mathbb{Z}^d, 0 \leq x \cdot \ell < a\}$ is connected. We denote $\partial B$ as the boundary of $B$, so $\partial B = \{x \in \mathbb{Z}^d / B, \ \exists y \in B, \ |y - x| = 1\}$.

If $\mathbb{P}[X_{\rho_1^{(0)}} = x, \rho_1^{(0)} < \eta(X_0)] > 0$, then the collection of $x \in \mathbb{Z}^d$ from which we obtain this inequality, it is the same of $H = \partial\{z \in \mathbb{Z}^d, z \cdot \ell < a\}$. We have that thanks to the connectedness of the set $\{x \in \mathbb{Z}^d, 0 \leq x \cdot \ell < a\}$.

We will see now that:

$$w \cdot v = 0 \text{ and } w \cdot x = 0, \text{ for any } x \in H \ . \tag{22}$$

Consider $x \in H$, then $\mathbb{P}[X_{\rho_1^{(0)}} = x, \rho_1^{(0)} < \eta(X_0)] > 0$. Now thanks to the connectedness of the set $\{x \in \mathbb{Z}^d, 0 \leq x \cdot \ell < a\}$, we have for all $n \geq 0$ that

$$\mathbb{P}[X_{\rho_1^{(0)}} = x, n \leq \rho_1^{(0)} < \eta(X_0)] > 0 \ . \tag{23}$$

Thus we obtain as a result for $n \geq 0$ and $x \in H$

$$\begin{aligned} &\mathbb{P}[X_{\tau_1} = x, n < \tau_1 = \rho_1^{(0)}, \eta(X_0) = \infty] = \\ &\quad = \mathbb{P}[X_{\rho_1^{(0)}} = x, n < \rho_1^{(0)}, \eta(X_{\rho_1^{(0)}}) = \infty, \eta(X_0) = \infty] \\ &\quad = \mathbb{P}[X_{\rho_1^{(0)}} = x, n < \rho_1^{(0)} < \eta(X_0), \eta(X_{\rho_1^{(0)}}) = \infty] \\ &\quad = \mathbb{P}[X_{\rho_1^{(0)}} = x, n < \rho_1^{(0)} < \eta(X_0)]\mathbb{P}[\eta(X_{\rho_1^{(0)}}) = \infty] \\ &\quad = \mathbb{P}[X_{\rho_1^{(0)}} = x, n < \rho_1^{(0)} < \eta(X_0)]\mathbb{P}[\eta(X_0) = \infty] > 0 \, . \end{aligned} \tag{24}$$

In the third equality in (24) we use the Strong Markov Property and Independence. The last inequality we obtain by (23) and Proposition 2.5.

Then in view of (21), for an arbitrary $n \geq 0$ and $x \in H$, we have,

$$n|w \cdot v| \leq |w \cdot x| \ ,$$

which implies $w \cdot v = 0$. Thus coming back to (21), we can deduce (22).

Hence if we take the limits of points in $H$, we see that, $w \cdot y = 0$, for any $y \in \mathbb{R}^d$, orthogonal to $\ell$. By this reason, since $v \cdot \ell > 0$ and (22), we obtain that $w = 0$. Then we finish the proof that $A$ is non-degenerate. □

2.3. **The behavior of the increments of the regeneration times.** To prove Proposition 2.1 we need to state and prove several auxiliary results. First we show that the probability that a $p$-GERW $X = \{X_n\}_{n\geq 0}$ visits less than $n^{1/2+\alpha}$ distinct sites until time $n$ decays as a stretched exponential for all $\alpha \in (0, 1/6)$ (see, Proposition 2.3). This result can then be used to show that $X_n \cdot \ell$ is at least of order $n^{\frac{1}{2}+\alpha}$ with high probability (see, Proposition 2.4). This establish super-diffusive behaviour for $X$, although still sub-ballistic, it is all we need to obtain a key result for the regeneration structure, namely that the probability of $\{\eta(X_0) = \infty\}$ is bounded from below by a constant whose behavior according to the choice of $p$ can be explicitly described (see, Proposition 2.5). Finally, in Proposition 2.6 we obtain some additional estimates related to the regeneration structure and we can then prove Proposition 2.1. The above strategy is analogous to that in [15], but several novel ideas had to be implemented to deal with the randomness of the cookie environment.

In this section we state the auxiliary results mentioned above and their proofs are postponed to Section 2.4.

Given a stochastic process $\{X_n\}_{n\geq 0}$ on the lattice $\mathbb{Z}^d$, we denote its range at time $n$ by

$$\mathcal{R}_n^X := \{x \in \mathbb{Z}^d : X_k = x \text{ for some } 0 \leq k \leq n\} \ ,$$

i.e., the set of sites visited by the process up to time $n$. The next result states that if $X$ is a $p$-GERW, then the probability that $|\mathcal{R}_n^X| \leq n^{1/2+\alpha}$ decays as a stretched exponential in $n$ for every $\alpha \in (0, 1/6)$, where $|A|$ denotes the number of elements of a set $A$.

**Proposition 2.3.** *Let $X = \{X_n\}_{n\geq 0}$ be a $p$-GERW. Then, for all $0 < \alpha < 1/6$ there exist positive constants $\gamma_1$, $\gamma_2$ , which depend on $d$, $K$, $h$, and $r$, such that*

$$\mathbb{P}[|\mathcal{R}_n^X| < n^{\frac{1}{2}+\alpha}] < \exp\{-\gamma_1 n^{\gamma_2}\} \ ,$$

*for all $n \geq 1$.*

Since the proof of Proposition 2.3 is rather lengthy, it will be deferred to Appendix B.

**Remark 2.2.** *It is important to notice that $\gamma_1$ and $\gamma_2$ do not depend on $p$. In particular, Proposition 2.3 holds true for $p = 0$ which is the case where the* *random walk is a d-martingale with zero mean vector satisfying a uniform elliptic condition. Our result also refines the proof Proposition 4.1 in [15] stated for* $p = 1$ *and some* $\alpha > 0$*, in that it quantifies the maximal value of* $\alpha$ *for which the statement holds true. For* $p \in (0, 1]$ *the determination of an upper bound for* $\alpha$ *might not seem relevant, since the ballisticity will imply that* $\mathcal{R}_n^X$ *is* $\Theta(n)$*, but we do believe that our proof might be useful to discuss generalizations of the* $p$-GERW*, for instance when the probability of having a cookie on the first visit is time dependent.*

**Remark 2.3.** *Note that the event* $\{|\mathcal{R}_n^X| < n^{\frac{1}{2}+\alpha}\}$ *for all* $0 < \alpha < 1/6$ *can be written as* $\{|\mathcal{R}_n^X| < n^{\frac{2}{3}-\varepsilon}\}$ *for all* $\varepsilon > 0$.

To simplify the statement of the next result, we will consider a slight generalization of the $p$-GERW. For a fixed set $A \subset \mathbb{Z}^d$ we say that $\{X_n\}_{n\geq 1}$ is a $p$-GERW with excitation-allowing set $A$, if it satisfies Condition I, Condition III in Section 1.1 and the following variation of Condition II:

**Condition II*.** *If there exists* $\lambda > 0$ *such that:*

- *almost surely on the event* $\{X_k \neq X_n \text{ for all } k < n\}$*, either*
$$\mathbb{E}[X_{n+1} - X_n|\mathcal{F}_n] \cdot \ell \geq \lambda \ , \ \text{ if } \pi(X_n) \leq p \ \text{ and } \ X_n \in A \ ,$$
*or*
$$\mathbb{E}[X_{n+1} - X_n|\mathcal{F}_n] = 0 \ , \ \text{ if } \pi(X_n) > p \ \text{ or } \ X_n \notin A \ .$$
- *almost surely on the event* $\{\exists\, k < n \text{ such that } X_k = X_n\}$*,*
$$\mathbb{E}[X_{n+1} - X_n|\mathcal{F}_n] = 0 \ .$$

We can think of set A as the set of sites where there is the possibility of having cookies. The definition will be useful to deal with time translations of the $p$-GERW, where $A$ will represent the set of sites not yet visited by the process.

Set $H(a, b) \subset \mathbb{Z}^d$ for $a < b$ as:
$$H(a, b) := \{x \in \mathbb{Z}^d : x \cdot \ell \in [a, b]\} \ ,$$
which represents the strip in direction $\ell$ between levels $a$ and $b$. Roughly speaking, the next proposition states that if the number of sites outside the excitation-allowing set in a strip with length of order $n^{\frac{1}{2}+\alpha}$, $0 < \alpha < 1/6$, containing the origin, is also of order $n^{\frac{1}{2}+\alpha}$, then $X_n \cdot \ell$ is at least of order $n^{\frac{1}{2}+\alpha}$ with high probability.

**Proposition 2.4.** *Fix* $0 < \alpha < 1/6$ *and suppose that* $\{X_n\}_{n\geq 1}$ *is a* $p$-GERW *with excitation-allowing set* $A \subset \mathbb{Z}^d$*. If for some* $n \geq 1$
$$\left|(\mathbb{Z}^d \setminus A) \cap H\left(-n^{\frac{1}{2}+\alpha}, \frac{2\lambda}{3} n^{\frac{1}{2}+\alpha}\right)\right| \leq \frac{1}{3} n^{\frac{1}{2}+\alpha} \ , \tag{25}$$
*then, for some positive constants* $\gamma_3$*,* $\gamma_4$ *depending on* $d$*,* $K$*,* $r$*,* $\lambda$*,* $\alpha$ *and* $p$*, we have*
$$\mathbb{P}\left[X_n \cdot \ell < \frac{p}{3} \lambda n^{\frac{1}{2}+\alpha}\right] < 6n \exp\{-\gamma_3 n^{\gamma_4}\} \ , \tag{26}$$

*where*

$$\gamma_3 = \min\left\{\gamma_1, \frac{1}{2K^2}, \frac{\lambda^2}{18K^2}, \frac{\varepsilon^2 p}{3}, \frac{((1/3 - 2/3(1-\varepsilon))\lambda p)^2}{2K^2}\right\},$$
$$\gamma_4 = \min\{\gamma_2, 2\alpha, 1/2+\alpha\},$$

$\varepsilon \in (1/2, 1)$ *and* $\gamma_1$, $\gamma_2$ *are the same as in Proposition 2.3.*

For every $\ell \in \mathbb{S}^{d-1}$, let $\mathbb{M}_\ell$ denote the positive half-space in direction $\ell$, that is, $\mathbb{M}_\ell = \{x \in \mathbb{Z}^d : x \cdot \ell > 0\}$. The next result provides a lower bound on the probability of $\{\eta(X_0) = \infty\}$, which will allow us to prove that regeneration times $\{\tau_k\}_{k\geq 1}$ are almost surely finite.

**Proposition 2.5.** *Fix* $0 < \alpha < 1/6$ *and let* $X$ *be a* $p$-GERW *in direction* $\ell$ *with excitation-allowing set* $A \subset \mathbb{Z}^d$ *such that* $\mathbb{M}_\ell \subset A$. *There exists* $\psi > 0$ *depending on* $d$, $K$, $h$, $r$, $\lambda$, $\alpha$ *and* $p$ *such that*

$$\mathbb{P}[\eta(X_0) = \infty] \geq \mathbb{P}[X_n \cdot \ell > 0 \text{ for all } n \geq 1] \geq \psi,$$

*where* $\psi = h^{C\left(\frac{3}{\lambda p}\right)^{\frac{1}{\delta-1}}} c$, $c \in (0,1)$, $\delta = (2-\alpha)(1/2+\alpha)$,

$$C > \lceil r^{-1} \rceil \left(\frac{K}{3}\right)^{\frac{1}{\delta-1}} \eta,$$
$$\eta = \left(\frac{2-\alpha}{\gamma_3 \varphi_1}\right)^{\frac{1}{\varphi_1}}, \quad \varphi_1 = \min\{\alpha, (2-\alpha)\gamma_4\},$$

*and* $\gamma_3$, $\gamma_4$ *are as in Proposition 2.4.*

We now state the last auxiliary result which will be used in the proof of Proposition 2.1. It provides bounds on probabilities associated to the regeneration times.

**Proposition 2.6.** *Let* $\{\tau_k\}_{k\geq 0}$ *be the regeneration times for a* $p$-GERW *in direction* $\ell$. *Take* $\gamma_3$ *and* $\gamma_4$ *as in Proposition 2.4 and* $\psi$ *as in Proposition 2.5, then*

(i) $\sup_{j,k\geq 1} \mathbb{P}\big[\eta_j^{(k)} < \infty \big| \mathcal{G}_j^{(k)}\big] < 1 - \psi$, *a.s.*,

(ii) $\sup_{k\geq 1} \mathbb{P}\big[(X_{\tau_k+n} - X_{\tau_k}) \cdot \ell < \frac{p}{3}\lambda n^{\frac{1}{2}+\alpha} \big| \mathcal{G}_0^{(k)}\big] < \frac{e^{-\gamma_3 n^{\gamma_4}}}{\psi}$, *a.s.*,

(iii) $\sup_{j\geq 1, k\geq 0} \mathbb{P}\big[n \leq \eta_j^{(k)} - \rho_j^{(k)} < \infty \big| \mathcal{G}_0^{(k)}\big] < 12 e^{-\gamma_3 n^{\gamma_4}}$, *a.s.*.

**Remark 2.4.** *In the proof of (ii) in Proposition 2.6, we will need a small adaptation of Proposition 2.4. If we want to estimate* $\{(X_{\tau_k+n} - X_{\tau_k}) \cdot \ell < \frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}\}$, *where* $\tau_k$ *is a regeneration time, we can still apply Proposition 2.4 to bound the probability of this event. Important to notice that conditional to be in position* $X_{\tau_k}$ *at time* $\tau_k$, *the process will not go below that position in direction* $\ell$ *and it has a completely unexplored environment forward of it. Then in condition (25) we only need to consider the intersection of the strip* *with* $\{x \in \mathbb{Z}^d : x \cdot \ell \geq X_{\tau_k} \cdot \ell\}$. *With this in mind, the reader just have to follow straightforwardly the proof of Proposition 2.4.*

The proof of Proposition 2.6 will follow closely the proof of Proposition 4.4 in [15] (with minor adjustments due to the fact that the renewal structure is slightly different) and it is deferred to Appendix A.

Using Proposition 2.6, we can now prove Proposition 2.1 basically in the same way as in [4] and [15]. For the sake of completeness the proof is provided in Appendix A. Below we give a sketch of this proof.

**Proof sketch of Proposition 2.1.** The idea is to define the following events:

$$G_n := \{(X_{\tau_k+n} - X_{\tau_k}) \cdot \ell \leq u_n\}\,,$$

$$B_n := \bigcap_{j=1}^{v_n} \{\eta_j^{(k)} < \infty\} \quad \text{and} \quad F_n := \bigcup_{j=1}^{v_n} \{w_n \leq \eta_j^{(k)} - \rho_j^{(k)} < \infty\}\,,$$

where, for each positive integer $n$, $u_n = \lfloor n^{a_1} \rfloor$, $v_n = \lfloor n^{a_2} \rfloor$ and $w_n = \lfloor n^{a_3} \rfloor$, with $a_1$, $a_2$ and $a_3$ positive real numbers such that $a_1 < 1/2+\alpha$, and $a_2+a_3 < a_1$. We choose $n$ large enough such that $(K+1)v_n(w_n+1)+2+K \leq u_n$ and $u_n < (p\lambda/3)n^{1/2+\alpha}$. Then we show that

$$G_n^c \cap B_n^c \cap F_n^c \subset \{\tau_{k+1} - \tau_k \leq n\}\,,$$

and now we will be able to control the probability of the event $\{\tau_{k+1} - \tau_k \leq n\}$. Hence we have

$$\mathbb{P}[\tau_{k+1} - \tau_k > n | \mathcal{G}_0^{(k)}] \leq \mathbb{P}[G_n | \mathcal{G}_0^{(k)}] + \mathbb{P}[B_n | \mathcal{G}_0^{(k)}] + \mathbb{P}[F_n | \mathcal{G}_0^{(k)}]\,. \tag{27}$$

For each sum portion in (27), we can use Proposition 2.6 part $ii)$, $i)$ and $iii)$, respectively, to control those probabilities. Thus we obtain for each one:

(i) $\mathbb{P}[B_n | \mathcal{G}_0^{(k)}] \leq (1-\psi)^{\lfloor n^{a_2} \rfloor}$;

(ii) $\mathbb{P}[G_n | \mathcal{G}_0^{(k)}] \leq \dfrac{e^{-\gamma_3 n^{\gamma_4}}}{\psi}$;

(iii) $\mathbb{P}[F_n | \mathcal{G}_0^{(k)}] \leq 2\lfloor n^{a_2} \rfloor e^{-\gamma_3 \lfloor n^{a_3} \rfloor^{\gamma_4}}$.

Finally we finish the proof using the above upper bounds in (27).

## 2.4. **Proof of auxiliary results.**

2.4.1. *Proof of Proposition 2.4.* Let us begin observing that the process $(X_n \cdot \ell, n \geq 0)$ is a $\mathcal{F}$-submartingale and thus $(-X_n \cdot \ell, n \geq 0)$ is a $\mathcal{F}$-supermartingale. Adaptability and integrability follows from the definitions and Condition I. Moreover we have two possible situations by Condition II,

$$\mathbb{E}[(X_{n+1} - X_n) \cdot \ell | \mathcal{F}_n] = 0 \;\text{ or }\; \mathbb{E}[(X_{n+1} - X_n) \cdot \ell | \mathcal{F}_n] \geq \lambda\,.$$

Thus, we have $\mathbb{E}[X_{n+1} \cdot \ell | \mathcal{F}_n] \geq X_n \cdot \ell$.

As a first step we show that

$$\mathbb{P}\left[\max_{k\le n} X_k\cdot\ell > \frac{2}{3}\lambda n^{\frac12+\alpha},\ X_n\cdot\ell < \frac{p}{3}\lambda n^{\frac12+\alpha}\right] \le n e^{-C_1 n^{2\alpha}}\,, \tag{28}$$

for $C_1>0$. Note that

$$\begin{aligned}&\left\{\max_{k\le n} X_k\cdot\ell > \frac{2}{3}\lambda n^{\frac12+\alpha},\ X_n\cdot\ell < \frac{p}{3}\lambda n^{\frac12+\alpha}\right\}\\ &\subset \bigcup_{k=1}^{n}\left\{X_n\cdot\ell - X_k\cdot\ell < \left(\frac{p}{3}-\frac{2}{3}\right)\lambda n^{\frac12+\alpha}\right\}\,,\end{aligned}$$

and by Azuma's inequality for supermartingales with increments uniformly bounded by $K$ (see Lemma 1 of [20]), for every $k=\{1,\dots,n-1\}$ it holds that

$$\begin{aligned}\mathbb{P}\left[X_n\cdot\ell - X_k\cdot\ell < \left(\frac{p}{3}-\frac{2}{3}\right)\lambda n^{\frac12+\alpha}\right] &\le \mathbb{P}\left[X_n\cdot\ell - X_k\cdot\ell < -\frac{1}{3}\lambda n^{\frac12+\alpha}\right]\\ &\le \exp\left(-\frac{\left(\frac13\right)^2\lambda^2 n^{1+2\alpha}}{2(n-k)K^2}\right) \le \exp\left(-\frac{\lambda^2 n^{2\alpha}}{18K^2}\right)\,.\end{aligned}$$

Then (28) follows from the usual union bound with $C_1 = \left(\frac13\lambda\right)^2/2K^2$. Moreover, again using Azuma's inequality (for supermartingales), we also have that

$$\mathbb{P}\left[\min_{k\le n} X_k\cdot\ell < -n^{\frac12+\alpha}\right] \le n\exp\{-C_2 n^{2\alpha}\}\,, \tag{29}$$

for $C_2 = 1/2K^2$.

Now let $D_k = \mathbb{E}[X_{k+1}-X_k|\mathcal{F}_k]$ and $Y_n = X_n - \sum_{k=0}^{n-1} D_k$. It follows that $Y_n$ is a martingale with bounded increments. Let $G$ be the following event

$$G := \left\{|\mathcal{R}_n^X| \ge n^{\frac12+\alpha}\right\} \cap \left\{X_k \in H\left(-n^{\frac12+\alpha}, \frac{2}{3}\lambda n^{\frac12+\alpha}\right),\ \text{for all } k\le n\right\}\,.$$

Using the hypotheses (25), on $G$ we have at least $|\mathcal{R}_n^X| - \frac13 n^{\frac12+\alpha} \ge \frac23 n^{\frac12+\alpha}$ sites visited on the excitation-allowing $A$. Therefore, there exists a Binomial random variable $W$ with parameters $N=\frac23 n^{\frac12+\alpha}$ and $p$ such that on $G$

$$\left(\sum_{k=0}^{n-1} D_k\right)\cdot\ell \ge \lambda W\,.$$

In order to prove (26), we write that probability as

$$\mathbb{P}\left[\left\{X_n\cdot\ell < \frac{p}{3}\lambda n^{\frac12+\alpha}\right\}\cap G\right] + \mathbb{P}\left[\left\{X_n\cdot\ell < \frac{p}{3}\lambda n^{\frac12+\alpha}\right\}\cap G^c\right] \tag{30}$$

and we control both terms separately. We start with the second term. Set

$$E = \left\{X_n\cdot\ell < \frac{p}{3}\lambda n^{\frac12+\alpha}\right\},\ M = \left\{|\mathcal{R}_n^X| < n^{\frac12+\alpha}\right\}\,,$$

$$J = \left\{\min_{k\le n} X_k\cdot\ell < -n^{\frac12+\alpha}\right\}\ \text{and}\ T = \left\{\max_{k\le n} X_k\cdot\ell > \frac{2}{3}\lambda n^{\frac12+\alpha}\right\}\,.$$

It follows that

$$\mathbb{P}\left[E\cap G^c\right]=\mathbb{P}\left[(E\cap M)\cup(E\cap J)\cup(E\cap T)\right]$$
$$\leq \mathbb{P}\left[E\cap M\right]+\mathbb{P}\left[E\cap J\right]+\mathbb{P}\left[\max_{k\leq n} X_k\cdot\ell>\frac{2}{3}\lambda n^{\frac{1}{2}+\alpha},\ X_n\cdot\ell<\frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}\right],$$

and from Proposition 2.3 and (29) and (28), we obtain

$$\begin{aligned}\mathbb{P}\left[E\cap G^c\right]&\leq \mathbb{P}\left[|\mathcal{R}_n^X|<n^{\frac{1}{2}+\alpha}\right]+\mathbb{P}\left[\min_{k\leq n} X_k\cdot\ell<-n^{\frac{1}{2}+\alpha}\right]+ne^{-C_1n^{2\alpha}}\\ &\leq e^{-\gamma_1 n^{\gamma_2}}+ne^{-C_2n^{2\alpha}}+ne^{-C_1n^{2\alpha}}\ .\end{aligned}\tag{31}$$

As regards the first term in (30), let $\varepsilon\in(1/2,1)$, $B=\{W\leq Np(1-\varepsilon)\}$, and write $\mathbb{P}[E\cap G]$ as

$$\mathbb{P}\left[\left\{X_n\cdot\ell<\frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}\right\}\cap G\cap B\right]+\mathbb{P}\left[\left\{X_n\cdot\ell<\frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}\right\}\cap G\cap B^c\right]$$
$$\leq\ \mathbb{P}\left[B\right]+\mathbb{P}\left[\left\{X_n\cdot\ell<\frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}\right\}\cap G\cap B^c\right].$$

To bound $\mathbb{P}[B]$ we use the Chernoff bound (cf., e.g., Theorem 4.5 of [16]) to obtain

$$\mathbb{P}\left[B\right]\leq\exp\Big\{-\frac{\varepsilon^2}{2}Np\Big\}=\exp\Big\{-C_3n^{\frac{1}{2}+\alpha}p\Big\}\ ,\tag{32}$$

where $C_3=\varepsilon^2/3$. To upper bound $\mathbb{P}\left[\left\{X_n\cdot\ell<\frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}\right\}\cap G\cap B^c\right]$, we use that $Y_n=X_n-\sum_{k=0}^{n-1}D_k$ is martingale with bounded increments and apply Azuma's inequality (see, for example, Theorem 2.19 in [6]). Thus, denoting $F=\left\{X_n\cdot\ell<\frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}\right\}\cap G\cap B^c$ we obtain

$$\begin{aligned}\mathbb{P}\left[F\right]\ &\leq\ \mathbb{P}\Big[X_n\cdot\ell-\Big(\sum_{k=1}^{n-1}D_k\Big)\cdot\ell<\frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}-\lambda Np(1-\varepsilon)\Big]\\ &\leq\ \mathbb{P}\Big[X_n\cdot\ell-\Big(\sum_{k=1}^{n-1}D_k\Big)\cdot\ell<p\lambda n^{\frac{1}{2}+\alpha}\Big(\frac{1}{3}-\frac{2}{3}(1-\varepsilon)\Big)\Big]\ .\end{aligned}\tag{33}$$

Hence, we have that $C_4:=-(1/3-2/3(1-\varepsilon))>0$, since $\varepsilon\in(1/2,1)$. By (33) and Azuma's inequality we obtain

$$\begin{aligned}\mathbb{P}[F]&\leq\mathbb{P}\Big[X_n\cdot\ell-\Big(\sum_{k=1}^{n-1}D_k\Big)\cdot\ell<-C_4p\lambda n^{\frac{1}{2}+\alpha}\Big]\\ &\leq 2\exp\Big(-\frac{C_4^2p^2\lambda^2n^{2\alpha}}{2K^2}\Big)=2\exp\big(-C_5p^2n^{2\alpha}\big)\ ,\end{aligned}\tag{34}$$

where $C_5=(C_4^2\lambda^2)/2K^2$.

Inequality (26) then follows from (31), (32) and (34) which imply that

$$\mathbb{P}\left[X_n\cdot\ell<\frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}\right]\leq$$
$$e^{-\gamma_1n^{\gamma_2}}+ne^{-C_2n^{2\alpha}}+ne^{-C_1n^{2\alpha}}+e^{-C_3n^{\frac{1}{2}+\alpha}p}+2e^{-C_5p^2n^{2\alpha}}\leq 6ne^{-\gamma_3n^{\gamma_4}}\ ,$$

where

$$\gamma_3 = \min\left\{\gamma_1, \frac{1}{2K^2}, \frac{\lambda^2}{18K^2}, \frac{\varepsilon^2 p}{3}, \frac{((1/3 - 2/3(1-\varepsilon))\lambda p)^2}{2K^2}\right\} \quad \text{and}$$
$$\gamma_4 = \min\{\gamma_2, 2\alpha, 1/2+\alpha\} \ .$$

□

2.4.2. *Proof of Proposition 2.5.* The main idea of this proof is to define an event to guarantee that the process $X$ will advance in direction $\ell$ in such a way that $X_n \cdot \ell > 0$, for all $n \geq 1$. Then we will obtain a lower bound for this event using Proposition 2.4 and Azuma's inequality. As it can be noticed from the statement, one of our main concerns is to make explicit the dependency of the bound on the parameter $p$.

*Proof of Proposition 2.5.* Since on $\{X_n \cdot \ell > 0 \text{ for all } n \geq 1\}$ the process doesn't visit the sites in $\mathbb{Z}^d/\mathbb{M}_\ell$, it is sufficient to prove the proposition for $A = \mathbb{Z}^d$.

Without loss of generality we consider $r \leq 1$ in Condition III. Define

$$U_0 = \left\{(X_{k+1} - X_k)\cdot \ell \geq r \ , \text{ for all } \quad k = 0, 1, ..., \lceil r^{-1}\rceil m - 1\right\}.$$

Observe that in $U_0$, $X_{\lceil r^{-1}\rceil m} \cdot \ell \geq m$ and by (UE1) of Condition III we have

$$\mathbb{P}[U_0] \geq h^{\lceil r^{-1}\rceil m} \ . \tag{35}$$

Consider the following time translation of the process $X$: $W_k = X_{\lceil r^{-1}\rceil m+k}$, $k \geq 0$. Then $W$ is a $p$-GERW with excitation-allowing set

$$A' = \mathbb{Z}^d/\{X_0, \ldots, X_{\lceil r^{-1}\rceil m-1}\}$$

starting at $W_0 = y_0 := X_{\lceil r^{-1}\rceil m}$.

Set $\delta = (2-\alpha)(1/2+\alpha)$ and

$$m = \frac{C}{\lceil r^{-1}\rceil}\left(\frac{3}{\lambda p}\right)^{\frac{1}{\delta-1}},$$

where $C > 0$ is a constant depending on $\alpha$, $K$, $\lambda$ and $r$, such that

$$C > (\lceil r^{-1}\rceil^\delta \lambda)^{\frac{1}{\delta-1}} \vee \left(\lceil r^{-1}\rceil \left(\frac{K}{3}\right)^{\frac{1}{\delta-1}} \eta\right) \quad \text{for}$$

$$\eta = \left(\frac{2-\alpha}{\gamma_3\varphi_1}\right)^{\frac{1}{\varphi_1}} \quad \text{with} \quad \varphi_1 = \min\{\alpha, (2-\alpha)\gamma_4\},$$

and $\gamma_3$, $\gamma_4$ as in the statement of Proposition 2.4. Note that for all $\alpha$ used in Proposition 2.3, i.e., $0 < \alpha < 1/6$, we have that $\delta > 1$.

The left-hand side of (25) with the set $A' - y_0$ is bounded above by $\lceil r^{-1}\rceil m$. Note that, for all $n \geq m^{2-\alpha}$,

$$\frac{1}{3}n^{\frac{1}{2}+\alpha} \geq \frac{1}{3}m^{(2-\alpha)(\frac{1}{2}+\alpha)} \geq \left(\frac{m^{\delta-1}}{3\lceil r^{-1}\rceil}\right)\lceil r^{-1}\rceil m \geq \lceil r^{-1}\rceil m \ ,$$

where the last inequality follows from $C > \left(\lceil r^{-1}\rceil^{\delta}\lambda\right)^{\frac{1}{\delta-1}}$. Thus (25) with excitation-allowing set $A' - y_0$ is satisfied for all $n \geq m^{2-\alpha}$.

Denote $m_0 = 0$, $m_1 = m$ and, for $k \geq 1$, $m_{k+1} = \frac{p}{3}\lambda m_k^{\delta}$. The sequence of $(m_k, k \geq 1)$ is increasing. The latter can be proved by induction since

$$\frac{m_2}{m_1} = \frac{\lambda p}{3} m^{\delta-1} = \left(\frac{C}{\lceil r^{-1}\rceil}\right)^{\delta-1} > \frac{K\eta^{\delta-1}}{3} > \frac{1}{3}\left(\frac{2-\alpha}{\gamma_3\varphi_1}\right)^{\frac{\delta-1}{\varphi_1}} > 1 ,$$

for all $\alpha \in (0, 1/6)$, and assuming $m_k/m_{k-1} > 1$, we have,

$$\frac{m_{k+1}}{m_k} = \frac{\frac{\lambda p}{3} m_k^{\delta}}{\frac{\lambda p}{3} m_{k-1}^{\delta}} = \left(\frac{m_k}{m_{k-1}}\right)^{\delta} > 1 .$$

For every $k \geq 1$ consider the following events

$$G_k = \left\{ \min_{\lfloor m_{k-1}^{2-\alpha}\rfloor < j \leq m_k^{2-\alpha}} \left(W_j - W_{\lfloor m_{k-1}^{2-\alpha}\rfloor}\right)\cdot \ell > -m_k \right\} ,$$
$$U_k = \left\{ W_{\lfloor m_k^{2-\alpha}\rfloor} \cdot \ell \geq m_{k+1} \right\} .$$

**Claim:** The following set inclusion holds:

$$\left\{X_n \cdot \ell > 0, \text{ for all } n \geq 1\right\} \supset \left(\bigcap_{k=1}^{\infty} (G_k \cap U_k)\right) \cap U_0 . \tag{36}$$

We will postpone the proof of the Claim to the end of the proof of Proposition 2.5.

As seen in proof of Proposition 2.4, the process $(X_n \cdot \ell, n \geq 0)$, is a $\mathcal{F}$-submartingale, so $(W - y_0)\cdot \ell$ is also $\mathcal{F}$-submartingale. Write

$$G_k^c = \bigcup_{j=\lfloor m_{k-1}^{2-\alpha}\rfloor+1}^{m_k^{2-\alpha}} \left\{ \left(W_j - W_{\lfloor m_{k-1}^{2-\alpha}\rfloor}\right)\cdot \ell \leq -m_k \right\} ,$$

and by Azuma's inequality (for supermartingales)

$$\begin{aligned}\mathbb{P}\left[\left(W_j - W_{\lfloor m_{k-1}^{2-\alpha}\rfloor}\right)\cdot \ell \leq -m_k\right] &= \mathbb{P}\left[\left(W_{\lfloor m_{k-1}^{2-\alpha}\rfloor} - W_j\right)\cdot \ell \geq m_k\right] \\ &\leq \exp\left(-\frac{m_k^2}{2K^2(j - \lfloor m_{k-1}^{2-\alpha}\rfloor)}\right) \leq \exp\left(-\frac{m_k^2}{2K^2 m_k^{2-\alpha}}\right) \leq \exp\left(-\frac{m_k^{\alpha}}{2K^2}\right).\end{aligned}$$

Thus

$$\mathbb{P}[G_k|U_0] \geq 1 - \left(m_k^{2-\alpha} - \lfloor m_{k-1}^{2-\alpha}\rfloor\right) e^{-\frac{m_k^{\alpha}}{2K^2}} \geq 1 - m_k^{2-\alpha} e^{-\frac{m_k^{\alpha}}{2K^2}} ,$$

for every $j = \{\lfloor m_{k-1}^{2-\alpha}\rfloor + 1, \ldots, m_k^{2-\alpha}\}$. Since the process $W - y_0$ satisfies Conditions I, II, III and the set $A' - y_0$ fulfills (25) for all $n \geq m^{2-\alpha}$, by Proposition 2.4, it holds that

$$\mathbb{P}[U_k|U_0] = \mathbb{P}\left[W_{\lfloor m_k^{2-\alpha}\rfloor} \cdot \ell \geq \frac{\lambda p}{3} m_k^{(2-\alpha)(\frac{1}{2}+\alpha)}\right] \geq 1 - 6 m_k^{2-\alpha} e^{-\gamma_3 m_k^{(2-\alpha)\gamma_4}} .$$

Now, write

$$\mathbb{P}\left[\left(\bigcap_{k=1}^{\infty}(G_k\cap U_k)\right)\cap U_0\right]=\mathbb{P}[U_0]\Big(1-\sum_{k=1}^{\infty}\mathbb{P}[G_k^c|U_0]+\mathbb{P}[U_k^c|U_0]\Big),$$

which is bounded from below by

$$\begin{aligned} &h^{\lceil r^{-1}\rceil m}\left(1-\sum_{k=1}^{\infty}\left(m_k^{2-\alpha}e^{-\frac{m_k^{\alpha}}{2K^2}}+6m_k^{2-\alpha}e^{-\gamma_3 m_k^{(2-\alpha)\gamma_4}}\right)\right)\\ &\geq h^{\lceil r^{-1}\rceil m}\left(1-7\sum_{k=1}^{\infty}m_k^{2-\alpha}e^{-\gamma_3 m_k^{\varphi_1}}\right). \end{aligned} \tag{37}$$

Now we are going to analyze the series $\sum_{k=1}^{\infty} m_k^{2-\alpha}e^{-\gamma_3 m_k^{\varphi_1}}$. Note that $m$ is large enough so that the sequence $(m_k^{2-\alpha}e^{-\gamma_3 m_k^{\varphi_1}}, k\geq 1)$ is decreasing. Indeed, $m$ is bigger than the inflection point $\left(\frac{2-\alpha}{\gamma_3\varphi_1}\right)^{\frac{1}{\varphi_1}}$ of the function $z(x)=x^{2-\alpha}e^{-\gamma_3 x^{\varphi_1}}$, $x>0$:

$$m=\left(\frac{C}{\lceil r^{-1}\rceil}\right)\left(\frac{3}{\lambda p}\right)^{\frac{1}{\delta-1}}>\left(\frac{K}{3}\right)^{\frac{1}{\delta-1}}\eta\left(\frac{3}{\lambda p}\right)^{\frac{1}{\delta-1}}\geq\left(\frac{K}{\lambda p}\right)^{\frac{1}{\delta-1}}\eta\geq\left(\frac{2-\alpha}{\gamma_3\varphi_1}\right)^{\frac{1}{\varphi_1}}.$$

Thus we have,

$$\sum_{k=1}^{\infty}m_k^{2-\alpha}e^{-\gamma_3 m_k^{\varphi_1}}\leq\int_{m_1}^{\infty}x^{2-\alpha}e^{-\gamma_3 x^{\varphi_1}}\,dx\,. \tag{38}$$

By a change of variables, we write,

$$\int_{m_1}^{\infty}x^{2-\alpha}e^{-\gamma_3 x^{\varphi_1}}\,dx=\varphi_1^{-1}\gamma_3^{\frac{\alpha-3}{\varphi_1}}\,\Gamma\left(\frac{3-\alpha}{\varphi_1},\gamma_3 m_1^{\varphi_1}\right), \tag{39}$$

where $\Gamma$ is the incomplete gamma function[3]. As mentioned above $m$ is large enough so that the sequence $(m_k^{2-\alpha}e^{-\gamma_3 m_k^{\varphi_1}}, k\geq 1)$ is decreasing. Thus, in order to obtain that (39) is smaller than $1/7$, we may increase $m$ even further by choosing a sufficiently bigger $C$. Thus, with such a suitable chosen $C$ we obtain

$$\begin{aligned} \sum_{k=1}^{\infty}m_k^{2-\alpha}e^{-\gamma_3 m_k^{\varphi_1}} &\leq\int_{m_1}^{\infty}x^{2-\alpha}e^{-\gamma_3 x^{\varphi_1}}\,dx\\ &=\varphi_1^{-1}\gamma_3^{\frac{\alpha-3}{\varphi_1}}\,\Gamma\left(\frac{3-\alpha}{\varphi_1},\gamma_3 m_1^{\varphi_1}\right)<\frac{1}{7}\,. \end{aligned} \tag{40}$$

[3] $\Gamma(s,x)=\int_x^{\infty}t^{s-1}e^{-t}dt$.

Using (40) in (37), we obtain that,

$$\mathbb{P}\left[\left(\bigcap_{k=1}^{\infty}(G_k\cap U_k)\right)\cap U_0\right]\geq h^{\lceil r^{-1}\rceil m}\left(1-7\sum_{k=1}^{\infty}m_k^{2-\alpha}e^{-\gamma_3 m_k^{\varphi_1}}\right)$$
$$\geq h^{\lceil r^{-1}\rceil\frac{C}{\lceil r^{-1}\rceil}\left(\frac{3}{\lambda p}\right)^{\frac{1}{\delta-1}}}c=\psi\,,$$

where $c$ is a positive constant such that $c\in(0,1)$. Proposition 2.5 then follows from (36) which we are going to prove below.

*Proof of the Claim:* First observe that

$$\Big(\bigcap_{k=1}^{\infty}(G_k\cap U_k)\Big)\cap U_0=\bigcap_{k=1}^{\infty}(G_k\cap U_{k-1})\,.$$

i) In $G_1\cap U_0$, we have that $X_n\cdot\ell>0$ for all $n\in(0,\lceil r^{-1}\rceil m+m^{2-\alpha}]$. Indeed, clearly $X_n\cdot\ell>0$ for all $n\in(0,\lceil r^{-1}\rceil m]$ by $U_0$ and as already seen $X_{\lceil r^{-1}\rceil m}\geq m$. Now suppose, towards a contradiction, that in $G_1\cap U_0$ there exists at least a $k\in(\lceil r^{-1}\rceil m,\lceil r^{-1}\rceil m+m^{2-\alpha}]$ such that $X_k\cdot\ell\leq 0$. Thus, we would have $\big(X_k-X_{\lceil r^{-1}\rceil m}\big)\cdot\ell\leq -m$, which contradicts $G_1$.

ii) Now in $G_k\cap U_{k-1}$ for $k\geq 2$ we have $X_n>0$, for all $n\in(\lceil r^{-1}\rceil m+m_{k-1}^{2-\alpha},\lceil r^{-1}\rceil m+m_k^{2-\alpha}]$. Indeed, suppose that in $G_k\cap U_{k-1}$ there exists at least a $k\in(\lceil r^{-1}\rceil m+m_{k-1}^{2-\alpha},\lceil r^{-1}\rceil m+m_k^{2-\alpha}$ such that $X_k\cdot\ell\leq 0$. So we would have $X_k\cdot\ell-W_{\lfloor m_{k-1}^{2-\alpha}\rfloor}\cdot\ell\leq -m_k$, which contradicts $G_k$. □

## 3. Proof of the main theorems for the $p_n$-GERW

We start this section remembering that if $\{X_n\}_{n\geq 0}$ is a $p_n$-GERW in direction $\ell$ we can write it as in (1), hence we have

$$X_n=\sum_{i=1}^{n}\left(1_{\{E_{i-1}\}}\xi_i+1_{\{E_{i-1}^c\}}1_{\{U_i>p_i\}}\xi_i+1_{\{E_{i-1}^c\}}1_{\{U_i\leq p_i\}}\gamma_i\right),$$

where $\{U_i\}_{i\geq 1}$ is a sequence of i.i.d. random variables with uniform distribution in $[0,1]$, $E_i=\{\exists\, k<i\ \text{ such that }\ X_k=X_i\}$ for all $i\geq 1$, $\{\xi_i,\mathcal{F}_i\}_{i\geq 1}$ is an increment of a $d$-martingale with zero mean and $\{\gamma_i,\mathcal{F}_i\}_{i\geq 1}$ is random vector such that $\mathbb{E}[\gamma_i\cdot\ell|\mathcal{F}_{i-1}]\geq\lambda$ .

We can rewrite (1) as

$$\begin{aligned}X_n&=\sum_{i=1}^{n}\left(1_{\{E_{i-1}\cup\{U_i>i^{-\beta}\}\}}\xi_i+1_{\{E_{i-1}^c\cap\{U_i\leq i^{-\beta}\}\}}\gamma_i\right)\\&=\sum_{i=1}^{n}\left(\xi_i+1_{\{E_{i-1}^c\cap\{U_i\leq i^{-\beta}\}\}}\gamma_i-1_{\{E_{i-1}^c\cap\{U_i\leq i^{-\beta}\}\}}\xi_i\right).\end{aligned}\tag{41}$$

Before we provide the proofs of the mains results for the $p_n$-GERW, let us point out that, for simplicity, henceforth we will work with

$$B^n_\cdot := \frac{X_{\lfloor n \cdot \rfloor}}{n^{1/2}} \ .$$

The second sum portion in $\hat{B}^n_t$ (see (5)) makes this process continuous in $t \in [0, \infty)$. By Proposition 10.4 in Chapter 3 in [7] if we have the convergence in distribution for $B^n_\cdot$ automatically we obtain it for $\hat{B}^n_\cdot$.

3.1. **Proof of the convergence in distribution of the $p_n$-GERW* with $\beta > 1/2$.** The $p_n$-GERW* can be written as in (1). Remembering, by the conditions of the model, we have that the sequence $\{U_i\}_{i\geq 1}$ will be uncorrelated, rather than independent, with both sequences $\{\gamma_i\}_{i\geq 1}$ and $\{\xi_i\}_{i\geq 1}$.

We rewrite a version of the Theorem 7.1.4 from [7] where the authors state a convergence in distribution of a $d$-martingale to a process with independents Gaussian increments. We will denote $\xrightarrow{\mathcal{D}}$ as convergence in distribution.

**Theorem 3.1** (see [7], Theorem 7.1.4)**.** *Let $(\phi_k, k \geq 1)$ be a sequence of $\mathbb{R}^d$-valued random vectors such that $\mathbb{E}[\phi_k | \mathcal{F}_{k-1}] = 0$ where $\mathcal{F}_k = \sigma(\phi_l, l \leq k)$. Define,*

$$M_{\lfloor nt \rfloor} = \sum_{i=1}^{\lfloor nt \rfloor} \phi_i \quad \text{ and } \quad A_{\lfloor nt \rfloor} = \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \phi_i \phi_i^T \ .$$

*Assume that the following conditions hold:*

i)

$$\lim_{n \to \infty} n^{-1/2} \mathbb{E}\left[ \sup_{1 \leq k \leq n} |M_{k-1} - M_k| \right] = 0 \ .$$

ii) *For each $t \geq 0$,*

$$A_{\lfloor nt \rfloor} \to C(t) \quad \text{as } n \to \infty \ ,$$

*in probability.*

*Then $M_{\lfloor n \cdot \rfloor}/n^{1/2} \xrightarrow{\mathcal{D}} Z_\cdot$ , where $Z$ is a process with independent Gaussian increments.*

We present below a result that will be important in the proof of Theorem 1.4.

**Lemma 3.1.** *Let the process $X$ be a $p_n$-GERW* in direction $\ell$, in $\mathbb{Z}^d$, with $d \geq 2$, $p_n = \mathcal{C} n^{-\beta} \wedge 1$, with $\beta > 1/2$. Using the representation (41) for $X$, let us consider the corresponding process defined as*

$$D_{\lfloor nt \rfloor} = \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} 1_{\{E^c_{i-1} \cap \{U_i \leq i^{-\beta}\}\}} (\gamma_i - \xi_i) \ .$$

*It holds that $D_{\lfloor n \cdot \rfloor}$ converges in $C_{\mathbb{R}^d}[0, \infty)$ to the identically zero function in probability.*

The proof of Lemma 3.1 will be postponed to the end of this section.

The main idea behind the proof of Theorem 1.4 is to use the decomposition (41) and to analyze separately the sum portions suitably rescaled. We will see that the sum portion corresponding to the part of $d$-martingale will converge in distribution and the other will converge to zero in probability. Thus, using Slutsky's Theorem we obtain the desired result.

*Proof of Theorem 1.4.* Without loss of generality, we shall assume $\mathcal{C} = 1$. By the last equality in (41) we can rewrite the the process $B_t^n$ as

$$B_t^n = \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i + \frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} 1_{\{E_{i-1}^c \cap \{U_i \leq i^{-\beta}\}\}} (\gamma_i - \xi_i)\,. \tag{42}$$

Now we will analyze separately the two portion sums of (42).

The second sum portion in (42) by Lemma 3.1 we have that

$$\frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor nt \rfloor} 1_{\{E_{i-1}^c\}} 1_{\{U_i \leq i^{-\beta}\}} \gamma_i - 1_{\{E_{i-1}^c\}} 1_{\{U_i \leq i^{-\beta}\}} \xi_i \to 0 \quad \text{as } n \to \infty\,, \tag{43}$$

in probability in $C_{\mathbb{R}^d}[0,\infty)$.

For the first sum potion in (42) we use Theorem 3.1 (Theorem 7.1.4 from [7]) and we obtain

$$\frac{1}{n^{1/2}} \sum_{i=1}^{\lfloor n \cdot \rfloor} \xi_i \xrightarrow{\mathcal{D}} Z. \quad \text{as } n \to \infty\,, \tag{44}$$

where $Z$ is a unique, in distribution, process with independent Guassian increments by Theorem 7.1.1 from [7].

Then by (43), (44), and Slutsky's Theorem (see Theorem 11.4 from [8]) we finish the proof. ☐

**Remark 3.1.** *As we already point out in Remark 1.2 the item $i$) of Condition $I^*$ and the sequence $\{p_n\}_{n\geq 1}$ can be more general. Specifically, if we had $\sum_{i=1}^{\lfloor nt \rfloor} p_i \mathbb{E}[||\gamma_i||] = o(\sqrt{n})$, then it will be possible to prove that the second sum portion in (42) goes to zero in probability. Hence we finish the proof with Slutsky's Theorem (Theorem 11.4 from [8]).*

From the Theorem 1.4 it is possible to obtain Corollary 1.1, which states that the $p_n$-ERW converges in distribution to a standard Brownian Motion.

The proof of Corollary 1.1 follows basically the same techniques used in the proof of Theorem 1.4. The main difference is in (44) where instead of using Theorem 3.1 (Theorem 7.1.4 from [7]), we apply Donsker's Theorem (see Theorem 8.2 from [5] or Theorem 5.1.2 part ($c$) from [7]) and we obtain the desired result.

3.1.1. *Proof of Lemma 3.1.* Let us define the following processes in $C_{\mathbb{R}^d}[0,T]$

$$D^{\gamma}_{\lfloor n\cdot\rfloor} := \frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor nt\rfloor} 1_{\{E^c_{i-1}\}}1_{\{U_i\le i^{-\beta}\}}\gamma_i \quad \text{and}$$

$$D^{\xi}_{\lfloor n\cdot\rfloor} := \frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor nt\rfloor} 1_{\{E^c_{i-1}\}}1_{\{U_i\le i^{-\beta}\}}\xi_i \,.$$

First we will prove that the process $D^{\gamma}_{\lfloor n\cdot\rfloor}$ converges in probability to zero in $C_{\mathbb{R}^d}[0,T]$ for all $T>0$. One can see that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le T}\left\|D^{\gamma}_{\lfloor nt\rfloor}\right\| > \varepsilon\right) \le \mathbb{P}\left(\sup_{0\le t\le T}\sum_{i=1}^{\lfloor nt\rfloor}\left\|1_{\{E^c_{i-1}\cap\{U_i\le i^{-\beta}\}\}}\gamma_i\right\| > \varepsilon n^{\frac{1}{2}}\right)\\
&\le \mathbb{P}\left(\sum_{i=1}^{\lfloor nT\rfloor}\left\|1_{\{E^c_{i-1}\cap\{U_i\le i^{-\beta}\}\}}\gamma_i\right\| > \varepsilon n^{\frac{1}{2}}\right) \le \mathbb{P}\left(\sum_{i=1}^{\lfloor nT\rfloor}\left\|1_{\{U_i\le i^{-\beta}\}}\gamma_i\right\| > \varepsilon n^{\frac{1}{2}}\right)\\
&\le \frac{1}{n^{1/2}\varepsilon}\sum_{i=1}^{\lfloor nT\rfloor}\frac{1}{i^{\beta}}\mathbb{E}\left[\|\gamma_i\|\right] \le \frac{1}{n^{1/2}\varepsilon}\sum_{i=1}^{\lfloor nT\rfloor}\frac{\mathbb{E}\left[\|\gamma_i\|\right]}{i^{\theta}}\times\frac{1}{i^{\beta-\theta}}\,.
\end{aligned} \tag{45}$$

In the first inequality in (45) we use the triangle inequality and in the forth Markov's Inequality. By Condition I*, we know that for all $i\ge 1$ and all $\theta<\beta-1/2$ there exists a positive constant $L$ such that

$$\frac{\mathbb{E}\left[\|\gamma_i\|\right]}{i^{\theta}} \le L\,. \tag{46}$$

Then using (46) on the last inequality in (45), we obtain that

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\le t\le T}\left\|D_{\lfloor nt\rfloor}\right\| > \varepsilon\right) &\le \frac{1}{n^{1/2}\varepsilon}\sum_{i=1}^{\lfloor nT\rfloor}\frac{\mathbb{E}\left[\|\gamma_i\|\right]}{i^{\theta}}\times\frac{1}{i^{\beta-\theta}}\\
&\le \frac{L}{n^{1/2}\varepsilon}\sum_{i=1}^{\lfloor nT\rfloor}\frac{1}{i^{\beta-\theta}} \le \frac{L}{n^{1/2}\varepsilon}\times c'\lfloor nT\rfloor^{1-\beta+\theta}\\
&\le \frac{Lc'}{\varepsilon}\times\frac{\lfloor nT\rfloor}{n}\times\frac{n^{1/2}}{\lfloor nT\rfloor^{\beta-\theta}}\,,
\end{aligned} \tag{47}$$

where $c'$ is a positive constant, since $\sum_{i=1}^{\lfloor nt\rfloor}\frac{1}{i^{\beta}} = O(\lfloor nt\rfloor^{1-\beta})$.

Since $\beta > 1/2+\theta$, by (46) we obtain that

$$\frac{\sum_{i=1}^{\lfloor nt\rfloor}1_{\{E^c_{i-1}\}}1_{\{U_i\le i^{-\beta}\}}\gamma_i}{n^{1/2}} \to 0 \quad \text{as } n\to\infty\,,$$

in probability in the space $C_{\mathbb{R}^d}[0,T]$ for all $T>0$.

All the computation we did above can be similarly done for $D^{\xi}_{\lfloor n\cdot\rfloor}$, since we have part $i$) from Condition I*, and we will reach the same conclusion, namely that

$$\frac{\sum_{i=1}^{\lfloor nt\rfloor} 1_{\{E^c_{i-1}\}}1_{\{U_i\le i^{-\beta}\}}\xi_i}{n^{1/2}} \to 0 \quad \text{as } n\to\infty\,,$$

in probability in the space $C_{\mathbb{R}^d}[0,T]$ for all $T>0$.

Finally we have that $D_{\lfloor n\cdot\rfloor}$ converges in probability to zero in $C_{\mathbb{R}^d}[0,T]$ for all $T>0$. Since this convergence in $C_{\mathbb{R}^d}[0,T]$ for all $T>0$ means that $D_{\lfloor n\cdot\rfloor}$ converges in probability to zero in $C_{\mathbb{R}^d}[0,\infty)$ (see Lemma C.2), we obtain the desired result.

□

3.2. **Proof of the convergence in distribution of the $p_n$-ERW with $\beta = 1/2$ and $d=2$.** Let us denote $K_n$ the set of times in which the process visits a site for the first time and it eats a cookie (i.e., it gains a drift) until time $n$. Henceforth, without loss of generality, we assume $\mathcal{C}=1$, i.e., $p_n = n^{-1/2}$. We can write $K_n$ as

$$K_n = \{i\in\{1,2,\dots,n\} : 1_{\{E^c_{i-1}\}}1_{\{U_i\le i^{-1/2}\}} = 1\}\,.$$

Now we set a sequence of $\mathcal{F}$-stopping times $\{\varphi_i\}_{i\ge1}$, corresponding to the times the $p_n$-ERW visits a new site. We can write the cardinality of the set $K_n$ as follows:

$$|K_n| = \sum_{i=1}^{n} 1_{\{E^c_{i-1}\}}1_{\{U_i\le i^{-1/2}\}} = \sum_{j=1}^{|\mathcal{R}^X_n|} 1_{\{U_{\varphi_j}\le\varphi_j^{-1/2}\}}\,. \tag{48}$$

Below we present a important auxiliary result which will be useful in the proof of Theorem 1.5.

**Lemma 3.2.** *Let $|K_n|$ be defined as in (48). We have that the sequence of processes $|K_{\lfloor n\cdot\rfloor}|/n^{1/2}$ converges in $C_{\mathbb{R}}[0,\infty)$ to the identically zero function in probability.*

The proof of Lemma 3.2 will be postponed at the end of this section.

*Proof of Theorem 1.5.* The idea of the proof of Theorem 1.5 is similar to the one used in Theorem 1.4, however to extend the result of Lemma 3.1 in the case $d=2$ and $\beta=1/2$ we will use Proposition 1.1 (see, proof of Lemma 3.2).

By the last equality in (41) and the definition of $K_n$, we can rewrite the the process $B^n_t$ as

$$
\begin{aligned}
B^n_t &= \frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor nt\rfloor}\xi_i + \frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor nt\rfloor} 1_{\{E^c_{i-1}\cap\{U_i\le i^{-1/2}\}\}}(\gamma_i-\xi_i) \\
&= \frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor nt\rfloor}\xi_i + \frac{1}{n^{1/2}}\sum_{i\in K_{\lfloor nt\rfloor}}(\gamma_i-\xi_i) \qquad (49)\\
&= \frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor nt\rfloor}\xi_i + \frac{|K_{\lfloor nt\rfloor}|}{n^{1/2}}\sum_{i\in K_{\lfloor nt\rfloor}}\frac{(\gamma_i-\xi_i)}{|K_{\lfloor nt\rfloor}|}\,.
\end{aligned}
$$

Applying Donsker's Theorem (Theorem 5.1.2 part $(c)$ from [7]) we obtain that the first sum portion of (49) converges in $C_{\mathbb{R}^2}[0,\infty)$ to a Brownian Motion in distribution, i.e.,

$$
\frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor n\cdot\rfloor}\xi_i \xrightarrow{\mathcal{D}} W. \quad \text{as } n\to\infty\,, \tag{50}
$$

where $W$ is a Brownian Motion in dimension 2 with zero mean vector and covariance matrix $\mathbb{E}[\xi_1\xi_1^T]$.

By Lemma 3.2 we have that

$$
\frac{|K_{\lfloor n\cdot\rfloor}|}{n^{1/2}} \to 0 \quad \text{as } n\to\infty\,, \tag{51}
$$

in the space $C_{\mathbb{R}}[0,\infty)$ in probability. We shall now consider two cases: $\lim_{n\to\infty}|K_{\lfloor nt\rfloor}|<\infty$ almost surely and $\lim_{n\to\infty}|K_{\lfloor nt\rfloor}|=+\infty$ almost surely.

If $\lim_{n\to\infty}|K_{\lfloor nt\rfloor}|<\infty$, then there exists a positive constant $L$ such that

$$
\lim_{n\to\infty}\sum_{i\in K_{\lfloor nt\rfloor}}\frac{\|\gamma_i-\xi_i\|}{|K_{\lfloor nt\rfloor}|}\le 2LK\,, \tag{52}
$$

where $K$ is from Condition I. Hence from (51) and (52) we have

$$
\frac{|K_{\lfloor nt\rfloor}|}{n^{1/2}}\sum_{i\in K_{\lfloor nt\rfloor}}\frac{(\gamma_i-\xi_i)}{|K_{\lfloor nt\rfloor}|}\to 0 \quad \text{as } n\to\infty\,, \tag{53}
$$

in the space $C_{\mathbb{R}^2}[0,\infty)$ in probability. Then from (50), (55) and Slutsky's Theorem (see Theorem 11.4 from [8]) we have our result.

If $|K_{\lfloor nt\rfloor}|\to\infty$, almost surely as $n\to\infty$, since the sequence of random vectors $\{\gamma_{\varphi_i}-\xi_{\varphi_i}\}_{i\ge1}$ is i.i.d. having the same distribution as $\{\gamma_i-\xi_i\}_{i\ge1}$, which is i.i.d. too (see Lemma C.3), we can use Theorem 8.2 item $iii$, page 303, in [8] and obtain

$$
\sum_{i\in K_{\lfloor nt\rfloor}}\frac{(\gamma_i-\xi_i)}{|K_{\lfloor nt\rfloor}|} = \sum_{i=1}^{|K_{\lfloor nt\rfloor}|}\frac{(\gamma_{\varphi_i}-\xi_{\varphi_i})}{|K_{\lfloor nt\rfloor}|}\to\mathbb{E}[\gamma_1-\xi_1] \quad \text{as } n\to\infty \text{ a.s..} \tag{54}
$$

Thus from (51) and (54) we have

$$\frac{|K_{\lfloor nt\rfloor}|}{n^{1/2}}\sum_{i\in K_{\lfloor nt\rfloor}}\frac{(\gamma_i-\xi_i)}{|K_{\lfloor nt\rfloor}|}\to 0 \quad \text{as } n\to\infty\,, \tag{55}$$

in the space $C_{\mathbb{R}^2}[0,\infty)$ in probability. Hence from (50), (55) and Slutsky's Theorem (see Theorem 11.4 from [8]) we have our result.

□

3.2.1. *Proof of Lemma 3.2.* The first step here will be to prove that the process converges in $C_{\mathbb{R}}[0,T]$ for all $T>0$ to the identically zero function in probability.

For every $\varepsilon>0$ and $\delta>0$, by Markov's inequality we have that

$$\begin{aligned}
&\mathbb{P}\left[\sup_{0\le t\le T}|K_{\lfloor nt\rfloor}|>\varepsilon\sqrt{n}\right]=\mathbb{P}\left[|K_{\lfloor nT\rfloor}|>\varepsilon\sqrt{n}\right]\\
&=\mathbb{P}[G\cap\{|\mathcal{R}^X_{\lfloor nT\rfloor}|>\delta\lfloor nT\rfloor\}]+\mathbb{P}[G\cap\{|\mathcal{R}^X_{\lfloor nT\rfloor}|\le\delta\lfloor nT\rfloor\}]\\
&\le\mathbb{P}[|\mathcal{R}^X_{\lfloor nT\rfloor}|>\delta\lfloor nT\rfloor]+\mathbb{P}\left[\sum_{i=1}^{\lceil\delta nT\rceil}1_{\{U_i\le i^{-1/2}\}}>\varepsilon\sqrt{n}\right]\\
&\le\mathbb{P}[|\mathcal{R}^X_{\lfloor nT\rfloor}|>\delta\lfloor nT\rfloor]+\frac{1}{\varepsilon\sqrt{n}}\sum_{i=1}^{\lceil\delta nT\rceil}\frac{1}{i^{1/2}}\,.
\end{aligned}\tag{56}$$

We set the event $G_n:=\{\sup_{0\le t\le T}|K_{\lfloor nt\rfloor}|>\varepsilon\sqrt{n}\}$ and from (56) we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\ \mathbb{P}\left[G_n\right]\le\limsup_{n\to\infty}\left(\mathbb{P}[|\mathcal{R}^X_{\lfloor nT\rfloor}|>\delta\lfloor nT\rfloor]+\frac{1}{\varepsilon\sqrt{n}}\sum_{i=1}^{\lceil\delta nT\rceil}\frac{1}{i^{1/2}}\right)\\
&\le\limsup_{n\to\infty}\mathbb{P}[|\mathcal{R}^X_{\lfloor nT\rfloor}|>\delta\lfloor nT\rfloor]+\limsup_{n\to\infty}\left(\frac{1}{\varepsilon\sqrt{n}}\sum_{i=1}^{\lceil\delta nT\rceil}\frac{1}{i^{1/2}}\right)\le\frac{c'(\delta T)^{1/2}}{\varepsilon}\,.
\end{aligned}\tag{57}$$

The last inequality in (57) follows from Proposition 1.1 and the fact that $\sum_{i=1}^{\lceil\delta n\rceil}\frac{1}{i^{1/2}}=\Theta(\lceil\delta n\rceil^{1/2})$. Thus, for every $\varepsilon$ fixed, since $\delta$ is arbitrary, we conclude that the process $|K_{\lfloor n\cdot\rfloor}|/n^{1/2}$ converges in $C_{\mathbb{R}}[0,T]$ for all $T>0$ to the identically zero function in probability.

The convergence in $C_{\mathbb{R}}[0,T]$ for all $T>0$ means that $|K_{\lfloor n\cdot\rfloor}|/n^{1/2}$ converges in $C_{\mathbb{R}}[0,\infty)$ to the identically zero function in probability (see, Lemma C.2).

□

3.3. **Proof of the convergence in distribution of the $p_n$-ERW with $\beta=1/2$ and $d\ge 4$.** We begin this section defining a useful coupling of the $p_n$-ERW and an aperiodic random walk. Before we start, let us just remember that he $p_n$-ERW has a bias parameter $q\in(1/2,1]$.

The $p_n$-ERW which we consider in this section has a restriction on the drift direction. Specifically, the drift direction could be any direction of the unitary sphere, as long as, it could be represented by at most $d-3$ canonical directions. If we denote the set $D_k \subset \{e_1, \ldots, e_d\}$, where $k = |D_k|$ and $1 \leq k \leq d-3$, a $p_n$-ERW with a drift direction $\ell_{D_k} \in \mathbb{S}^{d-1}$ means that the direction $\ell_{D_k}$ has to be write as $\ell_{D_k} = \sum_{i=1}^{k} \alpha_i x_i$, where $\alpha_i \in [0,1]$ and $x_i \in D_k$ for all $1 \leq i \leq k$.

We set $(\Omega, \mathcal{G}, P)$ as the probability space where all the following random variables are defined. Let $\{U_i'\}_{i\geq 1}$ be a sequence of i.i.d. random variables uniformly distributed in [0,1]. We set the sequences $\{\xi_i\}_{i\geq 1}$ and $\{\gamma_i\}_{i\geq 1}$ as before (see Section 1.1.2) such that the drift direction of $\{\gamma_i\}_{i\geq 1}$ is $\ell_{D_k}$; both independent of the sequence $\{U_i'\}_{i\geq 1}$. We also define $\mathcal{P}_{D_k}$ and $\mathcal{P}_{D_k^c}$ as the projections respectively on $D_k$ and $D_k^c$, where $D_k^c := \{e_1, e_2, \ldots, e_d\}/D_k$. As one last but important restriction, we suppose that $\xi_i, \gamma_i \in \mathcal{P}_{D_k}(\mathbb{Z}_d) \cup \mathcal{P}_{D_k^c}(\mathbb{Z}_d)$, and moreover $\mathcal{P}_{D_k^c}(\xi_i)$ and $\mathcal{P}_{D_k^c}(\gamma_i)$ are identically distributed.

We define the sequences $\{Y_i\}_{i\geq 0}$ and $\{Z_i\}_{i\geq 0}$ of random vectors in $\mathbb{Z}^d$ in the following way. First, $Y_0 = 0$ and $Z_0 = 0$. Then for $n \geq 1$ we set recursively, if we have $B_n := \{Y_n \notin \mathcal{R}_{n-1}^Y\}$, then

$$Y_{n+1} = Y_n + 1_{B_n} 1_{\{U'_{n+1} \leq p_{n+1}\}} \gamma_{n+1} + 1_{B_n} 1_{\{U'_{n+1} > p_{n+1}\}} \xi_{n+1} + 1_{B_n^c} \xi_{n+1} .$$

Now for the process $\{Z_i\}_{i\geq 0}$ and $n \geq 1$, we have

$$Z_{n+1} = Z_n + \mathcal{P}_{D_k^c}(1_{B_n}\xi_{n+1} + 1_{B_n^c}(1_{\{U'_{n+1} > p_{n+1}\}}\xi_{n+1} + 1_{\{U'_{n+1} \leq p_{n+1}\}}\gamma_{n+1})) .$$

Then we conclude the following properties:

- $\{Y_i\}_{i\geq 0}$ is a $p_n$-ERW in $\mathbb{Z}^d$ with drift direction $\ell_{D_k}$.
- $\{Z_i\}_{i\geq 0}$ is an aperiodic random walk (lazy random walk) in $\mathbb{Z}^d$. Besides that the process $Z$ behaves like a lazy random walk in $\mathbb{Z}^{d-k}$.
- For all $e_j \in D_k^c$ and $i \geq 0$, we have $Y_i \cdot e_j = Z_i \cdot e_j$.

Hence we will show that if the process $\{Z_i\}_{i\geq 0}$ visits a new site then the process $\{Y_i\}_{i\geq 0}$ will reach a new site too.

**Lemma 3.3.** *If the process $\{Z_i\}_{i\geq 0}$ visits a new site then $\{Y_i\}_{i\geq 0}$ visits too.*

As a direct consequence of Lemma 3.3 we obtain that $|\mathcal{R}_n^Y| \geq |\mathcal{R}_n^Z|$ for all $n \geq 0$. The proof of Lemma 3.3 will be deferred to end of this section.

Henceforth, without loss of generality, we shall assume $\mathcal{C} = 1$, i.e., $p_n = n^{-1/2}$. Let us recall that $K_n$ is the set of times the process visits a site for the first time and it eats a cookie and $|K_n|$ is its cardinality. Let $X$ be a $p_n$-ERW in $\mathbb{Z}^d$, where $d \geq 4$. Let us define the following random variable

$$|J_n'| := \sum_{i=1}^{\delta n} 1_{\{U_i \leq i^{-1/2}\}} , \tag{58}$$

where $\delta \in (\pi_d, 1)$ and $\pi_d$ denote the probability of a random walk with i.i.d. increments (with zero mean and finite variance) given by the corresponding $\{\xi_i\}_{i\geq 0}$ never returning to the origin.

If $\{\varphi_i\}_{i\geq 1}$ denotes the sequence of $\mathcal{F}$-stopping times corresponding to the times the $p_n$-ERW visits a new site, then we have

$$|K_n| = \sum_{j=1}^{|\mathcal{R}_n^X|} 1_{\{U_{\varphi_j} \leq \varphi_j^{-1/2}\}} ,$$

and

$$|J_n| := \sum_{i=1}^{\delta n} 1_{\{U_{\varphi_i} \leq \varphi_i^{-1/2}\}} \preceq |J_n'| , \tag{59}$$

since $\{\varphi_i\}_{i\geq 1}$ is a sequence of of $\mathcal{F}$-stopping times and the sequence $\{U_i\}_{i\geq 1}$ is i.i.d..

Now we denote the following random variable

$$\sum_{i=n-\delta' n+1}^{n} 1_{\{U_i \leq i^{-1/2}\}} = \sum_{i=1}^{n} 1_{\{U_i \leq i^{-1/2}\}} - \underbrace{\sum_{i=1}^{n-\delta' n} 1_{\{U_i \leq i^{-1/2}\}}}_{:=|F_n'|} , \tag{60}$$

where $\delta' \in (0, \pi_{d-k})$ and $\pi_{d-k}$ denotes the probability of a random walk with i.i.d. increments (with zero mean and finite variance) given by the corresponding lazy random walk of the coupling never returning to the origin.

Since $\{\varphi_i\}_{i\geq 1}$ is a sequence of $\mathcal{F}$-stopping times and the sequence $\{U_i\}_{i\geq 1}$ is i.i.d., one can note that

$$\sum_{i=1}^{n} 1_{\{U_i \leq i^{-1/2}\}} - |F_n'| \preceq \sum_{i=1}^{\delta' n} 1_{\{U_{\varphi_i} \leq \varphi_i^{-1/2}\}} := |V_n| . \tag{61}$$

The random variables $|J_n'|$ and $|F_n'|$ will be important to compute the constants $c_1$ and $c_2$, as we will see in the proof of Theorem 1.6. Informally speaking, what we obtain in Theorem 1.6 is that every limit point of the $p_n$-ERW in direction $\ell_{D_k}$ suitably rescaled will be in a kind a "cone" region, with high probability (see, Figure 3.1).

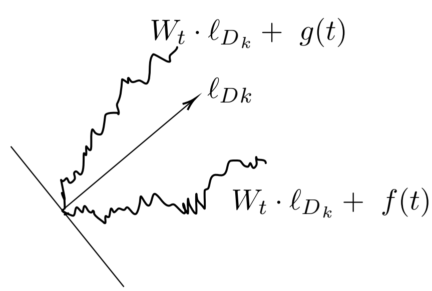


FIGURE 3.1. "Cone" region representation around the direction $\ell_{D_k}$.

Now in the next two Lemmas we will see the asymptotic behavior of $|J'_n|$ and $|F'_n|$ respectively. We will conclude that both random variables converges when divided by $n^{1/2}$.

**Lemma 3.4.** *Let $\{|J'_n|\}_{n\geq 1}$ be defined as in (58), $\{|F'_n|\}_{n\geq 1}$ be defined as in (60), $\delta$ and $\delta'$ are positive constants such that $\delta \in (\pi_d, 1)$ and $\delta' \in (0, \pi_{d-k})$. Then we have*

i)
$$\frac{\mathbb{E}[|J'_n|]}{n^{1/2}} \to 2\delta^{1/2} \quad \text{as } n \to \infty \text{ a.s.}.$$

ii) *For any $\varepsilon > 0$, we obtain*
$$\mathbb{P}[||J'_n| - \mathbb{E}[|J'_n|]| > \varepsilon n^{1/2}] \to 0 \quad \text{as } n \to \infty\,.$$

iii)
$$\frac{\mathbb{E}[|F'_n|]}{n^{1/2}} \to 2(1-\delta')^{1/2} \quad \text{as } n \to \infty \text{ a.s.}.$$

iv) *For any $\varepsilon > 0$, we obtain*
$$\mathbb{P}[||F'_n| - \mathbb{E}[|F'_n|]| > \varepsilon n^{1/2}] \to 0 \quad \text{as } n \to \infty\,.$$

v)
$$\frac{\mathbb{E}[\sum_{i=1}^n 1_{\{U_i \leq i^{-1/2}\}}]}{n^{1/2}} \to 2 \quad \text{as } n \to \infty \text{ a.s.}.$$

vi) *For any $\varepsilon > 0$, we obtain*
$$\mathbb{P}\left[\left|\sum_{i=1}^n 1_{\{U_i \leq i^{-1/2}\}} - \mathbb{E}\left[\sum_{i=1}^n 1_{\{U_i \leq i^{-1/2}\}}\right]\right| > \varepsilon n^{1/2}\right] \to 0 \quad \text{as } n \to \infty\,.$$

The proof of Lemma 3.4 will be postponed at the end of this section. From Lemma 3.4 we obtain the following Corollary.

**Corollary 3.1.** *Let $\{|J'_n|\}_{n\geq 1}$ be defined as in (58), $\{|F'_n|\}_{n\geq 1}$ be defined as in (60), $\delta$ and $\delta'$ are positive constants such that $\delta \in (\pi_d, 1)$ and $\delta' \in (0, \pi_{d-k})$. Then we have*

i)
$$\frac{|J'_n|}{n^{1/2}} \to 2\delta^{1/2} \quad \text{as } n \to \infty \text{ in probability.}$$

ii)
$$\frac{|F'_n|}{n^{1/2}} \to 2(1-\delta')^{1/2} \quad \text{as } n \to \infty \text{ in probability.}$$

iii)
$$\frac{\sum_{i=1}^n 1_{\{U_i \leq i^{1/2}\}}}{n^{1/2}} \to 2 \quad \text{as } n \to \infty \text{ in probability.}$$

Relying on Corollary 3.1, we are able to prove the following result about the sequence of processes corresponding to the $|J'_n|$ and $|F'_n|$.

**Lemma 3.5.** *Let* $|J'_{\lfloor n\cdot\rfloor}|$ *and* $|F'_{\lfloor n\cdot\rfloor}|$ *be a sequence of processes in* $C_{\mathbb{R}}[0,\infty)$. *Then we have the following*

i)

$$\frac{|J'_{\lfloor n\cdot\rfloor}|}{n^{1/2}} \to 2(\delta\cdot)^{1/2} \text{ as } n\to\infty\,,$$

*in distribution in the space* $C_{\mathbb{R}}[0,\infty)$.

ii)

$$\frac{\sum_{i=1}^{\lfloor n\cdot\rfloor} 1_{\{U_i\leq i^{-1/2}\}} - |F'_{\lfloor n\cdot\rfloor}|}{n^{1/2}} \to 2(\cdot)^{1/2}(1-(1-\delta')^{1/2}) \text{ as } n\to\infty\,,$$

*in distribution in the space* $C_{\mathbb{R}}[0,\infty)$.

The proof of Lemma 3.5 will be postponed at the end of this section.

The next result states that the sequence of processes $|K_{\lfloor n\cdot\rfloor}|/n^{1/2}$ is tight in $C_{\mathbb{R}}[0,\infty)$.

**Lemma 3.6.** *The sequence* $|K_{\lfloor n\cdot\rfloor}|/n^{1/2}$ *is tight in the space* $C_{\mathbb{R}}[0,\infty)$.

A direct consequence of Lemma 3.6 is that every sub-sequence of $|K_{\lfloor n\cdot\rfloor}|/n^{1/2}$ will converge in distribution to a limit point. In the next result we will bound those limits points. The proof of Lemma 3.6 will be postponed at the end of this section.

**Proposition 3.1.** *If* $H_{\cdot}$ *is a limit point of* $|K_{\lfloor n\cdot\rfloor}|/n^{1/2}$, *then*

$$\mathbb{P}\left[\forall t\in[0,\infty) : 2t^{1/2}(1-(1-\delta')^{1/2}) \leq H_t \leq 2(t\delta)^{1/2}\right] = 1\,,$$

*where* $\delta'$ *and* $\delta$ *are positive constants such that* $\delta'\in(0,\pi_{d-k})$ *and* $\delta\in(\pi_d,1)$.

The proof of Proposition 3.1 will be postponed at the end of this section.

Now we have all the auxiliaries results to prove Theorem 1.6. For the proof of Theorem 1.6 the main idea is to use the decomposition (41) and then analyze separately the sum portions suitably rescaled. We will see that both processes are tight in $C_{\mathbb{R}^d}[0,T]$, consequently we will obtain that the process $B^n_{\cdot}$ is tight too. Finally we will describe the processes which dominate stochastically the limit points of $B^n_{\cdot}$. The idea here is similar to the proof of Theorem 1.5. The main difference is that the sum portion that represents the drift direction will not go to zero. For this reason the random variables $|J'_n|$ and $|F'_n|$ will be important, to control this sum portion.

*Proof of Theorem 1.6.* By the last equality in (41) and the definition of $K_n$, we can rewrite the process $B^n_t$ as

$$
\begin{aligned}
B^n_t &= \frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor nt\rfloor}\xi_i + \frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor nt\rfloor} 1_{\{E^c_{i-1}\cap\{U_i\le i^{-1/2}\}\}}(\gamma_i-\xi_i) \\
&= \frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor nt\rfloor}\xi_i + \frac{1}{n^{1/2}}\sum_{i\in K_{\lfloor nt\rfloor}}(\gamma_i-\xi_i) \qquad (62)\\
&= \frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor nt\rfloor}\xi_i + \frac{|K_{\lfloor nt\rfloor}|}{n^{1/2}}\sum_{i\in K_{\lfloor nt\rfloor}}\frac{(\gamma_i-\xi_i)}{|K_{\lfloor nt\rfloor}|}\,.
\end{aligned}
$$

By Donsker's Theorem we obtain that the first sum portion of (62) converges in $C_{\mathbb{R}^d}[0,\infty)$ to a Brownian Motion in distribution

$$
\frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor n\cdot\rfloor}\xi_i \xrightarrow{\mathcal{D}} W. \quad \text{as } n\to\infty\,, \tag{63}
$$

where $W$ is a Brownian Motion in dimension $d$ with zero mean vector and covariance matrix $\mathbb{E}[\xi_1\xi_1^T]$.

Now we will see that the second sum portion in (62) is a sequence of stochastic processes which is tight. For that we will need two results from [5], Theorem 7.3 and Corollary on page 83. Let us denote

$$
D_{\lfloor nt\rfloor} := \frac{1}{n^{1/2}}\sum_{i=1}^{\lfloor nt\rfloor} 1_{\{E^c_{i-1}\cap\{U_i\le i^{-1/2}\}\}}(\gamma_i-\xi_i)\,.
$$

The first condition on Theorem 7.3 in [5] is satisfied, since the process $D_{\lfloor n\cdot\rfloor}$ is tight for $t=0$ for all $n\ge 1$. To prove that the process satisfies the second condition on Theorem 7.3 in [5] we will use the Corollary on page 83 in [5].

This Corollary states that Condition $(ii)$ of Theorem 7.3 in [5] holds if, for each positive $\varepsilon$ and $\eta$, there exists a $\phi\in(0,1)$, and an integer $n_0$ such that

$$
\frac{1}{\phi}P_n\left[f\in C_{\mathbb{R}^d}[0,T] : \sup_{t\le s\le t+\phi}|f(s)-f(t)|\ge\varepsilon\right]\le\eta \quad \forall n\ge n_0\,, \tag{64}
$$

where the probability measure $P_n$ on $C_{\mathbb{R}^d}[0,T]$ is the distribution of $D_{\lfloor n\cdot\rfloor}$.

Let us denote the set $A_t(\varepsilon,\phi) := \{f \in C_{\mathbb{R}^d}[0,T] : \sup_{t\le s\le t+\phi}|f(s) - f(t)| \ge \varepsilon\}$. Hence we obtain the following

$$\begin{aligned} P_n[A_t(\varepsilon,\phi)] &= \mathbb{P}\left[D_{\lfloor n\cdot\rfloor} \in A_t(\varepsilon,\phi)\right] = \mathbb{P}\left[\sup_{t\le s\le t+\phi}\left\|D_{\lfloor ns\rfloor} - D_{\lfloor nt\rfloor}\right\| \ge \varepsilon\right] \\ &= \mathbb{P}\left[\sup_{t\le s\le t+\phi}\left\|\sum_{i=\lfloor nt\rfloor+1}^{\lfloor ns\rfloor} 1_{\{E_{i-1}^c\cap\{U_i\le i^{-1/2}\}\}}(\gamma_i - \xi_i)\right\| \ge \varepsilon n^{\frac{1}{2}}\right]. \end{aligned} \tag{65}$$

Now we only analyze the process inside the probability measure in (65). We will find an upper bound for this process for all the trajectory. For all $s \in [t, t+\phi]$ we obtain

$$\begin{aligned} \left\|\sum_{i=\lfloor nt\rfloor+1}^{\lfloor ns\rfloor} 1_{\{E_{i-1}^c\cap\{U_i\le i^{-1/2}\}\}}(\gamma_i - \xi_i)\right\| &\le \sum_{i=\lfloor nt\rfloor+1}^{\lfloor ns\rfloor}\left\|1_{\{U_i\le i^{-1/2}\}}(\gamma_i - \xi_i)\right\| \\ \le \sum_{i=\lfloor nt\rfloor+1}^{\lfloor ns\rfloor}\left\|1_{\{U_i\le i^{-1/2}\}}(\gamma_i - \xi_i)\right\| &\le \sum_{i=\lfloor nt\rfloor+1}^{\lfloor ns\rfloor} 1_{\{U_i\le i^{-1/2}\}}2K\,. \end{aligned} \tag{66}$$

In the second inequality in (66) we have by triangle inequality and the last one by Condition I. Then from (64), (65) and (66) we have

$$\begin{aligned} \frac{1}{\phi}P_n[A_t(\varepsilon,\phi)] &\le \frac{1}{\phi}\mathbb{P}\left[\sup_{t\le s\le t+\phi}\left(\sum_{i=\lfloor nt\rfloor+1}^{\lfloor ns\rfloor} 1_{\{U_i\le i^{-1/2}\}}2K\right) \ge \varepsilon n^{\frac{1}{2}}\right] \\ &\le \frac{1}{\phi}\mathbb{P}\left[\sum_{i=\lfloor nt\rfloor+1}^{\lfloor n(t+\phi)\rfloor} 1_{\{U_i\le i^{-\frac{1}{2}}\}}2K \ge \varepsilon n^{\frac{1}{2}}\right] \\ &\le \frac{1}{\phi}\exp\left(\frac{-\varepsilon n^{\frac{1}{2}}}{2K}\right)\mathbb{E}\left[\exp\left(\sum_{i=\lfloor nt\rfloor+1}^{\lfloor n(t+\phi)\rfloor} 1_{\{U_i\le i^{-\frac{1}{2}}\}}\right)\right]. \end{aligned} \tag{67}$$

To obtain the second inequality in (67), we exponentiate both sides of the inequality inside the probability and then we use Markov's inequality.

We set $c = \varepsilon/2K$ an we continue the computation in (67)

$$\begin{aligned}
\frac{1}{\phi}P_n[A_t(\varepsilon,\phi)] &\le \frac{1}{\phi}e^{-cn^{\frac{1}{2}}}\prod_{i=\lfloor nt\rfloor+1}^{\lfloor n(t+\phi)\rfloor}\mathbb{E}\left[\exp\left(1_{\{U_i\le i^{-\frac{1}{2}}\}}\right)\right]\\
&\le \frac{1}{\phi}e^{-cn^{\frac{1}{2}}}\prod_{i=\lfloor nt\rfloor+1}^{\lfloor n(t+\phi)\rfloor}\left(1+\frac{e-1}{i^{\frac{1}{2}}}\right) \le \frac{1}{\phi}e^{-cn^{\frac{1}{2}}}\prod_{i=\lfloor nt\rfloor+1}^{\lfloor n(t+\phi)\rfloor}\exp\left(\frac{e-1}{i^{\frac{1}{2}}}\right)\\
&\le \frac{1}{\phi}e^{-cn^{\frac{1}{2}}}\exp\left(\sum_{i=\lfloor nt\rfloor+1}^{\lfloor n(t+\phi)\rfloor}\frac{e-1}{i^{\frac{1}{2}}}\right)\\
&\le \frac{1}{\phi}\exp\!\left(-cn^{\frac{1}{2}}\right)\exp\left(2(e-1)(\sqrt{n(t+\phi)}-\sqrt{nt}\right)\,.
\end{aligned} \tag{68}$$

The second inequality in (68) we obtain by the moment generating function and the third one by the fact that $x+1<e^x$ for all $x<1$.

At least remains to us explain how we bounded the summation in the last inequality in (68). For that we did the following

$$\sum_{i=\lfloor nt\rfloor+1}^{\lfloor n(t+\phi)\rfloor}\frac{e-1}{i^{\frac{1}{2}}} \le (e-1)\int_{nt}^{n(t+\phi)}x^{-1/2}dx \le 2(e-1)\left(\sqrt{n(t+\phi)}-\sqrt{nt}\right)\,.$$

Now if for each positive $\varepsilon$ and $\eta$, there exists a $\phi\in(0,1)$, and an integer $n_0$ such that

$$\frac{1}{\phi}\exp\!\left(-cn^{\frac{1}{2}}\right)\exp\left(2(e-1)\sqrt{n}(\sqrt{t+\phi}-\sqrt{t}\right)\le\eta\quad\forall n\ge n_0\,, \tag{69}$$

then we have that (64) is satisfied.

One can see that, since we have $\phi\in(0,1)$, for all $\hat{\varepsilon}>0$, there exists a $\phi'>\phi$ such that $|\sqrt{t+\phi'}-\sqrt{t}|<\hat{\varepsilon}$. Now we can choose $\hat{\varepsilon}=c/4(e-1)$ and we obtain, for a large enough $n$, that (69) is fulfilled for all $\eta$. Consequently we have that (64) is satisfied. Thus we can conclude the sequence of stochastic process $\{D_{\lfloor n\cdot\rfloor}\}_{n\ge1}$ is tight on $C_{\mathbb{R}^d}[0,T]$.

Since the process $B^n_\cdot$ is the sum of two tight processes in $C_{\mathbb{R}^d}[0,T]$, we obtain that $B^n_\cdot$ is a tight process in $C_{\mathbb{R}^d}[0,T]$ for all $T>0$ as a simple exercise (see Lemma C.1).

Now by Theorem 4.10 in [10] Chapter 2 one can see that since $B^n_\cdot$ is tight in $C_{\mathbb{R}^d}[0,T]$ for all $T>0$ with the topology of uniform convergence in the compacts then $B^n_\cdot$ is tight in $C_{\mathbb{R}^d}[0,\infty)$.

Now we will analyze other aspect of the second sum portion in (62) and by Proposition 3.1 we obtain that

$$\mathbb{P}\left[\forall t\in[0,\infty): 2t^{1/2}(1-(1-\delta')^{1/2})\le H_t\le 2(t\delta)^{1/2}\right]=1\,, \tag{70}$$

where $H_t$ is a limit point of a sub-sequence of $|K_{\lfloor n\cdot\rfloor}|/n^{1/2}$.

From (70) we have that $|K_{\lfloor nt\rfloor}|\to\infty$ as $n\to\infty$ almost surely. Then observe that, the sequence of random vectors $\{\gamma_{\varphi_i}-\xi_{\varphi_i}\}_{i\ge1}$ is i.i.d. and has the same distribution of $\{\gamma_i - \xi_i\}_{i\geq 1}$, which is i.i.d. too (see Lemma C.3). We can use Theorem 8.2 item *iii*, page 303, in [8] and obtain

$$\sum_{i\in K_{\lfloor nt\rfloor}} \frac{(\gamma_i - \xi_i)}{|K_{\lfloor nt\rfloor}|} \to \mathbb{E}[\gamma_1 - \xi_1] = \mathbb{E}[\gamma_1] \quad \text{as } n\to\infty \text{ a.s..} \tag{71}$$

From the properties of the sequence $\{\gamma_n\}_{n\geq 1}$ we have that $\lambda \leq \mathbb{E}[\gamma_i \cdot \ell_{D_k}] \leq K$ for all $i \geq 1$. Then we set $\mathbb{E}[\gamma_i \cdot \ell_{D_k}] = \mu_\gamma$ for all $i \geq 1$, since $\{\gamma_n\}_{n\geq 1}$ is an i.i.d. sequence of random vectors.

Thus from (70) and (71) we obtain that

$$\frac{|K_{\lfloor nt\rfloor}|}{n^{1/2}} \sum_{i\in K_{\lfloor nt\rfloor}} \frac{(\gamma_i - \xi_i)\cdot \ell_{D_k}}{|K_{\lfloor nt\rfloor}|},\ t\geq 0,$$

is tight in $C_{\mathbb{R}}[0,\infty)$ and any of its limit points $\widetilde{H}_t$ satisfies

$$\mathbb{P}\left[\forall t\in[0,\infty) : 2\mu_\gamma t^{\frac{1}{2}}(1-(1-\delta')^{\frac{1}{2}}) \leq \widetilde{H}_t \leq 2\mu_\gamma (t\delta)^{1/2}\right] = 1. \tag{72}$$

Since $B^n_\cdot$ is tight process in $C_{\mathbb{R}^d}[0,\infty)$ by Prohorov's Theorem (see Theorem 5.1 in [5]) we have that $B^n_\cdot$ is relatively compact and consequently every sub-sequence converges to a limit point.

Then for every limit point $Y_\cdot$ of a sub-sequence of $B^n_\cdot$, by (63) and (72) we obtain

$$W_t \cdot \ell_{D_k} + f(t) \preceq Y_t \cdot \ell_{D_k} \preceq W_t \cdot \ell_{D_k} + g(t)\,,$$

where $f(t) = \underbrace{2(1-(1-\delta')^{1/2})\mu_\gamma}_{c_1} t^{1/2}$ and $g(t) = \underbrace{2\mu_\gamma(\delta)^{1/2}}_{c_2} t^{1/2}$ for all $t \in [0,\infty)$.

The reader can check that $0 < c_1 \leq c_2$. Thus we finish the proof.

□

3.3.1. *About the restrictions in Theorem 1.6.* Let us stress that Theorem 1.6 has some restrictions on the dimension and in the drift direction. Those restrictions are due to the technique we used to find a lower bound for range of the $p_n$-ERW, namely the coupling of the $p_n$-ERW and the lazy random walk. The coupling of the $p_n$-ERW and a lazy random walk had a fundamental part in the proof of Theorem 1.6. Indeed, with this technique we could find a lower bound for the range of the $p_n$-ERW and describe the functions $g(t)$ and $f(t)$. However this method breaks down when, for example, we have a $p_n$-ERW in direction $\ell \in \mathbb{S}^{d-1}$ in $\mathbb{Z}^d$, and $\ell$ has to be writen with more than $d-3$ canonical directions of $\mathbb{Z}^d$. This is due to the fact that for the proof of Theorem 1.6 to work, we need that the lazy random walk, which is used in the coupling, must be at least 3 dimensional.

If Conjecture 1.2 holds true (see, Section 1.3), it would be possible to prove Theorem 1.6 for a $p_n$-ERW in $\mathbb{Z}^d$, with $d\geq 3$ and in direction $\ell \in \mathbb{S}^{d-1}$, that is, for all directions in the unit sphere, then completely avoiding the use of the coupling that we described at the beginning of this section. Specifically, if Conjecture 1.2 holds, it would be possible to compute a lower bound for the range of the $p_n$-ERW in direction $\ell$, and $\ell$ can be any in $\mathbb{S}^{d-1}$, thus extending Theorem 1.6 to a $p_n$-ERW in dimension 3.

3.3.2. *Proof of Lemma 3.3.* The idea of the proof is quite simple, we show, using the coupling, that is impossible the lazy random walk reaches a new site and the $p_n$-ERW do not.

*Proof.* Suppose that in time $n$ the process $\{Z_i\}_{i\geq 0}$ reaches a new site and $\{Y_i\}_{i\geq 0}$ do not.

Then the direction in which the process $\{Z_i\}_{i\geq 0}$ moves it was not $e_1$. By the definition of the process $\{Z_i\}_{i\geq 0}$.

From the hypothesis we have that $Z_n \notin \mathcal{R}^Z_{n-1}$. From the coupling we have that $(Z_n\cdot e_2, Z_n\cdot e_3,\dots,Z_n\cdot e_d) = (Y_n\cdot e_2, Y_n\cdot e_3,\dots,Y_n\cdot e_d)$. Since the process $\{Y_i\}_{i\geq 0}$ reaches a already visited site then $Y_n\in\mathcal{R}^Y_{n-1}$ and there exist a $0\leq k\leq n-1$ such that $Y_k = Y_n$.

Hence, we have $(Z_n\cdot e_2, Z_n\cdot e_3,\dots,Z_n\cdot e_d) = (Y_k\cdot e_2, Y_k\cdot e_3,\dots,Y_k\cdot e_d)$ and by the coupling we obtain that $(Z_k\cdot e_2, Z_k\cdot e_3,\dots,Z_k\cdot e_d) = (Y_k\cdot e_2, Y_k\cdot e_3,\dots,Y_k\cdot e_d)$. Since, by hypothesis $Z_n\neq Z_k$, then in $Z_n$, the process would have reached a new site moving in direction $e_1$, which is an absurd. □

3.3.3. *Proof of Lemma 3.4.* We begin with the proof of part $i$). One can see that

$$\frac{\mathbb{E}[|J'_n|]}{n^{1/2}} = \frac{1}{n^{1/2}}\mathbb{E}\left[\sum_{i=1}^{\delta n} 1_{\{U_i\leq i^{-1/2}\}}\right] = \frac{1}{n^{1/2}}\sum_{i=1}^{\delta n} i^{-1/2}\ .$$

Then we have that

$$\lim_{n\to\infty}\left(\frac{1}{n^{1/2}}\sum_{i=1}^{\delta n} i^{-1/2}\right) = 2\delta^{1/2}\ .$$

Hence we finish the proof of part $i$).

For the proof of part $ii$) we first use Chebyshev's inequality and obtain

$$\mathbb{P}[||J'_n| - \mathbb{E}[|J'_n|]| > \varepsilon n^{1/2}] \leq \frac{1}{\varepsilon^2 n}\mathrm{Var}\left[\sum_{i=1}^{\delta n} 1_{\{U_i\leq i^{-1/2}\}}\right]\ . \tag{73}$$

Then we use the fact that the random variables $\{U_i\}_{i\geq 1}$ are independent and have

$$\frac{1}{\varepsilon^2 n}\mathrm{Var}\left[\sum_{i=1}^{\delta n} 1_{\{U_i\leq i^{-1/2}\}}\right] = \frac{1}{\varepsilon^2 n}\sum_{i=1}^{\delta n} i^{-1/2}(1-i^{-1/2}) \to 0 \quad \text{as } n\to\infty\,. \tag{74}$$

Thus by (73) and (74) we obtain that

$$\mathbb{P}[||J'_n| - \mathbb{E}[|J'_n|]| > \varepsilon n^{1/2}] \to 0 \quad \text{as } n\to\infty\,,$$

and finish the proof of part $ii$).

Now we prove part $iii$). One can see that

$$\frac{\mathbb{E}[|F'_n|]}{n^{1/2}} = \frac{1}{n^{1/2}}\mathbb{E}\left[\sum_{i=1}^{n-\delta' n} 1_{\{U_i \le i^{-1/2}\}}\right] = \frac{1}{n^{1/2}} \sum_{i=1}^{n-\delta' n} i^{-1/2} \, .$$

Then we have that

$$\lim_{n\to\infty} \left( \frac{1}{n^{1/2}} \sum_{i=1}^{n-\delta' n} i^{-1/2} \right) = 2(1-\delta')^{1/2} \, .$$

Hence we finish the proof of part $i$).

For the proof of part $ii$) we first use Chebyshev's inequality and obtain

$$\mathbb{P}[||F'_n| - \mathbb{E}[|F'_n|]| > \varepsilon n^{1/2}] \le \frac{1}{\varepsilon^2 n}\mathrm{Var}\left[\sum_{i=1}^{n-\delta' n} 1_{\{U_i \le i^{-1/2}\}}\right] \, . \tag{75}$$

Then we use the fact that the random variables $\{U_i\}_{i\ge 1}$ are independent and have

$$\frac{1}{\varepsilon^2 n}\mathrm{Var}\left[\sum_{i=1}^{n-\delta' n} 1_{\{U_i \le i^{-1/2}\}}\right] = \frac{1}{\varepsilon^2 n} \sum_{i=1}^{n-\delta' n} i^{-1/2}(1-i^{-1/2}) \to 0 \quad \text{as } n \to \infty \, . \tag{76}$$

Thus by (75) and (76) we obtain that

$$\mathbb{P}[||F'_n| - \mathbb{E}[|F'_n|]| > \varepsilon n^{1/2}] \to 0 \quad \text{as } n \to \infty \, ,$$

and finish the proof of part $iv$).

We begin with the proof of part $v$). One can see that

$$\frac{\mathbb{E}[\sum_{i=1}^{n} 1_{\{U_i \le i^{-1/2}\}}]}{n^{1/2}} = \frac{1}{n^{1/2}} \sum_{i=1}^{n} i^{-1/2} \, .$$

Then we have that

$$\lim_{n\to\infty} \left( \frac{1}{n^{1/2}} \sum_{i=1}^{n} i^{-1/2} \right) = 2 \, .$$

Hence we finish the proof of part $v$).

For the proof of part $vi$) we use Chebyshev's inequality and the fact that the random variables $\{U_i\}_{i\ge 1}$ are independent. Then we obtain

$$\begin{aligned} &\mathbb{P}\left[\left|\sum_{i=1}^{n} 1_{\{U_i \le i^{-1/2}\}} - \mathbb{E}\left[\sum_{i=1}^{n} 1_{\{U_i \le i^{-1/2}\}}\right]\right| > \varepsilon n^{1/2}\right] \\ &\le \frac{1}{\varepsilon^2 n}\mathrm{Var}\left[\sum_{i=1}^{n} 1_{\{U_i \le i^{-1/2}\}}\right] = \frac{1}{\varepsilon^2 n}\sum_{i=1}^{n} i^{-1/2}(1-i^{-1/2}) \to 0 \, , \end{aligned} \tag{77}$$

as $n$ goes to infinity. Thus we finish the proof of part $vi$). $\square$

3.3.4. *Proof of Lemma 3.5.* Since by Corollary 3.1 we have the convergence of the finite dimensional distribution, to prove this Lemma, both parts $(i)$ and $ii)$), by Theorem 7.1 in [5] only remains to us prove that both sequences of processes are tight. For that we will proceed in the same way we did in the proof of Theorem 1.6, when we show the second sum portion in (62) is tight in $C_{\mathbb{R}}[0,T)$.

We start the proof of this Lemma with part $i$). One can notice that the process $|J'_{\lfloor n\cdot\rfloor}|/n^{1/2}$ is tight for $t=0$ for all $n\geq 1$. Thus first condition on Theorem 7.3 in [5] is satisfied.

Then let $P_n$ be a probability measure on $C_{\mathbb{R}}[0,T)$ and the distribution of $|J'_{\lfloor n\cdot\rfloor}|/n^{1/2}$. We denote the set $A_t(\varepsilon,\phi):=\{f\in C_{\mathbb{R}}[0,T]:\sup_{t\leq s\leq t+\phi}|f(s)-f(t)|\geq\varepsilon\}$. Hence we have

$$\begin{aligned}\frac{1}{\phi}P_n[|J'_{\lfloor n\cdot\rfloor}|/n^{1/2}\in A_t(\varepsilon,\phi)] &= \frac{1}{\phi}\mathbb{P}\left[\sup_{t\leq s\leq t+\phi}||J'_{\lfloor ns\rfloor}|-|J'_{\lfloor nt\rfloor}||\geq\varepsilon n^{\frac{1}{2}}\right]\\ &= \frac{1}{\phi}\mathbb{P}\left[\sum_{i=\delta\lfloor nt\rfloor+1}^{\delta\lfloor n(t+\phi)\rfloor}1_{\{U_i\leq i^{-\frac{1}{2}}\}}\geq\varepsilon n^{\frac{1}{2}}\right].\end{aligned}$$

From this point the computation is exactly the same we did in the proof of Theorem 1.6, when we show the second sum portion in (62) fulfills the second condition of Theorem 7.3 in [5].

Then we have that for each positive $\varepsilon$ and $\eta$, there exists a $\phi\in(0,1)$, and an integer $n_0$ such that

$$\frac{1}{\phi}P_n\left[f\in C_{\mathbb{R}}[0,T]:\sup_{t\leq s\leq t+\phi}|f(s)-f(t)|\geq\varepsilon\right]\leq\eta\quad\forall n\geq n_0\,.$$

Ergo by Theorem 7.3 in [5] we obtain that the sequence $|J_{\lfloor n\cdot\rfloor}|/n^{1/2}$ is a tight in $C_{\mathbb{R}}[0,T]$ for all $T>0$ with the topology of uniform convergence in compacts and moreover by Theorem 2.4.10 in [10] is tight sequence of processes in $C_{\mathbb{R}}[0,\infty)$.

The proof of part $ii$) is similar with the difference that we analyze separately $\sum_{i=1}^{\lfloor n\cdot\rfloor}1_{\{U_i\leq i^{-\frac{1}{2}}\}}/n^{1/2}$ and $|F'_{\lfloor n\cdot\rfloor}|/n^{1/2}$. Using the same computation of part $i$), it is possible to conclude that $\sum_{i=1}^{\lfloor n\cdot\rfloor}1_{\{U_i\leq i^{-\frac{1}{2}}\}}/n^{1/2}$ and $|F'_{\lfloor n\cdot\rfloor}|/n^{1/2}$ are tight processes in $C_{\mathbb{R}}[0,T]$ for all $T>0$. Thus by Lemma C.1 we obtain that the all process is tight in $C_{\mathbb{R}}[0,T]$ for all $T>0$. Finally by Theorem 2.4.10 in [10], since we have in $C_{\mathbb{R}}[0,T]$ the topology of uniform convergence in compacts, we obtain the desired result.

□

3.3.5. *Proof of Lemma 3.6.* For this proof we will see that the sequence of processes $|K_{\lfloor n\cdot\rfloor}|/n^{1/2}$ will satisfies the two conditions of Theorem 7.3 in [5] and we will have tightness for the sequence of processes.

One can notice that the process $|K_{\lfloor n\cdot \rfloor}|/n^{1/2}$ is tight for $t=0$ for all $n \geq 1$. Thus first condition on Theorem 7.3 in [5] is satisfied.

Then let $P_n$ be a probability measure on $C_{\mathbb{R}}[0,T)$ and the distribution of $|K_{\lfloor n\cdot \rfloor}|/n^{1/2}$. We denote the set $A_t(\varepsilon,\phi) := \{f \in C_{\mathbb{R}}[0,T] : \sup_{t\leq s\leq t+\phi} |f(s)-f(t)| \geq \varepsilon\}$. Hence we have

$$\begin{aligned}\frac{1}{\phi}P_n[|K_{\lfloor n\cdot \rfloor}|/n^{1/2} \in A_t(\varepsilon,\phi)] &= \frac{1}{\phi}\mathbb{P}\left[\sup_{t\leq s\leq t+\phi} ||K_{\lfloor ns\rfloor}| - |K_{\lfloor nt\rfloor}|| \geq \varepsilon n^{\frac{1}{2}}\right] \\ &= \frac{1}{\phi}\mathbb{P}\left[\sum_{i=\lfloor nt\rfloor+1}^{\lfloor n(t+\phi)\rfloor} 1_{\{E_i^c \cap \{U_i \leq i^{-\frac{1}{2}}\}\}} \geq \varepsilon n^{\frac{1}{2}}\right] \\ &\leq \frac{1}{\phi}\mathbb{P}\left[\sum_{i=\lfloor nt\rfloor+1}^{\lfloor n(t+\phi)\rfloor} 1_{\{U_i \leq i^{-\frac{1}{2}}\}} \geq \varepsilon n^{\frac{1}{2}}\right]\end{aligned}$$

From this point the computation is exactly the same we did in the proof of Theorem 1.6, when we show the second sum portion in (62) fulfills the second condition of Theorem 7.3 in [5].

Then we have that for each positive $\varepsilon$ and $\eta$, there exists a $\phi \in (0,1)$, and an integer $n_0$ such that

$$\frac{1}{\phi}P_n\left[f \in C_{\mathbb{R}}[0,T] : \sup_{t\leq s\leq t+\phi} |f(s)-f(t)| \geq \varepsilon\right] \leq \eta \quad \forall n \geq n_0 \,.$$

Ergo by Theorem 7.3 in [5] we obtain that the sequence $|K_{\lfloor n\cdot \rfloor}|/n^{1/2}$ is a tight in $C_{\mathbb{R}}[0,T]$ for all $T>0$ with the topology of uniform convergence in compacts and moreover by Theorem 2.4.10 in [10] is tight sequence of processes in $C_{\mathbb{R}}[0,\infty)$.

□

3.3.6. *Proof of Proposition 3.1.* We will divide this proof in two parts: the first is concerned with the upper bound while the second with the lower bound.

Let us denote the following event

$$A_n^{c,M} := \left\{\forall t \in [c,M] : \frac{|K_{\lfloor nt\rfloor}}{n^{1/2}} \leq 2(\hat{\delta}t)^{1/2}\right\} \,,$$

where $c$, $M$ and $\hat{\delta}$ are positive constants such that $M > c$ and $\hat{\delta} \in (\delta, 1]$.

Hence we have that

$$\begin{aligned}
&\mathbb{P}[A_n^{c,M}] \geq \mathbb{P}[A_n^{c,M} \cap \{\forall t \in [c,M] : |\mathcal{R}^X_{\lfloor nt \rfloor}| \leq \delta \lfloor nt \rfloor\}] \\
&\geq \mathbb{P}\left[\left\{\forall t \in [c,M] : \frac{|J_{\lfloor nt \rfloor}|}{n^{1/2}} \leq 2(\hat{\delta} t)^{1/2}\right\} \bigcap \{\forall t \in [c,M] : |\mathcal{R}^X_{\lfloor nt \rfloor}| \leq \delta \lfloor nt \rfloor\}\right] \\
&\geq \mathbb{P}\left[\left\{\forall t \in [c,M] : \frac{|J'_{\lfloor nt \rfloor}|}{n^{1/2}} \leq 2(\hat{\delta} t)^{1/2}\right\} \bigcap \{\forall t \in [c,M] : |\mathcal{R}^X_{\lfloor nt \rfloor}| \leq \delta \lfloor nt \rfloor\}\right].
\end{aligned} \tag{78}$$

The last inequality in (78) follows from (59). Now by Lemma 3.5 part $i$) we obtain that

$$\frac{|J'_{\lfloor n \cdot \rfloor}|}{n^{1/2}} \to 2(\delta \cdot)^{1/2} \text{ as } n \to \infty, \tag{79}$$

in probability, since it converges in distribution to a continuous function in $t$ (see [5] page 27). Thus by Proposition 1.1 and (79) we obtain for every $M > c > 0$ the following holds

$$\mathbb{P}\left[\forall t \in [c,M] : \frac{|K_{\lfloor nt \rfloor}|}{n^{1/2}} \leq 2(\hat{\delta} t)^{1/2}\right] \to 1 \text{ as } n \to \infty,$$

since in the last inequality in (78) we have an intersection of two events whose probability converges to 1 as $n$ goes to infinity.

Since we have an arbitrary $\delta \in (\pi_d, 1)$, we can obtain the desired result

$$\mathbb{P}\left[\forall t \in [c,M] : \frac{|K_{\lfloor nt \rfloor}|}{n^{1/2}} \leq 2(\delta t)^{1/2}\right] \to 1 \text{ as } n \to \infty,$$

Now suppose that we have monotonic decreasing and increasing sequences $\{c_j\}_{j\geq 1}$ and $\{M_j\}_{j\geq 1}$ respectively, such that $c_j \to 0$ and $M_j \to \infty$ as $j$ goes to infinity. Let $\{H_t\}_{t\geq 0}$ be a limit point in distribution of a sub-sequence of $\frac{|K_{\lfloor nt \rfloor}|}{n^{1/2}}$, which is a tight process by Lemma 3.6, and define

$$A := \left\{\forall t \in [0,\infty) : H_t \leq 2(\hat{\delta} t)^{1/2}\right\},$$

and

$$A^{c_i,M_i} = \left\{\forall t \in [c_i, M_i] : H_t \leq 2(\hat{\delta} t)^{1/2}\right\}$$

and we have that $A = \cap_{i=1}^{\infty} A^{c_i,M_i}$. Thus by Portmanteu Theorem we have for every $i \in \mathbb{Z}^+$

$$\mathbb{P}[A^{c_i,M_i}] \geq \limsup_{n\to\infty} \mathbb{P}[A_n^{c_i,M_i}] = 1.$$

Hence, $\mathbb{P}[A^{c_i,M_i}] = 1$ for all $i \in \mathbb{Z}^+$ and we obtain

$$\mathbb{P}[A] = \mathbb{P}\left[\bigcap_{j=1}^{\infty} A^{c_j,M_j}\right] = 1. \tag{80}$$

Finishing the first part of the proof. Let us define the following events

$$B_n^{c,M} := \left\{ \forall t \in [c,M] : 2t^{1/2}(1-(1-\delta'')^{1/2}) \le \frac{|K_{\lfloor nt \rfloor}|}{n^{1/2}} \right\},$$

$$H_n^{c,M} := \left\{ \forall t \in [c,M] : 2t^{\frac{1}{2}}(1-(1-\delta'')^{\frac{1}{2}}) \le \frac{\sum_{j=1}^{\lfloor nt \rfloor} 1_{\{U_j \le j^{-1/2}\}} - |F_{\lfloor nt \rfloor}|}{n^{\frac{1}{2}}} \right\}$$

$$R_{\lfloor nt \rfloor} := \{\forall t \in [c,M] : |\mathcal{R}^X_{\lfloor nt \rfloor}| \ge \delta' \lfloor nt \rfloor\},$$

where $c$, $M$ and $\delta''$ are positive constants such that $M > c$ and $\delta'' \in (0, \delta')$.

Thus we have that

$$\begin{aligned} \mathbb{P}[B_n^{c,M}] &\ge \mathbb{P}[B_n^{c,M} \cap \{\forall t \in [c,M] : |\mathcal{R}^X_{\lfloor nt \rfloor}| \ge \delta' \lfloor nt \rfloor\}] \\ &\ge \mathbb{P}\left[ \left\{ \forall t \in [c,M] : 2t^{1/2}(1-(1-\delta'')^{1/2}) \le \frac{|V_{\lfloor nt \rfloor}|}{n^{1/2}} \right\} \cap R_{\lfloor nt \rfloor} \right] \\ &\ge \mathbb{P}\left[ H_n^{c,M} \cap R_{\lfloor nt \rfloor} \right] \ge \mathbb{P}[H_n^{c,M} \cap \{|\mathcal{R}^Z_{\lfloor nt \rfloor}| \ge \delta' \lfloor nt \rfloor\}]. \end{aligned} \tag{81}$$

The third inequality in (81) follows from (61) and the last inequality by the coupling with the lazy random walk.

Now by Lemma 3.5 part *ii*) we obtain that

$$\frac{\sum_{j=1}^{\lfloor n\cdot \rfloor} 1_{\{U_j \le j^{-1/2}\}} - |F_{\lfloor n\cdot \rfloor}|}{n^{\frac{1}{2}}} \to 2(\cdot)^{1/2}(1-(1-\delta')^{1/2}) \text{ as } n \to \infty, \tag{82}$$

in probability, since it converges in distribution to a continuous function in $t$ (see [5] page 27). Thus by Theorem 3.2 and (82) we obtain for every $M > c > 0$ the following holds

$$\mathbb{P}\left[ \forall t \in [c,M] : 2t^{1/2}(1-(1-\delta'')^{1/2}) \le \frac{|K_{\lfloor nt \rfloor}|}{n^{1/2}} \right] \to 1 \text{ as } n \to \infty,$$

since in the last inequality in (81) we have an intersection of two events whose probability converges to 1 as $n$ goes to infinity.

Finally, we can use the same argument used to obtain (80) and we finish the proof.

□

3.4. **On the range of $p_n$-ERW in $d \ge 2$ and $\beta = 1/2$.** In this section we provide the proof of Proposition 1.1.

Before we start, we introduce some auxiliary results that will be required to analyze the asymptotic behavior of the $p_n$-ERW in $\mathbb{Z}^d$, with $d \ge 2$ for $\beta = 1/2$.

Let $\{\xi_i\}_{i \ge 1}$ be i.i.d. $\mathbb{Z}^d$-valued random variables with zero mean vector and finite variance. We set the process $Y = \{Y_n\}_{n \ge 0}$ such that $Y_0 = 0$ and

$$Y_n = \sum_{i=1}^{n} \xi_i .$$

Hence we have that $\{Y_n\}_{n\geq 0}$ is a random walk in $\mathbb{Z}^d$. For $m \leq n$, let us define

$$\mathcal{R}^{Y}_{[m,n]} := \{Y_m, Y_{m+1}, \dots, Y_n\} ,$$

and simply $\mathcal{R}_n = \mathcal{R}_{[0,n]}$. This is the range of the process $\{Y_n\}_{n\geq 0}$ or the range of the random walk.

Our first auxiliary result comes from Theorem 1 in [9], which provides upper and lower bound to the range of a random walk with i.i.d. increments in $\mathbb{Z}^d$. Henceforth, we denote by $\pi_d$ the probability of the random walk $Y$ in $\mathbb{Z}^d$ never returning to the origin.

**Theorem 3.2** (see [9], Theorem 1)**.** *Let $Y = \{Y_n\}_{n\geq 0}$ be an aperiodic random walk in $\mathbb{Z}^d$, with $d \geq 2$, then it holds that*

$$\lim_{n\to\infty} \mathbb{P}[|\mathcal{R}^Y_n| \geq \theta n] = 1 , \quad \textit{for every } \theta < \pi_d , \tag{L}$$

$$\mathbb{P}[|\mathcal{R}^Y_n| \geq \theta' n] \leq e^{-c_{\theta'} n} , \quad \textit{for every } \theta' > \pi_d \textit{ and } n \textit{ sufficiently large} , \tag{U}$$

*where $c_{\theta'}$ is a positive constant that depends of $\theta'$ (note that for $d = 2$, we have that $\pi_d = 0$, whereas for $d \geq 3$, $\pi_d \in (0, 1]$.*

**Remark 3.2.** *It is important to notice that Theorem 1 from [9] is for aperiodic random walks. However, as the authors of [9] explained at the beginning of Section 2 in [9] page 188, this condition can be relaxed without loss of generality.*

Suppose that the $p_n$-ERW at time $j$ visits a new site and eats a cookie and only after $k$ further steps it visits a new site again and eats a new cookie. Then, by the characteristics of the model, we know that between time $j + 2$ and $j + k$ every new site visited by the process it was visited by a random walk with i.i.d. increments. The main idea behind the proof of Proposition 1.1 relies on this observation. Specifically, we can think that in each time window described above our process behaves as independent random walks with i.i.d. increments. Then we use the ranges of these processes to upper bound the range of the $p_n$-ERW.

*Proof of Proposition 1.1.* Let us denote $\{N_i\}_{i\geq 0}$ as the sequence of times such that $N_0 \equiv 0$ and

$$N_i = \inf\{k > N_{i-1} : Z_k = 1\} ,$$

where $Z_k = 1_{\{U_k \leq k^{-1/2}\}}$, $k \geq 1$, is a sequence of independent random variables with Bernoulli distribution of parameter respectively $k^{-1/2}$ for each $k$.

We denote $\Delta N_i = N_i - N_{i-1}$ and

$$M_n = \inf\left\{ i \geq 1 : \sum_{j=1}^{i} \Delta N_j \geq n \right\} .$$

First observe that

$$
\begin{aligned}
|\mathcal{R}_n^X| &= \sum_{t=0}^{n} 1_{\{X_t \neq X_l, \forall l<t\}} \\
&\leq \sum_{t=0}^{N_1} 1_{\{X_t \neq X_l, \forall l<t\}} + \sum_{t=N_1+1}^{N_2} 1_{\{X_t \neq X_l, \forall l<t\}} + \cdots + \sum_{t=N_{M_n-1}+1}^{N_{M_n}} 1_{\{X_t \neq X_l, \forall l<t\}} \\
&\leq M_n + \sum_{j=1}^{M_n} \sum_{t=N_{j-1}+2}^{N_j} 1_{\{X_t \neq X_l, \forall l<t\}} \leq M_n + \sum_{j=1}^{M_n} \sum_{t=N_{j-1}+2}^{N_j} 1_{\{X_t \neq X_l, \forall l \in [N_{j-1}+1, t)\}} \,.
\end{aligned}
$$

One can see that in each time interval $[N_{j-1}+2, N_j]$ the process $X$ behaves like a random walk with i.i.d. increments. In order to have some control on the length of these intervals, or equivalently on $\{\Delta N_j\}_{j\geq 1}$, we proceed as follows: Let $\varepsilon \in (0,1)$ and note that

$$
|\mathcal{R}_n^X| \leq n^\varepsilon + |\mathcal{R}_{[n^\varepsilon, n]}^X| \,. \tag{83}
$$

Thus we could just redefine $N_0 \equiv n^\varepsilon$ and apply the very same decomposition as before to obtain

$$
\begin{aligned}
|\mathcal{R}_{[n^\varepsilon, n]}^X| &\leq M_n + \sum_{j=1}^{M_n} \sum_{t=N_{j-1}+2}^{N_j} 1_{\{X_t \neq X_l, \forall l \in [N_{j-1}, t)\}} \\
&\leq M_n + \sum_{j=1}^{M_n} |\mathcal{R}_{[N_{j-1}+2, N_j]}^Y| \,.
\end{aligned} \tag{84}
$$

where $\{Y_n\}_{n\geq 0}$ is a random walk with i.i.d. increments in $\mathbb{Z}^2$, whose increments will be defined by the sequence $\{\xi_i\}_{i\geq 1}$.

Now, for any $k \in \{1, 2, \ldots, n\}$ fixed, we define the random set

$$
A_{n,k} := \{j \in \{1, 2, \ldots, M_n\} : \Delta N_j \leq k\} \,,
$$

and we can write

$$
\begin{aligned}
\sum_{j=1}^{M_n} |\mathcal{R}_{[N_{j-1}+2, N_j]}^Y| &= \sum_{j \in A_{n,k}} |\mathcal{R}_{[N_{j-1}+2, N_j]}^Y| + \sum_{j \in A_{n,k}^c} |\mathcal{R}_{[N_{j-1}+2, N_j]}^Y| \\
&\leq (k-1)|A_{n,k}| + \sum_{j \in A_{n,k}^c} |\mathcal{R}_{[N_{j-1}+2, N_j]}^Y| \,.
\end{aligned}
$$

Using the simple inequality

$$
M_n = |A_{n,k}| + |A_{n,k}^c| \leq |A_{n,k}| + \frac{M_n}{k} \leq |A_{n,k}| + \frac{n}{k}
$$

and (84), we obtain the inequality

$$
|\mathcal{R}_{[n^\varepsilon, n]}^X| \leq \frac{n}{k} + k|A_{n,k}| + \sum_{j \in A_{n,k}^c} |\mathcal{R}_{[N_{j-1}+2, N_j]}^Y|. \tag{85}
$$

To control the right hand side of (85) we start with estimates on $|A_{n,k}|$. By coupling arguments, one can see that $G \preceq \Delta N_j$ for all $j \in \{1, 2, \ldots, M_n\}$, where $G$ has a Geometric distribution with parameter $n^{-\varepsilon/2}$ and $\preceq$ means stochastic dominance. Thus we have

$$\mathbb{P}[\Delta N_j \leq k] \leq 1 - \left(1 - \frac{1}{n^{\varepsilon/2}}\right)^k . \tag{86}$$

Then we can write $|A_{n,k}| = \sum_{j=1}^{M_n} 1_{\{\Delta N_j \leq k\}}$ and we obtain for a fixed $a$

$$\mathbb{P}\left[\sum_{j=1}^{M_n} 1_{\{\Delta N_j \leq k\}} > a\right] \leq \mathbb{P}\left[\sum_{j=1}^{n} 1_{\{\Delta N_j \leq k\}} > a\right] \leq \binom{n}{a} \left(\mathbb{P}[G \leq k]\right)^a$$
$$\leq \left(\frac{ne}{a}\right)^a \left(1 - \left(1 - \frac{1}{n^{\frac{\varepsilon}{2}}}\right)^k\right)^a \leq \left(\frac{ne}{a}\right)^a \left(1 - \exp\left\{\frac{-k}{n^{\frac{\varepsilon}{2}}}\right\}\right)^a \leq \left(\frac{ne}{a} \times \frac{k}{n^{\frac{\varepsilon}{2}}}\right)^a .$$

Now we set $a = n^{1-\varepsilon/4}$ and obtain

$$\mathbb{P}\left[\sum_{j=1}^{M_n} 1_{\{\Delta N_j \leq k\}} > n^{1-\varepsilon/4}\right] \leq \left(\frac{ek}{n^{\frac{\varepsilon}{4}}}\right)^{n^{1-\frac{\varepsilon}{4}}} . \tag{87}$$

We choose $k = \lceil \log^2(n) \rceil$. With this choice, the deterministic first term in (85) divided by $n$ converges to zero, moreover the sum in $n$ of the probabilities of the events $\{|A_{n,\lceil \log^2(n) \rceil}| > n^{1-\varepsilon/4}\}$ is finite. By Borel-Cantelli the second term in (85) divided by $n$ converges to zero almost surely.

Now we will analyze the third term in (85). First we will obtain an upper bound on the probability of the event that there exist at least one $j \in A^c_{n,\lceil \log^2(n) \rceil}$, such that $|\mathcal{R}^Y_{[N_{j-1}+2, N_j]}| > \gamma \Delta N_j$, where $\gamma \in (\pi_d, 1]$.

First we observe that

$$|A^c_{n,\lceil \log^2(n) \rceil}| \leq \frac{n}{\lceil \log^2(n) \rceil} , \tag{88}$$

for all $n$.

Now for all $i > \lceil \log^2(n) \rceil$, we obtain by Theorem 3.2 that

$$\begin{aligned} \mathbb{P}[|\mathcal{R}^Y_{i-2}| > \gamma i] &\leq \mathbb{P}[|\mathcal{R}^Y_{i-2}| > \gamma(i-2)] \leq \exp(-c_\gamma(i-2)) \\ &\leq \exp\big(-c_\gamma(\lceil \log^2(n) \rceil - 2)\big) , \end{aligned} \tag{89}$$

for all $\gamma \in (\pi_d, 1]$.

Remember that, for all $j \in A^c_{n,\lceil \log^2(n) \rceil}$, then we have $\Delta N_j > \lceil \log^2(n) \rceil$. Hence by (88) and (89) we obtain

$$\begin{aligned} &\mathbb{P}[\exists j \in A^c_{n,\lceil \log^2(n) \rceil} : |\mathcal{R}^Y_{[N_{j-1}+2, N_j]}| > \gamma \Delta N_j] \\ &\leq \frac{n}{\lceil \log^2(n) \rceil} \exp\left(-c_\gamma \lceil \log^2(n) - 2 \rceil\right) \\ &\leq \exp\left(\log\left(\frac{n}{\lceil \log^2(n) \rceil}\right) - c_\gamma \lceil \log^2(n) - 2 \rceil\right) . \end{aligned} \tag{90}$$

Now, with the control of the probability of the event $\{\exists j \in A^c_{n,\lceil \log^2(n)\rceil} : |\mathcal{R}^Y_{[N_{j-1}+2,N_j]}| > \gamma \Delta N_j\}$ in (90), we can conclude that

$$\mathbb{P}\left[\sum_{j\in A^c_{n,k}} |\mathcal{R}^Y_{[N_{j-1}+2,N_j]}| > \gamma n\right] \le \exp\left(\log\left(\frac{n}{\lceil \log^2(n)\rceil}\right) - c_\gamma \lceil \log^2(n) - 2\rceil\right). \tag{91}$$

Thus we obtain that the sum in $n$ of the probabilities of the events $\{\sum_{j\in A^c_{n,k}} |\mathcal{R}^Y_{[N_{j-1}+2,N_j]}| > \gamma n\}$ is finite. Thus, by Borell-Cantelli the third term in (85) divide by $n$ is bigger than $\gamma$ only finitely many times almost surely.

Hence with (87), (91) and (83), we can finish the proof setting $\delta = \gamma$.

□

3.5. **Positive probability of never returning to the origin (along direction $\ell$) for $p_n$-GERW with $\beta < 1/6$ and $d \ge 2$.** For the proof of Theorem 1.7 we will need the next result which is basically the same of Proposition 2.4, however now it applies to the $p_n$-GERW, with $p_n = n^{-\beta}$ and $\beta < 1/6$.

**Proposition 3.2.** *Fix $\beta < \alpha < 1/6$ and suppose that $X$ is a $p_n$-GERW with excitation-allowing set $A \subset \mathbb{Z}^d$ and $p_n = (n_0 + n)^{-\beta}$, where $n_0$ is a non negative integer. If for some $n \ge n_0$*

$$\left|(\mathbb{Z}^d \setminus A) \cap H\left(-n^{\frac{1}{2}+\alpha}, \frac{2\lambda}{3} n^{\frac{1}{2}+\alpha}\right)\right| \le \frac{1}{3} n^{\frac{1}{2}+\alpha}, \tag{92}$$

*then, for some positive constants $\vartheta_1$, $\vartheta_2$ depending on $d$, $K$, $r$, $\lambda$, $\alpha$ and $\beta$, we have*

$$\mathbb{P}\left[X_n \cdot \ell < \frac{1}{3}\lambda n^{\frac{1}{2}+\alpha-\beta}\right] < 6n \exp\left\{-\vartheta_1 n^{\vartheta_2}\right\}, \tag{93}$$

*where*

$$\vartheta_1 = \min\left\{\gamma_1, \frac{1}{2K^2}, \frac{\lambda^2}{18K^2}, \frac{\delta_0^2}{2^\beta \times 3}, \frac{((1/3 - 2^{1-\beta}/3(1-\delta_0))\lambda)^2}{2K^2}\right\} \quad \text{and}$$
$$\vartheta_2 = \min\{\gamma_2, 2(\alpha - \beta)\},$$

*$\delta_0 \in (0, 0.43)$ and $\gamma_1$, $\gamma_2$ are the same as in Proposition 2.3.*

**Remark 3.3.** *Let us point out that in the proof of Proposition 3.2, $\alpha - \beta$ plays the same role as $\alpha$ in Proposition 2.4 and therefore $\alpha - \beta \in (0, 1/6)$.*

We will postpone the proof of Proposition 3.2 to end of this section.

The proof of Theorem 1.7 is similar to that of Proposition 2.5. However, to deal with the time dependence of the sequence $\{p_n\}_{n\ge1}$, here we use Proposition 3.2 (which is a extension of Proposition 2.4 in the time-dependent case). Our main concern in Theorem 1.7 is to make explicit the dependence of the bound $\psi$ on the parameters $\beta$ and $q_0$.

Before we provide the proof of Theorem 1.7, below we give a more detailed statement (than the one provided in Section 3) making explicit the dependence of the constant $\psi$ on the various parameters.

**Theorem 1.7.** *Let $X$ be a $p_n$-GERW in direction $\ell$, in $\mathbb{Z}^d$ with $d \geq 2$, where $p_n = (q_0 + n)^{-\beta}$, with $\beta < 1/6$, $q_0$ is a non negative integer and excitation-allowing set $A \subset \mathbb{Z}^d$ such that $\mathbb{M}_\ell \subset A$. There exists $\psi > 0$ depending on $d$, $K$, $h$, $r$, $\lambda$, $\alpha$ and $\beta$ such that*

$$\mathbb{P}\left[\eta(X_0) = \infty\right] \geq \mathbb{P}\left[X_n \cdot \ell > 0 \text{ for all } n \geq 1\right] \geq \psi\,,$$

*where $\psi = h^{\lceil r^{-1} \rceil C\left(\frac{3}{\lambda}\right)^{\frac{1}{\delta-1}}} c$, $c \in (0,1)$, $\delta = (2 - \alpha + \beta)(1/2 + \alpha - \beta)$,*

$$C = K^{\frac{1}{\delta-1}} \left(\eta + \lceil r^{-1} \rceil^{\frac{1}{\delta-1}}\right) + q_0\,,$$

$$\eta = \left(\frac{2-\alpha+\beta}{\vartheta_1 \varphi_1}\right)^{\frac{1}{\varphi_1}}\,, \quad \varphi_1 = \min\left\{\alpha - \beta, (2 - \alpha + \beta)\vartheta_2\right\}\,,$$

*and $\vartheta_1$, $\vartheta_2$ are as in Proposition 3.2.*

*Proof of Theorem 1.7.* Since on $\{X_n \cdot \ell > 0 \text{ for all } n \geq 1\}$ the process doesn't visit the sites in $\mathbb{Z}^d/\mathbb{M}_\ell$, it is sufficient to prove the proposition for $A = \mathbb{Z}^d$.

Without loss of generality we consider $r \leq 1$ in Condition III. Define

$$U_0 = \left\{(X_{k+1} - X_k) \cdot \ell \geq r\,, \text{ for all } \quad k = 0, 1, ..., \lceil r^{-1} \rceil m - 1\right\}.$$

Observe that in $U_0$, $X_{\lceil r^{-1} \rceil m} \cdot \ell \geq m$ and by (UE1) of Condition III we have,

$$\mathbb{P}\left[U_0\right] \geq h^{\lceil r^{-1} \rceil m}\,. \tag{94}$$

Consider the following time translation of the process $X$: $W_k = X_{\lceil r^{-1} \rceil m + k}$, $k \geq 0$. Then $W$ is a $p_n$-GERW with excitation-allowing set

$$A' = \mathbb{Z}^d / \{X_0, \ldots, X_{\lceil r^{-1} \rceil m - 1}\}$$

starting at $W_0 = y_0 := X_{\lceil r^{-1} \rceil m}$ and for some time $k$, we have $p_k = (q_0 + \lceil r^{-1} \rceil m + k)^{-\beta}$.

Set $\delta = (2 - \theta)(1/2 + \theta)$, where $\theta = \alpha - \beta$ and

$$m = C\left(\frac{3}{\lambda}\right)^{\frac{1}{\delta-1}},$$

where $C > 0$ is a constant depending on $\alpha$, $\beta$, $K$, $\lambda$, $q_0$ and $r$, such that

$$C = K^{\frac{1}{\delta-1}} \left(\eta + \lceil r^{-1} \rceil^{\frac{1}{\delta-1}}\right) + q_0 \quad \text{for}$$

$$\eta = \left(\frac{2-\alpha+\beta}{\vartheta_1 \varphi_1}\right)^{\frac{1}{\varphi_1}} \quad \text{with} \quad \varphi_1 = \min\left\{\alpha - \beta, (2 - \alpha + \beta)\vartheta_2\right\},$$

and $\vartheta_1$, $\vartheta_2$ as in the statement of Proposition 3.2. Note that for all $\alpha$ used in Proposition 2.3 and $\beta$ i.e., $0 < \beta < \alpha < 1/6$, we have that $\delta > 1$.

The left-hand side of (92) with the set $A'-y_0$ is bounded above by $\lceil r^{-1}\rceil m$. Note that, for all $n \geq m^{2-\alpha-\beta}$,

$$\frac{1}{3}n^{\frac{1}{2}+\alpha} \geq \frac{1}{3}n^{\frac{1}{2}+\alpha-\beta} \geq \frac{1}{3}m^{\delta} \geq \lceil r^{-1}\rceil m\ . \tag{95}$$

We obtain the last inequality in (95) by the following fact,

$$\frac{m^{\delta-1}}{3\lceil r^{-1}\rceil} = \frac{3C^{\delta-1}}{3\lceil r^{-1}\rceil\lambda} > \frac{K\lceil r^{-1}\rceil}{\lceil r^{-1}\rceil\lambda} = \frac{K}{\lambda} > 1\ . \tag{96}$$

The second inequality follows from the definition of $C$, since $C > (K\lceil r^{-1}\rceil)^{\frac{1}{1-\delta}}$. Thus (92) with excitation-allowing set $A'-y_0$ is satisfied for all $n \geq m^{2-\alpha+\beta}$.

Denote $m_0 = 0$, $m_1 = m$ and, for $k \geq 1$, $m_{k+1} = \frac{1}{3}\lambda m_k^{\delta}$. The sequence $(m_k, k \geq 1)$ is increasing. This can be proved by induction since

$$\frac{m_2}{m_1} = \frac{\lambda}{3}m^{\delta-1} = C^{\delta-1} > 1,$$

for all $\theta \in (0, 1/6)$ and assuming $m_k/m_{k-1} > 1$ we have

$$\frac{m_{k+1}}{m_k} = \frac{\frac{\lambda}{3}m_k^{\delta}}{\frac{\lambda}{3}m_{k-1}^{\delta}} = \left(\frac{m_k}{m_{k-1}}\right)^{\delta} > 1,$$

For every $k \geq 1$ consider the following events

$$G_k = \left\{\min_{\lfloor m_{k-1}^{2-\theta}\rfloor < j \leq m_k^{2-\theta}} \left(W_j - W_{\lfloor m_{k-1}^{2-\theta}\rfloor}\right)\cdot \ell > -m_k\right\}\ ,$$
$$U_k = \left\{W_{\lfloor m_k^{2-\theta}\rfloor}\cdot \ell \geq m_{k+1}\right\}\ .$$

As we saw in the proof of Proposition 2.5 the following holds:

$$\left\{X_n\cdot \ell > 0,\ \text{for all } n \geq 1\right\} \supset \left(\bigcap_{k=1}^{\infty}(G_k \cap U_k)\right) \cap U_0\ . \tag{97}$$

As seen in proof of Proposition 2.4, the process $\{X_n \cdot \ell\}_{n\geq 0}$, is a $\mathcal{F}$-submartingale, so $(W-y_0)\cdot \ell$ is also $\mathcal{F}_{\cdot+\lceil r^{-1}\rceil m}$-submartingale. Write

$$G_k^c = \bigcup_{j=\lfloor m_{k-1}^{2-\theta}\rfloor+1}^{m_k^{2-\theta}} \left\{\left(W_j - W_{\lfloor m_{k-1}^{2-\theta}\rfloor}\right)\cdot \ell \leq -m_k\right\}.$$

and by Azuma's inequality (for supermartingales with bounded increments)

$$\mathbb{P}\left[\left(W_j - W_{\lfloor m_{k-1}^{2-\theta}\rfloor}\right)\cdot \ell \leq -m_k\right] = \mathbb{P}\left[\left(W_{\lfloor m_{k-1}^{2-\theta}\rfloor} - W_j\right)\cdot \ell \geq m_k\right]$$
$$\leq \exp\left(-\frac{m_k^2}{2K^2(j-\lfloor m_{k-1}^{2-\theta}\rfloor)}\right) \leq \exp\left(-\frac{m_k^2}{2K^2 m_k^{2-\theta}}\right) \leq \exp\left(-\frac{m_k^{\theta}}{2K^2}\right).$$

Thus, we have,

$$\mathbb{P}[G_k|U_0] \geq 1 - \left(m_k^{2-\theta} - \lfloor m_{k-1}^{2-\theta}\rfloor\right)e^{-\frac{m_k^{\theta}}{2K^2}} \geq 1 - m_k^{2-\theta}e^{-\frac{m_k^{\theta}}{2K^2}}\ .$$

Suppose that $n_0 = q_0 + \lceil r^{-1}\rceil m$, where $n_0$ is from Proposition 3.2. The sequence $(m_k, k \geq 1)$ is increasing, then if $m^{2-\theta} \geq n_0$ we obtain that $m_k^{2-\theta} \geq n_0$ for all $k \geq 1$. Observe that

$$m^{1-\theta} - \lceil r^{-1}\rceil \geq \left(\frac{3K\lceil r^{-1}\rceil}{\lambda}\right)^{\frac{1-\theta}{\delta-1}} - \lceil r^{-1}\rceil > 1\,,$$

since for all $\theta \in (0, 1/6)$ we have $(1-\theta)/(\delta-1) \geq 3.75$.

Ergo we obtain

$$m^{2-\theta} - \lceil r^{-1}\rceil m = m\underbrace{\left(m^{1-\theta} - \lceil r^{-1}\rceil\right)}_{>1} \geq m \geq q_0\,. \tag{98}$$

Since the process $W - y_0$ satisfies Conditions I, II, III, the set $A' - y_0$ fulfills (92) for all $n \geq m^{2-\theta}$ and $m_k^{2-\theta} \geq q_0 + \lceil r^{-1}\rceil m$ for all $k \geq 1$ by Proposition 3.2, it holds that

$$\mathbb{P}[U_k|U_0] = \mathbb{P}\left[W_{\lfloor m_k^{2-\theta}\rfloor} \cdot \ell \geq \frac{\lambda}{3} m_k^{(2-\theta)(\frac{1}{2}+\theta)}\right] \geq 1 - 6m_k^{2-\theta}e^{-\vartheta_1 m_k^{(2-\theta)\vartheta_2}}.$$

Now, write

$$\mathbb{P}\left[\left(\bigcap_{k=1}^{\infty}(G_k \cap U_k)\right) \cap U_0\right] = \mathbb{P}[U_0]\Big(1 - \sum_{k=1}^{\infty}\mathbb{P}[G_k^c|U_0] + \mathbb{P}[U_k^c|U_0]\Big),$$

which is bounded from below by

$$\begin{aligned} &h^{\lceil r^{-1}\rceil m}\left(1 - \sum_{k=1}^{\infty}\left(m_k^{2-\theta}e^{-\frac{m_k^{\theta}}{2K^2}} + 6m_k^{2-\theta}e^{-\vartheta_1 m_k^{(2-\theta)\vartheta_2}}\right)\right) \\ &\geq h^{\lceil r^{-1}\rceil m}\left(1 - 7\sum_{k=1}^{\infty} m_k^{2-\theta}e^{-\vartheta_1 m_k^{\varphi_1}}\right). \end{aligned} \tag{99}$$

Now we are going to analyze the series $\sum_{k=1}^{\infty} m_k^{2-\theta}e^{-\vartheta_1 m_k^{\varphi_1}}$. Note that $m$ is large enough so that the sequence $(m_k^{2-\theta}e^{-\vartheta_1 m_k^{\varphi_1}}, k \geq 1)$ is decreasing. Indeed, $m$ is bigger than the inflection point $\left(\frac{2-\theta}{\vartheta_1\varphi_1}\right)^{\frac{1}{\varphi_1}}$ of the function $z(x) = x^{2-\theta}e^{-\vartheta_1 x^{\varphi_1}}$, $x > 0$:

$$m = C\left(\frac{3}{\lambda p}\right)^{\frac{1}{\delta-1}} > K^{\frac{1}{\delta-1}}\eta\left(\frac{3}{\lambda}\right)^{\frac{1}{\delta-1}} = \eta\left(\frac{3K}{\lambda}\right)^{\frac{1}{\delta-1}} \geq \left(\frac{2-\theta}{\vartheta_1\varphi_1}\right)^{\frac{1}{\varphi_1}}.$$

Thus we have,

$$\sum_{k=1}^{\infty} m_k^{2-\theta}e^{-\vartheta_1 m_k^{\varphi_1}} \leq \int_{m_1}^{\infty} x^{2-\theta}e^{-\vartheta_1 x^{\varphi_1}}\,dx\,. \tag{100}$$

By a change of variables, we write,

$$\int_{m_1}^{\infty} x^{2-\theta_1}e^{-\vartheta_1 x^{\varphi_1}}\,dx = \varphi_1^{-1}\vartheta_1^{\frac{\theta-3}{\varphi_1}}\Gamma\left(\frac{3-\theta}{\varphi_1}, \vartheta_1 m_1^{\varphi_1}\right), \tag{101}$$

where $\Gamma$ is the incomplete gamma function[4]. As mentioned above $m$ is large enough so that the sequence $(m_k^{2-\theta} e^{-\vartheta_1 m_k^{\varphi_1}}, k \geq 1)$ is decreasing. Thus, in order to obtain that (101) is smaller than $1/7$, we may increase $m$ even further by choosing a sufficiently bigger $C$. Thus, with such a suitable chosen $C$ we obtain

$$\sum_{k=1}^{\infty} m_k^{2-\theta} e^{-\vartheta_1 m_k^{\varphi_1}} \leq \int_{m_1}^{\infty} x^{2-\theta} e^{-\vartheta_1 x^{\varphi_1}} dx \qquad (102)$$
$$= \varphi_1^{-1} \vartheta_1^{\frac{\theta-3}{\varphi_1}} \Gamma\left(\frac{3-\theta}{\varphi_1}, \vartheta_1 m_1^{\varphi_1}\right) < \frac{1}{7}.$$

Using (102) in (99), we obtain that,

$$\mathbb{P}\left[\left(\bigcap_{k=1}^{\infty} (G_k \cap U_k)\right) \cap U_0\right] \geq h^{\lceil r^{-1} \rceil m} \left(1 - 7 \sum_{k=1}^{\infty} m_k^{2-\theta} e^{-\vartheta_1 m_k^{\varphi_1}}\right)$$
$$\geq h^{\lceil r^{-1} \rceil C \left(\frac{3}{\lambda}\right)^{\frac{1}{\delta-1}}} c = \psi,$$

where $c$ is a positive constant such that $c \in (0,1)$. Theorem 1.7 then follows from (97).

□

3.5.1. *Proof of Proposition 3.2*. The idea of this proof is similar to the Proposition 2.4.

*Proof of Proposition 3.2*. Let us begin remembering that the process $(X_n \cdot \ell, n \geq 0)$ is a $\mathcal{F}$-submartingale and thus $(-X_n \cdot \ell, n \geq 0)$ is a $\mathcal{F}$-supermartingale as we saw in the beginning of the proof of Proposition 2.4.

As a first step we show that

$$\mathbb{P}\left[\max_{k \leq n} X_k \cdot \ell > \frac{2}{3} \lambda n^{\frac{1}{2}+\alpha},\ X_n \cdot \ell < \frac{1}{3} \lambda n^{\frac{1}{2}+\alpha-\beta}\right] \leq n e^{-C_1 n^{2\alpha}}, \qquad (103)$$

for $C_1 > 0$. Note that

$$\left\{\max_{k \leq n} X_k \cdot \ell > \frac{2}{3} \lambda n^{\frac{1}{2}+\alpha},\ X_n \cdot \ell < \frac{1}{3} \lambda n^{\frac{1}{2}+\alpha-\beta}\right\}$$
$$\subset \bigcup_{k=1}^{n} \left\{X_n \cdot \ell - X_k \cdot \ell < \left(\frac{1}{3n^{\beta}} - \frac{2}{3}\right) \lambda n^{\frac{1}{2}+\alpha}\right\},$$

and by Azuma's inequality for supermartingales with increments uniformly bounded by $K$ (see Lemma 1 of [20]), for every $k = \{1, \ldots, n-1\}$ it holds

[4] $\Gamma(s,x) = \int_x^{\infty} t^{s-1} e^{-t} dt$.

that

$$\mathbb{P}\left[X_n \cdot \ell - X_k \cdot \ell < \left(\frac{1}{3n^\beta} - \frac{2}{3}\right)\lambda n^{\frac{1}{2}+\alpha}\right] \le \mathbb{P}\left[X_n \cdot \ell - X_k \cdot \ell < -\frac{1}{3}\lambda n^{\frac{1}{2}+\alpha}\right]$$
$$\le \exp\left(-\frac{\left(\frac{1}{3}\right)^2 \lambda^2 n^{1+2\alpha}}{2(n-k)K^2}\right) \le \exp\left(-\frac{\lambda^2 n^{2\alpha}}{18K^2}\right).$$

Then (103) follows from the usual union bound with $C_1 = \left(\frac{1}{3}\lambda\right)^2/2K^2$. Moreover, again using Azuma's inequality (for supermartingales), we also have that

$$\mathbb{P}\left[\min_{k\le n} X_k \cdot \ell < -n^{\frac{1}{2}+\alpha}\right] \le n\exp\{-C_2 n^{2\alpha}\}, \tag{104}$$

for $C_2 = 1/2K^2$.

Now let $D_k = \mathbb{E}[X_{k+1} - X_k | \mathcal{F}_k]$ and $Y_n = X_n - \sum_{k=0}^{n-1} D_k$. It follows that $Y_n$ is a martingale with bounded increments. Let $G$ be the following event

$$G := \left\{|\mathcal{R}_n^X| \ge n^{\frac{1}{2}+\alpha}\right\} \cap \left\{X_k \in H\left(-n^{\frac{1}{2}+\alpha}, \frac{2}{3}\lambda n^{\frac{1}{2}+\alpha}\right),\ \text{for all } k \le n\right\}.$$

Using the hypotheses (92), on $G$ we have at least $|\mathcal{R}_n^X| - \frac{1}{3}n^{\frac{1}{2}+\alpha} \ge \frac{2}{3}n^{\frac{1}{2}+\alpha}$ sites visited on the excitation-allowing $A$. Therefore, there exists a random variable $W$ such that on $G$

$$\left(\sum_{k=0}^{n-1} D_k\right) \cdot \ell \ge \lambda W,$$

and

$$\mathbb{E}[W] > \frac{2}{3}n^{\frac{1}{2}+\alpha} \times \frac{1}{(n_0+n)^\beta} \ge \frac{2}{3}n^{\frac{1}{2}+\alpha} \times \frac{1}{(n+n)^\beta} > \frac{2^{1-\beta}}{3}n^{\frac{1}{2}+\alpha-\beta}. \tag{105}$$

In order to prove (93), we write that probability as

$$\mathbb{P}\left[\left\{X_n \cdot \ell < \frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}\right\} \cap G\right] + \mathbb{P}\left[\left\{X_n \cdot \ell < \frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}\right\} \cap G^c\right] \tag{106}$$

and we control both terms separately. We start with the second term. Set

$$E = \left\{X_n \cdot \ell < \frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}\right\},\ M = \left\{|\mathcal{R}_n^X| < n^{\frac{1}{2}+\alpha}\right\},$$

$$J = \left\{\min_{k\le n} X_k \cdot \ell < -n^{\frac{1}{2}+\alpha}\right\} \text{ and } T = \left\{\max_{k\le n} X_k \cdot \ell > \frac{2}{3}\lambda n^{\frac{1}{2}+\alpha}\right\}.$$

It follows that

$$\mathbb{P}[E \cap G^c] = \mathbb{P}[(E\cap M) \cup (E \cap J) \cup (E \cap T)]$$
$$\le \mathbb{P}[E\cap M] + \mathbb{P}[E\cap J] + \mathbb{P}\left[\max_{k\le n} X_k \cdot \ell > \frac{2}{3}\lambda n^{\frac{1}{2}+\alpha},\ X_n \cdot \ell < \frac{p}{3}\lambda n^{\frac{1}{2}+\alpha}\right],$$

and from Proposition 2.3 and (104) and (103), we obtain

$$\begin{aligned}\mathbb{P}\left[E\cap G^c\right] &\le \mathbb{P}\left[|\mathcal{R}_n^X| < n^{\frac{1}{2}+\alpha}\right] + \mathbb{P}\left[\min_{k\le n} X_k\cdot\ell < -n^{\frac{1}{2}+\alpha}\right] + ne^{-C_1 n^{2\alpha}}\\ &\le e^{-\gamma_1 n^{\gamma_2}} + ne^{-C_2 n^{2\alpha}} + ne^{-C_1 n^{2\alpha}}\ .\end{aligned} \tag{107}$$

As regards the first term in (106), let $\delta_0 \in (0, 0.43)$, $\mu_W = \mathbb{E}[W]$, $B = \{W \le \mu_W(1-\delta_0)\}$, and write $\mathbb{P}[E\cap G]$ as

$$\begin{aligned}&\mathbb{P}\left[\left\{X_n\cdot\ell < \frac{1}{3}\lambda n^{\frac{1}{2}+\alpha-\beta}\right\}\cap G\cap B\right] + \mathbb{P}\left[\left\{X_n\cdot\ell < \frac{1}{3}\lambda n^{\frac{1}{2}+\alpha-\beta}\right\}\cap G\cap B^c\right]\\ &\qquad\le \mathbb{P}\left[B\right] + \mathbb{P}\left[\left\{X_n\cdot\ell < \frac{1}{3}\lambda n^{\frac{1}{2}+\alpha-\beta}\right\}\cap G\cap B^c\right].\end{aligned}$$

To bound $\mathbb{P}[B]$ we use the Chernoff bound (cf., e.g., Theorem 4.5 of [16]) to obtain

$$\mathbb{P}\left[B\right] \le \exp\Big\{-\frac{\delta_0^2}{2}\mu_W\Big\} \le \exp\Big\{-C_3' n^{\frac{1}{2}+\alpha-\beta}\Big\}\ , \tag{108}$$

where $C_3' = \delta_0^2/(3{\times}2^\beta)$. To upper bound $\mathbb{P}\left[\left\{X_n\cdot\ell < \frac{1}{3}\lambda n^{\frac{1}{2}+\alpha-\beta}\right\}\cap G\cap B^c\right]$, we use that $Y_n = X_n - \sum_{k=0}^{n-1} D_k$ is martingale with bounded increments and apply Azuma's inequality (see, for example, Theorem 2.19 in [6]). Thus, denoting $F = \left\{X_n\cdot\ell < \frac{1}{3}\lambda n^{\frac{1}{2}+\alpha-\beta}\right\}\cap G\cap B^c$ we obtain

$$\begin{aligned}\mathbb{P}\left[F\right] &\le \mathbb{P}\Big[X_n\cdot\ell - \Big(\sum_{k=1}^{n-1} D_k\Big)\cdot\ell < \frac{1}{3}\lambda n^{\frac{1}{2}+\alpha-\beta} - \lambda\mu_W(1-\delta_0)\Big]\\ &\le \mathbb{P}\Big[X_n\cdot\ell - \Big(\sum_{k=1}^{n-1} D_k\Big)\cdot\ell < \lambda n^{\frac{1}{2}+\alpha-\beta}\Big(\frac{1}{3} - \frac{2^{1-\beta}}{3}(1-\delta_0)\Big)\Big].\end{aligned} \tag{109}$$

The last inequality in (109) we obtain using (105).

Hence, we have that $C_4' := -(1/3 - 2^{1-\beta}/3(1-\delta_0)) > 0$, since $\delta_0 \in (0, 0.29)$. By (109) and Azuma's inequality we obtain

$$\begin{aligned}\mathbb{P}[F] &\le \mathbb{P}\Big[X_n\cdot\ell - \Big(\sum_{k=1}^{n-1} D_k\Big)\cdot\ell < -C_4'\lambda n^{\frac{1}{2}+\alpha-\beta}\Big]\\ &\le 2\exp\Big(-\frac{(C_4')^2\lambda^2 n^{2\alpha-\beta}}{2K^2}\Big) = 2\exp\Big(-C_5' n^{2\alpha-\beta}\Big)\ ,\end{aligned} \tag{110}$$

where $C_5' = (C_4'\lambda)^2/2K^2$.

Inequality (93) then follows from (107), (108) and (110) which imply that

$$\begin{aligned}&\mathbb{P}\left[X_n\cdot\ell < \frac{1}{3}\lambda n^{\frac{1}{2}+\alpha-\beta}\right] \le\\ &e^{-\gamma_1 n^{\gamma_2}} + ne^{-C_2 n^{2\alpha}} + ne^{-C_1 n^{2\alpha}} + e^{-C_3' n^{\frac{1}{2}+\alpha-\beta}} + 2e^{-C_5' n^{2\alpha-\beta}} \le 6ne^{-\vartheta_1 n^{\vartheta_2}}\ ,\end{aligned}$$

where

$$\vartheta_1 = \min\left\{\gamma_1, \frac{1}{2K^2}, \frac{\lambda^2}{18K^2}, \frac{\delta_0^2}{2^\beta \times 3}, \frac{((1/3 - 2^{1-\beta}/3(1-\delta_0))\lambda)^2}{2K^2}\right\} \quad \text{and}$$

$$\vartheta_2 = \min\{\gamma_2, 2(\alpha - \beta)\}\ .$$

□

## Appendix A. Proof of Proposition 2.1,2.2 and 2.6

The proof of Proposition 2.1 closely follows Proposition 1 in [4] and Proposition 2.1 in [15].

*Proof of Proposition 2.1.* First we consider the case $k \geq 1$. Fix $0 < \alpha < 1/6$ and let $a_1$, $a_2$ and $a_3$ be positive real numbers such that $a_1 < 1/2 + \alpha$, and $a_2 + a_3 < a_1$. For each positive integer $n$, we denote $u_n = \lfloor n^{a_1} \rfloor$, $v_n = \lfloor n^{a_2} \rfloor$ and $w_n = \lfloor n^{a_3} \rfloor$. Now we choose $n$ large enough such that $(K+1)v_n(w_n+1) + 2 + K \leq u_n$ and $u_n < (p\lambda/3)n^{1/2+\alpha}$. Let us define the following events:

$$G_n := \{(X_{\tau_k+n} - X_{\tau_k}) \cdot \ell \leq u_n\},$$

$$B_n := \bigcap_{j=1}^{v_n} \{\eta_j^{(k)} < \infty\} \quad \text{and} \quad F_n := \bigcup_{j=1}^{v_n} \{w_n \leq \eta_j^{(k)} - \rho_j^{(k)} < \infty\},$$

where, in the definition of $F_n$, we use the convention that $\eta_j^{(k)} - \rho_j^{(k)} = \infty$ whenever $\eta(X_{\rho_j^{(k)}}) = \infty$.

Our first step will be to show that

$$G_n^c \cap B_n^c \cap F_n^c \subset \{\tau_{k+1} - \tau_k \leq n\}, \tag{111}$$

Afterwards we will estimate separately the probabilities of $G_n$, $B_n$ and $F_n$ to finish the proof.

On the event $B_n^c$, we can define

$$M = \inf\{1 \leq j \leq v_n : \eta_j^{(k)} = \infty\},$$

and from definition we have $\tau_{k+1} = \rho_M^{(k)}$. Hence, we shall prove that $\{\rho_M^{(k)} - \tau_k \leq n\}$ always happens in $G_n^c \cap B_n^c \cap F_n^c$, then we have (111).

For each natural $m \geq \tau_k$, define

$$r_n = \max\{(X_j - X_{\tau_k}) \cdot \ell : \tau_k \leq j \leq m\}. \tag{112}$$

By the definition of $M$ we have $\eta_{M-1}^{(k)} < \infty$. Set $\eta_0^{(k)} = \tau_k$ and write

$$\sum_{j=1}^{M-1} \Big( (r_{\eta_j^{(k)}} - r_{\rho_j^{(k)}}) + (r_{\rho_j^{(k)}} - r_{\eta_{j-1}^{(k)}}) \Big) = r_{\eta_{M-1}^{(k)}} - r_{\eta_0^{(k)}} = r_{\eta_{M-1}^{(k)}}. \tag{113}$$

We are going to analyze separately each term on the right hand side of the of (113), that is, $(r_{\eta_j^{(k)}} - r_{\rho_j^{(k)}})$ and $(r_{\rho_j^{(k)}} - r_{\eta_{j-1}^{(k)}})$. By the definition of $\rho_j^{(k)}$ we get directly

$$r_{\rho_j^{(k)}} = \big(X_{\rho_j^{(k)}} - X_{\tau_k}\big) \cdot \ell. \tag{114}$$

By Condition I, we have that each jump of the process has a maximum range $K$, thus for each $1 \leq j \leq M-1$, we have that $(r_{\eta_j^{(k)}} - r_{\rho_j^{(k)}}) \leq K(\eta_j^{(k)} - \rho_j^{(k)})$. Now in the event $F_n^c$ we have $\eta_j^{(k)} - \rho_j^{(k)} < w_n$ for $1 \leq j \leq M-1$, thus $(r_{\eta_j^{(k)}} - r_{\rho_j^{(k)}}) \leq Kw_n$. For $(r_{\rho_j^{(k)}} - r_{\eta_{j-1}^{(k)}})$, from what we saw in (114), Condition I and by the definition of $\rho_j^{(k)}$, we have $(r_{\rho_j^{(k)}} - r_{\eta_{j-1}^{(k)}}) \le K+1$ for each $1 \le j \le M-1$. From those inequalities and (113) we obtain

$$r_{\eta_{M-1}^{(k)}} \le \sum_{j=1}^{M-1} ((Kw_n) + (K+1)) \le v_n(K+1)(w_n+1)\,.$$

Since we are considering $n$ large enough such that $v_n(K+1)(w_n+1)+2+K \le u_n$, then $r_{\eta_{M-1}^{(k)}} + 2 + K \le u_n$ on $B_n^c \cap F_n^c$.

Now on the event $G_n^c \cap B_n^c \cap F_n^c$, we have $(X_{\tau_k+n} - X_{\tau_k}) \cdot \ell > r_{\eta_{M-1}^{(k)}} + 2 + K$. Set $i = \min\{j \le n : (X_{\tau_k+j} - X_{\tau_k}) \cdot \ell > r_{\eta_{M-1}^{(k)}} + 1\}$. Since $\tau_{k+1} = \rho_M^{(k)}$ and $\rho_M^{(k)}$ is the first time the process move forward at least one position in direction $\ell$ from $r_{\eta_{M-1}^{(k)}}$, i.e. the maximum the walk reaches in direction $\ell$ until time $\eta_{M-1}^{(k)}$, we have that $\rho_M^{(k)} - \tau_k = i \le n$ which gives (111).

Using (111), we have that

$$\mathbb{P}[\tau_{k+1} - \tau_k > n | \mathcal{G}_0^{(k)}] \le \mathbb{P}[G_n | \mathcal{G}_0^{(k)}] + \mathbb{P}[B_n | \mathcal{G}_0^{(k)}] + \mathbb{P}[F_n | \mathcal{G}_0^{(k)}]\,. \tag{115}$$

Now we will bound these three probabilities. By Proposition 2.6 part (ii)

$$\begin{aligned}\mathbb{P}[G_n | \mathcal{G}_0^{(k)}] &= \mathbb{P}[(X_{\tau_k+n} - X_{\tau_k}) \cdot \ell \le u_n | \mathcal{G}_0^{(k)}] \\ &\le \mathbb{P}[(X_{\tau_k+n} - X_{\tau_k}) \cdot \ell \le \frac{p}{3} \lambda n^{\frac{1}{2}+\alpha} | \mathcal{G}_0^{(k)}] \le \frac{e^{-\gamma_3 n^{\gamma_4}}}{\psi}\,.\end{aligned} \tag{116}$$

By Proposition 2.6 part (iii) we have

$$\begin{aligned}\mathbb{P}[F_n | \mathcal{G}_0^{(k)}] &= \mathbb{P}\Big[\bigcup_{j=1}^{v_n} \{w_n \le \eta_j^{(k)} - \rho_j^{(k)} < \infty\} \Big| \mathcal{G}_0^{(k)}\Big] \\ &\le \sum_{j=1}^{v_n} \mathbb{P}\left[\{w_n \le \eta_j^{(k)} - \rho_j^{(k)} < \infty\} | \mathcal{G}_0^{(k)}\right] \\ &\le \sum_{j=1}^{v_n} 2e^{-\gamma_3 w_n^{\gamma_4}} = 2\lfloor n^{a_2} \rfloor e^{-\gamma_3 \lfloor n^{a_3} \rfloor^{\gamma_4}}\,.\end{aligned} \tag{117}$$

Finally using Proposition 2.6 part (i) we have

$$\begin{aligned}&\mathbb{P}[B_n | \mathcal{G}_0^{(k)}] = \mathbb{P}\Big[\bigcap_{j=1}^{v_n} \{\eta_j^{(k)} < \infty\} \Big| \mathcal{G}_0^{(k)}\Big] \\ &= \mathbb{P}\Big[\eta_{v_n}^{(k)} < \infty \Big| \mathcal{G}_0^{(k)}, \bigcap_{j=1}^{v_n-1} \{\eta_j^{(k)} < \infty\}\Big] \mathbb{P}\Big[\bigcap_{j=1}^{v_n-1} \{\eta_j^{(k)} < \infty\} \Big| \mathcal{G}_0^{(k)}\Big]\,.\end{aligned} \tag{118}$$

From the definition of the regeneration times

$$\{\eta_{v_n}^{(k)} < \infty\} = \{\eta_{v_n}^{(k)} < \infty\} \cap \{\rho_{v_n}^{(k)} < \infty\} = \emptyset\,,$$

then we have

$$\mathbb{P}\Big[\eta^{(k)}_{v_n} < \infty \Big| \mathcal{G}^{(k)}_0, \bigcap_{j=1}^{v_n-1}\{\eta^{(k)}_j < \infty\}\Big] = \mathbb{P}\Big[\{\rho^{(k)}_{v_n} < \infty\}\Big|\mathcal{G}^{(k)}_0, \bigcap_{j=1}^{v_n-1}\{\eta^{(k)}_j < \infty\}\Big]$$
$$\times \mathbb{P}\Big[\{\eta^{(k)}_{v_n} < \infty\}\Big|\mathcal{G}^{(k)}_0, \{\rho^{(k)}_{v_n} < \infty\}, \bigcap_{j=1}^{v_n-1}\{\eta^{(k)}_j < \infty\}\Big]$$
$$\leq \mathbb{P}\left[\{\eta^{(k)}_{v_n} < \infty\}|\mathcal{G}^{(k)}_{v_n}\right] .$$

Using the last inequality and Proposition 2.6 part (i) in (118), we obtain

$$\begin{aligned}
&\mathbb{P}[B_n|\mathcal{G}^{(k)}_0] \leq \mathbb{P}\Big[\{\eta^{(k)}_{v_n} < \infty\}\Big|\mathcal{G}^{(k)}_{v_n}\Big]\mathbb{P}\Big[\bigcap_{j=1}^{v_n-1}\{\eta^{(k)}_j < \infty\}\Big|\mathcal{G}^{(k)}_0\Big] \\
&\leq \prod_{j=1}^{v_n}\mathbb{P}\Big[\eta^{(k)}_j < \infty\Big|\mathcal{G}^{(k)}_j\Big] \leq \prod_{j=1}^{v_n}(1-\psi) = (1-\psi)^{v_n} = (1-\psi)^{\lfloor n^{a_2}\rfloor} ,
\end{aligned} \tag{119}$$

where we have used induction in second inequality. Next, using (116), (117) and (119) in (115) we obtain

$$\begin{aligned}
\mathbb{P}[\tau_{k+1} - \tau_k > n|\mathcal{G}^{(k)}_0] &\leq \mathbb{P}[G_n|\mathcal{G}^{(k)}_0] + \mathbb{P}[B_n|\mathcal{G}^{(k)}_0] + \mathbb{P}[F_n|\mathcal{G}^{(k)}_0] \\
&\leq \frac{e^{-\gamma_3 n^{\gamma_4}}}{\psi} + 12\lfloor n^{a_2}\rfloor e^{-\gamma_3\lfloor n^{a_3}\rfloor^{\gamma_4}} + (1-\psi)^{\lfloor n^{a_2}\rfloor} \\
&\leq \frac{e^{-\gamma_3 n^{\gamma_4}}}{\psi} + 12\lfloor n^{a_2}\rfloor e^{-\gamma_3\lfloor n^{a_3}\rfloor^{\gamma_4}} + e^{-\psi\lfloor n^{a_2}\rfloor} ,
\end{aligned}$$

finishing the proof for $k \geq 1$.

It remains to prove the result for $k = 0$, i.e., there exist positive constants $\tilde{C}$ and $\zeta$ such that for some $n \geq 1$

$$\mathbb{P}[\tau_1 > n] \leq \tilde{C} e^{-n^{\zeta}} . \tag{120}$$

The proof of (120) is analogous as that for $k \geq 1$, the only difference is in events. To prove (120) we consider

$$G_n = \{X_n \cdot \ell \leq u_n\} ,$$
$$B_n = \bigcap_{j=1}^{v_n}\{\eta^{(0)}_j < \infty\} \quad \text{and} \quad F_n = \bigcup_{j=1}^{v_n}\{w_n \leq \eta^{(0)}_j - \rho^{(0)}_j < \infty\} .$$

Then we can conclude the proof. □

The proof of Proposition 2.2 follows closely that of Proposition 2.2 in [15].

*Proof of Proposition 2.2.* (*i*)*:* We first prove the case $k = 1$. For each $n \geq 1$, we construct a set of trajectories $\Lambda_n$ of the form $\{x_0, \dots, x_n\}$ such that $x_n \cdot \ell > \sup_{0\leq i\leq n-1} x_i \cdot \ell$. For a element $\gamma \in (\mathbb{Z}^d)^{\mathbb{N}}$, we will denote $\gamma_n$ as a projection to the $n$ first coordinates and $\pi(\gamma_n)$ as the $n$ first Bernoulli's trials.

Let $b \in \Lambda_n$, then we have that $\mathcal{G}_0^{(1)}$ is generated by the disjoint collection of the form $\{\tau_1 = n\} \cap \{\gamma_n = b\} \cap \{\pi(\gamma_n) = \pi(b)\}$. Thus we obtain,

$$\begin{aligned}
&\mathbb{P}\left[X_{\tau_1+\cdot} \in A|\mathcal{G}_0^{(1)}\right] = \\
&= \sum_{n=1}^{\infty} 1_{\{\tau_1=n\}}(\varpi)\sum_{\gamma_n\in\Lambda_n} 1_{\{\varpi_n=\gamma_n\}}\mathbb{P}\left[X_{\tau_1+\cdot} \in A|\tau_1 = n, \varpi_n = \gamma_n, \pi(\varpi_n) = \pi(\gamma_n)\right] \\
&= \sum_{n=1}^{\infty} 1_{\{\tau_1=n\}}(\varpi)\sum_{\gamma_n\in\Lambda_n} 1_{\{\varpi_n=\gamma_n\}}\mathbb{P}\left[X_{\tau_1+\cdot} \in A|\eta(X_n) = \infty, \varpi_n = \gamma_n, \pi(\varpi_n) = \pi(\gamma_n)\right] \\
&= \sum_{n=1}^{\infty} 1_{\{\tau_1=n\}}(\varpi)\mathbb{P}\left[X_{\tau_1+\cdot} \in A|\eta(X_n) = \infty, \mathcal{F}_n\right] .
\end{aligned} \tag{121}$$

The third equality in (121) follows from observing that in the event $\{\tau_1 = n\}$,

$$\{\varpi_n = \gamma_n, \pi(\varpi_n) = \pi(\gamma_n)\} = \{\eta(X_n) = \infty, \varpi_n = \gamma_n, \pi(\varpi_n) = \pi(\gamma_n)\} .$$

The fourth equality in (121) follows from noticing that $1_{\{\tau_1=n\}}(\varpi)1_{\{\varpi_n=\gamma_n\}} = 0$, if $\gamma_n \notin \Lambda_n$. As a matter of fact, if $\gamma_n \notin \Lambda_n$, by the property of the regeneration times, it is not possible the trajectory $\gamma$ has $\tau_1 = n$, since in the first $n$ coordinates it must exists a $0 \le j \le n-1$ such that $x_j \cdot \ell > x_n \cdot \ell$.

The case $k \ge 2$ can be proved in a similar way. Indeed, the sequence of regeneration time is increasing and in the event $\{\tau_k = n\}$ we always have $\{\eta(X_n) = \infty\}$.

*(ii):* This proof is similar to Proposition 2.2 part (ii) in [15]. For a element $\gamma \in (\mathbb{Z}^d)^{\mathbb{N}}$, we will denote $\gamma_n$ as a projection to the $n$ first coordinates and $\pi(\gamma_n)$ as the $n$ first Bernoulli's trials. First we consider the case $k = 1$ and $j = 1$.

For an $u > 0$, let us denote by $T_u$ the first time the process reaches or exceeds this position in direction $\ell$, i.e., $T_u = \inf\{k \ge 1, X_k \cdot \ell \ge u\}$. For each $n \ge 1$ and $m \ge 1$, we construct a set of trajectories $\Lambda_{n,m}$ of the form $\{x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}\}$ satisfying the following properties:

(i) $x_n \cdot \ell > \sup_{0\le i\le n-1} x_i \cdot \ell$ .

(ii) For each $0 \le i \le n-1$, we have,

$$\min_{T_{x_i\cdot\ell}<t\le n-1} x_t \cdot \ell \le x_i \cdot \ell .$$

(iii) $x_{n+m} \cdot \ell \ge x_n \cdot \ell + 1 > \sup_{0\le i\le n+m-1} x_i \cdot \ell$.

One can see that, for this set of trajectories $\Lambda_{n,m}$, if $\eta(x_n) = \infty$ then $\rho_1^{(1)} = n + m$ and $\tau_1 = n$. Let $b \in \Lambda_{n,m}$, thus $\mathcal{G}_1^{(1)}$ is generated by the disjoint collection of the form $\{\tau_1 = n\} \cap \{\gamma_{n+m} = b\} \cap \{\pi(\gamma_{n+m}) = \pi(b)\}$.

Hence we obtain,

$$\begin{aligned}
&\mathbb{P}[X_{\rho_1^{(1)}+\cdot} \in A|\mathcal{G}_1^{(1)}] \qquad (122)\\
&= \sum_{n=1}^{\infty} 1_{\{\tau_1=n\}}(\varpi) \sum_{m=1}^{\infty} 1_{\{\rho_1^{(1)}=n+m\}}(\varpi) \sum_{\gamma_{n+m}\in\Lambda_{n,m}} 1_{\{\varpi_{n+m}=\gamma_{n+m}\}}\\
&\times \mathbb{P}[X_{n+m+\cdot} \in A|\tau_1 = n, \varpi_{n+m} = \gamma_{n+m}, \pi(\varpi_{n+m}) = \pi(\gamma_{n+m})] \,.
\end{aligned}$$

Thus, in the event $\{\tau_1 = n\}$, we have $\{\varpi_{n+m} = \gamma_{n+m}, \pi(\varpi_{n+m}) = \pi(\gamma_{n+m})\} = \{\eta(X_n) = \infty, \varpi_{n+m} = \gamma_{n+m}, \pi(\varpi_{n+m}) = \pi(\gamma_{n+m})\}$. Now observe that,

$$1_{\{\tau_1=n\}}(\varpi)1_{\{\rho_1^{(1)}=n+m\}}(\varpi)1_{\{\varpi_{n+m}=\gamma_{n+m}\}} = 0 \,,$$

if $\gamma_{n+m} \notin \Lambda_{n,m}$. One can see that if $\gamma_{n+m} \notin \Lambda_{n,m}$, by the property of the regeneration times, it is not possible the trajectory $\gamma$ has $\tau_1 = n$, by the fact that in the first $n$ coordinates we have a $0 \le j \le n-1$ such that $x_j\cdot\ell > x_n\cdot\ell$. Besides that, if we have a $n+1 \le j \le n+m-1$, such that $x_j\cdot\ell \ge x_n\cdot\ell+1$, then $\rho_1^{(1)} = j$ by the definition of $\rho_1^{(1)}$. Going back to (122), we have

$$\begin{aligned}
&\mathbb{P}[X_{\rho_1^{(1)}+\cdot} \in A|\mathcal{G}_1^{(1)}]\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sum_{\gamma_{n+m}\in\Lambda_{n,m}} 1_{\{\tau_1=n\}}(\varpi)1_{\{\rho_1^{(1)}=n+m\}}(\varpi)1_{\{\varpi_{n+m}=\gamma_{n+m}\}}\\
&\times \mathbb{P}[X_{n+m+\cdot} \in A|\eta(X_n) = \infty, \mathcal{F}_{n+m}]\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} 1_{\{\tau_1=n\}}(\varpi)1_{\{\rho_1^{(1)}=n+m\}}(\varpi)\mathbb{P}[X_{n+m+\cdot} \in A|\eta(X_n) = \infty, \mathcal{F}_{n+m}] \,.
\end{aligned}$$

The case that $k = 1$ and $j > 1$ is similar. We just need to make the proper adjustments in $\Lambda_{n,m}$. For $k > 1$, we also prove in the similar way using the fact that, for every natural $n$ and $k > 1$, the event $\{\eta(X_n) = \infty\} \cap \{n < \tau_k\}$ is $\mathcal{G}_0^{(k)}$ measurable. □

The proof of Proposition 2.6 closely follows the proof of Proposition 4.4 in [15].

*Proof of Proposition 2.6. Proof of (i):* From the definition of $\eta_j^{(k)}$ we have that,

$$\mathbb{P}[\eta_j^{(k)} < \infty|\mathcal{G}_j^{(k)}] = \mathbb{P}[\eta(X_{\rho_j^{(k)}}) < \infty|\mathcal{G}_j^{(k)}] = 1 - \mathbb{P}[\eta(X_{\rho_j^{(k)}}) = \infty|\mathcal{G}_j^{(k)}] \,. \qquad (123)$$

The event $\{\eta(X_{\rho_j^{(k)}}) = \infty\}$ means that the process does not come back in direction $\ell$ to the position $X_{\rho_j^{(k)}}$, hence $X_{\rho_j^{(k)}+i}\cdot\ell > X_{\rho_j^{(k)}}\cdot\ell$ for all $i \in \mathbb{N}$.

Let $B$ be a Borel set of $(\mathbb{Z}^d)^{\mathbb{N}}$ such that $X_{\rho_j^{(k)}+i}\cdot\ell > X_{\rho_j^{(k)}}\cdot\ell$ for all $i\in\mathbb{N}$, then by part (ii) of Proposition 2.2 we have,

$$\begin{aligned}
&\mathbb{P}\Big[\eta(X_{\rho_j^{(k)}})=\infty|\mathcal{G}_j^{(k)}\Big]=\mathbb{P}\Big[X_{\rho_j^{(k)}+\cdot}\in B\Big|\mathcal{G}_j^{(k)}\Big]\\
&\quad=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}1_{\{\tau_k=n\}}(\varpi)1_{\{\rho_j^{(k)}=n+m\}}(\varpi)\mathbb{P}[X_{n+m+\cdot}\in B|\eta(X_n)=\infty,\mathcal{F}_{n+m}]\\
&\quad=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}1_{\{\tau_k=n\}}(\varpi)1_{\{\rho_j^{(k)}=n+m\}}(\varpi)\mathbb{P}\left[\eta(X_{n+m})=\infty|\eta(X_n)=\infty,\mathcal{F}_{n+m}\right].
\end{aligned}\tag{124}$$

Since we have that $\{\eta(X_{n+m})=\infty\}\subset\{\eta(X_n)=\infty\}$ then,

$$\begin{aligned}
\mathbb{P}\left[\eta(X_{n+m})=\infty|\eta(X_n)=\infty,\mathcal{F}_{n+m}\right]&=\frac{\mathbb{P}\left[\{\eta(X_{n+m})=\infty\}|\mathcal{F}_{n+m}\right]}{\mathbb{P}\left[\{\eta(X_n)=\infty\}|\mathcal{F}_{n+m}\right]}\\
&\geq\mathbb{P}\left[\eta(X_{n+m})=\infty|\mathcal{F}_{n+m}\right]\geq\psi\,.
\end{aligned}\tag{125}$$

The last inequality in (125) follows from Proposition 2.5 and the fact that $m+n$ is a maximum point in $\ell$ direction, so like in the origin the process has a completely new environment to explore forward in $\ell$ direction. Then using (125) in (124) we get

$$\mathbb{P}\left[\eta(X_{\rho_j^{(k)}})=\infty|\mathcal{G}_j^{(k)}\right]\geq\psi\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}1_{\{\tau_k=n\}}(\varpi)1_{\{\rho_j^{(k)}=n+m\}}(\varpi)\geq\psi\,.\tag{126}$$

Then, by (123) and (126) we conclude that $\mathbb{P}\left[\eta_j^{(k)}<\infty|\mathcal{G}_j^{(k)}\right]<(1-\psi)$.

*Proof of (ii):* Use Proposition 2.2 part (i) to write

$$\begin{aligned}
&\mathbb{P}\Big[(X_{\tau_k+n}-X_{\tau_k})\cdot\ell<\frac{p}{3}\lambda n^{\frac12+\alpha}\Big|\mathcal{G}_0^{(k)}\Big]=\\
&\quad=\sum_{m=1}^{\infty}1_{\{\tau_k=m\}}(\varpi)\mathbb{P}\left[(X_{m+n}-X_m)\cdot\ell<\frac{p}{3}\lambda n^{\frac12+\alpha}|\eta(X_m)=\infty,\mathcal{F}_m\right]\\
&\quad\leq\sum_{m=1}^{\infty}1_{\{\tau_k=m\}}(\varpi)\frac{\mathbb{P}\left[(X_{m+n}-X_m)\cdot\ell<\frac{p}{3}\lambda n^{\frac12+\alpha}|\mathcal{F}_m\right]}{\mathbb{P}\left[\eta(X_m)=\infty|\mathcal{F}_m\right]}\,,
\end{aligned}$$

which, by Propositions 2.4, Remark 2.4 and Proposition 2.5, is bounded from above by

$$\sum_{m=1}^{\infty}1_{\{\tau_k=m\}}(\varpi)\frac{e^{-\gamma_3 n^{\gamma_4}}}{\mathbb{P}\left[\eta(X_m)=\infty|\mathcal{F}_m\right]}\leq\frac{e^{-\gamma_3 n^{\gamma_4}}}{\psi}\,.$$

*Proof of (iii):* By definition of $\eta_j^{(k)}$, we have that,

$$\mathbb{P}\left[n \leq \eta_j^{(k)} - \rho_j^{(k)} < \infty | \mathcal{G}_0^{(k)}\right] = \mathbb{P}\left[n \leq \eta(X_{\rho_j^{(k)}}) - \rho_j^{(k)} < \infty | \mathcal{G}_0^{(k)}\right] \tag{127}$$
$$= \sum_{i=n}^{\infty} \mathbb{P}\left[\eta(X_{\rho_j^{(k)}}) - \rho_j^{(k)} = i | \mathcal{G}_0^{(k)}\right] = \sum_{i=n}^{\infty} \mathbb{P}\left[\eta(X_0) = i\right] \leq \sum_{i=n}^{\infty} \mathbb{P}\left[X_i \cdot \ell < 0\right].$$

The third equality in (127) holds since the system is invariant and by the definition of $\rho_j^{(k)}$ we know the process in direction $\ell$ sees a new environment. Then to go below that position in direction $\ell$, it is the same to go under $X_0$ in direction $\ell$. Hence, using Proposition 2.4, we obtain,

$$\mathbb{P}\left[n \leq \eta_j^{(k)} - \rho_j^{(k)} < \infty | \mathcal{G}_0^{(k)}\right] \leq \sum_{i=n}^{\infty} \mathbb{P}\left[X_i \cdot \ell < 0\right] \leq \sum_{i=n}^{\infty} \mathbb{P}\left[X_i \cdot \ell < \frac{p\lambda}{3} i^{\frac{1}{2}+\alpha}\right]$$
$$\leq 6 \sum_{i=n}^{\infty} i e^{-\gamma_3 i^{\gamma_4}} \leq 6 \left( n e^{-\gamma_3 n^{\gamma_4}} + \int_{\gamma_3 n^{\gamma_4}}^{\infty} x e^{-x} dx \right) \leq 12 e^{-\gamma_3 n^{\gamma_4}} .$$

□

## Appendix B. Proof of Proposition 2.3

This section is devoted to prove Proposition 2.3. We will start by stating some auxiliaries lemmas whose proofs are deferred to the end of this Appendix.

Let $X$ be a $p$-GERW. We denote by $L_n(m)$ its local time up to time $n$ on the $m$-th strip in direction $\ell$ which is defined as

$$L_n(m) := \sum_{j=0}^{n} 1_{\{X_j \cdot \ell \in [m, m+1)\}} , \ m \in \mathbb{Z} .$$

The next result provides a control on the tail of the local times $L_n(m)$.

**Lemma B.1.** *Let $X$ be a $p$-GERW, for any $\delta > 0$ there exists a constant $\gamma_1'$ depending of $K$, $h$ and $r$ such that for all $m$ we have,*

$$\mathbb{P}\left[L_n(m) \geq n^{\frac{1}{2}+2\delta}\right] \leq \exp\left\{-\gamma_1' n^{\delta}\right\} .$$

### B.1. Some auxiliary results for $d$-dimensional martingales.

In this brief section we lists a few results for $d$-dimensional martingales which will be used in the proof of Proposition 2.3.

**Lemma B.2.** *Let $Y$ be a $d$-dimensional martingale with uniformly $K$-bounded increments, i.e., $\sup_{n\geq 1} \|Y_{n+1} - Y_n\| \leq K$. Then, for all $b \geq 0$ and for all $n \geq 1$, it holds that*

$$\mathbb{E}\big(\|Y_{n+1}\|^b | \mathcal{F}_n\big) \geq \|Y_n\|^b 1_{\{\|Y_n\| > K/(\sqrt{2}-1)\}} .$$

The next result is a bound on the number of visits of a $d$-dimensional martingale to any site in $\mathbb{Z}^d$.

**Lemma B.3.** *Let $Y$ be a $d$-dimensional martingale which satisfies Condition I and (UE2) in Condition III and suppose that $Y_0 = x_0$. Then, for any $\delta > 0$ and $\phi > 0$, there exists a constant $\gamma_3' > 0$ depending on $K$, $\phi$, $r$ and $h$ such that for all $x_0, y_0 \in \mathbb{Z}^d$ and for all $n$, we have,*

$$\mathbb{P}\left[\sum_{j=1}^{n} 1_{\{Y_j = y_0\}} > n^{\phi+\delta}\right] \le \exp\left\{-\gamma_3' n^{\delta}\right\}.$$

Let us denote by $\tau_B^X$ the hitting time to a set $B$ for a process $X$, i.e.,

$$\tau_B^X := \min\{n \ge 0 : X_n \in B\},$$

and denote by $B(x, q)$ a discrete ball centered in $x \in \mathbb{Z}^d$ and with radius $q$, i.e., $B(x, q) := \{y \in \mathbb{Z}^d : \|x - y\| \le q\}$. The next result implies that a $d$-dimensional martingale $Y$ hits with high probability sets that contains enough points close to the origin of the process.

**Lemma B.4.** *Let $Y$ be a $d$-dimensional martingale which satisfies Condition I and (UE2) in Condition III. Assume that $Y_0 = x$. Consider an arbitrary $\delta > 0$, a set $U$ and suppose that $|B(x, m^{1/2}) \setminus U| \le m^{1-\phi-2\delta}$ for some $m$ and $0 < \phi \le 1$. Then there exists a constant $\gamma_4' > 0$ depending on $d$, $K$, $\phi$, $h$ and $r$ such that,*

$$\mathbb{P}\left[\tau_U^Y \ge m^{1-\delta}\right] \le \exp\left\{-\gamma_4' m^{\delta}\right\}.$$

B.2. **Proof of Proposition 2.3.** Using Lemma B.1 and Lemma B.4 we now prove Proposition 2.3[5].

*Proof of Proposition 2.3.* Consider $b \in (0, 1)$ and $\varepsilon > 0$. Set $e_w$ the strip width exponent and define

$$H_j^n := H\big(2(j-1)n^{e_w}, 2(j+1)n^{e_w}\big),\ n \ge 1,\ j \ge 1,$$

so that $H_j^n$ is strip of width $4n^{e_w}$ in direction $\ell$. The strip $H_j^n$ will be called a trap if $|\mathcal{R}_n^X \cap H_j^n| \ge n^{e_t}$, where $e_t = 2e_w(1-(b/2)) - 2\varepsilon$ is the trap exponent. Set

$$G = \{|\mathcal{R}_n^X| \ge n^{\frac{1}{2}+e_w(1-b)-4\varepsilon}\}.$$

We are going to prove that

$$\mathbb{P}[G] \ge 1 - \left((2Kn+1)\, e^{-\gamma_1' n^{\frac{\varepsilon}{2}}} + \frac{n^{1-2e_w+\varepsilon}}{2} e^{-\gamma_4' n^{\varepsilon}}\right), \tag{128}$$

for every $\varepsilon > 0$ sufficiently small. This establishes Proposition 2.3 since, as we will see, for every $0 < \alpha < 1/6$, we can choose $e_w$, $b$ and $\varepsilon$ such that $\alpha < e_w(1-b) - 4\varepsilon$ (i.e., $\{|\mathcal{R}_n^X| < n^{\frac{1}{2}+\alpha}\} \subset G^c$) and (128) holds.

Thus we have to prove (128). Let us first introduce the event

$$G_1 = \left\{L_n(k) \le n^{\frac{1}{2}+\varepsilon} \text{ for all } k \in [-Kn, Kn]\right\}.$$

[5] Let us point out that Lemma B.2 is used to prove Lemma B.3 which, in turn, is used to prove Lemma B.4.

By Lemma B.1, it holds that

$$\mathbb{P}[G_1] \geq 1 - (2Kn+1)\exp\left\{-\gamma_1' n^{\frac{\varepsilon}{2}}\right\}. \tag{129}$$

Now, let us define $\sigma_0 = 0$ and inductively

$$\sigma_{k+1} = \min\{j \geq \sigma_k + \lfloor n^{2e_w - \varepsilon} \rfloor : |\mathcal{R}_j^X \cap B(X_j, n^{e_w})| \leq n^{e_t}\}, \tag{130}$$

(formally, if such $j$ does not exist, we put $\sigma_{k+1} = \infty$). Consider the event

$$G_2 = \Big\{\text{at least one new point is hit on each of the time intervals}$$
$$[\sigma_{j-1}, \sigma_j), j = 1, ..., \frac{1}{2} n^{1-2e_w+\varepsilon}\Big\},$$

where hitting a new point means to visit a not-yet-visited site. Note that on $G_2^c$, the process does not hit a new point in time interval $[\sigma_{j-1}, \sigma_j)$ for some $j = 1, ..., \frac{1}{2} n^{1-2e_w+\varepsilon}$. When this happens, the process $X$ evolves as a d-dimensional martingale during time interval $[\sigma_{j-1}, \sigma_j)$ and for $Y_\cdot = X_{\sigma_{j-1}+\cdot}$

$$\tau^Y_{\left(\mathcal{R}^X_{\sigma_{j-1}}\right)^c} \geq \sigma_j - \sigma_{j-1} \geq n^{2e_w - \varepsilon}.$$

To control the probability of $G_2^c$, we will apply Lemma B.4. For this sake we point out that $2e_w \geq e_t$ and introduce $\tilde{\delta}$ in $(0,1)$ such that

$$e_t = 2e_w(1-\tilde{\delta}).$$

This allows us to write

$$\frac{\tilde{\delta}}{2} = \frac{1}{2} - \frac{e_t}{4e_w} = \frac{1}{2} - \frac{1}{2}\Big(1 - \frac{b}{2}\Big) - \frac{\varepsilon}{2e_w} = \frac{b}{4} - \frac{\varepsilon}{2e_w}.$$

Setting $\delta = \frac{\varepsilon}{2e_w}$, for every $b \in (0,1)$, we can choose $\varepsilon$ sufficiently small such that

$$\frac{\tilde{\delta}}{2} > \delta > 0.$$

Then, we are ready to apply Lemma B.4 with the following choice of parameters: $\delta = \frac{\varepsilon}{2e_w}$ (as above), $m = n^{2e_w}$, $\phi = \tilde{\delta} - 2\delta$, and choosing $U = \left(\mathcal{R}^X_{\sigma_{j-1}}\right)^c$. Note that by the definition of $\sigma_{j-1}$, we have that $|\mathcal{R}^X_{\sigma_{j-1}} \cap B(X_{\sigma_{j-1}}, n^{e_w})| \leq n^{e_t}$, which for our choice of parameters implies

$$\begin{aligned} |B(X_{\sigma_{j-1}}, n^{e_w}) \setminus \left(\mathcal{R}^X_{\sigma_{j-1}}\right)^c| &= |B(X_{\sigma_{j-1}}, (n^{2e_w})^{1/2}) \setminus \left(\mathcal{R}^X_{\sigma_{j-1}}\right)^c| \leq n^{e_t} \\ &= \left(n^{2e_w}\right)^{\frac{e_t}{2e_w}} = \left(n^{2e_w}\right)^{1-\tilde{\delta}} = \left(n^{2e_w}\right)^{1-\phi-2\delta}, \end{aligned}$$

and thus we can use Lemma B.4 to conclude that

$$\mathbb{P}[\tau^Y_{\left(\mathcal{R}^X_{\sigma_{j-1}}\right)^c} \geq \sigma_j - \sigma_{j-1} \geq n^{2e_w-\varepsilon}] \leq e^{-\gamma_4' n^{\epsilon}},$$

for every $j = 1, ..., \frac{1}{2} n^{1-2e_w+\varepsilon}$. Thus

$$\mathbb{P}[G_2] \geq 1 - \frac{1}{2} n^{1-2e_w+\varepsilon} e^{-\gamma_4' n^{\varepsilon}}. \tag{131}$$

Next, assuming that $n$ is large enough so that $8n^{1-\varepsilon} < n/2$, we will show that $(G_1 \cap G_2) \subset G$. Suppose that both $G_1$ and $G_2$ occur, but $|\mathcal{R}_n^X| < n^{\frac{1}{2}+e_w(1-b)-4\varepsilon}\}$. Let us denote by $\hat{L}_j$ the total number of visits to $H_j^n$ up to time $n$, i.e.,

$$\hat{L}_j = \sum_{k=2(j-1)n^{e_w}}^{2(j+1)n^{e_w}-1} L_n(k)\,,$$

On $\{|\mathcal{R}_n^X| < n^{\frac{1}{2}+e_w(1-b)-4\varepsilon}\}$ we have

$$n^{\frac{1}{2}+e_w(1-b)-4\varepsilon} > |\mathcal{R}_n^X| \geq n^{e_t}|\{j : H_j^n \text{ is a trap}\}|\,,$$

thus the number of traps is at most

$$2n^{\frac{1}{2}+e_w(1-b)-4\varepsilon-e_t} = 2n^{\frac{1}{2}-e_w-2\varepsilon}\,.$$

On $G_1$, we have,

$$\sum_{j\in\mathbb{Z}} \hat{L}_j 1_{\{H_j^n \text{ is a trap}\}} \leq 4n^{e_w} \times 2n^{\frac{1}{2}-e_w-2\varepsilon} \times n^{\frac{1}{2}+\varepsilon} = 8n^{1-\varepsilon}\,.$$

Now observe that, since for $j \leq n$ we have $\mathcal{R}_j^X \subset \mathcal{R}_n^X$, if $|\mathcal{R}_j^X \cap B(X_j, n^{e_w})| > n^{e_t}$ then $X_j$ must be in a trap. Since $n$ is such that $8n^{1-\varepsilon} < n/2$, we obtain that, on the event

$$\Big\{\sum_{j\in\mathbb{Z}} \hat{L}_j 1_{\{H_j \text{ is a trap}\}} \leq 8n^{1-\varepsilon}\Big\}\,,$$

the total time (up to time $n$) spent in non-traps is at least $n - 8n^{1-\varepsilon} > n/2$. From the definition (130), the latter implies that $\sigma_{\frac{n^{1-2e_w+\varepsilon}}{2}} < n$. Indeed, up to time $\sigma_{\frac{n^{1-2e_w+\varepsilon}}{2}}$ we can have at most $n/2$ instances $j$ such that $|\mathcal{R}_j^X \cap B(X_j, n^{e_w})| \leq n^{e_t}$. Therefore, on the event $G_2$ we have that $|\mathcal{R}_n^X| \geq \frac{1}{2}n^{1-2e_w+\varepsilon}$. Recall that we assumed that $G_1$ and $G_2$ occur, but $|\mathcal{R}_n^X| < n^{\frac{1}{2}+e_w(1-b)-4\varepsilon}\}$. Since for $e_w < \frac{1}{6}$ (and $n$ sufficiently large) it holds that $\frac{1}{2}n^{1-2e_w+\varepsilon} > n^{\frac{1}{2}+e_w(1-b)-4\varepsilon}$, for every $b \in (0,1)$ and $\varepsilon > 0$, we obtain a contradiction. Then, indeed, $(G_1 \cap G_2) \subset G$, and (128) follows from (129) and (131),

$$\begin{aligned}\mathbb{P}[G] &\geq \mathbb{P}[G_1 \cap G_2] \geq 1 - (\mathbb{P}[G_1] + \mathbb{P}[G_2]) \\ &\geq 1 - \left((2Kn+1)\, e^{-\gamma_1' n^{\frac{\varepsilon}{2}}} + \frac{n^{1-2e_w+\varepsilon}}{2} e^{-\gamma_4' n^{\varepsilon}}\right)\,.\end{aligned}$$

To conclude the proof of Proposition 2.3, just note that for every $\alpha < 1/6$, we can find $e_w < 1/6$ and $b \in (0,1)$ and $\varepsilon$ (sufficiently small), such that $\alpha < e_w(1-b) - 4\varepsilon$.

□

B.3. **Proof of the Lemmas.** In this section we provide the proof of the Lemmas used to prove Proposition 2.3.

The proof of Lemma B.1 follows closely that of Lemma 5.1 from [15].

*Proof of Lemma B.1.* Without loss of generality we can consider $m = -1$. As we will see in this proof, it will be only require the uniform elliptic condition, then we can choose any $m$ and the proof will use the same techniques. We denote $\hat{t}_0 = 0$ and

$$\hat{t}_{k+1} = \min\{j > \hat{t}_k : X_j \cdot \ell \in [-1, 0)\}\,.$$

With this notation we have $L_n(-1) = \max\{k : \hat{t}_k \le n\}$.

By Condition III there exist a positive constant $C_1$ and a natural number $i_0 \ge 1$ such that for any stopping time $T$ we have

$$\mathbb{P}[(X_{T+i_0} - X_T) \cdot \ell \ge 2 | \mathcal{F}_T] \ge C_1\,. \tag{132}$$

As we saw in the proof of Proposition 2.4 the process $\{X_n \cdot \ell\}_{n \ge 0}$ is a $\mathcal{F}_n$-submartingale. By the optional stopping theorem for any positive integer $j$ and $x \in \mathbb{Z}^d$ such that $x \cdot \ell \ge 1$, we have

$$\mathbb{E}[(X_{T_n \wedge T_0}) \cdot \ell | \mathcal{F}_{\hat{t}_j + i_0}, X_{\hat{t}_j + i_0} = x] \ge x \cdot \ell \ge 1\,, \tag{133}$$

where $T_n = \tau^X_{H(n^{1/2+\delta}, +\infty)} \circ \theta_{\hat{t}_j + i_0}$ and $T_0 = \tau^X_{H(-\infty, 0)} \circ \theta_{\hat{t}_j + i_0}$ and $\theta$ is the canonical time shift on the space of trajectories. Now we obtain,

$$\begin{aligned}
&\mathbb{E}[(X_{T_n \wedge T_0}) \cdot \ell | \mathcal{F}_{\hat{t}_j + i_0}, X_{\hat{t}_j + i_0} = x] = \\
&= \mathbb{E}[(X_{T_n} \cdot \ell) 1_{\{T_n < T_0\}} + (X_{T_0} \cdot \ell) 1_{\{T_0 < T_n\}} | \mathcal{F}_{\hat{t}_j + i_0}, X_{\hat{t}_j + i_0} = x] \\
&\le (n^{\frac{1}{2} + \delta} + K) \mathbb{P}[T_n < T_0 | \mathcal{F}_{\hat{t}_j + i_0}, X_{\hat{t}_j + i_0} = x]\,.
\end{aligned} \tag{134}$$

With (133) and (134) we have,

$$\mathbb{P}[T_n < T_0 | \mathcal{F}_{\hat{t}_j + i_0}, X_{\hat{t}_j + i_0} = x] \ge \frac{1}{n^{\frac{1}{2} + \delta} + K}\,. \tag{135}$$

We set the events $E = \{(X_{\hat{t}_j + i_0} - X_{\hat{t}_j}) \cdot \ell \ge 2\}$ and $F_y = \{X_{\hat{t}_j + i_0} = y\}$ where $y \in \mathbb{Z}^d$. Then we obtain

$$\begin{aligned}
&\mathbb{P}[X_{\hat{t}_j + l} \cdot \ell > 0 \;\forall\; i_0 \le l \le \tau^X_{H(n^{1/2+\delta}, +\infty)} \circ \theta_{\hat{t}_j} | \mathcal{F}_{\hat{t}_j}] \\
&\ge \mathbb{P}[(X_{\hat{t}_j + i_0} - X_{\hat{t}_j}) \cdot \ell \ge 2, X_{\hat{t}_j + l} \cdot \ell > 0 \;\forall\; i_0 \le l \le i_0 + T_n | \mathcal{F}_{\hat{t}_j}] \\
&\ge \sum_{y \in \mathbb{Z}^d} \mathbb{P}[E \cap F_y | \mathcal{F}_{\hat{t}_j}] \mathbb{P}[X_{\hat{t}_j + l} \cdot \ell > 0 \;\forall\; i_0 < l \le i_0 + T_n | \{E \cap F_y\}, \mathcal{F}_{\hat{t}_j}]\,,
\end{aligned} \tag{136}$$

where the last inequality above holds since $\mathbb{P}[E \cap F_y | \mathcal{F}_{\hat{t}_j}] > 0$ only if $y \cdot \ell \ge 1$. Thus (133) holds, and using (135) in (136) we obtain

$$\begin{aligned}
&\mathbb{P}[X_{\hat{t}_j + l} \cdot \ell > 0 \;\forall\; i_0 \le l \le \tau^X_{H(n^{1/2+\delta}, +\infty)} \circ \theta_{\hat{t}_j} | \mathcal{F}_{\hat{t}_j}] \\
&\ge \frac{1}{n^{\frac{1}{2} + \delta} + K} \sum_{y \in \mathbb{Z}^d} \mathbb{P}[E \cap F_y | \mathcal{F}_{\hat{t}_j}] \ge \frac{C_1}{n^{\frac{1}{2} + \delta} + K} \ge C_2 n^{-\frac{1}{2} - \delta}\,,
\end{aligned} \tag{137}$$

where in the last inequality we use (132).

Set

$$B = \{X_{\hat{t}_j+l} \cdot \ell > 0,\ \forall\ i_0 \le l \le \tau^X_{H(n^{1/2+\delta},+\infty)} \circ \theta_{\hat{t}_j}\}\,.$$

We will find a lower bound on the probability of the event that the process spends more than $n$ steps outside the strip $[-1,0)$ in direction $\ell$.

$$\begin{aligned} \mathbb{P}[\hat{t}_{j+1} - \hat{t}_j > n|\mathcal{F}_{\hat{t}_j}] &\ge \mathbb{P}[\{\hat{t}_{j+1} - \hat{t}_j > n\} \cap B|\mathcal{F}_{\hat{t}_j}] \\ &\ge \mathbb{P}[B|\mathcal{F}_{\hat{t}_j}]\,\mathbb{P}[\hat{t}_{j+1} - \hat{t}_j > n|B, \mathcal{F}_{\hat{t}_j}] \\ &\ge \mathbb{P}[B|\mathcal{F}_{\hat{t}_j}]\,(1 - \mathbb{P}[\hat{t}_{j+1} - \hat{t}_j \le n|B, \mathcal{F}_{\hat{t}_j}]) \\ &\ge \mathbb{P}[B|\mathcal{F}_{\hat{t}_j}]\,(1 - \mathbb{P}[(-X_n + X_k)\cdot \ell \ge n^{1/2+\delta}]) \\ &\ge C_2 n^{-\frac{1}{2}-\delta}(1 - e^{-\frac{n^{2\delta}}{2K^2}}) \ge C_3 n^{-\frac{1}{2}-\delta}\,. \end{aligned} \tag{138}$$

In the forth inequality in (138), since we have the event $B$, we know that the process at time $k < n$ will be in $H(n^{1/2+\delta}, +\infty)$. Then in the last inequality in (138) we use (137) and Azuma's inequality for super-martingales.

Finally we have,

$$\begin{aligned} \mathbb{P}[L_n(-1) \ge n^{\frac{1}{2}+2\delta}] &= \mathbb{P}[\max\{k : \hat{t}_k \le n\} \ge n^{\frac{1}{2}+2\delta}] \\ &\le \mathbb{P}[\nexists\ j \le n^{\frac{1}{2}+2\delta} - 1 \text{ such that } \hat{t}_{j+1} - \hat{t}_j > n] \\ &\le \mathbb{P}\Big[\bigcap_{j=0}^{n^{\frac{1}{2}+2\delta}-1} \{\hat{t}_{j+1} - \hat{t}_j \le n\}\Big]\,, \end{aligned}$$

which is bounded above by

$$\begin{aligned} &\mathbb{P}\big[\hat{t}_1 \le n\big] \prod_{j=1}^{n^{\frac{1}{2}+2\delta}-1} \mathbb{P}\big[\hat{t}_{j+1} - \hat{t}_j \le n \big|\, \{\hat{t}_1 - \hat{t}_0 \le n\}\,, \dots, \{\hat{t}_j - \hat{t}_{j-1} \le n\}\,\big] \\ &\qquad \le \Big(1 - \frac{C_3}{n^{\frac{1}{2}+\delta}}\Big)^{n^{\frac{1}{2}+2\delta}} \le \Big(\Big(1 - \frac{C_3}{n^{\frac{1}{2}+\delta}}\Big)^{n^{\frac{1}{2}+\delta}}\Big)^{n^{\delta}} \le e^{-C_3 n^{\delta}}\,, \end{aligned}$$

finishing the proof. □

Now we prove Lemma B.2. This Lemma is similar to Lemma 5.2 in [15] but it is more general in that our statement holds true for every $b \ge 0$ (rather than claiming the existence of a $b \in (0,1)$ as in [15]) For the proof of Lemma B.2 we use Taylor expansion of first order, some standards inequalities and the fact that $Y$ is a $d$-dimensional martingale with $K$-bounded increments.

*Proof of Lemma B.2.* Note that for $b \ge 1$ and $b = 0$ the proof is straightforward. For $b \in (0,1)$, let us begin observing that for all $y \in \mathbb{R}^d$ such that $\|y\| > K/(\sqrt{2}-1)$ it holds that

$$\left|\frac{2y\cdot z + \|z\|^2}{\|y\|^2}\right| < 1\,, \forall z \in \mathbb{R}^d \text{ with } \|z\| \le K\,.$$

Let $b \in (0,2)$. For any real number $u \neq 0$ such that $|u| < 1$, there exists $\theta \in (0,1)$ such that if $\xi = (1-\theta)u$, it holds that

$$(1+u)^{b/2} = 1 + \frac{b}{2}(1+\xi)^{b/2-1}u \geq 1 + \frac{b}{2}e^{-(1-\theta)(1-b/2)u}u\,.$$

Due to the fact that $(1-b/2) > 0$, $1-\theta > 0$ and the assumption that $|u| < 1$, we obtain that

$$(1+u)^{b/2} \geq 1 + \frac{b}{2}e^{-(1-\theta)(1-b/2)u}u \geq 1 + \frac{b}{2}e^{-(1-\theta)(1-b/2)}u\,.$$

If we denote $P_{y,y+z} = \mathbb{P}(Y_{n+1} - Y_n = z | \mathcal{F}_n, Y_n = y)$, we can write

$$\mathbb{E}\left(\|Y_{n+1}\|^b - \|Y_n\|^b | \mathcal{F}_n, Y_n = y\right) = \sum_z P_{y,y+z}\left(\|y+z\|^b - \|y\|^b\right)$$

$$= \|y\|^b \sum_z P_{y,y+z}\left(\frac{\|y+z\|^b}{\|y\|^b} - 1\right) = \|y\|^b \sum_z P_{y,y+z}\left(\left(\frac{\|y+z\|^2}{\|y\|^2}\right)^{b/2} - 1\right)$$

$$= \|y\|^b \sum_z P_{y,y+z}\left(\left(1 + \frac{2y\cdot z + \|z\|^2}{\|y\|^2}\right)^{b/2} - 1\right)\,.$$

Note that, since $Y$ has uniformly $K$-bounded increments, the summation only runs over $z$ such that $\|z\| \leq K$. Therefore, for all $y$ such that $\|y\| > K/(\sqrt{2}-1)$, we have that $\left|\frac{2y\cdot z + \|z\|^2}{\|y\|^2}\right| < 1$, for all $z$ in the summation. Thus, we obtain that

$$\begin{aligned}
&\mathbb{E}\left(\|Y_{n+1}\|^b - \|Y_n\|^b | \mathcal{F}_n, Y_n = y\right) \\
&\geq \|y\|^b \sum_z P_{y,y+z}\left(1 + \frac{b}{2}e^{-(1-\theta)(1-b/2)}\frac{2y\cdot z + \|z\|^2}{\|y\|^2} - 1\right) \\
&= \|y\|^b \sum_z P_{y,y+z}\left(\frac{b}{2}e^{-(1-\theta)(1-b/2)}\frac{2y\cdot z + \|z\|^2}{\|y\|^2}\right) \\
&= \|y\|^b \sum_z P_{y,y+z}\left(\frac{b}{2}e^{-(1-\theta)(1-b/2)}\frac{\|z\|^2}{\|y\|^2}\right) \geq 0\,.
\end{aligned}$$

where, in the last equality, we used that $\sum_z zP_{y,y+z} = 0$, which holds since $Y$ is a $d$-dimensional martingale. ☐

The proof of Lemma B.3 closely follows that of Lemma 5.3 from [15]. Here we will use similar techniques that we apply in the proof of Lemma B.1 and B.2.

*Proof of Lemma B.3.* Since the process $Y$ is a $d$-dimensional martingale without loss of generality we may assume $x_0 = 0$. Moreover, we may also assume $y_0 = x_0 = 0$.

Lemma B.3 is trivial for $\phi \geq 1$. Moreover, we can assume $\phi < 1/2$.

Set $\gamma_2' = K/(\sqrt{2}-1)$ and define $\tilde{\tau} := \tau^Y_{\mathbb{Z}^d/B(0,\gamma_2'+1)}$ the first time the process $Y$ exits the ball of radius $\gamma_2'+1$ centered in the origin. Define

$\tilde{V} := \{Y_m \neq 0,\ \text{for all } 1 \leq m \leq \tilde{\tau}\}$. By Condition III, there exists $C_1 > 0$ depending on $r$ and $h$ such that,

$$\mathbb{P}[\tilde{V}] > C_1\ . \tag{139}$$

Let $C_2$ be a large constant to be chosen later. We have that $\|Y_n\|^{2\phi}$ is a supermartingale, so by optional stopping theorem,

$$\begin{aligned}\mathbb{E}\left[\|Y_{\tau^Y_{\mathbb{Z}^d/B(0,C_2n^{\frac{1}{2}})}\circ\theta_{\tilde{\tau}}\wedge\tau^Y_{B(0,\gamma_2')}\circ\theta_{\tilde{\tau}}}\|^{2\phi}|\mathcal{F}_{\tilde{\tau}},\tilde{V}\right] &\leq \|Y_{\tilde{\tau}}\|^{2\phi}\\ &\leq \|Y_{\tilde{\tau}}\|^{2\phi}1_{\{\|Y_{\tilde{\tau}}\|>\gamma_2'\}} + \underbrace{\|Y_{\tilde{\tau}}\|^{2\phi}1_{\{\|Y_{\tilde{\tau}}\|\leq\gamma_2'\}}}_{=0}\ .\end{aligned}$$

The above inequality together with Lemma *B.2* imply that

$$\mathbb{E}\left[\|Y_{\tau^Y_{\mathbb{Z}^d/B(0,C_2n^{\frac{1}{2}})}\circ\theta_{\tilde{\tau}}\wedge\tau^Y_{B(0,\gamma_2')}\circ\theta_{\tilde{\tau}}}\|^{2\phi}|\mathcal{F}_{\tilde{\tau}},\tilde{V}\right] = \|Y_{\tilde{\tau}}\|^{2\phi}1_{\{\|Y_{\tilde{\tau}}\|>\gamma_2'\}}\ .$$

Denote $T_1 = \tau^Y_{\mathbb{Z}^d/B(0,C_2n^{\frac{1}{2}})} \circ \theta_{\tilde{\tau}}$ and $T_2 = \tau^Y_{B(0,\gamma_2')} \circ \theta_{\tilde{\tau}}$, then we obtain

$$\begin{aligned}&\mathbb{E}\left[\|Y_{T_1}\|^{2\phi}1_{\{T_1<T_2\}}|\mathcal{F}_{\tilde{\tau}},\tilde{V}\right] + \underbrace{\mathbb{E}\left[\|Y_{T_2}\|^{2\phi}1_{\{T_2<T_1\}}|\mathcal{F}_{\tilde{\tau}},\tilde{V}\right]}_{\leq(\gamma_2')^{2\phi}} = \|Y_{\tilde{\tau}}\|^{2\phi}1_{\{\|Y_{\tilde{\tau}}\|>\gamma_2'\}}\\ &\implies \mathbb{E}\left[\|Y_{T_1}\|^{2\phi}1_{\{T_1<T_2\}}|\mathcal{F}_{\tilde{\tau}},\tilde{V}\right] + (\gamma_2')^{2\phi} \geq (\gamma_2'+1)^{2\phi}\ .\end{aligned} \tag{140}$$

Ergo by (140) we have

$$\left(C_2n^{\frac{1}{2}}+K\right)^{2\phi}\mathbb{P}\left[\tau^Y_{\mathbb{Z}^d/B(0,C_2n^{\frac{1}{2}})}\circ\theta_{\tilde{\tau}} < \tau^Y_{B(0,\gamma_2')}\circ\theta_{\tilde{\tau}}|\mathcal{F}_{\tilde{\tau}},\tilde{V}\right] + \left(\gamma_2'\right)^{2\phi} \geq \left(\gamma_2'+1\right)^{2\phi},$$

This implies

$$\mathbb{P}\left[\tau^Y_{\mathbb{Z}^d/B(0,C_2n^{\frac{1}{2}})}\circ\theta_{\tilde{\tau}} < \tau^Y_{B(0,\gamma_2')}\circ\theta_{\tilde{\tau}}|\mathcal{F}_{\tilde{\tau}},\tilde{V}\right] \geq \frac{(\gamma_2'+1)^{2\phi}-(\gamma_2')^{2\phi}}{\left(C_2n^{\frac{1}{2}}+K\right)^{2\phi}}\ . \tag{141}$$

Now, for any stopping time $T$ and $y \in \mathbb{Z}^d/B(0,C_2n^{\frac{1}{2}})$, we have,

$$\begin{aligned}\mathbb{P}\left[\tau^Y_0\circ\theta_T > n|\mathcal{F}_T, Y_T = y\right] &\geq \mathbb{P}\left[|Y_n - Y_0|\cdot\ell' < C_2n^{\frac{1}{2}}\right]\\ &\geq 1-2\exp\left(-\frac{\left(C_2n^{\frac{1}{2}}\right)^2}{2nK}\right) \geq 1-2\exp\left(-\frac{C_2^2}{2K}\right) \geq \frac{1}{2},\end{aligned} \tag{142}$$

where $\ell' \in \mathbb{S}^{d-1}$ in some fixed direction. The first inequality in (142) follows from noticing that if the martingale $Y$ is initially in position $y$ and it can not reach 0 in $n$ steps, then there exists a direction $\ell' \in \mathbb{S}^{d-1}$ along which we must have a distance at most $C_2n^{1/2}$ units between the initial position $Y_0$ and $Y_n$. In the second inequality we used Azuma's inequality (see, for example, Theorem 2.19 in [6]). Finally, the last inequality in (142) follows from choosing $C_2 \geq K^{1/2}\sqrt{2}\left(-\ln\left(1/4\right)\right)^{1/2}$; for example we can choose $C_2 = 2K$.

Setting $C_2 = 2K$, we can rewrite (141) as follows:

$$\begin{aligned}
&\mathbb{P}\left[\tau^{Y}_{\mathbb{Z}^d/B(0,C_2 n^{\frac{1}{2}})} \circ \theta_{\tilde{\tau}} < \tau^{Y}_{B(0,\gamma_2')} \circ \theta_{\tilde{\tau}} | \mathcal{F}_{\tilde{\tau}}, \tilde{V}\right] \geq \frac{(\gamma_2'+1)^{2\phi} - (\gamma_2')^{2\phi}}{\left(2Kn^{\frac{1}{2}} + K\right)^{2\phi}} \\
&\geq \frac{\left(K+\sqrt{2}-1\right)^{2\phi} - K^{2\phi}}{K^{2\phi}\left(\sqrt{2}-1\right)^{2\phi}} \frac{1}{(2n^{\frac{1}{2}}+1)^{2\phi}} \geq \frac{C_{K,b}}{n^{\phi}},
\end{aligned} \tag{143}$$

where $C_{K,\phi} = (\left(K+\sqrt{2}-1\right)^{2\phi} - K^{2\phi})/(4K^{2\phi}\left(\sqrt{2}-1\right)^{2\phi})$.

Recall that $T_1 = \tau^{Y}_{\mathbb{Z}^d/B(0,C_2 n^{\frac{1}{2}})} \circ \theta_{\tilde{\tau}}$, $T_2 = \tau^{Y}_{B(0,\gamma_2)} \circ \theta_{\tilde{\tau}}$, with $\tilde{\tau} = \tau^{Y}_{\mathbb{Z}^d/B(0,\gamma_2'+1)}$, and $\tilde{V} := \{Y_m \neq 0, \text{ for all } 1 \leq m \leq \tilde{\tau}\}$. By (139), (143) and (142), we have,

$$\begin{aligned}
\mathbb{P}\left[Y_m \neq 0, \text{ for all } m = 1, ..., n\right] &\geq \mathbb{P}\left[\tilde{V} \cap \{T_1 < T_2\} \cap \left\{\tau^{Y}_{0} \circ \theta_{T_1} > n\right\}\right] \\
&\geq \frac{C_{K,\phi} C_1}{2n^{\phi}}.
\end{aligned}$$

Let $L_n(0) = \sum_{j=1}^{n} 1_{\{Y_j = 0\}}$, $\hat{t}_0 = 0$ and $\hat{t}_{k+1} = \min\left\{j > \hat{t}_k : Y_j = 0\right\}$, thus we have $L_n(0) = \max\{k : \hat{t}_k \leq n\}$. We will apply the same technique used at the end of the proof of the Lemma B.1.

$$\mathbb{P}\big[\hat{t}_{k+1} - \hat{t}_k > n \big| \mathcal{F}_{\hat{t}_k}\big] = \mathbb{P}\big[Y_m \neq 0, \text{ for all } m = 1, ..., n\big] \geq \frac{C_{K,\phi} C_1}{2n^{\phi}}.$$

Thus,

$$\begin{aligned}
&\mathbb{P}\big[\text{there exist a } k \leq n^{\phi+\delta} - 1 \text{ such that } \hat{t}_{k+1} - \hat{t}_k > n\big] = \\
&= 1 - \mathbb{P}\Big[\bigcap_{k=0}^{n^{\phi+\delta}-1} \left\{\hat{t}_{k+1} - \hat{t}_k \leq n\right\}\Big] \\
&= 1 - \prod_{k=0}^{n^{\phi+\delta}-1} \mathbb{P}\big[\hat{t}_{k+1} - \hat{t}_k \leq n \big| \left\{\hat{t}_1 - \hat{t}_0 \leq n\right\}, \ldots, \left\{\hat{t}_k - \hat{t}_{k-1} \leq n\right\}\big] \\
&\geq 1 - \prod_{k=0}^{n^{\phi+\delta}-1} \left(1 - \frac{C_{K,\phi} C_1}{2n^{\phi}}\right) \geq 1 - \left(\left(1 - \frac{\frac{C_{K,\phi} C_1}{2}}{n^{\phi}}\right)^{n^{\phi}}\right)^{n^{\delta}} \geq 1 - e^{-C_3 n^{\delta}},
\end{aligned} \tag{144}$$

where $C_3 = C_{K,\phi} C_1/2$. Finally, we obtain

$$\begin{aligned}
\mathbb{P}\left[L_n(0) > n^{\phi+\delta}\right] &= \mathbb{P}\left[\max\left\{k : \hat{t}_k \leq n\right\} > n^{\phi+\delta}\right] \\
&\leq 1 - \mathbb{P}\left[\text{there exist a } k \leq n^{\phi+\delta} - 1 \text{ such that } \hat{t}_{k+1} - \hat{t}_k > n\right] \\
&\leq 1 - \left(1 - e^{-C_3 n^{\delta}}\right) \leq e^{-C_3 n^{\delta}}.
\end{aligned}$$

□

The proof of Lemma B.4 follows closely that of Lemma 5.4 from [15].

*Proof of Lemma B.4.* First begin providing a lower bound to the probability of the following event

$$F = \{Y_j \in B(x, m^{\frac{1}{2}}) \text{ for all } j \leq m^{1-\delta}\}\,.$$

On the event $F^c$, there must exist a direction $\ell' \in \mathbb{S}^{d-1}$ and a time $j \leq m^{1-\delta}$ such that $|Y_j - Y_0| \cdot \ell' \geq m^{\frac{1}{2}}$. Thus,

$$\mathbb{P}[F^c] \leq \mathbb{P}\left[\bigcup_{j=1}^{m^{1-\delta}} \{|Y_j - Y_0| \cdot \ell' \geq m^{\frac{1}{2}}\}\right] \leq \sum_{j=1}^{m^{1-\delta}} \mathbb{P}\left[|Y_j - Y_0| \cdot \ell' \geq m^{\frac{1}{2}}\right]\,.$$

Using Azuma's inequality for martingales with bounded increments (see, for example, Theorem 2.19 in [6]), we obtain that

$$\begin{aligned}\mathbb{P}[F] &\geq 1 - \sum_{j=1}^{m^{1-\delta}} 2\exp\left(-\frac{m}{2jK^2}\right) \geq 1 - 2m^{1-\delta}\exp\left(-\frac{m}{2m^{1-\delta}K^2}\right) \\ &\geq 1 - 2m^{1-\delta}\exp\left(-\frac{m^\delta}{2K^2}\right)\,.\end{aligned} \tag{145}$$

Next, we use Lemma B.3 to estimate a lower bound to the probability of all $y \in B(x, m^{1/2})$ be visited less than $m^{\phi+\delta}$ times up until time $m^{1-\delta}$. Let us denote the event $B = \{\text{for all } y \in B(x, m^{1/2}),\ \text{we have } \sum_{j=1}^{m^{1-\delta}} 1_{\{Y_j = y\}} < m^{\phi+\delta}\}$. Thus we have,

$$\begin{aligned}\mathbb{P}[B] &= \mathbb{P}\left[\bigcap_{y \in B(x, m^{1/2})} \left\{\sum_{j=1}^{m^{1-\delta}} 1_{\{Y_j = y\}} < m^{\theta+\delta}\right\}\right] \\ &\geq 1 - \sum_{y \in B(x,m^{1/2})} \mathbb{P}\left[\sum_{j=1}^{m^{1-\delta}} 1_{\{Y_j=y\}} \geq m^{\theta+\delta}\right] \geq 1 - C_1\mathbb{P}\left[\sum_{j=1}^{m^{1-\delta}} 1_{\{Y_j=y\}} \geq m^{\theta+\delta}\right] \\ &\geq 1 - C_1\mathbb{P}\left[\sum_{j=1}^{m} 1_{\{Y_j=y\}} \geq m^{\theta+\delta}\right] \geq 1 - C_1 e^{-\gamma_3' m^\delta}\,,\end{aligned} \tag{146}$$

where $C_1 = |B(x, m^{1/2})|$. In the third inequality above, we used the fact that $\left\{\sum_{j=1}^{m^{1-\delta}} 1_{\{Y_j=y\}} \geq m^{\phi+\delta}\right\} \subset \left\{\sum_{j=1}^{m} 1_{\{Y_j=y\}} \geq m^{\phi+\delta}\right\}$, while the last inequality follows from Lemma B.3.

Now, on the event $F \cap B$, we necessarily have that

$$\left|\mathcal{R}^Y_{m^{1-\delta}}\right| > \frac{m^{1-\delta}}{m^{\phi+\delta}} = m^{1-\phi-2\delta}\,.$$

Therefore, by (145) and (146) we have

$$\mathbb{P}\left[\left|\mathcal{R}^{Y}_{m^{1-\delta}}\right| > m^{1-\phi-2\delta}\right] \geq \mathbb{P}\left[F \cap B\right] \geq 1 - \left(\mathbb{P}\left[F^c\right] + \mathbb{P}\left[B^c\right]\right) \geq 1 - \left(2m^{1-\delta}\exp\left(-\frac{m^{\delta}}{2K^2}\right) + C_1 e^{-\gamma_3' m^{\delta}}\right). \tag{147}$$

Now, since by hypotheses the set $U$ is such that $|B(x, m^{1/2})/U| \leq m^{1-\phi-2\delta}$, on the event

$$G = \left\{\left|B(x, m^{\frac{1}{2}}) \cap \mathcal{R}^{Y}_{m^{1-\delta}}\right| > m^{1-\phi-2\delta}\right\},$$

we have $\{Y_1, ..., Y_{m^{1-\delta}}\} \cap U \neq \emptyset$, that is $\{\tau^Y_U \leq m^{1-\delta}\}$. With this observation we can finish the proof using (147),

$$\mathbb{P}\left[\tau^Y_U \leq m^{1-\delta}\right] \geq \mathbb{P}[G] \geq \mathbb{P}\left[F \cap B\right] \geq 1 - \left(2m^{1-\delta}\exp\left(-\frac{m^{\delta}}{2K^2}\right) + C_1 e^{-\gamma_3' m^{\delta}}\right).$$

□

## Appendix C. Auxiliary Results

In this section we will present and prove some general results that we needed in this text.

**Lemma C.1.** *Suppose that the sequences $X^n_\cdot$ and $Y^n_\cdot$ are tight processes in $C[0,T]$ for a $T > 0$. Then we have that the sequence $(X^n_\cdot + Y^n_\cdot)$ is a tight process in $C[0,T]$.*

*Proof.* Let us denote the probability measure $P_n$ on $C[0,T]$ as the distribution of $(X^n_\cdot + Y^n_\cdot)$. First we will prove that for each positive $\eta$, there exist an $a$ and an $n_0$ such that

$$P_n[f \in C[0,T] : |f(0)| \geq a] \leq \eta \quad \text{for all } n \geq n_0\,. \tag{148}$$

We denote the set $G_0(a) := \{f \in C[0,T] : |f(0)| \geq a\}$.

Since $X^n_\cdot$ and $Y^n_\cdot$ are tight processes in $C[0,T]$, by Theorem 7.3 in [5] there exist $a_x$, $a_y$, $n^x_0$ and $n^y_0$ such that

$$\begin{aligned} &\mathbb{P}[|X^n_0| \geq a_x] \leq \frac{\eta}{2} \quad \text{for all } n \geq n^x_0 \quad \text{and} \\ &\mathbb{P}[|Y^n_0| \geq a_y] \leq \frac{\eta}{2} \quad \text{for all } n \geq n^y_0\,. \end{aligned} \tag{149}$$

Now let us choose $a \geq a_x + a_y$. By triangle inequality and union bound we have

$$\begin{aligned} P_n[G_0(a)] &= \mathbb{P}[|X^n_0 + Y^n_0| \geq a] \leq \mathbb{P}[|X^n_0| + |Y^n_0| \geq a] \\ &\leq \mathbb{P}[\{|X^n_0| \geq a_x\} \cup \{|Y^n_0| \geq a_y\}] \leq \mathbb{P}[|X^n_0| \geq a_x] + \mathbb{P}[|Y^n_0| \geq a_y]\,. \end{aligned} \tag{150}$$

Thus for a $n_0 = \max\{n^x_0, n^y_0\}$, by (149), (150), we obtain (148).

We set $\omega_f(\delta)$ as the *modulus of continuity* of an arbitrary function $f(\cdot)$ on $[0,T]$. Now we shall prove that for each positive $\varepsilon$ and $\eta$, there exist a $\delta \in (0,1)$ and an $m_0$ such that

$$P_n[f \in C[0,T] : \omega_f(\delta) \geq \varepsilon] \leq \eta \quad \text{for all } n \geq m_0 \,. \tag{151}$$

We denote the set $H_t(\varepsilon,\delta) := \{f \in C[0,T] : \omega_f(\delta) \geq \varepsilon\}$.

Since $X^n_\cdot$ and $Y^n_\cdot$ are tight processes in $C[0,T]$, by Theorem 7.3 in [5] there exist $\delta_x \in (0,1)$, $\delta_y \in (0,1)$, $m_0^x$ and $m_0^y$ such that

$$\begin{aligned} &\mathbb{P}\left[\sup_{|s-t|\geq\delta_x} |X^n_s - X^n_t| \geq \frac{\varepsilon}{2}\right] \leq \frac{\eta}{2} \quad \text{for all } n \geq m_0^x \quad \text{and} \\ &\mathbb{P}\left[\sup_{|s-t|\geq\delta_y} |Y^n_s - Y^n_t| \geq \frac{\varepsilon}{2}\right] \leq \frac{\eta}{2} \quad \text{for all } n \geq m_0^y \,. \end{aligned} \tag{152}$$

Then we obtain by triangle inequality and union bound the following

$$\begin{aligned} P_n[H_t(\varepsilon,\delta)] &= \mathbb{P}\left[\sup_{|s-t|\geq\delta} |X^n_s + Y^n_s - X^n_t - Y^n_t| \geq \varepsilon\right] \\ &\leq \mathbb{P}\left[\sup_{|s-t|\leq\delta} |X^n_s - X^n_t| + \sup_{|s-t|\leq\delta} |Y^n_s - Y^n_t| \geq \varepsilon\right] \\ &\leq \mathbb{P}\left[\left\{\sup_{|s-t|\leq\delta} |X^n_s - X^n_t| \leq \frac{\varepsilon}{2}\right\} \bigcup \left\{\sup_{|s-t|\leq\delta} |Y^n_s - Y^n_t| \leq \frac{\varepsilon}{2}\right\}\right] \\ &\leq \mathbb{P}\left[\sup_{|s-t|\leq\delta} |X^n_s - X^n_t| \leq \frac{\varepsilon}{2}\right] + \mathbb{P}\left[\sup_{|s-t|\leq\delta} |Y^n_s - Y^n_t| \leq \frac{\varepsilon}{2}\right] . \end{aligned} \tag{153}$$

We choose now a $\delta = \min\{\delta_x, \delta_y\}$ and $m_0 = \max\{m_0^x, m_0^y\}$, thus by (152) and (153) we obtain (151). Hence by Theorem 7.3 in [5] we finish the proof. □

**Lemma C.2.** *Suppose that we have a process $Y_{\lfloor n\cdot\rfloor}$ that converges in probability to zero in the space $C_{\mathbb{R}^d}[0,T]$ with the uniform metric for all $T>0$. Then $Y_{\lfloor n\cdot\rfloor}$ converges in probability to zero in the space $C_{\mathbb{R}^d}[0,\infty)$ equipped with the following metric*

$$\rho(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \sup_{0\leq t\leq k} (||f(t)-g(t)|| \wedge 1) \,.$$

*Proof.* Since $Y_{\lfloor n\cdot\rfloor}$ converges in probability to zero in $C_{\mathbb{R}^d}[0,T]$, we have

$$\mathbb{P}\left[\sup_{0\leq t\leq T} ||Y_{\lfloor nt\rfloor}|| > \delta\right] \to 0 \quad \text{as } n\to\infty \,, \tag{154}$$

for any $\delta > 0$.

Now let $\varepsilon > 0$ and we choose a positive integer $N$ such that $\sum_{l\geq N} 2^{-l} < \varepsilon/2$. Thus we have the following

$$\begin{aligned}
&\mathbb{P}\left[\sum_{k=1}^{\infty} 2^{-k} \sup_{0\leq t\leq k} (||Y_{\lfloor nt\rfloor}|| \wedge 1) > \varepsilon\right] = \\
&= \mathbb{P}\left[\sum_{k=1}^{N} 2^{-k} \sup_{0\leq t\leq k} (||Y_{\lfloor nt\rfloor}|| \wedge 1) + \sum_{k=N+1}^{\infty} 2^{-k} \sup_{0\leq t\leq k} (||Y_{\lfloor nt\rfloor}|| \wedge 1) > \varepsilon\right] \\
&\leq \mathbb{P}\left[\sum_{k=1}^{N} \sup_{0\leq t\leq k} ||Y_{\lfloor nt\rfloor}|| + \sum_{k=N+1}^{\infty} 2^{-k} > \varepsilon\right].
\end{aligned} \tag{155}$$

Since we choose $N$ large enough by (155) we obtain that

$$\begin{aligned}
&\mathbb{P}\left[\sum_{k=1}^{\infty} \sup_{0\leq t\leq k} (||Y_{\lfloor nt\rfloor}|| \wedge 1) > \varepsilon\right] \\
&\leq \mathbb{P}\left[\left\{\sum_{k=1}^{N} \sup_{0\leq t\leq k} ||Y_{\lfloor nt\rfloor}|| > \frac{\varepsilon}{2}\right\} \bigcup \left\{\sum_{k=N+1}^{\infty} 2^{-k} > \frac{\varepsilon}{2}\right\}\right] \\
&\leq \mathbb{P}\left[\sum_{k=1}^{N} \sup_{0\leq t\leq k} ||Y_{\lfloor nt\rfloor}|| > \frac{\varepsilon}{2}\right] + \mathbb{P}\left[\sum_{k=N+1}^{\infty} 2^{-k} > \frac{\varepsilon}{2}\right] \\
&\leq \mathbb{P}\left[\bigcup_{k=1}^{N} \sup_{0\leq t\leq k} ||Y_{\lfloor nt\rfloor}|| > \frac{\varepsilon}{2N}\right] \leq \sum_{k=1}^{N} \mathbb{P}\left[\sup_{0\leq t\leq k} ||Y_{\lfloor nt\rfloor}|| > \frac{\varepsilon}{2N}\right].
\end{aligned} \tag{156}$$

We have the third and last inequalities in (156) by union bound.

Now one can see that all sum portions in the last inequality in (156) go to zero as $n$ tends to infinity by (154), ergo we have

$$\mathbb{P}\left[\sum_{k=1}^{\infty} 2^{-k} \sup_{0\leq t\leq k} (||Y_{\lfloor nt\rfloor}|| \wedge 1) > \varepsilon\right] \to 0 \quad \text{as } n \to \infty\,,$$

for any $\varepsilon > 0$. Hence we obtain the desired result. □

**Lemma C.3.** *Let $\{\phi_n\}_{n\geq 1}$ be a sequence of i.i.d. random vectors in $\mathbb{Z}^d$, with $d \geq 2$ and $\{\tau_k\}_{k\geq 1}$ a sequence of $\mathcal{F}$-stopping times, where $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 1}$ is a natural filtration, that is $\mathcal{F}_n = \sigma(\phi_1, \ldots, \phi_n)$. If the sequences $\{\phi_n\}_{n\geq 1}$ and $\{\tau_k\}_{k\geq 1}$ are independents then we have that the sequence $\{\phi_{\tau_j}\}_{j\geq 1}$ is i.i.d. and moreover $\phi_{\tau_k}$ has the same distribution of $\phi_1$.*

*Proof.* Our first step here will be to prove that for any $k \geq 1$, $\phi_{\tau_k}$ has the same distribution of $\phi_1$.

Let $A$ be a set in $\mathbb{Z}^d$ and we fix a $j \geq 1$. Then we have

$$
\begin{aligned}
&\mathbb{P}[\phi_{\tau_j} \in A | \mathcal{F}_{\tau_{j-1}}] = \sum_{n=1}^{\infty} \mathbb{P}[\{\phi_{\tau_j} \in A\} \cap \{\tau_j = n\} | \mathcal{F}_{\tau_{j-1}}] \\
&= \sum_{n=1}^{\infty} \mathbb{P}[\phi_{\tau_j} \in A | \{\tau_j = n\}, \mathcal{F}_{\tau_{j-1}}] \mathbb{P}[\tau_j = n | \mathcal{F}_{\tau_{j-1}}] \\
&= \sum_{n=1}^{\infty} \mathbb{P}[\phi_n \in A] \mathbb{P}[\tau_j = n | \mathcal{F}_{\tau_{j-1}}] = \mathbb{P}[\phi_1 \in A] \underbrace{\sum_{n=1}^{\infty} \mathbb{P}[\tau_j = n | \mathcal{F}_{\tau_{j-1}}]}_{=1} .
\end{aligned}
\tag{157}
$$

The third inequality in (157) we obtain by the Strong Markov Property and the fact that $\phi_j$ and $\tau_i$ are independent for all $i$ and $j$.

Thus we obtain that the sequence $\{\phi_{\tau_j}\}_{j \geq 1}$ is identically distributed, so remains to prove independence. Thereunto, let $B$ and $D$ be sets in $\mathbb{Z}^d$, then we have

$$
\begin{aligned}
&\mathbb{P}[\{\phi_{\tau_1} \in B\} \cap \{\phi_{\tau_2} \in D\}] = \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{P}[\{\{\phi_{\tau_1} \in B\} \cap \{\tau_1 = n\}\} \cap \{\{\phi_{\tau_2} \in D\} \cap \{\tau_2 = m\}\}] \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{P}[\phi_{\tau_2} \in D | \tau_1 = n, \tau_2 = m, \phi_{\tau_1} \in B] \mathbb{P}[\tau_1 = n, \tau_2 = m, \phi_{\tau_1} \in B] \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{P}[\phi_m \in D] \mathbb{P}[\tau_1 = n, \tau_2 = m, \phi_{\tau_1} \in B] .
\end{aligned}
\tag{158}
$$

The last inequality in (158) we obtain by the Strong Markov Property and the fact that $\phi_j$ and $\tau_i$ are independent for all $i$ and $j$.

Since the sequence $\{\phi_n\}_{n \geq 1}$ is i.i.d., using the Strong Markov Property and the fact that $\phi_j$ and $\tau_i$ are independent for all $i$ and $j$, we can continue the computation in (158) as the following

$$
\begin{aligned}
\mathbb{P}[\{\phi_{\tau_1} \in B\} \cap \{\phi_{\tau_2} \in D\}] &= \mathbb{P}[\phi_1 \in D] \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{P}[\tau_2 = m, \tau_1 = n, \phi_{\tau_1} \in B] \\
&= \mathbb{P}[\phi_1 \in D] \sum_{n=1}^{\infty} \mathbb{P}[\tau_1 = n, \phi_{\tau_1} \in B] \\
&= \mathbb{P}[\phi_1 \in D] \sum_{n=1}^{\infty} \mathbb{P}[\phi_n \in B] \mathbb{P}[\tau_1 = n] \\
&= \mathbb{P}[\phi_1 \in D] \mathbb{P}[\phi_1 \in B] = \mathbb{P}[\phi_{\tau_1} \in D] \mathbb{P}[\phi_{\tau_2} \in B] .
\end{aligned}
$$

Hence we finish the proof. □

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