ON MINIMUM CUTS OF CYCLES BY VERTICES AND VERTEX DISJOINT CYCLES

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RESUMO

Descrevemos uma nova família de dígrafos, denominados conexamente redutíveis, para a qual provamos que a cardinalidade mínima de um conjunto de vértices que interceptam todos os ciclos iguala à máxima de um conjunto de ciclos disjuntos em vértices. Além disso, formulamos algoritmos polinomiais para os problemas de reconhecimento e determinação desses conjuntos, mínimo e máximo, para dígrafos dessa família. Resultados similares são conhecidos para os dígrafos totalmente redutíveis. Mais recentemente, uma outra família foi definida, os dígrafos ciclicamente redutíveis, que também possibilita a computação em tempo polinomial desses conjuntos mínimo e máximo. É conhecido o fato de que os dígrafos totalmente redutíveis não estão contidos nem contêm os ciclicamente redutíveis. Em contraste, provamos que os conexamente redutíveis contêm ambas as famílias existentes.

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ON MINIMUM CUTS OF CYCLES BY VERTICES AND MAXIMUM VERTEX DISJOINT CICLES

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ABSTRACT

We describe a new family of digraphs, named connectively reducible, for which we prove that the minimum cardinality of a set of vertices intersecting all cycles equals the maximum cardinality of a set of vertex disjoint cycles. In addition, formulate polynomial time algorithms for the problems of recognition and finding these minimum and maximum sets for digraphs of the family. Similar results hold for the currently existing families of fully reducible and cyclically reducible digraphs. Neither the fully reducible are contained nor contain the cyclically reducible. However, we show that the connectively reducible digraphs contain both of the existing families.

1. **INTRODUCTION**: Frank and Gyárfás [1] have shown that for fully reducible digraphs the minimum cardinality set of vertices intersecting all cycles equals the maximum cardinality set of vertex disjoint cycles. Furthermore, there are polynomial time algorithms for finding such a minimum set of vertices [4-6] for this family of digraphs, whereas the same problem is well known to be NP-hard in the general case [2-3]. More recently, Wang, Lloyd and Soffa [9] defined another family of digraphs, called cyclically reducible, which also enables the computation of the above sets in polynomial time. In addition, both these families of digraphs can be recognized in polynomial time [8-9]. However, as noted in [9], the fully reducible digraphs neither are contained nor contain the cyclically reducible ones. In the present paper, we define a new family of digraphs, named connectively reducible, and present the following results:

(i) A proof that the above min-max equality is valid for them.

(ii) A polynomial time algorithm which recognizes digraphs of this kind and finds the corresponding minimum and maximum sets, of vertices and cycles, respectively for digraphs of the family.

(iii) A proof that the connectivelly reducible digraphs contain both the fully and cyclically reducible ones.

The following is the plan of the paper. In Section 2, we present the concepts of critical vertices and cycles, in which are based the proposed results. These lead to the idea of critical sequences and connectively reducible digraphs, defined in Section 3. A characterization of the proposed family of digraphs is given in Section 4. The min-max theorem is proved in Section 5, whereas in the following we formulate the polynomial time algorithm for finding the minimum and maximum sets. The algorithm is based on the characterization previously described. The proofs that connectivelly reducible digraphs contain cyclically and fully reducible ones are presented in sections 7 and 8, respectively. Some further remarks form the last section.

Throughout the paper, D denotes a digraph with vertex set $V(D)$ and edge set $E(D)$. If $v \in V(D)$ and $V' \subseteq V(D)$ then $D-v$ and $D-V'$ represent the digraphs obtained from D by removing v and V', respectively. We use the term component meaning a strongly connected component of D. A component is trivial if it consists of a single vertex. $T(D)$ denotes the subset of vertices of D which are trivial components and $\bar{T}(D)=V(D)-T(D)$. A cycle cut or feedback vertex set of D is a subset of vertices, denoted $\alpha(D)$, intersecting all cycles of D. Two cycles which are vertex disjoint are simply called disjoint. The notation $\beta(D)$ represents a set of disjoint cycles. In an acyclic digraph, if there is a path from vertex v to w then v is an ancestor of w, and w a descendant of v; in addition if $v \neq w$ then v is a proper ancestor and w a proper descendant. Finally, we employ the same notation to represent some operations in sets or sequences, the meaning being clear from the context.

2. CRITICAL VERTICES AND CYCLES

In this section we present the concept and properties of critical vertices and cycles of a digraph, in which are based the results later described.

Let D be a digraph and v,w vertices of it. The class of v in D is the subset of vertices $\{v\} \cup T(D-v)$, which we denote by $[v,D]$. The classes $[v,D]$ and $[w,D]$ are distinct when $[v,D] \neq [w,D]$.

A vertex $v \in V(D)$ is critical in D when the subgraph induced in D by $[v,D]$ has at least one cycle C. In this case, C is a critical cycle of v in D.

The first lemma relates critica vertices and cycles.

Lemma 1: Let v be a critical vertex and C a critical cycle of v in D. Then C contains v.

Proof: Suppose the contrary. Then there exists a cycle C' formed solely by vertices of some subset of $T(D-v)$. Consequently, every vertex $w \in V(C')$ belongs to a non trivial component of the subgraph induced in D by $\{v\} \cup T(D-v)$. The latter con tradicts $w \in T(D-v)$□

We now describe a condition for two classes to be distinct

Lemma 2: Let v,w be critical vertices in D. Then $v \in [w,D]$ if and only if $[v,D]=[w,D]$.

Proof: We consider $v \neq w$, otherwise the result is trivial. If $v \in [w,D]$ then v belongs to a trivial component of D-w, that is, every cycle passing through v contains also w. Since v is also critical, there exists a cycle C formed by a subset of trivial components of D-v. By lemma 1, C contains v. That is, w is a trivial component of D-v and then $w \in [v,D]$. Consider now a vertex $z \neq v,w$ such that $z \in [w,D]$. In this case, every cycle C' containing z passes through w. Since $w \in [v,D]$ we conclude that C' also contains v. Then $z \in [v,D]$ and hence $[v,D]=[w,D]$. The converse is immediate, since $v \notin [w,D]$ implies $[v,D]\neq[w,D]$, because $v \in [v,D]$□.

The next lemma asures that any critical cycle contains al critical vertices of its class.

Lemma 3: Let v,w be critical vertices in D such that $[v,D]=[w,D]$. Then a cycle contains v if and only if it contains also w.

Proof: Suppose there exists in D some cycle C containing v, but not w. Then C remains a cycle in D-w. Because C contains v, it follows that v can not be a trivial component of D-w. Consequently, $v \notin [w,D]$. Then we apply lemma 2 and conclude that $[v,D]\neq[w,D]$, which contradicts the hypothesis. Therefore C contains both v and w□.

There are certain vertices which may belong to more than one distinct class of a digraph. These vertices satisfy the following condition.

Lemma 4: Let v,w be critical vertices n D such that $[v,D]\neq[w,D]$, and $z \in [v,D] \cap [w,D]$. Then there is no critical cycle in D containing z.

Proof: Suppose the lemma false. Then there is a critical cycle C of v which contains z. Because $[v,D] \neq [w,D]$ we conclude by lemma 2 that $w \notin [v,D]$. Hence $w \notin V(C)$. On the other hand, $v \in V(C)$. Consequently, C remains a cycle in D-w. Since $z \in V(C)$, z can not be a trivial component of D-w, i.e., $z \notin [w,D]$, which contradicts the hypothesis □.

The next lemma describes a condition for two critical cycles to be disjoint.

Lemma 5: Let v,w be critical vertices in D, and C,C' critical cycles of v,w, respectively. Then C,C' are disjoint if and only if $[v,D] \neq [w,D]$.

Proof: Suppose C,C' disjoint and $[v,D] = [w,D]$. In this case, according to lemma 3, both cycles C,C' contain both vertices v,w. Then C,C' are not disjoint, a contradiction. That is, $[v,D] \neq [w,D]$, necessarily. Conversely, when $[v,D] \neq [w,D]$ we apply lemma 4 to conclude that no vertex of C or C' can belong to $[v,D] \cap [w,D]$. Therefore C,C' are disjoint □.

Now, we discuss the effect of removing critical vertices.

Lemma 6: Let v,w be critical vertices in D. Then $[v,D] \neq [w,D]$ if and only if w is critical in D-v.

Proof: If $[v,D] \neq [w,D]$ we must prove that w remains critical after removing v. Let C be a critical cycle of w in D. The idea consists of showing that C is also a critical cycle of w in D-v. Let z be a common vertex of $[v,D]$ and $[w,D]$. By lemma 4, we know that there is no critical cycle of D containing z. That is, $z \notin V(C)$. In addition, since every vertex $z' \in V(C)$ necessarily belongs to $[w,D]$ we conclude that $z' \notin [v,D]$. Therefore, C is preserved in D-v and w remains critical. The converse is simple, as follows. If w is critical in D-v then $w \notin [v,D]$, necessarily. Otherwise, if $w \in [v,D]$ either $w \notin V(D-v)$ or w becomes a trivial component in D-v. In none of these cases can w be a critical vertex in D-v, a contradiction. Now, when $w \notin [v,D]$ we apply lemma 2 and conclude that $[v,D] \neq [w,D]$ □.

Lemma 7: Let v,w,z be critical vertices in D. Then $[v,D] \neq [w,D]$ if and only if $[v,D-z] \neq [w,D-z]$.

Proof: Initially, we consider the hypothesis $[v,D] \neq [w,D]$. If $z \in [v,D]$ then $z \notin [w,D]$. Otherwise, z would be a critical vertex belonging to $[v,D]$ and $[w,D]$, simultaneously; then, by lemma 2, $[v,D]=[z,D]$ and $[w,D]=[z,D]$, i.e. $[v,D]=[w,D]$ a contradiction. Now, $[v,D]=[z,D]$ implies that v can not be a critical vertex in D-z, by lemma 6. Also, $[w,D] \neq [z,D]$ means that v must be a critical vertex in D-z, since no critical cycle of w in D can contain z, according to lemma 5. Therefore, $[v,D-z] \neq [w,D-z]$ and the lemma is valid for this case. If $z \in [w,D]$ we apply a similar argument. It remains to analyse the situation $z \notin [v,D],[w,D]$. Suppose the lemma false, that is, $[v,D-z]=[w,D-z]$ and let C be a critical cycle of v in D. Then $w \notin V(C)$, since it follows from the hypothesis that $w \notin [v,D]$. In addition, C must contain some vertex $x \in [z,D]$, $x \neq z$. Otherwise, C would remain as a critical cycle of v in D-z; and because $[v,D-z]=[w,D-z]$ we conclude by lemma 3 that C also contains w, a contradiction. Consequently, in fact $x \in [z,D]$. In addition, since C is a critical cycle of v in D we know that $x \in [v,D]$. In the present situation, v and z are two critical vertices in D such that $[v,D] \neq [z,D]$ and x is a common vertex of $[v,D]$ and $[z,D]$. By lemma 4, we can see that there is no critical cycle in D containing x. Therefore, C does not exist, which contradicts the fact that v is a critical vertex. Consequently, $[v,D-z] \neq [w,D-z]$ and the proof of necessity is completed. Conversely, let the hypothesis $[v,D-z] \neq [w,D-z]$. There are four cases to consider:

(i) $z \in [v,D],[w,D]$.

Then by lemma 2, $[v,D]=[w,D]=[z,D]$. In this case, $[v,D-z]=[v,D]-\{z\}$ and $[w,D-z]=[w,D]-\{z\}$. That is, $[v,D-z]=[w,D-z]$, contradicting the hypothesis. Therefore, this case does not occur.

(ii) $z \in [v,D]$ and $z \notin [w,D]$.

That is, $[v,D] \neq [w,D]$ and the lemma holds

(iii) $z \notin [v,D]$ and $z \in [w,D]$.

Similar to (ii).

(iv) $z \notin [v,D], [w,D]$.

Then $[v,D] \neq [z,D]$ and $[w,D] \neq [z,D]$. Let C and C' be critical cycles of v and w in D, respectively. By lemma 6, we conclude that v and w remain critical in D-z and therefore C and C' are critical also in D-z. We now apply lemma 5 to D-z and find out that C and C' are disjoint. Next, using again lemma 5, but to the digraph D instead, we finally conclude that $[v,D] \neq [w,D]$. This completes the proof □.

3. CRITICAL SEQUENCES

In order to describe the class of connectively reducible digraphs we need the following definitions.

Let D be a digraph and $S=\{v_1,\ldots,v_k\}$ a sequence of vertices of it. The value k is the length of S, while the symbol S_j denotes the subsequence $\{v_1,\ldots,v_j\}$, for any j, $1 \leq j \leq k$. We also write S_o to represent the empty sequence $\emptyset$. The notation $D(S_j)$ means the digraph induced in D by the subset of vertices $\overline{T}(D-S_j)$. Then, for example, $D(S_o)$ is the digraph formed by the non trivial components of D. The digraph $D(S_j)$ is ca ed the resulting of S_j. If each vertex v_j is critical in $D(S_{j-1})$ then S is a critical sequence of D, $1 \leq j \leq k$. In this case, additionally, if D(S) does not contain any critical vertices then S is a complete critical sequence, or simply, complete sequence. Next, a vertex $v \in V(D)$ is strongly non critical if there is no critical sequence of D containing v. Finally, D is connectively reducible when the subgraph induced in it by the subset of all strongly non critical vertices is acyclic.

For example, the digraph of figure 1 has only one critical vertex, namely v. In addition, {v} is its only critical sequence, while the removal of this vertex destroys all cycles. Therefore, it is connectively reducible.

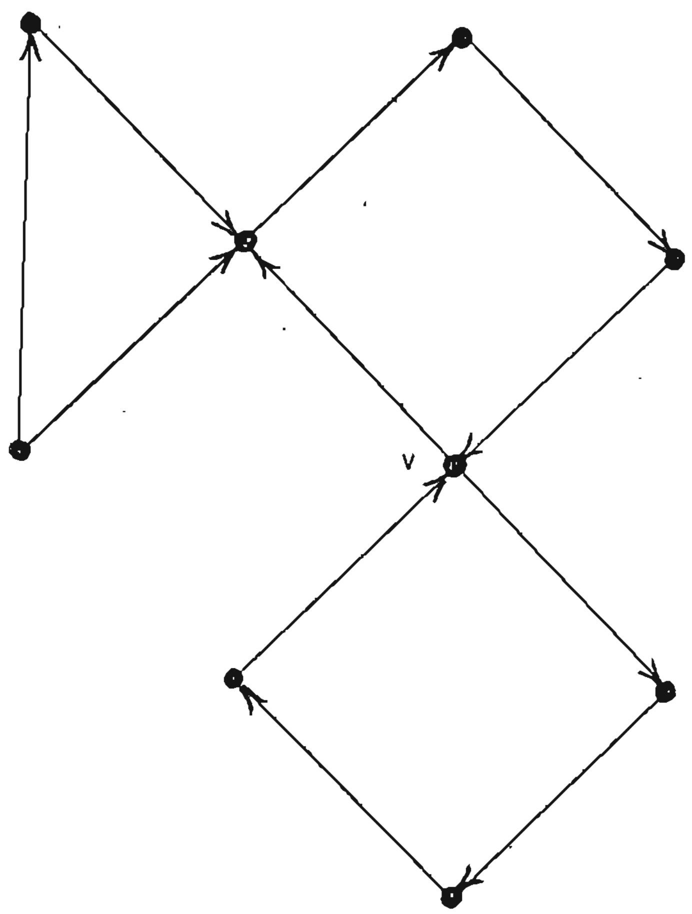

FIGURE 1: A CONNECTIVELY REDUCIBLE DIGRAPH

Next, we establish relations between critical vertices and resulting digraphs.

Lemma 8: Let D be a digraph, S a critical sequence of it and v,w critical vertices in D such that $[v,D]=[w,D]$. Then $v \in V(D(S))$ implies:

(i) $[v,D(S)] = [w,D(S)]$,

(ii) $w \in V(D(S)$ and

(iii) v,w remain critical vertices in D(S)

Proof: Let $S=\{v_1,\dots,v_k\}$. We use induction in k. If k=0 the lemma is trivially true. When k>0, assume it valid for all critical sequences of length at most k-1. Let $v \in V(D(S_k))$. Then $v \in V(D(S_{k-1}))$ and we can apply the induction hypothesis to conclude that

(i) $[v,D(S_{k-1})] = [w,D(S_{k-1})]$,

(ii)' $w \in V(D(S_{k-1}))$, and

(iii)' v,w are both critical in $D(S_{k-1})$

We can now observe that vertices v,w,v_k are all critical in $D(S_k)$. Therefore, we can apply (i)' to lemma 7 and conclude that $[v,D(S_k)]=[w,D(S_k)]$, because $\bar{T}(D(S_k))=\bar{T}(D(S_{k-1})-v_k$ This proves (i). Next, since $v \in V(D(S_k))$ we can apply (i) to write $[v,D(S_k)]=[w,D(S_k)]$, which leads to $w \in V(D(S_k))$ assuring (ii)

Finally,

$$[v, D(S_{k-1})] \neq [v_k, D(S_{k-1})],$$

otherwise $v \notin V(D(S_k))$, a contradiction. Therefore, we can apply lemma 6 to obtain that v is critica in $D(S_k)$. Similary for w. The proof of iii is now completed □.

We now introduce the concept of representatives of D.

Let D be a digraph and $R(D)=\{v_1,...,v_k\}$ some subset of critical vertices of it. R(D) is a <u>critical representative subset</u>, or simply a <u>representative</u>, of D when the following conditions are both satisfied:

(i) $i \neq j \Rightarrow [v_i, D] \neq [v_j, D]$.

(ii) $w \in V(D)$ is a critical vertex of D $\Rightarrow$
$[v_i, D] = [w, D]$, for some i, $1 \leqslant i \leqslant k$.

In other words, a representative of D is a maximum cardinality subset formed by critical vertices belonging to distinct classes of D.

The next lemma shows a relation between representatives and critical sequences of a digraph.

<u>Lemma 9</u>: Let S be a sequence formed by vertices of a representative of D, in any arbitrary order. Then S is a critical sequence of D.

<u>Proof</u>: Let $S=\{v_1,...,v_k\}$. We employ induction n k. f k=0 there is nothing to prove. Otherwise, suppose the lemma holds for all sequences of length at most k-1. Since the vertices of S belong to a representative of D we know that each v_i is

critica in $D(S_0)$ and $[v_i,D] \neq [v_j,D]$, $i \neq j$. Consequently, we can apply lemma 6 to conclude that v_k s critical in $D(S_1)$. In addition, t follows from lemma 7 that $[v_k,D(S_1)] \neq [v_j,D(S_1)]$, for $1 \leq j < k$. Repeating iteratively this argument it results that v_k is still critical in $D(S_{k-1})$. In this situation, we can use the induction hypothesis to conclude that $\{v_1,\dots,v_k\}$ is a critical sequence of D □.

4. CHARACTERIZATION OF CONNECTIVELY REDUCIBLE DIGRAPHS

Consider solving the recognition problem for connectively reducible digraphs. A first idea might be to apply the definition and recognize as a member of this family every digraph D whose subgraph induced by its strongly non critical vertices is acyclic. To use this strategy we would need previously to devise a method for finding the set of all strongly non critical vertices. It seems difficult to solve the latter problem directly from the definition, since to identify these special vertices we would need to generate all possible complete sequences of D, whose number can grow exponentially with $|V(D)|$. In this section, we prove a convenient characterization for this family, which enables to recognize connectively reducible digraphs after constructing just one complete sequence.

Theorem 1: All complete sequences of a digraph D have the same resulting digraph.

Proof: Let S be an arbitrary complete sequence. The proof consists of defining a canonical sequence S' and showing that D(S)=D(S'), as below detailed. We start by S'.

Constructing S': Let R(D) be a representative of D. A canonical sequence S' of D is recursively defined as follows. If $R(D)=\emptyset$ then $S'=\emptyset$. Otherwise, S' is formed by the vertices of R(D), in an arbitrary order, followed by a canonical sequence of D-R(D).

To verify that the above construction always finds a complete sequence of D, we use induction in the length k' of S'. If k'=0 the result is correct, since S'=∅ means R(D)=∅ and there can be no critical sequence without critical vertices. Otherwise, assume the construction correct for lengths at most k-1. From the definition, we know that S' is formed by the vertices of R(D), followed by a canonical sequence of D-R(D), which we now denote by S". The leading vertices of S', i.e. R(D), form a critical sequence of D, according to lemma 9. On the other hand, using the induction hypothesis we conclude that S" is a complete sequence of D-R(D). At this point we can apply the definition and asure that S' is a complete sequence of D, which proves the correctness of the above construction.

To show that D(S)=D(S') the idea consists of transforming S into S' through the application of some different operations. Each operation can result in alterations in S. In this case, we must guarantee that the resulting digraph of the sequence remained the same. If we asure the invariance of D(S) through the process we obtain D(S)=D(S'), which would prove the theorem.

We use four different operations to transform S. Two of them replace certain vertices of S by others, while the remaining operations simply change the order of the sequence.

Now, we describe the transformation from S into S' together with the proofs of invariance of D(S) in the process The current sequence S is denoted by $\{v_1,\ldots,v_k\}$, while R(D) is precisely the representative of D which S' contains.

<u>Operation 1</u>: For each vertex $v \in R(D)$ verify if S contains some critical vertex $w \in [v,D]$. In the affirmative case, replace w by v in S.

The proof that S is mantained complete and D(S) preserved after the end of the above operations is simple. Let v,w according to the hypothesis, that is, $v \in R(D)$, $w \in [v,D]$, both critical vertices in D and $w=v_i$ for some i, $1 \leq i \leq k$. Then $w \in V(D(S_{i-1}))$ and applying lemma 8 we conclude that

$v \in V(D(S_{i-1}))$ and that v,w are both critical vertices in $D(S_{i-1})$ belonging to a same class in it. Therefore, by lemma 2, $[v,D(S_{i-1})]=[w,D(S_{i-1})]$. Hence, $D(S_{i-1})-v$ and $D(S_{i-1})-w$ coincide That is, S is mantained complete and D(S preserved after each of the vertex replacements.

After operation 1, S may not contain yet all vertices of R(D). The transformation to include in S the remaining desired vertices is given below.

Operation 2: For each $v \in R(D)-S$, determine the value $j \geq 1$ such that v is a critical vertex in $D(S_{j-1})$, but not in $D(S_j)$ and next replace v_j by v in S.

We now describe the proof of correctness of operation 2. We need to show that the new sequence S contains R(D), after all transformations. The argument is inductive. If $R(D)-S=\emptyset$ there is nothing to prove. Otherwise, choose $v \in R(D)-S$. Operation 2 identifies the value $j \geq 1$ satisfying

$$v \in V(D(S_{j-1})) - V(D(S_j))$$

We need to asure that such value j does exist. Since S is complete, $D(S_k)$ does not contain critical vertices. Because $v \in R(D)$, it follows that v is critical in $D(S_o)$ Thus, there exists necessarily j, $1 \leq j \leq k$, such that v is critical in $D(S_{j-1})$ but not in $D(S_j)$ There are two alternatives to consider, namely $v \in V(D(S_j))$ or not. In the first case, v s non critical in $D(S_j)$ by hypothesis. However, this can not occur. Because v_j and v are both critical in $D(S_{j-1})$ and $v_j \notin V(D(S_j))$ it fo lows that $[v,D(S_{j-1})] \neq [v_j,D(S_{j-1})]$. Hence, by applying lemma

6 we would conclude that v remains critical in $D(S_j)$, a contradiction. Therefore, the only possibi ity is $v \notin V(D(S_j))$ In this case, using again that v_j and v are critica in $D(S_{j-1})$ and lemma 6, we obtain $[v,D(S_{j-1})]=[v_j,D(S_{j-1})]$. That is,

$$D(S_j) = D(S_{j-1}) \quad [v,D(S_{j-1})] = D(S_{j-1}) \quad [v_j,D(S_{j-1})].$$

Hence, after replacing v_j by v, S is stil complete. Furthermore, for any vertex $w \in R(D)$, necessari y $w \neq v_j$ Because, if $w=v_j$ then $[v_j,D(S_{j-1})]=[w,D(S_{j-1})]$. In this case, applying succesively lemma 7 would lead us to $[v,D]=[w,D]$, which contradicts $v,w \in R(D)$. Therefore, each replacement of v_j by v in S increases by one the number of vertices of R(D) which appear in S. This completes the proof of operation 2.

After operations 1 and 2, S necessarily contains R(D). However, in order to transform S into S' we need the vertices of R(D) to appear in the leading positions of S. This is accomplished by the following.

Operation 3: If S contains some vertex $v_j \in R(D)$ such that $v_{j-1} \notin R(D)$, $j>1$, then interchange the positions of v_j and v_{j-1} in S. Repeat the operation unti no such $v_j \in R(D)$ exists in S

The proof of correctness of operation 3 consists in showing that after the last interchange of positions, S is still a complete sequence and that D(S) was preserved. In addition, the $|R(D)|$ leading vertices of the transformed sequence are precisely those of R(D). The argument is again inductive. If S is formed solely by vertices of R(D) there is nothing to prove. Otherwise, for each $v_i \in S-R(D)$ define displacement (v_i) as the number of vertices of $R(D)$ which are at the right side of v_i in S. If displacement $(v_i)=0$ for al $v_i \in S-R(D)$ then operation 3 is not performed and its correctness follows trivially. Otherwise, S contains necessarily a vertex $v_j \in R(D)$ such that $v_{j-1} \notin R(D)$, $1<j\leq k$. In this case, v_j is critical in $D(S_{j-1})$, and clearly also n $D(S_o)$ On the other hand

$$[v_j,D(S_{i-1})] \neq [v_j,D(S_i)], \text{ for all } 1\leq i<j$$

Because, otherwise, if for some i vertices v_i and v_j belong to a same class n $D(S_{i-1})$ then according to lemma 2 $v_j \notin V(D(S_i))$ which contradicts $v_j \in V(D(S_{j-1}))$. Similary, we conclude that v_j s critical in $D(S_{i-1})$, $1\leq i<j$. Consequently, v_{j-1} and v_j are both critical vertices and belonging to distinct classes in $D(S_{j-2})$. Now, we apply lemma 6 to certify that v_{j-1} is also critical in $D(S_{j-2})-v_j$ Therefore, we can interchange the positions of v_{j-1} and v_j in S and asure that the new sequence so obtained is still critical and complete. Besides, $D(S)$ is also preserved. Because $D(S_j)$ in both sequences, old and new, equals the digraph obtained by removing the trivial components of $D(S_{j-2}) - \{v_{j-1},v_j\}$. On the other hand, the change of positions between v_{j-1} and v_j asures that displacement (v_{j-1}) decreases by one unit. This completes the proof of correctness of operation 3

The leading vertices of S are now exactly these of $R(D)$. However, we need them in S with the same ordering as they are in S'. This is the purpose of the last operation below

<u>Operation 4</u>: Reorder the vertices of $R(D)$ n S, so as to obey the same ordering as they appear in S'.

The correction of it is simple. The sequence formed in S by the vertices of R(D) in its new ordering is itself critical, according to lemma 9. Besides, $D(S_{|R(D)|})$ is the digraph obtained by removing the trivial components of D-R(D). Therefore, the new sequence S is also complete and D(S) is mantained.

Consider now the sequence S after all above operations and let us complete the transformation from S into S'. In both sequences the |R(D)| leading vertices coincide, respectively. Now, remove R(D) both from S and S'. If D-R(D)=ø then S=S'. Otherwise, S-R(D) is a complete sequence of D-R(D). Also, S'-R(D) is a canonical sequence of it. In addition, D(S)=D(S-R(D)) and D(S')=D(S'-R(D)). Next, apply the four described operations to S-R(D) which would transform it into a new sequence having the same leading |R(D-R(D))| vertices as S'-R(D), while preserving its resulting digraph. Then remove R(D-R(D)) from both S-R(D) and S'-R(D) and so on iteratively. We can then conclude that any arbitrary complete sequence of D has the same resulting digraph as the canonical one. This completes the proof of theorem 1 □.

The next propositions follow directly from the above proof

<u>Corollary 1</u>: Al complete sequences of a digraph have the same length.

<u>Corollary 2</u>: Let D be a digraph and S an arbitrary complete sequence of it. Then D is connectively reducible if and only if D(S)=ø.

5. THE MIN-MAX THEOREM

<u>Theorem 2</u>: Let D be a connectively reducible digraph. Then $\min|\alpha(D)|=\max|\beta(D)|$

<u>Proof</u>: If D does not contain critical vertices then all its vertices are strongly non critical. In this case D is necessarily acyclic and the theorem is trivial. Otherwise, let $S=\{v_1,\ldots,v_k\}$ be a complete sequence of D, $k\geqslant 1$. Define the

subsets of vertices $\alpha(D)=\{v_1,\ldots,v_k\}$ and cycles $\beta(D)=\{C_1,\ldots,C_k\}$ where C_j is a critica cycle of v_j in $D(S_{j-1})$ $1\leqslant j\leqslant k$. First, we show that $\alpha(D)$ is a cycle cut of D. Since D is connectively reducible, $D(S_k)=\emptyset$, according to corollary 2. Consequently, for any cycle C of $D(S_o)$ there exists an index j, $1\leqslant j\leqslant k$, such that C is a cycle in $D(S_{j-1})$, but not in $D(S_j)$ Therefore C contains some vertex $w \in [v_j,D(S_{j-1})]$. Suppose that C does not contain v_j Then $w \notin T(D(S_{j-1})-v_j)$, that is $w \notin [v_j,D(S_{j-1})]$, a contradiction Hence, C contains v_j and we conclude that $\alpha(D)$ is in fact a cycle cut. Next, we examine $\beta(D)$. Suppose there exists a pair of distinct cycles C_p, $C_q \in \beta(D)$ containing a common vertex z. Without loss of generality, let p<q. Then $z \notin V(D(S_p))$ because $z \in \{v_p\} \cup T(D(S_{p-1})-v_p)$. That is, $z \notin \bar{T}(D-S_i)$, $i\geqslant p-1$, which contradicts $z \in V(D(S_{q-1}))$ and $z \in V(C_q)$ Therefore, C_p, C_q can not contain common vertices. Hence, $\alpha(D)$ and $\beta(D)$ are respectively a cycle cut and a set of vertex disjoint cycles of D, having the same cardinality. Therefore the first is minimum and the second maximum □.

6. THE ALGORITHM

A polynomial time algorithm for recognizing connectively reducible digraphs and finding minimum cycle cuts and maximum sets of disjoint cycles for digraphs of this family is a direct consequence of corollary 2 and theorem 2.

The algorithm below accepts as input an arbitrary digraph D and computes one of the following alternative results. Either it confirms that D is connectively reducible and simultaneously exhibits a minimum cycle cut and maximum set of disjoint cycles, or it reports that D is not connectively reducible

In the *initial step*, let $i:=0$, define the digraph $D_i:=D$, the sets $\alpha:=\beta:=\emptyset$ and unmark all vertices. In the *general step*, if there are no unmarked vertices the process terminates (D is connectively reducible iff D_i is acyclic; in the affirmative case, α and β are respectively a minimum cycle cut and maximum set of disjoint cycles of D). Otherwise, choose any unmarked vertex v, mark it and construct class $[v,D_i]$. Next, verify if the subgraph induced in D by the vertices of $[v,D_i]$ contains some cycle C. If it does contain, then include v in α, include C in β, define $D_{i+1}:=D_i-[v,D_i]$, unmark all vertices of D_{i+1} and increase i by 1. In any case, repeat the general step □

There is no difficulty to implement this algorithm in $O(n^2(n+m))$ time, $n=|V(D)|$ and $m=|E(D)|$.

7. CONNECTIVELY AND CYCLICALLY REDUCIBLE DIGRAPHS

In this section we show that the family of connectively reducible digraphs contains the cyclically reducible ones. We start by presenting the definitions of the latter.

Let D be a digraph. A vertex $w \in V(D)$ is *blocked in* D if there exists a path in D from w to some vertex $z \in \overline{T}(D)$. The *associated digraph* $A(v,D)$ *of* D *relative* to v is the subgraph induced in D by the subset of $V(D)$ that contains v and all vertices that are not blocked in D. A *W-sequence of* D is a sequence of vertices $\{v_1,\ldots,v_k\}$ such that there are cycles in each of the associated digraphs $A(v_i,D_{i-1})$, $1\leqslant i\leqslant k$, where $D_o=D$ and

$$D_i = D_{i-1} - V(A(v_i,D_{i-1}))$$

n addition, if D_k is acyclic then the W-sequence is *complete*.

Finally, a *cyclically reducible digraph* is precisely one that admits a complete W-sequence.

The following lemma relates the above associated digraphs and classes as defined in Section 2.

Lemm 10: Let D be a digraph and $w \in V(D)$. If w is a vertex of $A(v,D)$ then w belongs to $[v,D]$.

Proof: If w is a vertex of $A(v,D)$ then $w=v$ or w is not blocked in $D-v$. In the first case, the lemma holds. Consider then $w \neq v$. By definition, there exists no path in $D-v$ from w to some vertex $z \in \bar{T}(D-v)$. Therefore $w \in T(D-v)$, otherwise there is a contradiction if we choose z as a vertex located in the same component as w of $D-v$, and such that $(w,z) \in E(D)$. Therefore, using the definition of class we conclude that $w \in [v,D]$.

Finally,

Theorem 3: Let D be a cyclically reducible digraph. Then D is connectively reducible.

Proof: If D is acyclic the theorem is trivial. Otherwise, D admits a complete W-sequence $S=\{v_1,\ldots,v_k\}$, $k \geq 1$. The proof consists of showing that S is a complete critical sequence of D. The argument is inductive. Suppose the result true for all digraphs admitting W-sequences with fewer than k vertices. Since D is cyclically reducible, $A(v_1,D)$ has some cycle C. By lemma 10, all vertices of C belong to $[v_1,D]$. That is, v_1 is critical in D. In addition, the non trivial components of

$$D - V(A(v_1,D))$$

are identical as those of

$$D - [v_1,D],$$

because v_1 is critical in D and according to lemma 10

$$V(A(v_1,D)) \subseteq [v_1,D]$$

Therefore, by removing v_1 from D and applying the induction hypothesis to $D-v_1$ we conclude that S is a complete critical sequence of D. Furthermore, D_k is acyclic because D is cyclically reducible. Then the resulting digraph D(S) is empty, since

$$\bar{T}(D_i) = \bar{T}(D(S_i)),\ 0 \leq i \leq k,$$

that is, D is connectively reducible □

8. CONNECTIVELY AND FULLY REDUCIBLE DIGRAPHS

We prove in this section that the connectively reducible contain the fully reducible digraphs.

A flow digraph is a digraph D together with a distinguished vertex $s \in V(D)$, called root, that reaches all the vertices of D. We say that $w \in V(D)$ dominates $v \in V(D)$ when every path in D from s to v contains w. D is fully reducible if every cycle C of D contains some vertex $w \in V(C)$ which dominates all the vertices of C. In this case, we call w a dominator of C and also of D. The edge of C which is directed to the dominator of this cycle is called a back edge.

Theorem 4: Let D be a fully reducible digraph having root s. Then D is connectively reducible.

Proof: Let L be the set of back edges of D. The argument is by induction on $|L|$. If $|L|=0$ then D is acyclic and the theorem is trivial. Otherwise, suppose the result correct for all fully reducible digraphs with fewer than $|L|$ back edges. Let $w \in V(D)$ be a dominator of D located at a maximal distance of s in D-L. That is, in the acyclic digraph D-L no proper descendant of w is a dominator in D. Let C be a cycle containing the back edge (v,w) and $z \in V(C)$, $z \neq w$. Suppose there exists a cycle C' in D such that $z \in V(C')$, but $w \notin V(C')$. Let w' be the dominator of C'. Observe that w does not dominate w' in D, otherwise there would be a path in D-L starting in w and containing w', which contradicts w as a dominator of D at a maximal distance of s in D-L. Hence there exists a path in D from s to w' that does not contain w. Consequently, this path s-w' followed by the path w'-z in C' forms a path originated in the root of D and intersecting C in some vertex other than its dominator w, which contradicts D as fully reducible. Therefore, if $z \in V(C) \cap V(C')$ then necessarily $w \in V(C')$. In this case, every vertex of C becomes a trivial component in D-w. That is, w is a critical vertex in D, and C is a critical cycle of w in D. Removing w from D and taking the non trivial components of D-w we obtain the resulting digraph D({w}). Let S' be a complete critical sequence of D({w}). Note that D({w}) has fewer than |L| back edges, that is, this digraph is connectively reducible according to the induction hypothesis. By corollary 2, we conclude that the resulting digraph of S' in D({w}) is empty. Consequently, the sequence S formed by w followed by S' is a complete sequence in D satisfying $D(S)=\emptyset$. Therefore, D is connectively reducible □.

9. CONCLUSIONS

We have described a new family of digraphs D named connectively reducible and proved that

$$\min|\alpha(D) = \max|\beta(D)|$$

The proofs lead to polynomial time algorithms for finding the minimum set of vertices $\alpha(D)$ and maximum of cycles $\beta(D)$. Furthermore, we have also proved that the proposed family of digraphs contains two others for which similar properties hold, namely the fully reducible and connectively reducible digraphs.

Less is currently known about the equivalent problem for edges instead of vertices, regarding reducible digraphs. In fact, it is not known if in a fully reducible digraph the minimum cardinality set of edges intersecting all cycles equals the maximum cardinality set of edge disjoint cycles. Frank and Gyárfás [1] have conjectured that equality also holds in the edge case. Partial results in this direction were reported in [7].

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