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Moon tides: generalizing classical gravity to an oscillating sphere.

A Hodge decomposition point of view

Victor Pessanha Mendes de Oliveira

Dissertação de Mestrado apresentada ao Programa de Pós-Graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro - UFRJ, como parte dos requisitos necessários à obtenção do título de Mestre (Matemática).

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Resumo

Marés lunares: generalizando a gravidade clássica para uma esfera oscilante.

Um ponto de vista de decomposição de Hodge

Victor Pessanha Mendes de Oliveira

Orientadora: Stefanella Boatto

Coorientador: Luca Comisso

Resumo da Dissertação de Mestrado apresentada ao Programa de Pós-Graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro - UFRJ, como parte dos requisitos necessários à obtenção do título de Mestre em Ciências (Matemática).

Na Parte I desta tese, focamos em como generalizar a noção de um campo gravitacional não apenas para uma dada métrica fixa, mas também no caso de uma métrica oscilante, fazendo uso deste último como um “toy model” para marés lunares. Seguimos uma generalização da decomposição de Hodge em um contexto relativístico que permite uma dedução da dinâmica de massas pontuais, usando em parte resultados recentemente derivados para vórtices pontuais em superfícies diferenciáveis fechadas M dotadas de uma métrica g.

Na Parte II da tese, a conexão entre tranças e sistemas hamiltonianos é explorada com base nos trabalhos de Boyland, Aref e Stremler. Uma conexão entre as tranças formadas por um sistema de partículas pontuais e sua integrabilidade de Liouville é encontrada pela construção de uma noção de integrabilidade vinda da Teoria das Tranças. É dado um primeiro teorema que relaciona esta integrabilidade por tranças com a já conhecida de Liouville.

Palavras-chave: Dinâmica Gravitacional, Relatividade Geral, Dinâmica de Vórtices, Sistemas Integráveis, Teoria de Tranças, Dinâmica de partícula teste

Abstract

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Abstract da Dissertação de Mestrado apresentada ao Programa de Pós-Graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro - UFRJ, como parte dos requisitos necessários à obtenção do título de Mestre em Ciências (Matemática).

In Part I of this thesis, we focus on how to generalize the notion of a gravitational field not only for a given fixed metric but also in the case of an oscillating one. The problem was raised as a toy model for Moon tides. We follow a generalization of Hodge decomposition in a relativistic background which enables a deduction of the dynamics of point masses, using in part recently derived results for point vortices on closed differentiable surfaces M endowed with a metric g.

In Part II of the thesis, the connection between braids and Hamiltonian systems is explored based on the works of Boyland, Aref and Stremler. A connection between the braids formed by a point particle system and its Liouville integrability is found by the construction of an integrability notion coming from Braid Theory. A first theorem relating this braid integrability to the already know Liouville one is given.

Keywords: Gravitational Dynamics, General Relativity framework, Test particle dynamics, Vortex Dynamics, Braid Theory, Integrability

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Contents

Introduction

The main problem we will be interested in within Part I of this thesis is that of massive interacting particle motion within the surface of an ellipsoid of revolution.

The reason behind our interest in pursuing this problem comes from the modelling of atmospheric dynamics. Indeed, the triggering question was

Question 1: *How does a slowly oscillating metric (modelling the Moon's influence over the Earth's atmosphere) affects the dynamics of interacting particles (moving within Earth's atmosphere)?*

Atmospheric dynamics: tides and gravity

It is known by meteorologists and oceanographers that the Moon has a gravitational effect not only over Earth's seas but also over its atmosphere [Gil16]. Such an effect, coupled to the Moon's motion around the planet, is what gives rise to the *Moon tides* we experience. In particular, the study of atmospheric tides dates back to the end of the 18th century with Laplace's tidal equations giving a first rough description of their formation and dynamics [Ped13] (see Figure 1).

As can be noted on Figure 2, an observer sitting on the planet will see throughout the day two high tides and two low tides due to the revolution of the Earth around itself. This picture shown illustrates in a sense the *intrinsically felt* effect of the tides. That is, for someone on the planet, such a tidal effect can be modelled by a periodic perturbation of some agreed upon sea or atmosphere level, with the Earth's rotation being taken into account by means of such a height variation.

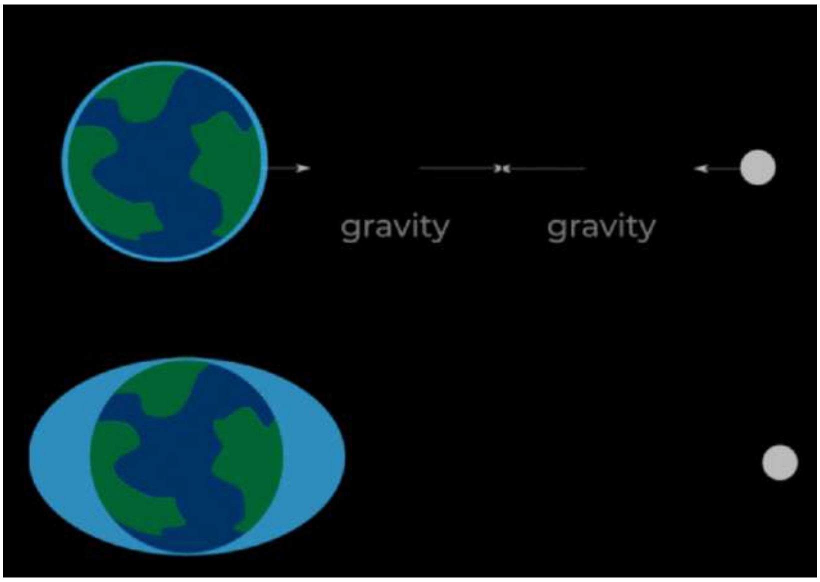


Figure 1: Picture taken from the NASA website [NAS] explaining the lunar oceanic tides. The gravitational pull from the Moon produces the above ellipsoidal looking shape of the ocean [Har21]. When coupled to the relative motion of the bodies (Moon and Earth) high and low tide effects arise, generating rich atmosphere and ocean dynamics.

Focusing on the atmospheric case, a massive particle sitting inside the atmosphere should perceive a certain increase and decrease in the density of the atmosphere twice a day. Although there are various factors and consequences for such an atmospheric tide [LC69], we here decide to focus on the gravitational part of it. Specifically the gravitational effect of lunar (or Moon) tides on the Earth's atmosphere motivated by Question 1 above.

To answer such a question in all generality, even if restricting our attention to the gravitational case only, would be quite complex. Indeed, an all encompassing model would for instance have to take into account for the different layers of the atmosphere and say how their different densities will impact the particle's motion, hence having to consider atmospheric dragging effects.

Besides this, lagging effects are also present in the atmosphere's motion. Indeed, as the Moon revolves around the planet, since the gravitational interaction propagates at a finite speed, it will take some time for the atmosphere to "perceive" the position

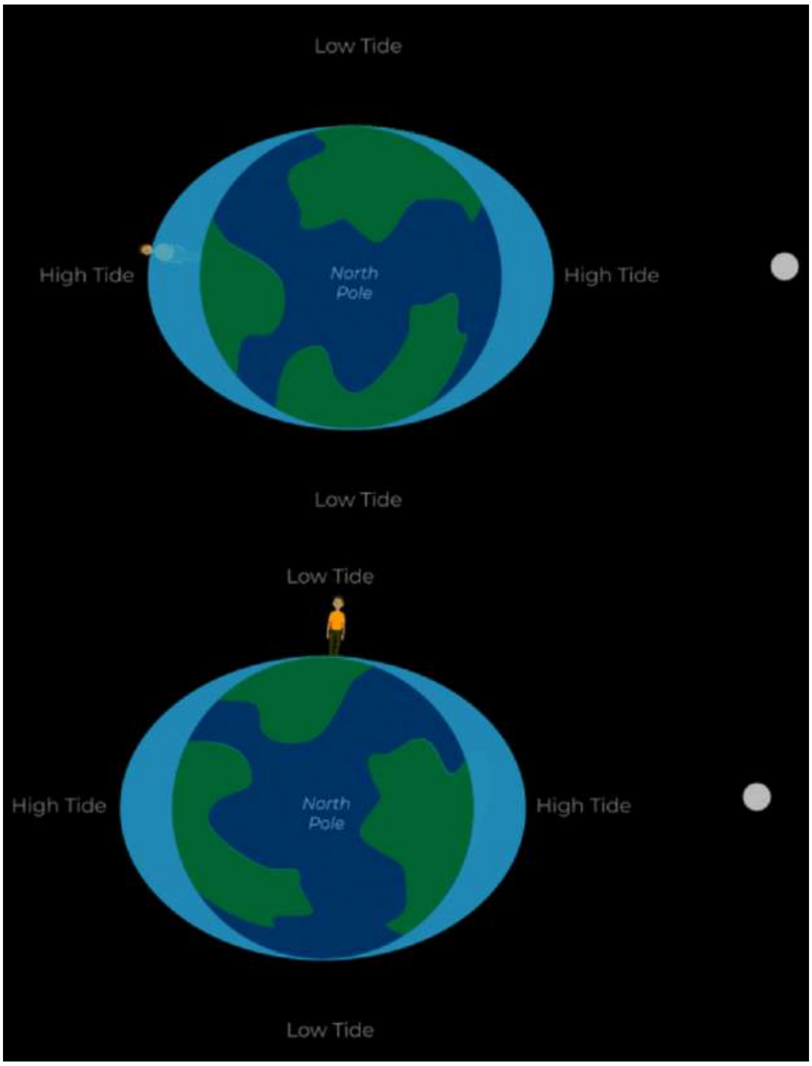


Figure 2: Picture taken from NASA website [NAS] depicting the high tide - low tide effect provoked by the relative motion of the Earth and the Moon. If one considers a frame in which the Moon is fixed, a spinning observer on the Earth experiences in a 24 hour period two high and two low tides.

displacement the Moon underwent (see Figure 3). In a sense, this effect is similar to the frame dragging effect in General Relativity felt by inertial observers when close to a spinning mass distribution.

To consider all these factors is quite the task. However, what we propose in Part I of this work is a simpler version of this situation. We construct a *toy model* that takes into account

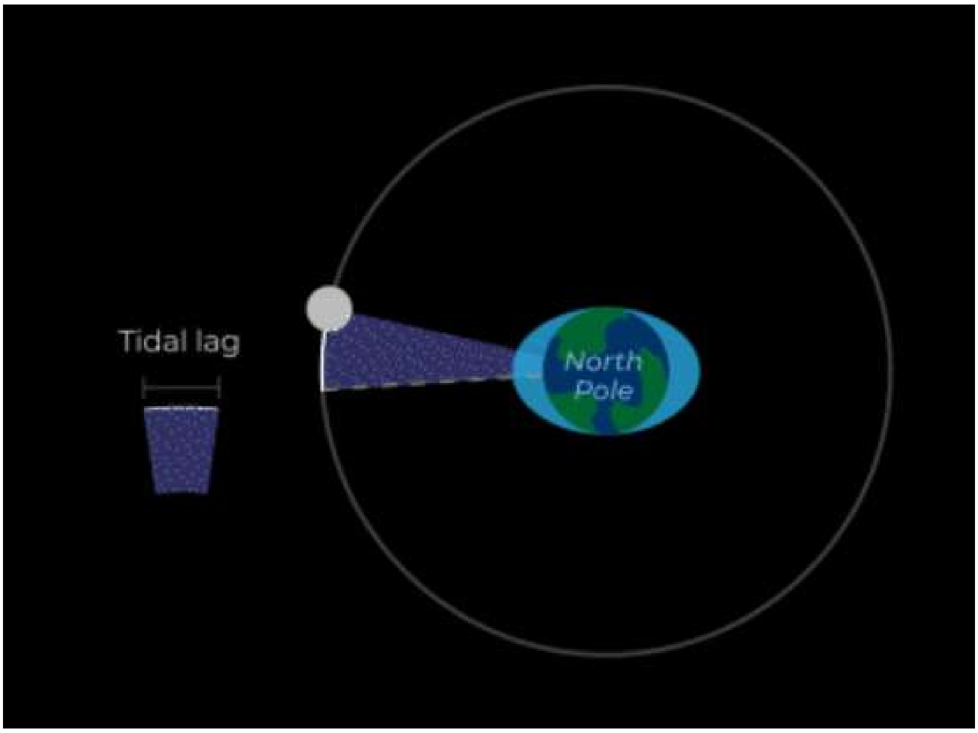


Figure 3: Picture taken from NASA website [NAS] illustrating the tidal lag felt by the atmosphere. Since the gravitational pull of the Moon takes a certain time to be felt by the atmosphere, when the former changes its position in the sky the latter takes a while to adjust to such a displacement.

1. the particle-particle interactions and,
2. the intrinsically felt motion of the atmosphere.

To do so we have to talk about the backbones of our model, which are due to James Maxwell and William Hodge.

Maxwell & Hodge: A brief historical background for theoretical modelling and particle dynamics

Throughout history and in a variety of disciplines, great advancements were maid in the direction of attaining a deeper understanding of the world around us by performing *experiments*.

Take James C. Maxwell (1831 - 1879), Scottish physicist responsible for the grand unification of electricity and magnetism, for instance. The very laws of classical electro-

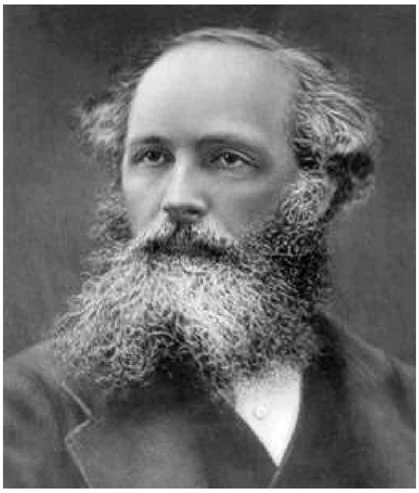


Figure 4: Picture of James Clark Maxwell, the Scottish physicist responsible for the "first grand unification", given by the junction of the electric and magnetic field into a single mathematical entity.

magnetism, which now bear his name, were only possible to be put together by means of the experiments performed by other scientists such as Michael Faraday. It would be hard to conceive of someone who would be able to simply argue by first principles that electric and magnetic fields ($\boldsymbol{E}$ and $\boldsymbol{B}$, respectively) are two sides of the same coin which behave according to

$$\begin{aligned} \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \\ \nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \quad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \end{aligned} \tag{1}$$

without ever having seen such a behavior more concretely. In a way, we can think of the above set of equations (called Maxwell's Field Equations) as a mathematical formulation coming from an *empirical model* which describes the behavior of these fields in a certain regime[1].

To be more precise in our terminology, when we talk about an ***empirical model*** we are referring to a model which steams primarily from the observation of certain physical phenomena. The collection of various data points of the same system makes its average

[1]if for example one deal with these field in some medium or fluid, the equations considered change [Alf42]

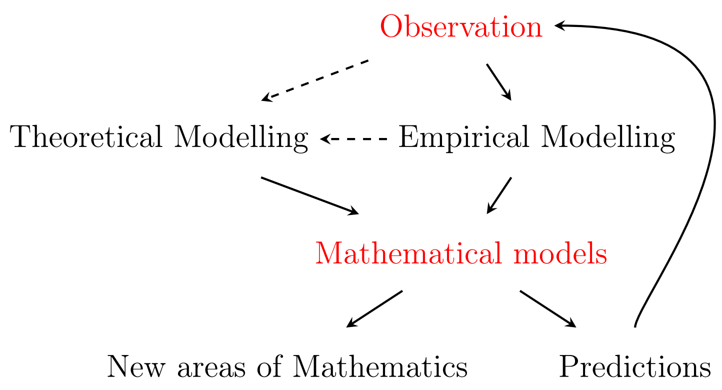


Figure 5: Diagram illustrating the process behind scientific thinking. Based on a set of observations we may come up with either an empirical or a theoretical model. These in turn generate a variety of mathematical models which in turn can be used to make predictions for the system's behavior (which are to be later check with further observations), or culminate in the birth of new mathematics.

behavior more evident, thus enabling us to conclude it's governing laws and possibly deduce its equations of motion, culminating in a more complete understanding of what goes on. When this stage is achieved, we say we have a ***mathematical model*** for our system.

In essence, an empirical model needs no prior assumptions regarding the functioning of the observed system. All that it's done are observations and assertions regarding the results found, based on which more robust (mathematical) models can be built. Such empirical models occur in a variety of disciplines, from fundamental physics to neuroscience [HH52].

In addition to this line of reasoning we have the so called ***theoretical models***. Their key difference when compared to the empirical one is in the presence of the so called *axioms* or *working hypothesis*. These working hypothesis are what make up for the theoretical part of the model and they are later tested by experiments to determine their validity range and hence overall accuracy of the theory. A model of a system which possess very few axioms and predicts a lot of "correct" behaviors for such a system (verified after experimentation) is considered robust, whereas one with many hypothesis and very few predictions is quite weak. That doesn't mean however that it should be entirely discarded since some further thinking could make it more precise and hence more descriptive of the reality we are able to access. Figure 5 shows a very rough sketch of how these different models come about and influence each other in science.

One example of a theoretical model comes from the English physicist Paul Dirac. In 1928, Dirac proposed an equation of motion for subatomic particles which contained a counter-intuitive solution representing a particle of negative energy [Dir28]. The deduction of such an equation was done solely on first principles and was based on previously established mathematical relations (which themselves had already been physically verified). It turned out that such an equation lead to the prediction of anti-particles whose experimental confirmation came some years after.

This serves to show that empirical and theoretical models go hand in hand and can actually influence each other in various ways, also culminating in the creation (or need for) new mathematics.

Moving on to our next important figure, we have the Scottish mathematician William Hodge.

Figure 6: Picture of the Scottish mathematician William V. D. Hodge, whose mathematical discoveries culminated in the field of Hodge Theory.

In the early 1930s British mathematician William Hodge, inspired by the works of George de Rham and Solomon Lefschetz, was starting to come up with what we now understand as Hodge Theory. Although his motivations lied more on the algebraic side of things, an important decomposition theorem relating the cohomology classes of manifolds and the kernel of the Laplace operator arose (see subsection A.2 for the proper definitions and examples).[2]

Theorem 0.0.1 (Hodge Decomposition Theorem). *Let* $\omega \in \Lambda^p(M)$*. Then* $\exists\, \alpha \in \Lambda^{p-1}(M)$, $\beta \in \Lambda^{p+1}(M)$ *and* $\gamma \in \mathcal{H}^p(M)$ *such that*

$$\omega = d\alpha + \delta\beta + \gamma \tag{2}$$

Where $d\gamma = \delta\gamma = 0$*, that is, upon considering the operator* $\Delta = d\delta + \delta d$*, we have that*

$$\Delta\gamma = 0 \tag{3}$$

To state it in words, the theorem says that given a differential $k-$form ω, we can always decompose it as a sum of three terms. One lying in the image of the exterior derivative d, one lying in the image of the codifferetial δ (see Eq.(A.2.21)) and one in the kernel of the Hodge-Laplacian $\Delta := d\delta + \delta d$. This decomposition generalizes the famous Helmholtz decomposition for vector fields on $\mathbb{R}^3$ to arbitrary differential forms on a Riemannian manifold (M, g) (Definition A.1.6), the latter being given by

Theorem 0.0.2 (Helmholtz-Hodge Decomposition). *Let* $\boldsymbol{V} \in \mathfrak{X}(\mathbb{R}^3)$ *be a vector field over the space manifold* $\mathbb{R}^3$*. Then, there exists* $\phi \in \mathcal{C}^\infty(\mathbb{R}^3)$, $\boldsymbol{W}, \boldsymbol{H} \in \mathfrak{X}(\mathbb{R}^3)$ *such that*

$$\boldsymbol{V} = grad(\phi) + curl(\boldsymbol{W}) + \boldsymbol{H}, \tag{4}$$

where the field $\boldsymbol{H}$ *satisfies*

$$div(\boldsymbol{H}) = curl(\boldsymbol{H}) = 0 \tag{5}$$

As discussed further in Chapter 1, the above can be used to extend the notion of gravity (or electromagnetism) as a central force. Indeed, when talking about a central

[2]The proof of this theorem apparently wasn't given by Hodge himself. In his original work a mistake had been found and posteriorly corrected by Hermann Weyl and Kunihiko Kodaira.

force vector field $\boldsymbol{V}$ on $\mathbb{R}^3$, we mean a vector field satisfying

$$\begin{aligned} \text{curl}(\boldsymbol{V}) &= 0, \\ \text{div}(\boldsymbol{V}) &= -\gamma\rho, \end{aligned} \tag{6}$$

for some constant γ and a function $\rho : \mathbb{R}^3 \to \mathbb{R}$. Under the assumption that $\boldsymbol{V}$ decays to zero at infinity, we have that

$$\boldsymbol{V} = -\text{grad}(\phi), \tag{7}$$

for some function $\phi : \mathbb{R}^3 \to \mathbb{R}$. By extending the notion of curl (Subsection A.2) we are able to think of the above result on a general Riemannian manifold (M, g). As we shall see on Chapter 2, the general result for $k-$forms can be written as

$$\omega = d\phi + \gamma, \tag{8}$$

also using some decaying assumptions on ω.

As a last comment, notice that we can rewrite Maxwell's field equations also in terms of these differential forms. By introducing the *Faraday tensor* [CRDB21]

$$F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{9}$$

seen as an element of $\Lambda^2(M)$, we have that Eq.(1) are written as

$$\begin{aligned} dF &= 0 \\ \delta F &= \mu_0 j \end{aligned} \tag{10}$$

where j is the one form associated to the current density $\boldsymbol{J}$.

Outline of Part I of the thesis

The objective of this first part is to construct and analyze a theoretical toy model that gives a first answer to **Question 1**. As the name suggests, the model does *not* encompass all the intricacies of atmospheric dynamics, and was envisioned with the aim of giving us some insight into how these oscillatory effects of the atmosphere can impact the motion of satellites.

Notice that a distinction is to be made between the objects to be modelled and the objects considered. The former are the satellite and atmosphere, while the latter are point particles and an oscillating surface. These simplifications are done with the intent of making the calculations more amenable and will be further improved in the future with the aim of making the model more realistic.

In Chapter 1 we lay out the axioms for our theoretical model, explaining more behind our reasoning and also illustrating the importance of the choice of ones axioms when constructing a theoretical model by drawing a parallel with Euclid's postulates and Newton's Laws of Motion for classical mechanics.

In Chapter 2 we lay out one instance in which Hodge Theory can be used to extend Classical mechanical results from $\mathbb{R}^3$ to any other Riemannian manifold. We also review the concept of first integral of motion and how they come about in a Hamiltonian system by considering certain symmetries of the system (illustrating this fact with two examples). Finally we defined the concept of a test particle and explain the gravitational regime we are considering a bit more formally.

In Chapter 3 compute the stress-energy tensor of the manifold considered, giving a possible physical interpretation of such a result. We also define the geometric means by which we model the lunar tides. In particular, we consider a perturbative parameter ε and a time dependent function $a(t;\varepsilon)$ that modifies the parametrization of a sphere of radius R to make it into an oblate ellipsoid, which serves as a first approximation for the atmosphere's behavior.

In Chapter 4 we perform the first calculations involving the lunar perturbation on the dynamics of a single particle living on the Earth's atmosphere drawing attention to the fact that two invariant submanifolds $\mathcal{V}_{\text{eq}}$ and $\mathcal{V}_{\text{mer}}$, plus a set of fixed points can be found. We perform a numerical study of the point-wise stability of such fixed points, restricting our analysis to the θp_θ subspace and finding numerical evidence indicating that in this subspace such fixed points are either of saddle or center type. More robust numerical and analytical calculations are left as part of a future work.

We also briefly analyze the efficacy of our integration method, namely `numba`'s LSODA, and see that for large values of ε the overall behavior of the solutions presents large errors, enabling us to find a parameter region in which the method could be more trust worthy.

On Chapter 5, we formulate the dynamical equations for the Kepler problem. We perform some simulations to see how the trajectory varies according to the parameter values, showing that we can fine tune the mass values in such a way as to generate a circular orbit. We also verify that it correctly extends the Riemannian notion we had for point particle dynamics on closed surfaces [BDS16].

Finally, on Chapter 6, we briefly outline the computational steps needed to numerically solve the system of equation for the full $2-$problem, where both masses are allowed to move under the gravitational interaction considered. Such a numerical integration is left for a future work.

Introduction for Part II of the thesis

On part II of this thesis, we will be interested in answering the following

Question 2: *Is there a way to think of the integrability of a point vortex/particle system in terms of braid theory?*

In principle an answer to this question could be given in terms of the *Yang-Baxter equation*. For a certain class of systems we need the validity of such equations to conclude the system's integrability [Lam15]. They can be thought of as a consistency condition that also appears in Braid Theory [d'A20a].

The approach we want to take here however is a bit more topological, rather than algebraic/geometric. In [BSA03], the authors made use of the Artin representation of the braid generated by the dynamics of point vortices on the cylinder to (topologically) classify the dynamics of advected particles. The real intent behind **Question 2** is thus to see if we could make use of this classification to say something about the behavior of the system in regards to its integrability.

Before outlining our end result, we first introduce the reader to a bit of the back end history concerning integrable systems, following up with that of braid theory and topological/vortex dynamics.

Solving and calculating: A brief background of Integrable Systems

Perhaps the first scientist to mathematically describe an (at the time unknown to be) integrable system was Isaac Newton. Thanks to some correspondences exchanged with his contemporary experimental physicist Robert Hooke, Newton (Figure 7) was motivated to go after the description of the motion of the objects we see in the sky. In particular, he wanted to understand what happened to a body that was subject to an inverse square law [Arn90].

Figure 7: On the left there is the picture of English polymath Isaac Newton. On the right a picture of his contemporary, the English experimental physicist Robert Hooke. The pictures gotten from their respective Wikipedia pages.

Independently from Leibniz, Newton created the mathematical tools necessary for such a description, which culminated in the invention of Calculus and hence of Mathematical Analysis. In the process, he not only laid out the postulates for classical mechanics

(Newton's 3 laws of motion), but also managed to solve the 2−body problem, showing that the mathematical results of his solutions matched the observations done by Johannes Kepler decades prior.

To be more precise, the 2−body problem in classical mechanics consists of finding and solving the equations of motion governing the movement of two massive (point) particles orbiting around each other subject to the force of gravity. When working over $\mathbb{R}^3$, We can show that such a problem can be reduced to that of single body problem moving under a suitable potential and from there get to the sought after solution. In doing so, we see that the allowable motions are conic curves (ellipses, circles, parabolas and hyperbolas), representing the different regimes of motion these two bodies may find themselves in. In this way we solve the system by completely *integrating* its equations of motion.

A natural next step (which indeed was the one taken by Newton himself) is to consider the 3 body problem under the same potential. As noted by Newton, describing the motion of this system is much more difficult than expected. One can not reduce the system to that of a two body system and then try to integrate its resulting equations like in the 2 body case. As it turns out, the 3 body problem happens to be generally *non-integrable*, which in practice means that for the majority of the initial configurations of the system, one can not mathematically predict its long time trajectory and behavior.

One of the first people to dive deeper into this strange behavior present in the 3 body problem was the French mathematical physicist Jules Henri Poincaré (Figure 8). Among his many contributions to a variety of fields in mathematics and physics, Poincaré's studies of the 3 body problem lead him to the discovery of what we now understand as a *chaotic system*. Through his development of qualitative analysis of ordinary differential equations (ODEs), he was able to better study the behavior of the solutions without ever having to explicitly find them in the first place.

Also during Poincaré's time the Irish mathematical physicist William Rowan Hamilton was developing a new way by which we could see classical mechanics. Hamilton managed

Figure 8: Pictures of Poincaré (left), Poisson (center) and Hamilton (right). Three of the most important mathematicians/physicists of the 19th to 20th century, whose studies made striking advancements in the fields of dynamical systems and classical mechanics. Their pictures were gotten from Wikipedia.

to describe a certain class of mechanical system entirely in terms of a function H defined by the sum of the kinetic terms (K) and interaction potential terms (V) that such a system had. More explicitly, given a pair of coordinates (q, p), such a function can generically be written as

$$H(q, p) = K(q, p) + V(q, p), \tag{11}$$

which is now suitably called the Hamiltonian of the system. Under the influence of the works of Siméon Denis Poisson, by making use of the *variational principle*, Hamilton got to the following equations of motion

$$\begin{aligned} \frac{dq}{dt} &= \frac{\partial H}{\partial p}, \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q} \end{aligned} \tag{12}$$

also called **Hamilton's equations of motion**, which can also be written in terms of *Poisson brackets* [Eq(2.43)].

Under this framework of Hamiltonian mechanics the French mathematician Joseph Liouville (Figure 9) was able to define the notion of an ***integrable system***. By making use of the definition of a first integral (Definition 2.3.1), Liouville concluded that a $2n$ degree of freedom Hamiltonian system was integrable by quadratures if n independent first integrals were found.

Figure 9: Picture of Joseph Liouville, one of the first mathematician to formally consider and define the notion of an integrable system. The picture was gotten from Wikipedia.

This notion, which we now refer to as **Liouville integrability**, is the central point of integrable systems. Over the centuries other definitions of integrability also appeared with various results connecting them [AHJ19a].

For instance, the Yang-Baxter equation is a consistency equation that in certain cases can be used to talk about the presence of such independent integrals of motion [Lam15]. Such an equation also appears in braid theory as part of the relations the braids ought to satisfy to generate the *braid group*, which we shall comment a bit about now.

Knots & Strands: A brief background on Braid Theory

The study of modern Braid Theory has its origins some 200 years ago with the German mathematician Carl Friedrich Gauss being one of the first people to consider their importance, as can be seen in some of his notes [Lam09] (Figure 10).

Prior to this, objects called *knots* were already being studied, with the main contributors to their understanding being the Scottish mathematical physicist Peter Tair and the Irish mathematician William Thomson (Lord Kelvin). The motivation for their study apparently was due to the belief some physicists had at the time that atoms could be

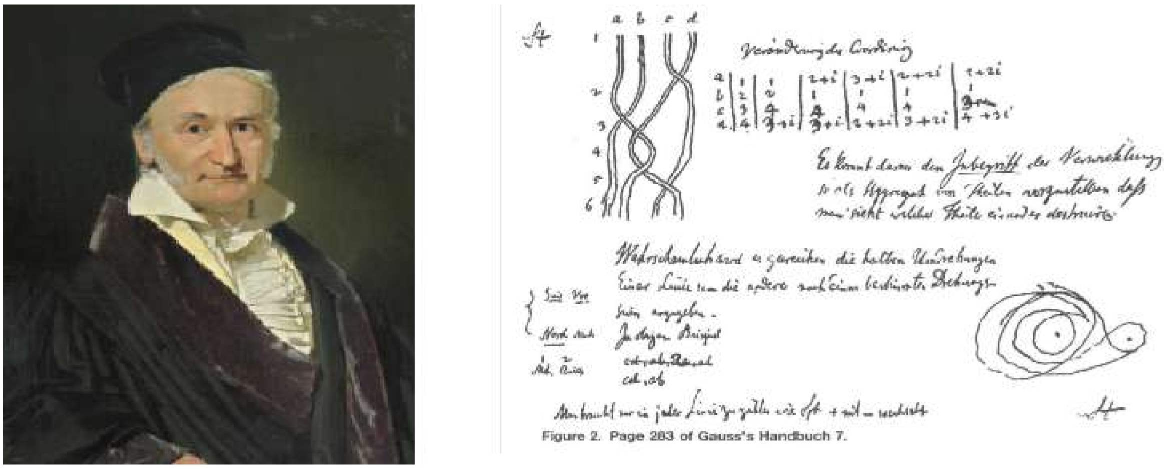


Figure 10: Picture of the German mathematician Carl Friedrich Gauss and his notes depicting the strands of a braid on the upper left corner. Pictues gotten from Wikipedia and Lambropoulou's presentation [Lam09].

modeled as curled up knots in space, and so the understanding of these knots (classified by their crossings) would yield an understanding of matter [KS04].

As we now know, this is clearly an oversimplification of what an atom should be thought of. Nevertheless, still back in those days, the American mathematician James Waddel Alexander manage to find a connection between knots and braids. In his paper [Ale23], Alexander gave a first proof of the fact that knots can be built from the closure of a braid.

The reason this is reasonable becomes clearer if one has in their mind what more specifically is a knot and a braid. A ***knot*** is simply an oriented embedding of a circle into $\mathbb{R}^3$, whereas a ***braid*** is a collection of strand that intertwine with each other in a possibly non trivial way. Each braid in composed of a collection of lines or *strands* that pass either in front or behind the other, yielding the tangled pattern illustrated on Figure 11.

By defining the closure of a braid as the junction of its respective initial and final points, we are able to close the braid in 3 dimensional space and thus obtain a knot as in Figure 12. Alexander's Theorem asserts that the converse to this construction also holds, hence, letting us say that every knot is the closure of a certain braid.

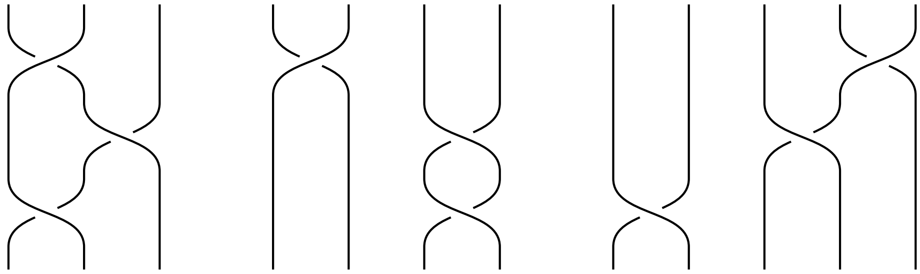


Figure 11: Example of some braids with 3, 4 and 5 strands.

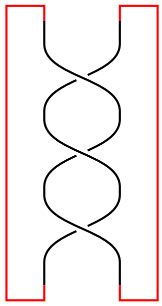


Figure 12: Figure depicting the closure of the braid (thick red lines). We joint the starting and final points of the braid as in the above fashion, thus creating a loop in $\mathbb{R}^3$ which can be seen as a knot.

The important fact about these braids is that they have a natural group structure to them. By concatenating the strands on top of each other we are able to generate new braids. Indeed, if we name the braid group on n strands as B_n, its generators, denoted σ_i, are defined by the over crossing of the ith strand by the $(i+1)$st one, as in Figure 13. We can then say that B_n is the finitely generated group given by

$$B_n = \langle \sigma_i \rangle \text{, with } \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \text{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 2 \tag{13}$$

where the constraint relations are geometrically given by Figure 14.

Among the many application of Braid Theory and the braid group B_n, one is of

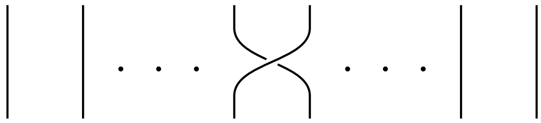
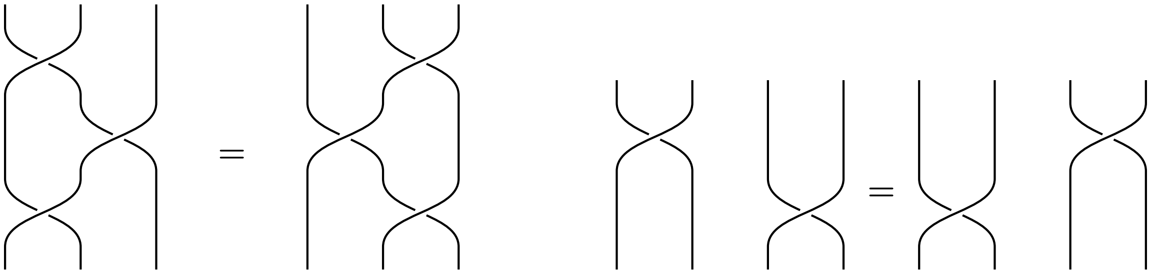


Figure 13: Generator σ_i of the braid group



(a) Relation for $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$ (b) Relation for $\sigma_i\sigma_j = \sigma_j\sigma_i$

Figure 14: Figure showing the constraint relations that are obeyed by the generators of the braid group.

interest. As done on [BSA03], we can classify the regimes of motion of a vortex dynamical system by considering the braids traced out by the vortices. By [BSA00], we can represent the braid groups is terms of matrices acting on the plane. Hence, under the usage of the Thurston-Nielsen classification, we can categorize the corresponding matrices generated by the vortex motions.

In what follow, let us briefly recall the history of vortex and topological dynamics, following up with an outline for Part II.

Topological dynamics and vortex motions

Topological dynamics is the area of mathematics concerned with various properties of dynamical systems, viewed from a general topological perspective [Aok89, GW06]. Topological dynamics has many connections with other fields of mathematics including ergodic theory, differential equations and differential topology [NH94].

Besides the classic works on topology and dynamics written by of Henri Poincaré, the American mathematician George Birkhoff also had great influence in the field of Topological Dynamics due to his proof on “Poincaré’s last geometric theorem” and his works on minimal sets [GH55, BG22].

One important concept in the theory is that of ***topological entropy***, introduced by Adler, Konheim and McAndrew in [AKM65]. It can be thought of as a measure of

Figure 15: Picture of the American mathematician George Birkhoff, whose works in part culminated in the creation of the field of Topological Dynamics.

complexity of the orbits of such a system. Zero topological entropy means that the system is simple, whereas positive topological entropy relates to the existence of some level of complexity in it. Some examples for dynamical systems with null and positive topological entropy can be found in [Moo].

The study of vortex dynamics on the other hand is an older matter. For instance, French physicist René Descartes considered that a vortex was a an object that induced a swirling motion of particles around a common center of rotation, the clearest example of this being cyclones and tornados on the atmosphere (Figure 16).

Figure 16: Satellite view of a cyclone over the Earth's surface. The middle point of the cyclone can be though of as the point vortex that generates it. The picture was gotten from Wikipedia.

The first vortex model to be developed was the ***point vortex*** one by Helmholtz. Later, the Hamiltonian formulation for such a vortex system was given by Kirchhoff. Until then, every calculation was done in the plane. However, as the need for more accurate models increased, other geometries started to be explored.

For instance, Bogomolov was one of the first to derive the equations of motion for vortices on the sphere [Bog77], meanwhile the cylindrical (or strip) case was first considered by Birkhoff and Fisher some years prior [BF59]. Other geometries such as the Hyperbolic one were explored, with general properties regarding the stability of certain vortex configurations being researched widely as well [MT13, Boa08, BS08, Kim99].

The main point that connects vortices with topological dynamics lies precisely in the equations for a *test particle*. As pointed out above, a vortex can only exists within a medium (usually thought of as a fluid) which itself is made of some material. When one considers an $N-$vortex system within this medium, its particles will move following the flow generated by the vortices. Evidently, as a simplifying though reasonable assumption, we suppose that the particles of this medium do ***not*** have an influence in the vortex motions. This is commonly referred to as the ***probe approximation***, and in it, the only thing that affects the particle's motion is the presence of other vortices.

In this way, by making use of the Hamiltonian formalism this system obeys, *assuming the trajectories of the vortices are already known* and are given by $z_j(t)$, we may write the equations of motion for an advected test particle as [BSA03]

$$\frac{dz^*}{dt} = \frac{1}{2\pi i} \sum_{1\leq j\leq n} \Gamma_j G(z, z_j(t)), \tag{14}$$

where $z = (x, y)$ is the position of the particle in complex coordinates, Γ_j is the circulation of the $j-$th vortex and $G(z, z')$ is the Green function coming from Laplace's equation.

The equation for the vortices however is found by using the assumption that *instantaneously, each vortex is a test particle in the field of the other vortices*. This reasoning then leads us to [BSA03]

$$\frac{dz_k^*}{dt} = \frac{1}{2\pi i} \sum_{j \neq k} \Gamma_j G(z_k, z_j), \tag{15}$$

in which case the positions z_j are to be found. By making use of these solutions, we are able to solve for $z(t)$ above and discover the specific path the advected particle will follow.

A more modern way to connect this behavior to Topological Dynamics, is done through the use of the *Thurston-Nielsen (TN) classification* for surface homeomorphisms. This result aims to classify the homeomorphisms of a compact orientable surface and it has connections with the mapping class group of surfaces also [MW21]. Given that the braid group also possesses such connections, we could in principle also give a topologically dynamic classification for the braids generated by the motion of these vortices (which are thought to describe the isotopy classes of the surface's homeomorphisms).

The TN classification result asserts that we can classify the motion of the advected particle (in terms of the braids formed by the real vortices) in 3 ways

TN 1 Periodic or finite order,

TN 2 Pseudo-Anosov,

TN 3 Reducible.

We will have **TN 1** when the homeomorphism composed with itself some number of times results in the identity. **TN 3** happens when the space possesses invariant curves which are fixed by the action of the homeomorphism, letting us then cut the space along these invariant curves and just analyze the behavior of the homeomorphism on their complement. Finally, **TN 2** happens when the homeomorphism has two transverse invariant foliations F^s and F^u, whose transverse measures are scaled by a factor of $1/\lambda$ and λ respectively [Thu88].

The factor of λ is called the expansion factor of the dynamics. Certain curves in the space are then stretched by this factor and can have an exponential increase in length (of order λ^n for some n). We can then use this construction to say that the *topological entropy* of the system is greater than or equal to $\log(\lambda)$ [BSA03].

Outline of Part II of the thesis

On Part II of the thesis we focused primarily on the study of Integrable systems (with an emphasis on point vortex systems) and braid theory. Our goal was to give a "topologically interesting" answer to **Question 2** and to do so we came up with an integrability notion on the basis of the braids formed by a system of interacting point particles (vortices or masses).

To be more precise the second part of the thesis is divided as follows: Chapter 7 is dedicated to a brief overview on the mathematical aspects of an integrable Hamiltonian system. We recall the Poisson bracket form of the equations of motion, and define the notion of a Liouville integrable system, showing that a 1 degree of freedom (d.o.f) Hamiltonian system is always integrable. We also outline the steps one has to take in order to construct the so called *Action-Angle* variables in an integrable Hamiltonian system, stating the famous Arnold-Jost Theorem.

On Chapter 8 we explore some basic definitions and results in Braid Theory. In particular, we mathematically define the notion of a knot and lay out three possible ways one can think of a braid, defining the Artin braid group B_n with its classical presentation in terms of the generators σ_i. We mention one important normal subgroup of B_n, namely the *pure braid group* P_n, illustrating its generators. We also mention Alexander's and Markov's theorems which give a first connection between braids and knots.

We also go over the *center* of the braid group $\mathrm{Z}(B_n)$, stating a theorem from Garside which shows that such a center subgroup is generated by the so called *full twist* element, a result which we shall use later on. At the end we talk about braid presentations in other manifolds M, constructing $P_n(M)$ and $B_n(M)$ as the fundamental classes of the collision-less configuration space generated by M (named $\tilde{\mathcal{C}}_n(M)$), and the S_n-reduced collision-less space (named $\mathcal{O}_n(M)$), respectively. We define the notion of a *topological strand* by making use of an elementary result in Algebraic Topology relating $P_n(M)$ to $\pi_1(M)$.

Finally, on Chapter 9, we start by focusing on the mathematical formulation behind the dynamics of point vortex systems, bringing to light the important difference between the *advected particle* and the *real particle* dynamics. In this way, by means of the Thurston-Nielsen classification and based on the vortex system studied by Boyland et al. [BSA03], we briefly emphasize the fact that *integrable Hamiltonian system can also present a topologically chaotic dynamic* when one studies the evolution of advected particle in the system. Then, by following the calculations done on [BSA03], we show once again that a system of 3 vortices on the cylinder is integrable, provided the net circulation is zero everywhere.

We then finish the chapter by giving various braid integrability notions for a system whose phase space dynamics is bounded. By proving at the end a theorem relating one of these notions to Liouville integrability, we henceforth guarantee a topological/braid-inspired obstruction to the former integrability notion of a point particle system, at least for bounded regimes of motion.

Part I

Relativistic framework for dynamics of point masses

Chapter 1

Theoretical Modelling

The main goal behind a mathematical model is to capture, even if in a very simple way, the characteristic behavior of the system we are studying. The prime trait of any such model is the fact that they are based on a set of *axioms* or *working hypothesis*.

Since our early years we are familiarized with the notion of *Euclidean Geometry*. Its building block is a set of 5 axioms [Hea56] with the last one, known as *Euclide's fifth postulate*, being the most famous. It reads

Postulate 5: Given a straight line and a point outside of it, there is a unique line passing through such a point that is parallel to the given one that can be drawn.

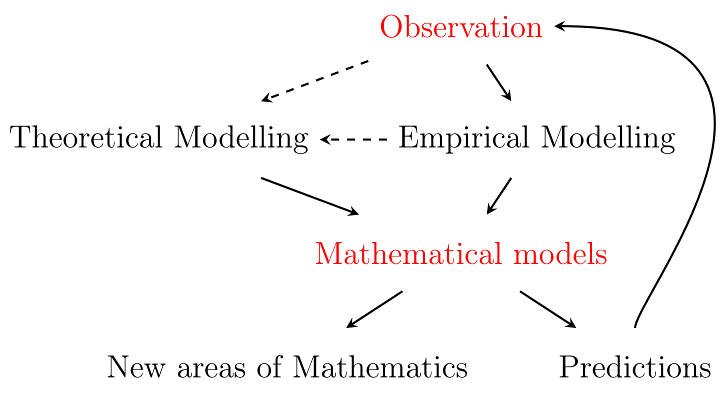


Figure 1.1: Diagram illustrating the process behind scientific thinking. Based on a set of observations we may come up with either an empirical or a theoretical model. These in turn generate a variety of mathematical models which in turn can be used to make predictions for the system's behavior (which are to be later check with further observations), or culminate in the birth of new mathematics.

(a) (b) (c)

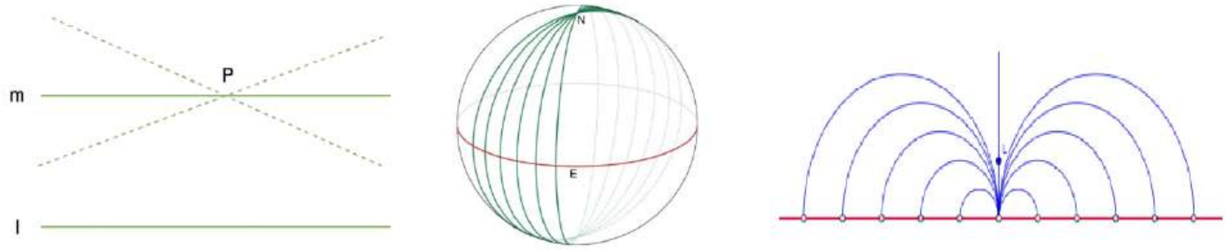


Figure 1.2: (a) Image of a line l with a point $p \notin l$ by which a line m parallel to l passes; (b) image of locally parallel lines that actually meet at some point. Given a line on the sphere, there is no other line parallel to it; (c) given a point L and a line passing through it, the are infinitely many parallel lines to the original one. In hyperbolic space, given a line and a point outside of it we can find infinitely many parallel lines to the given one which pass through that point.

For some 2000 years or so, such a postulate was subject of great debate amongst mathematician since, when compared to the other four axioms given by Euclide, his fifth one was indeed quite complicated, to the point where people would try to prove it by just using the first four. Evidently, as we know now, all attempts to prove the fifth postulate failed and around the 17th Century, mathematicians came up to terms with the fact that the fifth postulate was indeed an axiom, and hence, to prove it was a meaningless (and impossible) endeavor.

As it happens, in being an axiom one could then ask what type of geometry would we describe were the fifth postulate *not* true (Figure 1.2). As we now know, suitable modifications to the fifth postulate yield other geometry classes referred to as spherical and hyperbolic geometries, which in and of themselves also have very rich and interesting properties [Whi19, BP92].

In this same spirit, another example of physical interest are the ***axioms of mechan-***

ics, commonly referred to as *Newton's laws of motion*, (see Section 2.3) given by

1. Law of Inertia
2. $\boldsymbol{F} = m\boldsymbol{a}$
3. To every action there is an equal and opposite reaction

Focusing more on the first law, such a principle states that a free body (i.e, one which experiences no net external forces) shall remain moving with constant velocity in a uniform motion. These Laws of motion however were made to fit in with our daily experience in a locally $\mathbb{R}^3$ environment. And so, a natural question to ask is: *are these still valid in any other geometry?*

One important study case regarding the generality of these laws is the gravitational one. When working on a globally $\mathbb{R}^3$ space we are familiar with Kepler's Laws of planetary motion[1], in particular with his first law asserting that *all closed orbits of planets moving around the sun are elliptical.*

Not only this but the following assertions (As.) are all *equivalent* to each other

As.1 Kepler's Laws of planetary motion

As.2 One over distance squared law, i.e $\boldsymbol{E}_{\text{grav}} \propto \frac{1}{r^2}$

As.3 Gravity is a central force, i.e $\text{curl}(\boldsymbol{E}_{\text{grav}}) = 0$, $\text{div}(\boldsymbol{E}_{\text{grav}}) = -\gamma\rho$.

This however only holds for the special case of $\mathbb{R}^3$. Indeed, by taking the *intrinsic* approach (to be further detailed down below) to the definition of the gravitational field, one sees that on $\mathbb{R}^2$ these equivalences already fail [Sch14a].

This then presents an issue when one is considering an extension of *classical* gravity to a curved space M, namely, which point should we take as being more fundamental? Kepler's Laws? The inverse square law?

[1]**KL. 1** All closed orbits are elliptical.

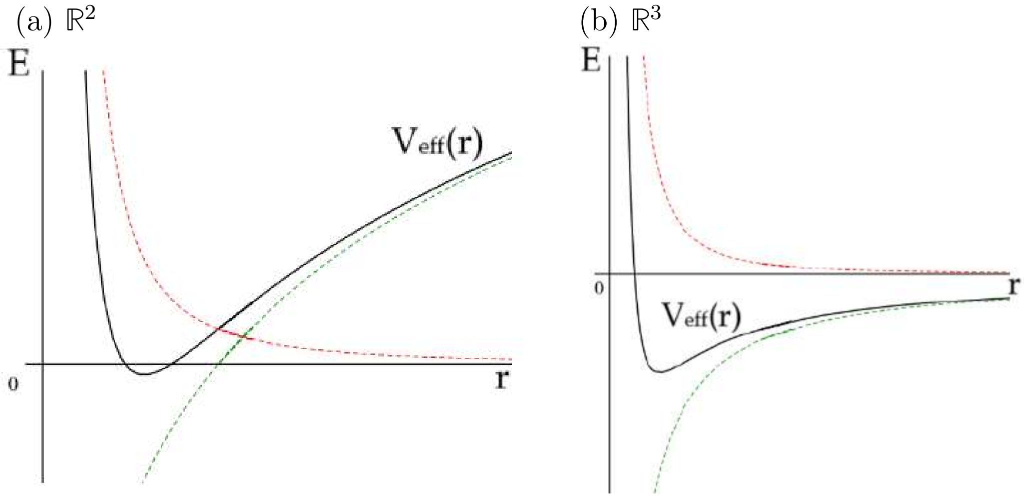


Figure 1.3: Figures taken from [Sch14b] in which it is shown that even though the orbits are all bounded, the only one which happens to be periodic is the circular one. Images (a) and (b), respectively, are depicting the effective gravitational potential on $\mathbb{R}^2$ and $\mathbb{R}^3$ gotten from solving Eq.(1.3).

In our case, the assertion we took to be our working hypothesis was that *gravity is a central force*. This means that given a mass distribution $\rho(r)$ responsible for generating a gravitational force field[2] $\boldsymbol{E}_{\text{grav}}$ on a manifold M, the following *field equations* will be satisfied

$$\operatorname{div}(\boldsymbol{E}_{\text{grav}}) = -\gamma\rho(r) \tag{1.1a}$$
$$\operatorname{curl}(\boldsymbol{E}_{\text{grav}}) = 0 \tag{1.1b}$$

for some suitable constant γ representing the strength of the interaction. An aspect to be observed about such type of equation is its “asymmetry” between the left and right hand sides of Eq.(1.1a). Indeed, the left side has to do with the acceleration field of a test particle, while the right hand side is the matter distribution of the source which generates such a field.

Under certain hypothesis on the topology/geometry of M, we may say that the grav-

KL. 2 A line that connects a planet to the Sun sweeps out equal areas in equal time intervals.
KL. 3 The square of a planet’s orbital period is proportional to the cube of the semi-major axis of its orbit.

[2]the reason for this notation now will become clearer on Chapter 5

itational field can be expressed as

$$\boldsymbol{E}_{\text{grav}} = -\nabla_M \phi, \tag{1.2}$$

where $\phi(r)$ is called the *potential function* and ∇_M is the gradient 1−form over M. Plugging this back in the first equation of Eq.(1.1) we see that the potential $\phi(r)$ satisfies *Poisson's equation*[3]

$$\Delta_M \phi = \gamma \rho(r), \tag{1.3}$$

where the operator Δ_M is just the Laplace-Beltrami or Laplace-de Rham operator (see Eq.(A.3.12)) over the manifold M. Naturally the behavior of solution to the above equation can differ according to the manifold we are working on. For instance, on the plane $\mathbb{R}^2$ all orbits will be bounded (Figure 1.3).

With this notion in our hands we see that $M = \mathbb{R}^3$ is just a particular case in which we may consider the gravitational interaction of two or more bodies and, as we vary M we shall also vary the solutions to the above equation. For instance, if we fix $\rho(r) = \delta(r)$, representing the mass distribution of a unit mass at the origin of M, we shall find the following solutions to Eq.(1.3) (see Table 1.1)

Gravitational Potentials			
M	$\mathbb{R}^3$	$\mathbb{R}^2$	$\mathbb{S}^2_p$
$\phi(r) \propto$	$-\frac{1}{r}$	$\frac{1}{2\pi}\log(r)$	$\frac{1}{2\pi}\log\left(\tan\left(\frac{r}{2}\right)\right)$

Table 1.1: List of gravitational potentials on a given geometry. $\mathbb{R}^3$ and $\mathbb{R}^2$ are the 3 and 2 dimensional Euclidean spaces, whereas $\mathbb{S}^2_p$ is the 2−sphere without a point. In all of the above r stands for the geodesic distance between each pair of bodies.

In particular, notice here that the restriction of the gravitational potential from $\mathbb{R}^3$ to $\mathbb{R}^2$ will **not** yield the same dynamics. To be more specific, the acceleration vector $\boldsymbol{a}$

[3] in the case of M being compact we need $\int_M \rho = 0$, due to Stoke's theorem.

produced by a certain particle is found by solving the equations

$$\mathrm{curl}_{\mathbb{R}^3}(\boldsymbol{a}) = 0, \tag{1.4a}$$
$$\mathrm{div}_{\mathbb{R}^3}(\boldsymbol{a}) = -\gamma\delta(r). \tag{1.4b}$$

Which, based on Eq.(1.3), is found by integrating the following expression

$$\Delta_{\mathbb{R}^3}\phi = \gamma\delta(r), \tag{1.5}$$

and then substituting its solution in the equation

$$\boldsymbol{a}(r) = -\nabla_{\mathbb{R}^3}\phi(r) = -\frac{\gamma}{r^2}\hat{\boldsymbol{r}}, \tag{1.6}$$

By now taking the $\mathbb{R}^2$ divergence and curl of the above equation (using polar coordinates), we find that

$$\mathrm{curl}_{|_{\mathbb{R}^2}}(\boldsymbol{a}_{|_{\mathbb{R}^2}}) = 0, \tag{1.7a}$$
$$\mathrm{div}_{|_{\mathbb{R}^2}}(\boldsymbol{a}_{|_{\mathbb{R}^2}}) = \frac{\gamma}{r^3}, \tag{1.7b}$$

Thus, we see that there is indeed a difference between Eq.(1.4) and Eq.(1.7), namely on the divergence of said vector field. Physically this means that, a mere restriction of a certain gravitational field to a given subspace of your configuration space will in general ***not*** yield another gravitational field obeying the same set of equations.

Such a difference also becomes evident when one looks at the restricted Laplace's equation

$$\Delta_{\mathbb{R}^2}\phi_{|_{\mathbb{R}^2}} = -\frac{1}{\gamma^2}\phi^3_{|_{\mathbb{R}^2}}, \tag{1.8}$$

which is now a *homogeneous non-linear* differential equation, contrasting the one with the Dirac delta source term.

Further notice that $\boldsymbol{a}_{|_{\mathbb{R}^2}}$ differs from the "correct", as in intrinsically found gravitational field, by a whole $1/r$ factor. Such a qualitative difference culminates in a very much different gravitational behavior, as one might expect. Indeed, as mentioned above, when $M = \mathbb{R}^2$, Kepler's laws of planetary motion are no longer valid. For $\boldsymbol{a}_{|_{\mathbb{R}^2}}$ however, they still hold but gravity ceases to be an *intrinsically defined central force*, that is, it no longer comes from some scalar potential defined by Poisson's equation.

The above serves to show that much like with Euclid's fifth postulate not all experimentally based assertions can be used as a self-consistent set of working hypothesis to build a theoretical model aiming to extend the dynamical behavior of a given system. It most importantly brings light to the power of ***choosing*** a set of axioms for a theory, given that a same problem can be extended in different ways, leading to completely different predictions and results.

1.1 Introducing our model

In our case, as described on the Introduction, the study we want to perform concerns the effect of atmospheric lunar tides on the dynamics of massive point particles moving on Earth's atmosphere. A part from answering such a question, this work also aims to extend the findings on [DB15a, BDS16], where the authors were interested in understanding how the topology and geometry of a given surface could influence the motion of a point particle.

Our working hypothesis (WH) will thus be based on the above papers, and will also have a novel ingredient to them. More specifically we have the following

WH 1: The atmosphere is a thickless shell on which the particles are going to move and hence will be modelled as a surface which for simplicity we shall take to be the $2-$sphere $\mathbb{S}^2$. We'll refer to such a surface as the *background* or *base manifold*.

WH 2: The massive point particles will ***not*** have an effect on the background manifold. Such an effect is usually called *back reaction* and, as commented earlier however, we will rely on the probe approximation for our particle dynamics.

WH 3: To model the particle-particle gravitational interaction we suppose that ***gravity is a central force***, and thus the gravitational field will be curl-free and have negative divergence proportional to the matter distribution. The field equations will thus be of the following format

$$dF = 0, \tag{1.9a}$$
$$\delta F = -\gamma j, \tag{1.9b}$$

for some real constant γ, some alternating $2-$form F and a $1-$form j. We'll call Eq.(1.9) the ***Gravitational Maxwell Equations***, since they are inspired on the field equations for the electromagnetic or Faraday tensor of Maxwell's theory of classical electrodynamics.

To further explain the rationale behind the choice of such axioms, we make the following points:

- The model itself, although concerned with the *intrinsic* interaction between the particles living on the surface also has a bit of an extrinsic catch to it. Indeed, in our case, the fact that we are working on an idealized atmosphere, its shape (and hence, overall curvature) is due to the existence of *outer* massive bodies (Earth + Moon) causing such a curvature intrinsically perceived by the particles.

  One can deduce the shape of the atmosphere in the relativistic setting by means of the geodesic deviation equation [Har21]. We thus will get inspiration from such a formalism and use some of the fundamental concepts of General Relativity to deduce the particle dynamics on the oscillating, as well as static, atmosphere.

- As discussed above, the influence of the Earth and the Moon on the point mass dynamics will be *entirely* captured by the metric tensor and a suitable perturbation thereof, to be discussed later on. The idea behind such a perturbation is the following: due to the Moon's rotation around the Earth, we naively expect a "dragging" effect of some sort to happen to the atmosphere. Such a drag should alter the gravitational potential and force, in a way as to allow for some gravito-magnetic like term [4]. That is, as the surface gets dragged along with the Moon's motion, a gravitational curl term should appear, so that the simple

$$\boldsymbol{a} = -\mathrm{grad}(\phi), \tag{1.10}$$

acceleration formula shouldn't cut it anymore. From Classical Electrodynamics and its analogy to classical Newtonian gravity, we already have in mind that the electric field $\boldsymbol{E}$ serves as an analog for the gravitational field $\boldsymbol{g}$.

The missing piece in this analogy is then the magnetic field $\boldsymbol{B}$ which comes as the curl of a suitable vector potential $\boldsymbol{A}$. As we shall see later on (Section 5.3) such a vector potential relates, via first order perturbation theory, to the induced spatial velocity of the particles when we take the Moon's influence into account.

Such an influence is measured by a perturbative parameter ε which, when set to zero, gives us the usual particle dynamics on the "unperturbed" base space $\mathbb{R} \times \mathbb{S}^2$ thus guaranteeing the model's self-consistency.

Observation 1.1 (Fixing the speed of light). As it is usually done in the relativistic setting, we also intend to make some simplifications regarding the values of c, the speed of light, avoiding the need of dragging them along the equations.

The simplification we make is to work with units in which c equals 1. Its original value is approximately given by[5]

$$c = 2.99 \times 10^8 \, \frac{m}{s}.$$

By setting $c = 1$ we are able to make the speed of light *dimensionless* upon considering

$$1\, m = \frac{1}{2.99 \times 10^8}\, s. \tag{1.11}$$

[4] for a physicist, the notion of a gravito-magnetic term is a bit different from the one mentioned here. See Appendix B.1.1

[5] we here use the S.I notation of m (meters), kg (kilograms) and s (seconds)

Such a modification however also has an impact on other constants of nature. For instance, the Gravitational constant G approximately reads

$$G = 6.67 \times 10^{-11} \frac{m^3}{\mathrm{kg}\, s^2}.$$

However, if we use $c = 1$ units instead, its value and dimension gets altered to

$$G = 6.67 \times 10^{-11} \frac{m^3}{\mathrm{kg}\, s^2} = \frac{6.67 \times 10^{-11}}{(2.99 \times 10^8)^3} \frac{s^3}{\mathrm{kg}\, s^2} \approx 2.49 \times 10^{-36} \frac{s}{\mathrm{kg}}.$$

Chapter 2

Classical Hodge Dynamics on Curved Spaces

We dedicate this chapter to the overview study of certain mathematical properties appearing in Classical Mechanics problems. Broadly speaking, we can think of Classical Mechanics as the area of physics concerned with the motion objects get into when acted upon by certain forces such as gravity, electromagnetic or even fictitious ones (for instance, when at a rotating frame). Some classical problems in the field are the gyroscope precession, falling cat and the $n-$body problem [But06, KS69, QD90].

As expected, the main interest one has when solving such problems is usually to find the so called *equations of motion*, a set of differential equations that, as the name suggests, once completely solved, revel to us how the object of study behaves under the influence of whatever force(s) we might be considering. A more mathematical approach to the subject is taken on Arnold's book [Arn89].

As a warm up for Section 2.2 we briefly overview the concepts of vectors, forms and their Hodge duals, as well as derivations of forms. A thorougher review of these topics is given in the first sections of Appendix A, which the reader is invited to check for further necessary clarifications and references.

2.1 An overview on the mathematical machinery

Starting with the usual real vector space $\mathbb{R}^n$ which we shall refer to as the *Euclidean space*. A vector will be understood as a point $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$. Given a basis $\{e_i\}$ for such a space, we can equivalently write $v = v^i e_i, v^i \in \mathbb{R}$. A *vector field* v over $\mathbb{R}^n$ will be thus nothing but a map $v : \mathbb{R}^n \to \mathbb{R}^n$, that associates to each point in $\mathbb{R}^n$, another point in this same space. The image point will thus be interpreted as a vector.

More generally though, if we are given an $n-$dimensional (real) manifold (see Definition A.1.1), a vector will ***not*** simply be a point on M. Instead, we have to consider the tangent space T_pM over M, which itself is a (real) vector space, so that now a point $v_p \in T_pM$ can be seen a vector over M, by abuse of language/notation.

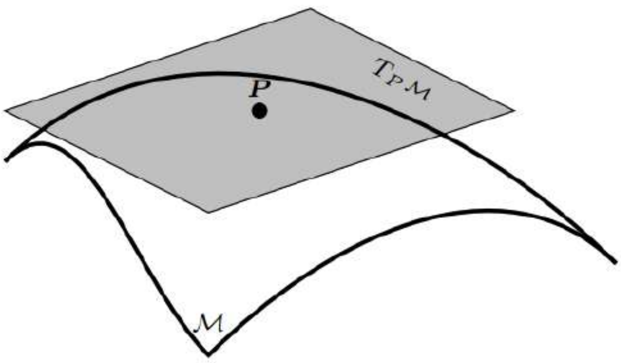


Figure 2.1: Image gotten from [FPS15] depicting the tangent space to a manifold at a point p. A tangent vector at p is an arrow connecting p to any other point in the tangent space.

Moreover, given some coordinates over M say $(x^1, \ldots, x^n)$ we can consider a basis of vectors over T_pM expressed as $\{\frac{\partial}{\partial x^i}\big|_p\}$ or $\{e_i(p)\}$, based on which we say $v_p = v^i(p)e_i(p)$, as in the Euclidean case. More generally, $\{e_i\}$ will be a basis for TM the *tangent bundle* (see Definition A.1.3) over M and so a generic vector over M will be expressed as $v = v^i e_i$[1] with $v^i \in C^\infty(M)$ (the space of infinitely differentiable functions over M).

We can further consider the following objects: take a vector $v \in TM$ and a linear

[1]throughout we use the Einstein summation convention for repeated, contracted indices.

function $f : TM \to \mathbb{R}$ that takes v to a real number $f(v)$. This function f is what we call a ***1−form***. It is uniquely determined by its values on the basis e_i since it acts linearly on TM, by definition. There is thus a very special set of 1−forms over M that obey the following condition

$$f_i(e_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{2.1}$$

The set $\{f_i\}$ is a basis for the space of 1−forms over M, denoted as $\Omega^1(TM)$ (or $\Omega^1(M)$). The more usual notation for such a basis is $\{e^i\}$ so that we can more compactly write $e^i(e_j) = \delta^i_j$, with δ^i_j being the Krönecker delta function, defined as the right-hand side of Eq.(2.1).

Much like the vectors $\{e_i\}$ generate each fibers of TM, the *co-vectors* $\{e^i\}$ generate the fibers of T^*M for each $p \in M$. The space T^*M is the *dual* to TM, and it's called the ***cotangent bundle*** over M. In this way, given some $p \in M$ and an $\alpha_p \in T^*_pM$, denoted by $\alpha_p = \alpha_i(p)e^i(p)$, for any $v_p \in T_pM$ we will have that

$$\alpha_p(v_p) = \alpha_i(p)e^i(p)(v^j(p)e_j(p)) = \alpha_j(p)v^i(p)\delta^i_j = \alpha_j(p)v^j(p).$$

Usually the point p is omitted since it makes the notation quite crowded. Other operations on forms can also be considered. For instance, we can take their symmetric or anti-symmetric product and make the resulting object act on multiple vectors. If we perform such products k times, we shall generate the so called *symmetric and alternating* ***k−forms*** respectively. The former set is denoted $S^k(TM)$ and the latter is $\Lambda^k(TM)$. Their basis are respectively given by $\{e^{i_1} \odot \cdots \odot e^{i_k}\}$ and $\{e^{i_1} \wedge \cdots \wedge e^{i_k}\}$ for *increasing* combinations of $i_1, \ldots, i_k = 1, \ldots, n$. For further details check the discussion following Example A.1.1.

Hodge star operator

Talking about forms and dual spaces, given an n dimensional Riemanian manifold (M, g) (Definition A.1.6), we can define a map $\star : \Lambda^k(TM) \to \Lambda^{n-k}(TM)$ called the *the*

Hodge star operator. Given a $k-$form $\alpha = \frac{1}{k!}\alpha_{i_1\dots i_k}e^{i_1}\wedge\dots\wedge e^{i_k}$, we define its *Hodge dual* $\star\alpha$ to be the form such that

$$\beta\wedge\star\alpha := \langle\beta,\alpha\rangle\,\mathrm{dVol}_M, \forall\beta\in\Lambda^k(TM), \tag{2.2}$$

where $\langle\beta,\alpha\rangle$ is the inner product between $k-$forms, defined to be

$$\langle\beta,\alpha\rangle := \frac{1}{k!}\alpha_{i_1\dots i_k}\beta^{i_1\dots i_k}$$

As more thoroughly deduced on Section A.2, such a dual form is given by

$$\star\alpha = \frac{\sqrt{g}\alpha^{i_1\dots i_k}\varepsilon_{i_1\dots i_k i_{k+1}\dots i_n}}{(n-k)!k!}e^{i_{k+1}}\wedge\dots\wedge e^{i_n}. \tag{2.3}$$

As a final recap, we mention the concept of *derivation* of a form. The main derivation we shall be interested in is $d:\Lambda^k(TM)\to\Lambda^{k+1}(TM)$, dubbed the *exterior derivative* [Eq.(A.2.2)] and defined to be:

$$d\alpha = \frac{1}{k!}\sum_{i_1,\dots,i_{k+1}}\partial_{[i_1}\alpha_{i_2\dots i_{k+1}]}e^{i_1}\wedge\dots\wedge e^{i_{k+1}}. \tag{2.4}$$

It can be though of as the generalization of the gradient of a function since

$$df = \partial_j f e^j = g^{ij}\partial_j f e_i, \tag{2.5}$$

which indeed is our notion of gradient.

Dual to it, we have the *codifferential operator*, $\delta:\Lambda^k(TM)\to\Lambda^{k-1}(TM)$ defined in a more intricate way by

$$\delta\alpha = (-1)^k\star^{-1}d\star\alpha,\quad \alpha\in\Lambda^k(TM). \tag{2.6}$$

As the reader might guess, $\star^{-1}\star = \star\star^{-1} = 1$ is nothing but the *inverse Hodge dual*. Below, we sum up the operations one can generalize by means of the metric, the Hodge star and the exterior derivative

Observation 2.1 (Differential operators over M). Given a Riemannian manifold (M, g), a function $f \in \Lambda^0(M)$ and a vector field $X \in \mathfrak{X}(M)$, we can first define the 1−form field $\omega_X \in \Lambda^1(M)$ by

$$\omega_X = X^\flat = g_{ij} X^j e^i, \tag{2.7}$$

based on which we have the following operators

$$\begin{aligned} \text{grad}(f) &= df, \\ \text{curl}(X) &= (\star^{-1} d\omega_X)^\sharp, \\ \text{div}(X) &= -\delta\omega_X, \end{aligned} \tag{2.8}$$

where the sharp operator is characterized by

$$\alpha^\sharp = g^{ij} \alpha_j e_j, \tag{2.9}$$

for any $\alpha \in \Lambda^k(M)$.

For more details on these quantities, check the first sections of Appendix A.

2.2 Hodge's Theorem and Classical Dynamics: A connection

In this section we briefly explore some of the intricacies one implicitly deals with when making use of the Hodge Decomposition Theorem (HDT) to define a given quantity or even solve a given problem. We bring attention to the fact that the HDT generalizes the Helmholtz decomposition for a bigger range of vector fields too, for instance, by having to consider the notion of a *harmonic vector field* to make the decomposition complete in all kinds of spaces. As mentioned in the Introduction, the Helmholtz decomposition is given by

Theorem 2.2.1 (Helmholtz-Hodge Decomposition). *Let* $\boldsymbol{V} \in \mathfrak{X}(\mathbb{R}^3)$ *be a vector field over the space manifold* $\mathbb{R}^3$*. Then, there exists* $\phi \in \mathcal{C}^\infty(\mathbb{R}^3)$, $\boldsymbol{W}, \boldsymbol{H} \in \mathfrak{X}(\mathbb{R}^3)$ *such that*

$$\boldsymbol{V} = grad(\phi) + curl(\boldsymbol{W}) + \boldsymbol{H}, \tag{2.10}$$

where the field $\boldsymbol{H}$ *satisfies*

$$div(\boldsymbol{H}) = curl(\boldsymbol{H}) = 0 \tag{2.11}$$

The Harmonic component in the Helmholtz decomposition

To start our discussion let's first consider the reverse problem of vector field decomposition[2], since it serves as a good motivation for the Helmholtz decomposition. Take a vector field $\boldsymbol{Q}$ on $\mathbb{R}^3$, a smooth function $\rho : \mathbb{R}^3 \to \mathbb{R}$ and consider the following set of equations

$$\nabla \cdot \boldsymbol{E} = -\gamma\rho, \tag{2.12a}$$
$$\nabla \times \boldsymbol{E} = \boldsymbol{Q}. \tag{2.12b}$$

Now, how do we find $\boldsymbol{E}$? One thing we can do is to make use of an *ansatz* that will help uncouple the above equations. Let's thus suppose that we can write $\boldsymbol{E}$ as the sum of three vector fields

$$\begin{gathered} \boldsymbol{E} = \boldsymbol{V}_1 + \boldsymbol{V}_2 + \boldsymbol{V}_3, \qquad \text{such that} \\ \nabla \cdot \boldsymbol{V}_1 = -\gamma\rho, \quad \nabla \cdot \boldsymbol{V}_2 = 0, \quad \nabla \cdot \boldsymbol{V}_3 = 0, \\ \nabla \times \boldsymbol{V}_1 = 0, \quad \nabla \times \boldsymbol{V}_2 = \boldsymbol{Q}, \quad \nabla \times \boldsymbol{V}_3 = 0 \end{gathered} \tag{2.13}$$

The last term, although seemingly unnecessary, can actually be non-trivial in a variety of situations. One such example is the vector field

$$\boldsymbol{V}_3 = y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}} \tag{2.14}$$

We shall see next that a *condition at infinity* (or a boundary condition in case we were dealing with a bounded region of $\mathbb{R}^3$) might eliminate such a term.

[2]that is, we give the divergence and the curl and ask for the vector field that generated it.

Moving on, by then using Eq.(2.13) we can separate the system of Eqs.(2.12) in two sets of equations

$$\nabla \cdot \boldsymbol{V}_1 = -\gamma\rho \tag{2.15a}$$
$$\nabla \times \boldsymbol{V}_2 = \boldsymbol{Q} \tag{2.15b}$$

We can solve Eqs.(2.15) by using the fact that the Laplace operator on $\mathbb{R}^3$ is found by using the following vector identity

$$\Delta \boldsymbol{V} = \nabla(\nabla \cdot \boldsymbol{V}) - \nabla \times (\nabla \times \boldsymbol{V}), \tag{2.16}$$

where notably we have that

$$\begin{aligned} \Delta \boldsymbol{V} =& \Delta V^x(x,y,z)\hat{\boldsymbol{x}} + \Delta V^y(x,y,z)\hat{\boldsymbol{y}} + \Delta V^z(x,y,z)\hat{\boldsymbol{z}}, \quad \text{with} \\ \Delta V^i(x,y,z) =& \frac{\partial^2 V^i}{\partial x^2} + \frac{\partial^2 V^i}{\partial y^2} + \frac{\partial^2 V^i}{\partial z^2} \end{aligned} \tag{2.17}$$

With the above formulae at hand we can not only rewrite Eq.(2.15) as

$$\Delta \boldsymbol{V}_1(x^i) = -\gamma\nabla\rho(x^i), \tag{2.18a}$$
$$\Delta \boldsymbol{V}_2(x^i) = -\nabla \times \boldsymbol{Q}(x^i), \tag{2.18b}$$

but also use the *Green Function*

$$\Delta G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \tag{2.19}$$

to solve for each of the components of Eq.(2.18) at once. In doing so, we are able to write $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ as

$$\begin{aligned} \boldsymbol{V}_1(\boldsymbol{x}) &= \nabla\left(-\gamma\int_{\mathbb{R}^3} G(\boldsymbol{x}, \boldsymbol{x}_0)\rho(\boldsymbol{x}_0)d\boldsymbol{x}_0\right) \\ \boldsymbol{V}_2(\boldsymbol{x}) &= \nabla \times \left(-\gamma\int_{\mathbb{R}^3} G(\boldsymbol{x}, \boldsymbol{x}_0)\boldsymbol{Q}(\boldsymbol{x}_0)d\boldsymbol{x}_0\right) \end{aligned} \tag{2.20}$$

Hence, by renaming the right hand side of the above as

$$\phi(\boldsymbol{x}) = -\gamma\int_{\mathbb{R}^3} G(\boldsymbol{x}, \boldsymbol{x}_0)\rho(\boldsymbol{x}_0)d\boldsymbol{x}_0, \;\; \boldsymbol{W}(\boldsymbol{x}) = -\gamma\int_{\mathbb{R}^3} G(\boldsymbol{x}, \boldsymbol{x}_0)\boldsymbol{Q}(\boldsymbol{x}_0)d\boldsymbol{x}_0, \tag{2.21}$$

we are able to conclude that

$$\boldsymbol{E}(\boldsymbol{x}) = \nabla\phi(\boldsymbol{x}) + \nabla \times \boldsymbol{W}(\boldsymbol{x}) + \boldsymbol{V}_3(\boldsymbol{x}) \tag{2.22}$$

which written in a more general way is precisely the form of Eq.(2.10). Now, in regards to the last term of the above, by again making use of Eq.(2.16) we can see that $\boldsymbol{V}_3(\boldsymbol{x})$ satisfies the following partial differential equation

$$\Delta \boldsymbol{V}_3 = 0, \tag{2.23}$$

which is precisely the one defining a so called ***Harmonic vector field***. As mentioned above, by requiring that $\boldsymbol{E}$ satisfy some boundary condition (or condition at infinity in our case) we can pin down the type of function $\boldsymbol{V}_3$ has to be. For instance if we ask that $\boldsymbol{E}(\boldsymbol{x}) \xrightarrow{\boldsymbol{x}\to\infty} 0$, then the only solution to Eq.(2.23) on $\mathbb{R}^3$ is the identically null one. If we let $\boldsymbol{E}(\boldsymbol{x})$ be unbounded however, we can have vector fields such as those of Eq.(2.14) as making up for the harmonic components of $\boldsymbol{E}$.

As a final remark, notice that one ought to take certain consistency conditions into account when performing the integrals of Eqs.(2.20). Indeed, notice that for those to be well defined, not only $\nabla\rho$ and $\nabla\times\boldsymbol{Q}$ have to be integrable over $\mathbb{R}^3$, but so do ρ and $\boldsymbol{Q}$.

In more precise terms, what this says is that the coordinate components of the vectors $\boldsymbol{V}_1(\boldsymbol{x})$ and $\boldsymbol{V}_2(\boldsymbol{x})$ belong to a function space denoted $H^1(\mathbb{R}^3)$ (or $W^{1,2}(\mathbb{R}^3)$)[3]. This is the space of functions $f : \mathbb{R}^3 \to \mathbb{R}$ that satisfy the following property

$$\|f\|^2_{H^1} := \int_{\mathbb{R}^3} |f|^2\, dx + \int_{\mathbb{R}^3} |\nabla f|^2\, dx < \infty. \tag{2.24}$$

Thus, in a sense we *can* have $\boldsymbol{V}_1(\boldsymbol{x})$ and $\boldsymbol{V}_2(\boldsymbol{x})$ be not simple vector fields over $\mathbb{R}^3$ but *vector distributions* instead, with the same being valid for $\boldsymbol{Q}$ and ρ, though with the latter technically being a *function distribution*.

A sufficient and necessary condition on ρ and $\boldsymbol{Q}$ however is that *they* vanish at infinity instead (which does not necessarily mean that $\boldsymbol{E}$ does so too). Hence, the utility behind the harmonic component present in $\boldsymbol{E}$'s decomposition [Eq.(2.13)] lies in its implicit description of the behavior such a vector field has at infinity (or, at a boundary). Further details on the HDT can be found on [BNPB12].

[3] see more about it on Section A.5.1

As mentioned previously, the generalization of this decomposition to forms is what constitutes the Hodge decomposition theorem. To state in words: given a differential $k-$form ω, we can always decompose it as a sum of three terms. One lying in the image of the exterior derivative d, one lying in the image of the co-differential δ and one in the kernel of the Hodge-Laplacian $\Delta := d\delta + \delta d$.

We now explore a bit of the connection between such a decomposition and Classical Mechanics in the context of classic Newtonian gravity.

Gravitational dynamics from the Hodge decomposition

As mentioned above, the HDT lets us write any $k-$form ω as a particular sum of other forms. When over a (pseudo-)Riemannian manifold (M, g), it relates to the Helmholtz decomposition via the musical isomorphisms (taking forms to vectors and vice-versa) which, together with the differential and co-differential operators lets us generalize the notions of divergence and curl.

$$X = \text{grad}(\phi) + \text{curl}(\boldsymbol{W}) + \boldsymbol{H} \xrightarrow{\flat} \omega_X = d\omega_\phi + \delta(\star^{-1}\omega_W) + \omega_H$$

The Hodge decomposition (HD) of some form $a \in \Lambda^k(M)$ is explicitly given by

$$a = -d\phi + \delta\beta + \gamma, \tag{2.25}$$

for $\phi \in \Lambda^{k-1}(TM)$, $\beta \in \Lambda^{k+1}(TM)$ and $\gamma \in \mathcal{H}^k(TM)$. Notice that the minus sign in the first gradient like term has, from a mathematical point of view, no effect on the decomposition and is in fact, from a physicists perspective, more desirable. A particular instance where this decomposition is useful is in the study of classically interacting particles.

As discussed on [BDS16], we can use the decomposition of Eq.(2.25) to talk about classical gravity on curved spaces, provided we think of it in the general setting as an attractive radial force. Say we are over a manifold M with metric g and we wish to consider a gravitational dynamics over it. Since the acceleration vector is essential in describing the evolution of the system, by taking $k = 1$ on Eq.(2.25), we have a decomposition for the acceleration 1−form $a = a_k e^k$, whose components relate to the acceleration vector by the equation $a_k = g_{kj} a^j$

The important assumption here is that over *any* manifold M *classical gravity is curl free*, so that (see Section A.3)

$$\star^{-1} \mathrm{d}a = 0. \tag{2.26}$$

Given that gravity is attractive, a "test ball" centered around a real mass density ρ on the manifold M will shrink by an amount proportional to the divergence of a. This can be modelled by demanding

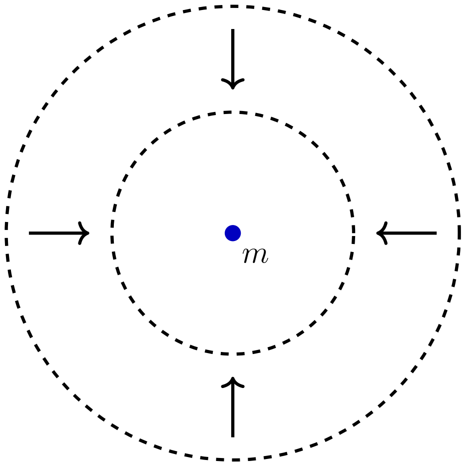


Figure 2.2: Figure depicting the divergence of the acceleration field produced by a point particle m at some point in M. A set of point particles located far from the mass gets drawn closer to it due to the gravitational pull.

$$\delta a = -\gamma \rho, \tag{2.27}$$

for some suitable constant γ. On putting these equations together, we readly see that

$$d\delta\beta = 0, \tag{2.28}$$

and so $\delta\beta$ is a ***closed*** form, i.e it's exterior derivative is identically zero. Since nothing was said about $H^1_{dR}(M)$ (M's first de Rham co-homology group) we *can not* just bluntly say that $\delta\beta$ is also ***exact***, i.e "$\delta\beta = d\alpha$". What we can say though, is that (see Eq.(A.2.8))

$$\delta\beta = d\alpha + \kappa,\ \kappa \in H^1_{dR}(M), \tag{2.29}$$

for some closed 1−form κ. This in turn changes the decomposition of Eq.(2.25) in the case of our acceleration 1−form α to the following

$$a = d(-\phi + \alpha) + \kappa + \gamma. \tag{2.30}$$

The overall test particle dynamics though remains the same. This is because from Eq.(2.29) we've got that

$$\delta d\alpha + \delta\kappa = 0,$$

and so, it also immediately follows that

$$\begin{aligned} -\gamma\rho = \delta a &= -\delta d\phi + \delta d\alpha + \delta\kappa + \delta\gamma \\ &= -\delta d\phi = -(d\delta + \delta d)\phi = -\Delta\phi \end{aligned}$$

$$\Rightarrow \Delta\phi = \gamma\rho, \tag{2.31}$$

which is precisely Poisson's equation for the gravitational field produced by a mass distribution ρ.

For the case of $M = \mathbb{R}^3$, $H^1_{dR}(\mathbb{R}^3) = 0$ and so we *do* have that $\delta\beta = d\alpha$ for some $\alpha \in \Lambda^0(\mathbb{R}^3)$. Since $d(d\alpha) = \delta(d\alpha) = 0$, we conclude that $d\alpha$ is harmonic and can be incorporated into the γ term from the decomposition. In doing so we end up with:

$$a = -d\phi + \tilde{\gamma}. \tag{2.32}$$

By Theorem A.2.3, since $H^1_{dR}(\mathbb{R}^3)$ trivial, so will be $\mathcal{H}^1(\mathbb{R}^3)$, and so $\tilde{\gamma} \equiv 0$, which yields us equation

$$a = -d\phi, \tag{2.33}$$

as expected. This first computation shows how the HDT can be used to give a natural extension of the notion of gravitational force within the Newtonian framework. We shall see on Chapter 5 how the Faraday tensor incorporating extension also yields a valid alternative for a classical gravity theory, now, in a metric varying space.

We now move on to discuss a crucial concept in physics, namely, that of integrals of motion and conserved quantities.

2.3 Symmetries and first integrals in Hamiltonian dynamics

First, let's give some contextualization behind the different formalisms used in classical mechanics. After that, we define what are first integrals (Definition 2.3.1) and on the next subsections compute some examples of those.

When dealing with classical mechanics three major formalisms come into play, namely, *Newtonian, Lagrangian* and *Hamiltonian.* Classical Newtonian mechanics, the one we mostly focused on above, bases itself on three key axioms, commonly referred to as *Newton's laws of motion.* These are

New.1 If not force is acted upon an object, it shall remain either static, or moving with uniform velocity (Newton's Law of inertia).

New.2 When acted upon by a force, an object will undergo a change of its linear momentum.

New.3 To every force that acts upon a body, a counter force of equal magnitude and opposite direction shall be provoked by the body on the actor of the former force.

Law **New.**2 yields the famous $\boldsymbol{F} = m\boldsymbol{a}$ that we all know when the mass doesn't change. Law **New.**3 is just the action-reaction principle and law **New.**1 gives us the principle of inertia (which may not be satisfied as stated above if one is on a curved surface such as the ellipsoid for instance [BDS16]).

When talking about (Classical) Lagrangian Mechanics, a major shift happens in the description of the movement of bodies. Instead of thinking about forces as the primary motors, we change our focus to talk about potentials instead. The main ingredient in Lagrangian mechanics is something called the *Lagrangian* of your system. On classical scenarios, it's generally given by

$$\mathcal{L}(\boldsymbol{q}_j, \dot{\boldsymbol{q}}_j, t) = K(\boldsymbol{q}_j, \dot{\boldsymbol{q}}_j, t) - V(\boldsymbol{q}_j, \dot{\boldsymbol{q}}_j, t), \tag{2.34}$$

with $\boldsymbol{q}_j$ and $\dot{\boldsymbol{q}}_j$ (the time derivative of $\boldsymbol{q}_j$) respectively being the *generalized position and momenta* of the $j-$th particle. The function K is the *kinetic energy* and V the *potential energy* of our system. In order to obtain the equations of motion, one has to use the *principle of least action*, which states that over the curve that represents the particle's trajectory, the functional

$$\mathcal{S}[\boldsymbol{q}, \dot{\boldsymbol{q}}] = \int_{t_0}^{t_1} \mathcal{L} dt, \tag{2.35}$$

is minimized (check [Arn89] for a derivation of this result). The equations of motion, now named the ***Euler-Lagrange equations***, are thus found to be

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_j} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_j} = 0, \tag{2.36}$$

Moving on, Hamiltonian dynamics closely relates to the Lagrangian one. As a matter of fact they're in a sense "dual" to each other, with this duality relation being given by the Legendre transform. We can define the *Hamiltonian* of our system by

$$H(q, p, t) = \sum_j q_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L} = \sum_j q_j p_j - \mathcal{L}. \tag{2.37}$$

Upon using Eq.(2.34) to talk about the form of $\mathcal{L}$, we see that H actually is nothing else but the *total energy* of the system, i.e

$$H = K + V. \tag{2.38}$$

The equations that describe the evolution of the system are now ***Hamilton's equations of motion***, given by

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}. \tag{2.39}$$

Here q_j are the *generalized positions* and p_j the *generalized momenta* of our system. To get to Hamilton's equations, we use the same reasoning as for the Lagrangian case. Indeed, the Lagrangian $\mathcal{L}$ can also be seen as the Legendre transform of the Hamiltonian H, so that after plugging it into Eq.(2.35) and varying with respect to p_i and q_i one gets to Eq.(2.39).

It can be shown that the *phase space* of our system, denoted by M, has the structure of a symplectic manifold (see Definition A.4.1) whose 2−form is given by $\omega = \sum_i dq_i \wedge dp_i$.

As briefly mentioned on Chapter 7, we can rewrite Eq.(2.39) in terms of vector fields. Indeed, given the above symplectic 2−form ω over M, we can think of the vector field $X_H = (\dot{q}_1, \ldots, \dot{q}_n, \dot{p}_1, \ldots, \dot{p}_n)$ and take notice of the following[4]

$$\iota_{X_H}\omega = \sum_i \dot{q}_i dp_i - \dot{p}_i dq_i = \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i = dH.$$

Hence, Hamilton's equations can also be written as

$$\iota_{X_H}\omega = dH. \tag{2.40}$$

On the same token, for some function $f \in \mathcal{C}^\infty(M)$ we define its *associate Hamiltonian vector field* X_f to be the one satisfying

$$\iota_{X_f}\omega = df. \tag{2.41}$$

Using Footnote 4, its not hard to see that

$$X_f = \frac{\partial f}{\partial p_i}\partial_{q_i} - \frac{\partial f}{\partial q_i}\partial_{p_i}. \tag{2.42}$$

Exploring this connection a bit more, we can define for any pair of functions $f, g \in \mathcal{C}^\infty(M)$ a bracket structure $\{\cdot, \cdot\}$ over the phase space given by

[4] we define the *inner product* of a $(k+1)$−form α with a vector (field) V to be the k−form $\iota_V\alpha$ given by

$$\iota_V\alpha(Y_1, \ldots, Y_k) = \alpha(V, Y_1, \ldots, Y_k),$$

for some set of vector fields Y_j with $j = 1, \ldots, k$.

$$\begin{aligned}\{f,g\} = \omega(X_g, X_f) &= \sum_{i,j,k} dq_i \wedge dp_i \left(\frac{\partial g}{\partial p_j}\partial_{q_j} - \frac{\partial g}{\partial q_j}\partial_{p_j}, \frac{\partial f}{\partial p_k}\partial_{q_k} - \frac{\partial f}{\partial q_k}\partial_{p_k} \right) \\ &= \sum_i -\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} + \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}.\end{aligned} \tag{2.43}$$

The bracket resulting from Eq.(2.43) is called a ***Poisson Bracket*** (see Definition A.4.2 for more properties). Note in particular that, if we compute the above bracket with a Hamiltonian H (that naturally satisfies Eq.(2.39)) we shall find that:

$$\{f,H\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} = \sum_i \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} + \frac{dq_i}{dt}\frac{\partial f}{\partial q_i} = \dot{f}(q,p). \tag{2.44}$$

And so, once our phase space M is endowed with a Poisson bracket and a Hamiltonian, we can use both of them together to talk about the time evolution of a given quantity defined over M! With this formalism in mind, we can finally consider

Definition 2.3.1 (First Integral). Given a symplectic manifold (M,ω) and a Hamiltonian $H : M \to \mathbb{R}$, we say that a function $f \in \mathcal{C}^\infty(M)$ is a ***first integral*** (with respect to H) if, and only if

$$\{f,H\} = 0. \tag{2.45}$$

In other words, the function f is constant along the integral curves of X_H.

Now for this next part, we shall retain our attention to the Hamiltonian formulation of classical mechanics. One instance of its mathematical formalization comes from Symplectic Geometry, which we chose to briefly detail on Section A.4.

Our next steps for the subsequent subsections will be to introduce the reader to the very important concept of *reduction* (based on which first integrals are found) and we leave Section 2.3.2 to show how one can compute these first integral from Nöether's Theorem (Theorem A.4.3).

2.3.1 A look on reduction

As the name suggests, the process of *reduction* has to do with diminishing the amount of some type of quantity. It turns out that the quantity to usually get reduced are the degrees of freedom a certain physical system has. Usually, reduction happens over the phase space of such system, as we shall see later on.

More specifically, the process of reduction can be seen as a first step towards integrating a given system, and the way it operates is given in the following

Theorem 2.3.1 (Reduction Procedure, [ABC+20a]). *Consider an autonomous Hamiltonian system with* $2n$ *degrees of freedom and Hamiltonian* $H(q_1, \ldots, q_n, p_1, \ldots, p_n)$*. Given* m *first integrals in involution* $\{F_j\}_{j=1\ldots m}$[5] *we can find a* ***canonical transformation***

$$(q_i, p_i) \to (\overline{q}_i, \overline{p}_i)$$

such that the Hamiltonian decomposes itself in the following fashion:

$$H = H_1(\overline{p_1}, \ldots, \overline{p}_m) + H_2(\overline{q}_{m+1}, \ldots, \overline{q}_n; \overline{p}_{m+1}, \ldots, \overline{p}_n) \tag{2.46}$$

When we talk about a canonical transformation in this context we essentially mean a transformation that doesn't change the form of the Hamiltonian equations of motion (see Definition 7.0.3). That is to say that in the new overlined coordinates, we still have that

$$\dot{\overline{q}}_i = \frac{\partial H}{\partial \overline{p}_i} \tag{2.47a}$$

$$\dot{\overline{p}}_i = -\frac{\partial H}{\partial \overline{q}_i} \tag{2.47b}$$

Based on Eq.(2.46) we see that Eq.(2.47b) is zero for the first m values of the index i. This in turn tells us that Eq.(2.47a) is constant for such values, so that the dynamics on $(q_1, \ldots, q_m, p_1, \ldots, p_m)$ is trivial. The other set of equations (those for $i = m + 1, \ldots, n$) have to be examined separately.

Most commonly, the way one performs the reduction process is by analyzing the symmetries of the system. That is, by considering which transformations can be made to it, so that the equations of motion remain unchanged. This set of symmetry transformations come up basically in two kinds

[5]check Appendix A.4 for further information on the terminology.

SYM.1 Symmetries of the Hamiltonian,

SYM.2 Symmetries of the equations of motion.

By **SYM.1** we mean phase space transformations that fix the Hamiltonian. So if we work over a Symplectic manifold (Definition A.4.1) M for instance, we would be interested in an (or a set of) invertible function(s) $f : M \to M$ such that

$$H(f(x)) = H(x), \; \forall x \in M. \tag{2.48}$$

These would in turn form a group G (see Definition C.1.1) with which we could perform a natural group action

$$\begin{aligned} \psi : G \times M &\to M \\ (f, x) &\mapsto f(x). \end{aligned}$$

As further explained on Sections A.4, the momentum map of such an action is an integral of motion of the system. This means in particular that "spurious" degrees of freedom exist, and can be dealt with be means of this map. On getting rid of them, we say we have perform a ***reduction*** of our system.

As for **SYM.2** on the other hand, imagine you are given a set of equations $\mathcal{S}$ that describe the time evolution of the particle system. It turns out that you might be able to perform some type of manipulation to your variables in a way that this set $\mathcal{S}$ doesn't change (whilst the Hamiltonian needn't stay the same now!).

Such an example would be the **Galilean group** in Newtonian Mechanics [Arn89]. On $\mathbb{R}^4$ this group is given be the set of transformations

$$(x, t) \to (x + tv, t), \tag{2.49a}$$
$$(x, t) \to (x + v, t + s), \tag{2.49b}$$
$$(x, t) \to (Rx, t), \tag{2.49c}$$

with $v \in \mathbb{R}^3, s \in \mathbb{R}$ and $R \in O(3)$. Physically, these are the possible transformations one could do to get from a stationary frame to another. It is possible to find instances where, under the Hamiltonian given by Eq.(2.38), a transformation like that of Eq.(2.49a) keeps the equations of motion unchanged[6] [BDS16], meanwhile the Hamiltonian changes.

It's not usually possible though to find integrals of motion from such a type of transformations, for they ***do not*** leave the Hamiltonian of the system unchanged. Hence, no sensible action and momentum map are to be found. Nonetheless, they do serve to perform coordinate transformations that may yield a simpler form for the Hamiltonian.

We now move on to some examples where reduction applies, also focusing on the particular and well studied case of the two body problem on $\mathbb{R}^3$.

2.3.2 Applications and Examples

Example 2.3.1 (*Masses on the disk*)**.** Consider a system point masses m_i, $i = 1, \dots, n$ sitting on the disk $\mathbb{D} \subset \mathbb{R}^2$, satisfying $\sum\limits_i m_i = 0$.

Due to the shape of such a region, the inner masses will naturally interact with its boundary. We can use the *method of images* to take this interaction into account, and consider the interaction potential provided by the *Circle Theorem* [MT68]. For one particle of mass m at position z_0, the potential is given by

$$\Phi = \frac{1}{2\pi} m \ln(z - z_0), \tag{2.50}$$

where we use complex coordinate $z = x + iy$ to describe the particle's position. The "effective" potential, representing the interaction with the boundary (obtained by the Circle Theorem) is in turn

$$\Psi = m \ln(z - z_0) + m \ln\left(\frac{a^2}{z} - \overline{z_0}\right). \tag{2.51}$$

The above can be rewritten as

[6] this invariance is more easily seen to manifest if one goes to the second order equations of motion, in this case, given by

$$\ddot{q}_j = -\frac{1}{m}\frac{\partial U}{\partial q_j},$$

upon supposing $K = \frac{1}{2m}\sum\limits_j p_j^2$

$$\Psi = m\ln(z-z_0) + m\ln\left(z - \frac{a^2}{\overline{z_0}}\right) - m\ln(z) + m\ln(\overline{z_0}), \tag{2.52}$$

the physical meaning of which is this: the first term represents the potential of a positive mass m sitting on z_0. The second one is also the potential of a mass m but which now sits *outside* the bounded region at $a^2/\overline{z_0}$. The Third term is the potential of a **negative** mass $-m$ at the origin, while the last one can be seen as a constant given its independence from the z variable.

Naturally the second mass m sitting outside the region and the negative mass $-m$ on its center are not *"real"* in the sense that, they appear as a manifestation of the interaction of the original mass m with the boundary at $r = a$.

Despite that, this conclusion is quite physically interesting. Indeed, by the fact that opposites sign masses repel and same sign ones attract, we tend to see an overall movement of the interior masses towards the boundary, very similar to the large scale expansion of the universe [Hub29].

To find the interaction Hamiltonian we use Eq.(2.38) together with the coordinate transformation $x_i = \sqrt{2I_i}\cos(\theta_i), y_i = \sqrt{2I_i}\sin(\theta_i)$ to obtain

$$\begin{aligned} H =& \frac{1}{2}\sum_{1\leq\alpha\leq n}\frac{1}{m_\alpha}\left(2I_\alpha p_{I,\alpha}^2 + \frac{p_{\theta,\alpha}^2}{2I_\alpha}\right) \\ &+ \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\alpha<\beta<n} m_\alpha m_\beta\big(\ln(I_\alpha + I_\beta - 2\sqrt{I_\alpha I_\beta}cos(\theta_\alpha - \theta_\beta))\big) \\ &+ \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} m_\alpha m_\beta \ln\left(I_\alpha I_\beta - 2a^2\sqrt{I_\alpha I_\beta}cos(\theta_\alpha - \theta_\beta) + \frac{a^4}{4}\right), \end{aligned} \tag{2.53}$$

with (x_i, y_i) referring to the position of the $i-$th particle

Let's now find, by means of Nöether's Theorem (see Theorem A.4.3), what the conserved quantities for this system are. To start, note that our configuration space $\mathcal{Q} = \mathbb{D}$ and the phase space $T^*\mathcal{Q} := M = \mathbb{D}\times\mathbb{R}^2$. The configuration space admits transformations that leave it invariant, namely, rotations around the origin. The set of such rotations actually forms a group under matrix multiplication, so that we can perform a group action

(Section A.4) on the configuration space that trivially extends to the phase space. This group is none other than the *special orthogonal group of* 2×2 *matrices* denoted as $SO(2)$[7].

We will chose to denote this action by $SO(2) \circlearrowright M$ and it shall be defined in the following fashion. Let $(q, p) \in M$ and $g \in SO(2)$. Given that the action on $\mathbb{D}$ is simple matrix multiplication, by Proposition A.4.1 together with the property $g^{-1} = g^T, \forall g \in SO(2)$, it follows that the above action is given by

$$(q, p) \mapsto (gq, gp) \tag{2.54}$$

based on which we can find the infinitesimal generator [Eq.(A.4.6)] $\mathfrak{u}$ of this action as being

$$\mathfrak{u} = a(q_1 \partial_{q_2} - q_2 \partial_{q_1}) + a(p_1 dq_1 - p_2 dq_2), \tag{2.55}$$

for some $u \in \mathfrak{so}(2) = \text{Lie}(SO(2))$ given by:

$$u = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \tag{2.56}$$

Moreover, by Proposition A.4.2, we have that the action is Hamiltonian and thus admits a co-momentum map $\mu^u = \iota_{\mathfrak{u}} \theta$, with θ being the canonical $1-$form over $M = T^*Q$. This leads us to the following

$$\begin{aligned} \mu^u(q, p) &= \theta_{(q,p)}(\mathfrak{u}) \\ &= \langle p, \pi_* \mathfrak{u} \rangle = \langle p, a(q_1 \partial_{q_2} - q_2 \partial_{q_1}) \rangle \\ &= a q_1 \langle p, \partial_{q_2} \rangle - a q_2 \langle p, \partial_{q_1} \rangle \\ &= a(q_1 p_2 - q_2 p_1) \end{aligned}$$

So that, by your usual vector multiplication on 2 dimensions, it follows that

$$\mu^u(q, p) = a q \times p \tag{2.57}$$

[7]check Appendix C for more details on this and other matrix groups.

So that, our momentum map $\mu = q \times p$, implying that *angular momentum* is conserved in the orbits of H. Note also that the Hamiltonian is invariant under $SO(2)$ action. This is because schematically we can written it as

$$H = \sum_i \frac{|p_{z_i}|^2}{2m_i} + \sum_{i<j} \Phi(z_i - z_j) + \sum_{i,j} \overline{\Phi} \left(\frac{a^2}{z_i} - \overline{z_j} \right), \tag{2.58}$$

meanwhile the $SO(2)$ action in complex coordinates is simply multiplication by a complex exponential, i.e $\tilde{z}_i = z_i e^{i\alpha}$ and $\tilde{p}_{z_i} = p_{z_i} e^{-i\alpha}$ for some $\alpha \in [0, 2\pi)$[8].

The kinetic energy is clearly unaltered since it only deals with the absolute value of the momenta. For the second term above, we see that it deals with the difference of two z_j's, so that a common $e^{i\alpha}$ factor can be collected. By a similar reasoning a $e^{-i\alpha}$ term can be collected from the third sum above. Given that the Φ's are actually complex logarithms, we see a contribution of $i\alpha$ coming from the second, and $-i\alpha$ coming from the third terms above. These in turn cancel, and thus we have the $SO(2)-$invariance of the Hamiltonian, as expected due to the rotational symmetry of our configuration space.

Again by Nöether's theorem, we have that μ^u, in this case the angular momentum, is a conserved quantity of motion, and hence it can be used to reduce the degrees of our system by 1 (for, it only has one component). Moreover, since the Hamiltonian is time independent, it is itself an integral of motion of our system.

The matter of integrability here is quite intricate however. By the zero net mass condition we imposed at the start, we cannot consider the motion of a single mass. As a matter of fact, we'd need to consider a system of *at least* 4 masses (counting here both real and imaginary) that interacting with each other.

If we get rid of this restriction, another term involving the total mass of the system will appear in the Hamiltonian of Eq.(2.53). This extra term will take into account the interaction of the real masses m_i with some negative image masses at the origin.

[8]This is the same as a canonical $U(1)$ action on a phase space with symplectic form $\omega = \frac{1}{2} dz \wedge dp_z + \frac{1}{2} d\overline{z} \wedge d\overline{p}_z$!

Hence, in this more generic case, even if we were to start with a single mass m at some point of $\mathbb{D} \setminus \{(0,0)\}$, we would already have to solve a sort gravitational $3-$body problem with logarithmic potential, a study that could very well produce an entire thesis on its own[9].

Another very well studied classical dynamical system is the $2-$body problem on $\mathbb{R}^3$, which we chose to be our next example.

Example 2.3.2 ($3-$*dimensional* $2-$*body problem*)**.** The (gravitational) two body problem in $\mathbb{R}^3$ is quite straight forward. You consider two masses m_1 and m_2 on $\mathbb{R}^3$, that interact with each other under the influence of the following potential

$$V(\boldsymbol{r}, \boldsymbol{r_0}) = -\frac{Gm_1m_2}{\|\boldsymbol{r} - \boldsymbol{r}_0\|} \tag{2.59}$$

where G is the gravitational constant and $\boldsymbol{r} = (r_x, r_y, r_z) \in \mathbb{R}^3$, with usual cartesian coordinates. The Hamiltonian for such an interaction is simply given by:

$$H = \sum_{i=1,2} \frac{\|p_i\|^2}{2m_i} - \frac{Gm_1m_2}{\|\boldsymbol{r}_2 - \boldsymbol{r}_1\|}. \tag{2.60}$$

Let's now try to reduce the system by finding its integrals of motion. The first thing to notice is that H exhibits *translational invariance*. That is, given a $\rho \in \mathbb{R}^3$, we see that:

$$H(\boldsymbol{r}_1 + \rho, \boldsymbol{r}_2 + \rho, \boldsymbol{p}_1, \boldsymbol{p}_2) = H(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2).$$

The above can be seen as a group action $G \circlearrowright \mathbb{R}^6 \times \mathbb{R}^6$ with $G = \mathbb{R}^3$ being an additive group (i.e, the group operation is addition). By naming $\mathfrak{g} = \mathrm{Lie}(G)$ (the Lie algebra of G), it follows that for some $\boldsymbol{u} \in \mathfrak{g} \simeq \mathbb{R}^3$, we have that

$$\mathfrak{u} = \left.\frac{d}{dt}\right|_{t=0} (\boldsymbol{r}_1 + e^{t\boldsymbol{u}}, \boldsymbol{r}_2 + e^{t\boldsymbol{u}}, \boldsymbol{p}_1, \boldsymbol{p}_2) = (\boldsymbol{u}, \boldsymbol{u}, 0, 0).$$

A slight abuse of notation lets us schematically write the symplectic form of $\mathbb{R}^6 \times \mathbb{R}^6$ as

[9] A particular instance of the $3-$body problem with logarithmic potential was studied in [MPS24]

$$\omega = d\boldsymbol{r}_1 \wedge d\boldsymbol{p}_1 + d\boldsymbol{r}_2 \wedge d\boldsymbol{p}_2, \tag{2.61}$$

where here we should actually be performing a sum over the components of the $\boldsymbol{r}_i$ and $\boldsymbol{p}_i$. The momentum map is found by the contraction

$$\iota_{\mathrm{u}}\omega = \boldsymbol{u} \cdot d\boldsymbol{p}_1 + \boldsymbol{u} \cdot d\boldsymbol{p}_2 = \boldsymbol{u} \cdot d(\boldsymbol{p}_1 + \boldsymbol{p}_2) = d(\boldsymbol{u} \cdot \boldsymbol{P}_{tot}).$$

So that the momentum map, here our first integral of motion, is nothing but the *total linear momentum* $\boldsymbol{P}_{tot}$. Given that we are dealing with a mass dynamics problem, physically we already have a notion for the momenta of our particles, namely

$$\boldsymbol{p}_i = m_i \dot{\boldsymbol{r}}_i, \tag{2.62}$$

with the over dot representing time derivative. From here we see that the total linear momentum is given by

$$\boldsymbol{P}_{tot} = m_1 \dot{\boldsymbol{r}}_1 + m_2 \dot{\boldsymbol{r}}_2. \tag{2.63}$$

We would thus like to define quantities $\boldsymbol{R}$ and M such that $\boldsymbol{P}_{tot} = M\dot{\boldsymbol{R}}$. The physically natural thing to do is thus to consider

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{M}, \quad M = m_1 + m_2, \tag{2.64}$$

called the *center of mass* of our system. This starts the reduction process. To keep it going, note that by the looks of Eq.(2.60), we see that it might be of our interest to work with the difference vector

$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1, \tag{2.65}$$

whose norm we simply denote by r, instead of both position vectors for each particle. Therefore, Eqs.(2.64) and (2.65) together give us a coordinate transformation

$$\begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{R} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ \frac{m_1}{M} & \frac{m_2}{M} \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{pmatrix}, \tag{2.66}$$

that can be made canonical by considering the inverse transpose of the above 2×2 matrix (see Proposition A.4.1), leaving us with

$$\begin{pmatrix} \boldsymbol{p}_r \\ \boldsymbol{p}_R \end{pmatrix} = \begin{pmatrix} -\frac{m_2}{M} & \frac{m_1}{M} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \end{pmatrix}. \tag{2.67}$$

By inverting the above, we can rewrite the Hamiltonian H to obtain the following

$$H = \frac{p_R^2}{2M} + \frac{p_r^2}{2\mu} - \frac{Gm_1m_2}{r}, \quad \mu = \frac{m_1m_2}{m_1+m_2} \tag{2.68}$$

where μ is called the ***reduced mass*** of our system, p_R and p_r are the norms of their respective vectors. Here we already see that the Hamiltonian really looks like the one from Eq.(2.46) with H_1 being the kinetic term coming from the center of mass and H_2 the Hamiltonian coming from the relative position of the particles.

As we commented after Theorem 2.3.1, the dynamics of the system is now entirely described by the Hamiltonian H_2, given by

$$H_2(r, p_r) = \frac{p_r^2}{2\mu} - \frac{Gm_1m_2}{r}. \tag{2.69}$$

Now, take a moment to appreciate the beauty of what just happened here. We started with a system leaving on 12 dimensional manifold $\mathbb{R}^6 \times \mathbb{R}^6$, described by the coordinates $(r_1^x, r_1^y, r_1^z, r_2^x, r_2^y, r_2^z, p_1^x, p_1^y, p_1^z, p_2^x, p_2^y, p_2^z)$. Due to the spacial translation invariance of its Hamiltonian, we managed to pin it down to a system leaving on a 6 dimensional manifold $(\mathbb{R}^3 \times \mathbb{R}^3)$[10] that still describes the evolution of our particles, though now with the coordinates $(r^x, r^y, r^z, p_r^x, p_r^y, p_r^z)$. As a matter of fact, the equations of motion now read

$$\dot{r}_k = \frac{\partial H_2}{\partial p_{r,k}}, \quad \dot{p}_{r,k} = -\frac{\partial H_2}{\partial r_k} \tag{2.70}$$

with $\boldsymbol{R}(t) = \boldsymbol{p}_{R,0}t + \boldsymbol{R}_0$, where $\boldsymbol{p}_{R,0}$ is a constant vector.

Getting back on track, our second first integral can be found by noticing that H_2 is invariant under $SO(3)$ group action. This means by Nöether's theorem that there exists

[10]this is because the original Hamiltonian was totally decoupled into two parts. One related to the conserved quantity, and the other related to the actual dynamics

a conserved quantity along the solution curves to Hamilton's equations of motion. The specific derivation of such a conserved quantity is a bit more intricate when compared to the $SO(2)$ case. Such computations were done on Example A.4.1 and, their take away is that the conserved quantity happens to be yet again the angular momentum (taken with respect to the pair $(\boldsymbol{r}, \boldsymbol{p}_r)$) given by

$$L = \boldsymbol{r} \times \boldsymbol{p}_r, \tag{2.71}$$

where, much like with center of mass case, here we see that $\boldsymbol{p} = \mu\dot{\boldsymbol{r}}$. The third and most obvious integral of motion is simply H_2 itself. Given that it is time independent, and it's the Hamiltonian describing the time evolution of our system, its own time evolution with respect to the canonical form coming from the coordinates $(\boldsymbol{r}, \boldsymbol{p}_r)$ is trivial. Indeed

$$\begin{aligned} \frac{dH_2}{dt} &= \sum_k \left(\frac{\partial H_2}{\partial r_k}\dot{r}_k + \frac{\partial H_2}{\partial p_{r,k}}\dot{p}_{r,k} \right) + \frac{\partial H_2}{\partial t} \\ &= \sum_k \left(\frac{\partial H_2}{\partial r_k}\frac{\partial H_2}{\partial p_{r,k}} - \frac{\partial H_2}{\partial p_{r,k}}\frac{\partial H_2}{\partial r_k} \right) = 0 \end{aligned}$$

Last but not least, there is yet another conserved quantity here called the *Laplace-Runge-Lenz vector* (LRL vector), defined to be

$$\boldsymbol{A} = \boldsymbol{p}_r \times \boldsymbol{L} - G\mu\frac{\hat{\boldsymbol{r}}}{r}. \tag{2.72}$$

The LRL vector can also be found by considering a clever group action, in this case, over the phase space $\mathbb{R}^4 \times \mathbb{R}^4$ and then projecting such an action in a certain way. For a thorough derivation of this result, the reader is invited to check [IKMS23].

We are now in position to talk more about a very important concept in physics that we were already implicitly used until now, namely, that of a *test particle*.

Chapter 3

Test particle dynamics

The idea of a *test particle* is very commonly and widely used in the treatment of physical problems. For instance, when considering passive traces in Fluid mechanics or idealized masses and charges in gravitational and electrodynamics, respectively. The actual distinction between *test* and *real* particles comes from the notion of a ***source term***, given by the following

Definition 3.0.1 (*Source*)**.** Consider a certain physical system $\mathcal{S}$. A **source** (or source term) will be a set of physical objects responsible for initiating, and possibly making persist, the dynamical evolution of the system $\mathcal{S}$. Evolution which is itself described by a suitable set of differential equations, named the *equations of motion* of $\mathcal{S}$.

Usually, the sources of a physical system $\mathcal{S}$ will be its own components. For instance, when we consider two masses that gravitationally interact with each other (as in Example 2.3.2), the sources for this dynamics are the masses themselves and the equations of motion are the ones originated from the Hamiltonian describing the system.

Though it may not seem like it, the concept of a test particle is even more fundamental then that of a source. That is because when we consider multi-particle interactions, the way we build this system's interaction Hamiltonian is by making the assumption that *each particle behaves as a test particle in the field of the other particles.* To clarify what we mean by this statement, let's go to

Definition 3.0.2 (*Test Particle*)**.** Given a physical system $\mathcal{S}$, a ***test particle*** $\mathfrak{p}$, will be a particle that moves under the influence of the system $\mathcal{S}$ as a source, but which has no influence in the dynamics of $\mathcal{S}$.

That is to say, a test particle is simply one which moves according to the *same* governing equations of $\mathcal{S}$ but which does ***not*** influence the system $\mathcal{S}$ in its motion. In physics, one usually refers to this as ***probe approximation***.

As an example, say we have a system $\mathcal{S}$ composed of a single mass m living in $\mathbb{R}^3$. Its position, given by $r_0(t)$, is a function of time which, implicitly, is assumed to be known a priori. The gravitational potential generated by m can be found by solving Poisson's equation

$$\Delta\phi(r, r_0) = m\delta(r - r_0). \tag{3.1}$$

And here is where the idea of test particle comes in. The potential $\phi(r, r_0)$ we find by solving Eq.(3.1) is the one we'll put into Newton's second Law (**New.**2)

$$\frac{d^2x^j}{dt^2} = -\nabla^j\phi, \quad j = 1, 2, 3. \tag{3.2}$$

The physical meaning of the above being: *what will be the motion of a particle subject to m's gravitational potential?* Since the initial assumption of $r_0(t)$ as being given was made, we are also implicitly assuming here that said particle has ***no*** effect on the motion of the mass m. i.e we are assuming that the particle is affected by m as a source *and* has no influence over it, which is precisely what a test particle should be according to the above definition.

The above example however is quite specific for the case of classical gravitation. Based on Definition 3.0.2 alone, we could have a variety of other types of test particles, whose underlying notion will of course be the same, but whose modelling might vary. In particular, the type of test particles we shall focus on throughout Part I of the thesis will be a ***relativistic test particles***.

To properly understand it, we need to have in mind that in the framework of General Relativity (GR), *gravity* is to be thought of as the *curvature of a space-time manifold* (see Definition B.0.1).

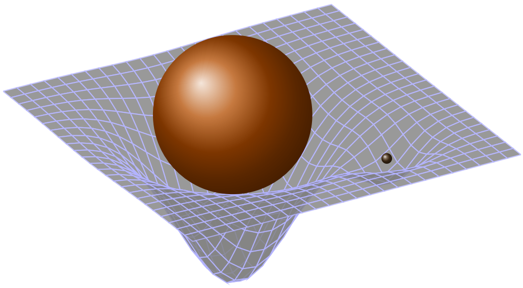


Figure 3.1: Visual representation of a spherically symmetric mass curving the fabric of space-time. A small enough mass could be seen as a fully relativistic test particle (see text below for a proper definition), since its effects on the curvature are negligible compared to those caused by the bigger mass. This in turn means that said particle moves along this new curved space-time's geodesics, whose metric is found by solving the vacuum Einstein field equations.

This in turn means that any *"real"* massive particles we consider in this gravitation framework will have an effect on the local (and potentially global) geometry of the space they sit in.

We see in this context that our physical system $\mathcal{S}$ is no longer the particle by itself, *but* it is now composed of the pair (particle, space-time) instead. This is because in GR, space-time is a dynamical quantity that functions as a background over which the matter fields get to sit on. Indeed, the object that is responsible for describing its evolution is the metric tensor $g_{\mu\nu}$, which evolves according to the Einstein field equations [Eq.(B.1.1)].

On the same token as before, we may be interested in knowing how a particle will move under the gravitational potential generated by a relativistic mass m. In this case, m will curve space-time and produce some non-trivial metric $g_{\mu\nu}$.

According to Proposition B.1.1, particles that undergo no other forces besides the gravitational one will move in geodesics. And here again we are dealing with yet another kind of test particle! This is because we are once again assuming that such a particle will

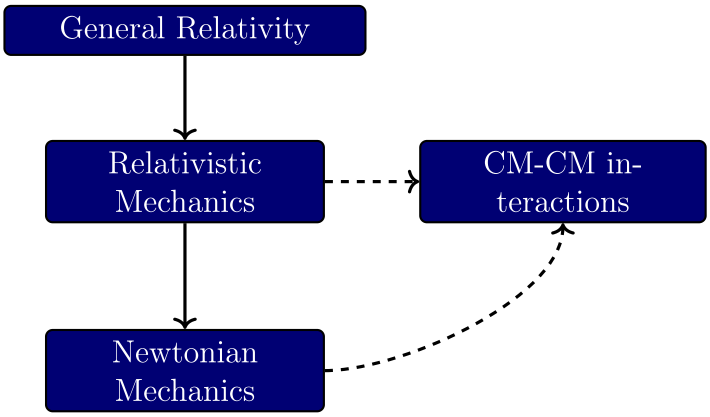


Figure 3.2: A rough sketch of the different interaction regimes one can consider in physics. As described in the text below, Classical matter interactions happening over a pre-established background space(-time) are a special limit of GR. In these cases the matter does **not** have an influence over the background's metric and hence, such a regime can be though of as a generalization of Classical (Newtonian) Mechanics to curved spaces. Relativistic regimes are in turn another possible extension of Newtonian mechanics, in which the laws of special relativity apply (see the end of Appendix B.1).

move in the distorted space-time generated by m, *and* it will *not* generate any further distortions whatsoever.

It is then important to be aware of the subtleties we encounter when working with gravity in other types of spaces, for different formalisms may apply. On one hand we have the ***fully relativistic approach***, which incorporates GR as the theory for gravitation and builds itself from there.

On the other, we have the approach we take in this thesis which we will call the ***relativistic approach***, where *special relativity* is used to establish some dynamical properties of the particles[1] and also where GR is used not as a theory for gravitation but as a *"sub-framework"*, used to define other relativistic quantities such as the energy-density tensor (Sections 3.2 & 3.3).

This reasoning makes sense because physics happens in "levels" or *regimes* one might say, in such a way that we could for instance consider classical matter (CM-CM) interac-

[1] by distinguishing between proper and coordinate times, considering the time-like condition for the tangent vector to the world line of the particles

tions happening over a fixed background (Figure 3.2). Under this approach, we are able to define *classical gravity over curved spaces* and see how such different geometries will antagonise our Euclidean intuition.

It thus becomes useful to distinguish between a ***fully relativistic test particle***, being one that does ***not*** affect neither the space-time geometry nor the real particles (for instance by some other type of physical interaction), and a ***relativistic test particle***, which is one that does ***not*** affect the geometry of the embedding space, but *does* affect the movement of the real particles through some type of interaction. It thus becomes clear that the condition of being a fully relativistic test particle is stronger then that of being a relativistic test particle[2].

3.1 A model for the dynamical equations

Following the above rationale, in an oscillating metric case, the natural framework in which to work with the equations of motion of our particle system is that of Relativistic mechanics, considering more specifically CM-CM gravitational interactions.

Our (test) particles will move according to the *geodesic equation* when subject to no force. Though, *since our treatment of gravity is classical*, the gravitational interaction between two masses will count as a force acting on them. This in turn changes the equations of motion slightly to what we shall refer to as the ***sourced geodesic equation***, given by

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = f^\mu, \tag{3.3}$$

where the 3−vector f^μ precisely represents the force term acting on the particle. For our case, this force term will be given by [Van06]

$$f^\mu = \frac{q}{m} g_{\alpha\beta} F^{\mu\alpha} \frac{dx^\beta}{ds} \tag{3.4}$$

[2]When considering the Kepler problem under our gravitational framework, we shall get back to the former concept.

where q represents the *gravitational charge* of the particle[3].

The above formula comes from the analogous force term a charged test particle would feel if present in a curved space-time over which an electromagnetic field is present. Such a term actually comes from the Lagrangian [Van06]

$$\mathcal{L} = \frac{1}{2m} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + q g_{\mu\nu}\dot{x}^{\mu}A^{\nu}(x). \tag{3.5}$$

Upon using the Euler-Lagrange equations on Eq.(3.5), we recover Eq.(3.3) with the force term of Eq.(3.4) by recalling that [Van06]

$$F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu}, \tag{3.6}$$

and that the indices are raised with the inverse metric $g^{\mu\nu}$.

On Chapter 5, we will get back to the above equations and focus on the study of the Kepler problem. For now, we compute the energy momentum tensor of the metric with and without the perturbations, giving along the way some possible physical interpretations for its components.

3.2 The vacuum manifold Stress-Energy tensor

As mentioned in the Introduction, the motivating problem for our study is one coming from atmospheric dynamics, in which we want to investigate the effects of high atmosphere perturbations, induced by Moon/solar tides, on the dynamics of vortex and masses. The geometry of our atmosphere is assumed to be intrinsically spherical for simplicity, and the manifold we will work with will be $\mathbb{R} \times \mathbb{S}^2$, in order to self-consistently define the relativistic quantities associated to the masses. The metric for such a manifold will be given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2\sin^2(\theta) \end{pmatrix}, \tag{3.7}$$

[3]the equivalence principle says that the gravitational mass (which can be thought of as a gravitational charge) of a particle should be equal to its inertial mass. It can be mathematically interesting however to consider cases in which these differ and, inspired by electromagnetism, are set to have opposite signs to each other.

where $(\theta, \phi) \in (0, \pi) \times [0, 2\pi)$ are the azimuthal and polar angles respectively. On what proceeds, we shall compute the quantity related to the matter and energy distributions within our space, namely the *stress-energy* or *energy-momentum tensor* $T_{\mu\nu}$, given by

$$T_{\mu\nu} = \frac{1}{\kappa} G_{\mu\nu} \tag{3.8}$$

where $G_{\mu\nu}$ is the Einstein tensor and κ is some number related to the Newtonian gravitational constant G. As can be deduced from Observation 3.1, the geometric interpretation behind a manifold's $T_{\mu\nu}$ is that of an average change in area of a geodesic ball centered at some point.

For simplicity we shall refer to it here as the *vacuum stress-energy tensor*, since our base manifold (or now, vacuum manifold, by the same reasoning) is at first free of any sources and matter[4].

Inspired by the interpretation of the additional term appearing in Poisson's equations when over a compact manifold [BDS16], we may say that the vacuum manifold's density would refer in a way to the distribution of an effective amount of matter required to keep the manifold's constant spacial slice's curvature and form static. As we shall soon see, for the case of $\mathbb{R} \times \mathbb{S}^2$ this amount is constant, meaning that such an effective matter distribution is uniformly distributed over the sphere.

Let's now see how one can compute the stress-energy tensor of a base manifold that we understand as an unperturbed classical vacuum. We will focus on the case of interest, namely $\mathbb{R} \times \mathbb{S}^2$, and explore a bit the higher dimensional analogs of such a manifold, having in mind the physical interpretation given above.

We start by considering our model of the atmosphere without any perturbations. This means that the intrinsic curvature of the atmosphere (considered as our classical vacuum manifold) will be due only to the presence of the Earth.

If we are to compute the stress-energy tensor of the this vacuum manifold, we need to

[4]Notice however that for a physicist, the notion of *vacuum* in GR refers to an *identically null* $T_{\mu\nu}$

calculate[5] the Einstein tensor $G_{\mu\nu}$ based on the metric of Eq.(3.7). This will yield us the following object

$$G_{\mu\nu}(\mathbb{S}^2 \times \mathbb{R}) = \begin{pmatrix} \frac{1}{R^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.9}$$

By the field equations we have that $T^{vac}(\mathbb{S}^2 \times \mathbb{R})$ is nothing but the above $G_{\mu\nu}(\mathbb{S}^2 \times \mathbb{R})$ with a $1/\kappa$ factor on its 00−component.

Constant Ricci scalar

The above calculation can be straightforwardly generalized to higher spatial dimensions, or even space-time manifolds given by the Lorenztian product of $\mathbb{R}$ with a constant constant Ricci scalar manifold Σ.

As further detailed at the end of Section A.1, we can compute a quantity called the *Ricci scalar* [Eq.(A.1.26)] by making use of the the Ricci tensor $R_{\mu\nu}$ and inverse metric tensor $g^{\mu\nu}$. In the case where R is a constant, one can show that the decomposition expressed at Eq.(A.1.29) reduces to the following [GHL04]

$$R_{\mu\nu\alpha\beta} = \frac{R}{n(n-1)}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}). \tag{3.10}$$

By contracting the appropriate indices we find that

$$R_{\mu\nu} = \frac{R}{n}g_{\mu\nu}, \tag{3.11}$$

Suppose we now want to deal with the manifold $M = \mathbb{R}\times\mathbb{S}^n$, whose metric is schematically given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & g_{\mathbb{S}^n} \end{pmatrix}. \tag{3.12}$$

It follows from the dependence of the Ricci scalar with the Christoffel symbols (and from their dependence with the above *time-independent* metric) that

$$R_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & R^{\mathbb{S}^n}_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{R}{n}g_{\mathbb{S}^n} \end{pmatrix}, \tag{3.13}$$

[5]the computations herein where done using the `Einsteinpy`/`GraviPy` module of `Python`

where we have used Eq.(3.11) (since the sphere has constant Ricci scalar). Now, using Eq.(B.1.3), we find that

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \begin{pmatrix} \frac{R}{2} & 0 \\ 0 & \frac{2-n}{2n}g_{\mathbb{S}^n}R \end{pmatrix}, \tag{3.14}$$

which explains why in the case of $\mathbb{S}^2$ the Einstein tensor is as in Eq.(3.9).

We thus see that the first non-trivial space terms for the Einstein (and thus energy-momentum) tensor appear in dimension four, when $n = 3$. Note that this result followed from the assumptions that i) R is constant, ii) the metric tensor is time-independent, and iii) the metric g_M of the manifold M, when considered in terms of infinitesimal line elements, splits as

$$ds_M^2 = -dt^2 + ds_{\mathbb{S}^n}^2. \tag{3.15}$$

Had we not assumed these conditions, the above result would not hold.

The physical meaning behind Eq.(3.14) should be similar to the one described at the end of the previous subsection. The 00−components can be intrinsically though of as a induced mass distribution over the space, responsible for causing the perceived curvature which is usually interpreted by physicists as a cosmological constant.

We end this section with the following

Observation 3.1 (Geometric significance of R)**.** Given a manifold M, take two geodesic curves $\gamma_0(t)$ and $\gamma_1(t)$ connected by a family of curves $x(s,t)$ such that $x(0,t) = \gamma_0(t)$, $x(1,t) = \gamma_1(t)$ and $\gamma_0(0) = \gamma_1(0)$. The measure of their separation as time goes on is given by the quantity

$$M^a{}_b = R^a{}_{cbd}\xi^c\xi^d, \quad \xi^a = \frac{\partial x^a(s,t)}{\partial s}. \tag{3.16}$$

This means for instance that the Riemann tensor can be used to quantify the rate of separation between two such curves. We can further think that, if we are given a ball $B(\varepsilon; p)$ of radius ε centered at p, the Riemann tensor gives us a measure of change along the ξ direction of this ball, as we move along the geodesics.

Moreover, upon taking the trace of Eq.(3.16), we see that

$$M = M^a{}_a = R_{cd}\xi^c\xi^d, \tag{3.17}$$

so that the Ricci tensor computes the *average* of such a rate of change, when we consider all possible pairs of geodesics that come out of p. Finally, the Ricci scalar R (found by taking the trace of R_{cd} with respect to the metric tensor) can be thought of as giving us the average area expansion of such a ball.

3.3 Earth's atmosphere with the Moon

The Moon's presence implies in the existence of tides over the Earth's surface and atmosphere. This in turn affects the dynamics of vortices and masses over the latter. What we do here is to model the atmosphere intrinsically, by thinking of it geometrically. In this framework, the tides would oscillations on this "geometric atmosphere" whose amplitude varies according to the passage of coordinate time t, as the phenomena is global an at the end of the day, we will have to arrive at a metric tensor that changes with time.

As in the previous section, we see the atmosphere as a surface of revolution, now of the form

$$x = a(t;\varepsilon)\sin(\theta)\cos(\varphi), \quad y = a(t;\varepsilon)\sin(\theta)\sin(\varphi), \quad z = \cos(\theta), \tag{3.18}$$

The fact that both coordinates x and y scale directly with the function $a(t,\varepsilon)$ means that the dynamics of our oscillations are a bit different from that of the actual atmosphere. In our case, the equator wobbles back and forth isotropically, starting from a minimal distance of R to the origin and going until the maximum distance defined by $a(t;\varepsilon)$. We have chosen to proceed with the study of such a dynamics rather then the atmospheric one(of a rotating ellipsoid of revolution) because the equations of motion and the metric tensor for the latter are much more complex to study. Indeed, we will leave the analysis of such a case for a future work.

The specific perturbation function we chose to work with is

$$a(t;\varepsilon) = R\left(1 + \frac{\epsilon}{2}\sin^2(\omega t)\right) \tag{3.19}$$

Recall now from Differential Geometry that given two manifolds M, N and an embedding $\iota : M \hookrightarrow N$, if g_N is the metric tensor over N then we have $g_M = \iota^* g_N$ is the (pull-back) metric over M. In our case M is the oscillating sphere and $N = \mathbb{R}^{1,3}$ (Minkowski space-time) with metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$. Upon writing it down into infinitesimals we have

$$\begin{aligned}\overline{ds}^2 &= \iota^*(-dt^2 + dx^2 + dy^2 + dz^2)\\ &= -dt^2 + (d(a(t;\varepsilon)\sin(\theta)\cos(\varphi)))^2 + (d(a(t;\varepsilon)\sin(\theta)\sin(\varphi)))^2 + (d(\cos(\theta)))^2.\end{aligned}$$

By using the chain and product rules we get to the following formula for $\overline{ds}^2$

$$\begin{aligned}\overline{ds}^2 = (-1 + (\partial_t a)^2 \sin(\theta))dt^2 + a\partial_t a \sin(2\theta) d\theta dt\\ + (a^2\cos^2(\theta)^2 + R^2\sin^2(\theta)^2)d\theta^2 + a^2\sin^2(\theta)d\varphi^2.\end{aligned} \tag{3.20}$$

Expanding the above with the formula from Eq.(3.19) we get

$$\begin{aligned}\overline{ds}^2 = -\left(1 - \frac{R^2\epsilon^2}{4}\sin^2(2\omega t)\sin^2(\theta)\right)dt^2 + R^2\left(1 + \frac{\epsilon}{2}\sin^2(\omega t)\right)\frac{\epsilon}{2}\sin(2\omega t)\sin(2\theta)dtd\theta\\ + R^2\left(\left(1 + \frac{\epsilon}{2}\sin^2(\omega t)\right)^2\cos^2(\theta) + \sin^2(\theta)\right)d\theta^2 + R^2\left(1 + \frac{\epsilon}{2}\sin^2(\omega t)\right)^2\sin^2(\theta)d\varphi^2.\end{aligned} \tag{3.21}$$

It is based on the above metric that we perform the one particle simulations present on Section 4. For the sake of completeness though, as it's usually done over flat Minkowski space-time (see Appendix B), we can also express the above perturbation to first order and consider

$$\overline{g}_{\mu\nu} = g_{\mu\nu}(\mathbb{R} \times \mathbb{S}^2) + \varepsilon h_{\mu\nu}(t, x), \tag{3.22}$$

for some small fixed parameter ε.

In matrix form, our perturbation will be written as

$$\overline{g}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0\\ 0 & R^2 & 0\\ 0 & 0 & R^2\sin^2(\theta)\end{pmatrix} + \varepsilon\begin{pmatrix} h_{00}(t,x) & h_{01}(t,x) & h_{02}(t,x)\\ h_{10}(t,x) & h_{11}(t,x) & h_{12}(t,x)\\ h_{20}(t,x) & h_{21}(t,x) & h_{22}(t,x)\end{pmatrix}. \tag{3.23}$$

The presence of *cross terms on the metric* is important because they are related to *dragging effects* of such a space. These effects are in turn expected to appear due to the motivation behind our problem for, in our case, the space-time dragging could be due to the Moon's motion about the planet, which we can intuitively think as "the atmosphere taking some time to notice the Moon's shift".

From Eq.(3.21), we get our first order perturbation tensor to be

$$h_{01}(t,\phi,\theta) = h_{10}(t,\phi,\theta) = \frac{R^2}{4}\sin(2\omega t)\sin(2\theta), \tag{3.24a}$$

$$h_{11}(t,\phi,\theta) = R^2\sin^2(\omega t)\cos^2(\theta), \tag{3.24b}$$

$$h_{22}(t,\phi,\theta) = R^2\sin^2(\omega t)\sin^2(\theta), \tag{3.24c}$$

with all other entries equal to zero.

3.3.1 On the matter Stress-Energy tensor

When dealing with relativistic test particles, we have that the Einstein field equations should assume the following format

$$\overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{R} = \kappa(T^{vac}_{\mu\nu} + T^{mat}_{\mu\nu}), \quad \left|T^{vac}_{\mu\nu}\right| \gg \left|T^{mat}_{\mu\nu}\right|, \tag{3.25}$$

with T^{vac} representing the ***effective vacuum stress-energy tensor*** generated by the background (Earth plus Moon) and T^{mat} coming from the matter distribution.

Since we are dealing with relativistic test particles on the CM-CM interaction regime, by definition, they *won't* have any effect on the background metric tensor. Therefore, on the left-hand-size of Eq.(3.25) the metric is ***fixed***, and so the main use of such an equation is to derive T^{vac}. That is to say

$$T^{vac}_{\mu\nu} := \frac{1}{\kappa}G_{\mu\nu}[\overline{g}], \tag{3.26}$$

with $\overline{g}(x,t)$ coming from Eq.(3.23). The explicit form of such a $T_{\mu\nu}$ can be numerically computed, being in general very cumbersome. Nonetheless, upon using Eq.(3.24) we can express its first order correction to be

$$
\begin{aligned}
T_{00}^{vac} &= \frac{1}{\kappa} + \frac{2\varepsilon}{\kappa}\sin^2(t)\cos^2(\theta) \\
T_{01}^{vac} &= T_{10}^{vac} = -\frac{\varepsilon}{4\kappa}(\cos(2(t-\theta)) - \cos(2(t+\theta))) \\
T_{11}^{vac} &= \frac{\varepsilon}{\kappa}\cos(2t)\sin^2(\theta) \\
T_{22}^{vac} &= -\frac{\varepsilon}{\kappa}\sin^4(\theta)\cos(2t)
\end{aligned}
$$

We in particular see a correction term for the “vacuum” energy of the base manifold $\mathbb{R}\times\mathbb{S}^2$, originally given by T_{00}.

In the next chapter we explore the consequences of our model in the single particle dynamics case, emphasizing the dynamics around two particular invariant submanifolds.

Chapter 4

Oscillating metric atmosphere: The test particle dynamics

Before going ahead into our discussion of the single particle dynamics, we more explicitly lay out the important

Assumption 1 (Small oscillation amplitudes). The amplitude of the metric perturbations, given by the parameter ε, is small with respect to the radius R of the sphere.

Now, as we saw in the previous chapter, the energy-momentum tensor for the classical oscillatory vacuum is given by Eq.(3.26). One interpretation we could assign to it is that, *it serves as an* ***effective matter density*** *that acts on the test particles*, steaming from the surface's intrinsic curvature which is in turn physically caused by the presence of exterior objects (Earth and Moon).

This can already be seen in the purely classical case too. For instance, in [BDS16] an extra $-1/A$ term, with A being the area of the closed surface, appears on Poisson's equation. It basically comes from a boundary condition that closed surfaces such as the sphere or the torus ought to satisfy. In our case, as said above, such an effective density comes from the fact that the atmosphere's intrinsic curvature and motion are due to massive bodies (Earth + Moon) sitting outside and pulling on it (Figure 4.1). The action of the background on the test particle is already taken into account by the equations of

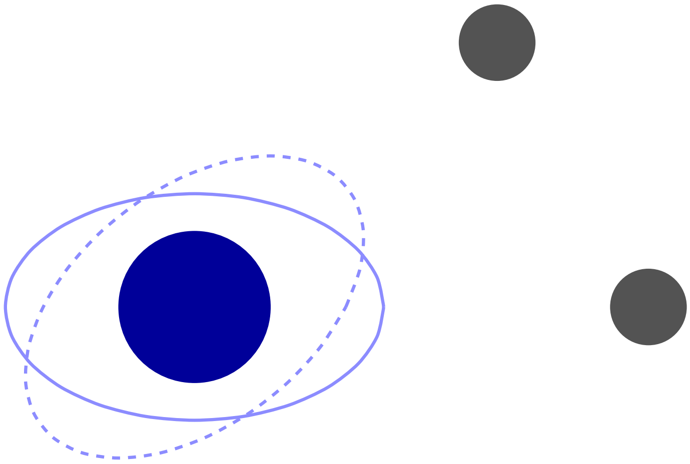


Figure 4.1: Exaggerated figure depicting the motion and shape of the atmosphere (light blue curve) when the Moon (dark gray disk) moves around the Earth (dark blue disk).

motion

$$\dot{\overline{V}}^{\mu} + \overline{\Gamma}^{\mu}_{\alpha\beta}\overline{V}^{\alpha}\overline{V}^{\beta} = 0, \tag{4.1}$$

where $\overline{V}^{\mu}$ is the perturbed velocity vector field; the over dot is proper time derivation and it is assumed that the Christoffel symbols $\overline{\Gamma}^{\mu}_{\alpha\beta}$ can be written as a perturbation series in ε (due to its smallness) as

$$\overline{\Gamma}^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\alpha\beta} + \varepsilon\Omega^{\mu}_{\alpha\beta} + \mathrm{O}(\varepsilon^2). \tag{4.2}$$

The explicit form of $\Omega^{\mu}_{\alpha\beta}$ can be checked on Section B.3. On the same token, we can further assume that the solution to Eq.(4.1) can also be written as a perturbation series on ε. By thus letting $\overline{V}^{\mu} = V^{\mu} + \varepsilon C^{\mu} + \mathrm{O}(\varepsilon^2)$, where V^{μ} is just the vector that satisfies the *unperturbed* background geodesic equation, namely

$$\dot{V}^{\mu} + \Gamma^{\mu}_{\alpha\beta}V^{\alpha}V^{\beta} = 0, \tag{4.3}$$

we can put together Eqs.(4.1, 4.3), obtaining to first order corrections in ε the following

$$\dot{C}^{\mu} + \Gamma^{\mu}_{\alpha\beta}C^{\alpha}C^{\beta} = -\Omega^{\mu}_{\alpha\beta}V^{\alpha}V^{\beta}. \tag{4.4}$$

On Eq.(4.4), we see an explicit appearance of the metric perturbations through its coupling with the unperturbed solution of Eq.(4.3), acting in turn as a force term for the motion of the deviation vector C^μ. This particular approach is useful because, if we take the curves $x^\mu(s)$ or $\overline{x}^\mu(s)$, then

$$\frac{dx^k}{ds} = V^k, \quad \frac{d\overline{x}^k}{ds} = V^k + \varepsilon C^k + \mathrm{O}(\varepsilon^2) \tag{4.5}$$

Then, upon choosing a τ-periodic geodesic[1] over the base manifold (whose coordinates are x^μ), we get the following estimate for the solution of Eq.(4.1):

$$\overline{x}^k(s_0 + \tau) - \overline{x}^k(s_0) = \varepsilon \int_{s_0}^{\tau + s_0} C^k(s)ds + \mathrm{O}(\varepsilon^2). \tag{4.6}$$

So that a deviation of order ε from $\tau-$periodicity is expected of the solution of Eq.(4.1). We can further proceed with this reasoning and obtain that

$$\overline{x}^k(s_0 + n\tau) - \overline{x}^k(s_0) = \varepsilon \sum_{i=0}^{n-1} \int_{s_0+i\tau}^{s_0+(i+1)\tau} C^k(s)ds + \mathrm{O}(\varepsilon^2), \;\; \forall n \in \mathbb{N}. \tag{4.7}$$

Hence, for most of the initial conditions we should be able to perceive after some finite time a noticeable difference between the particle's trajectory when the metric oscillates, versus when it doesn't.

For the present moment though, notice that upon computing the Christoffell symbols for the oscillating metric [Eq.(3.20)], we find that

$$\begin{aligned} &\overline{\Gamma}^0_{02} = \overline{\Gamma}^0_{12} = 0, \\ &\overline{\Gamma}^1_{02} = \overline{\Gamma}^1_{12} = 0, \\ &\overline{\Gamma}^2_{00} = \overline{\Gamma}^2_{01} = \overline{\Gamma}^2_{11} = \overline{\Gamma}^2_{22} = 0. \end{aligned}$$

The $0-$th component refers to the coordinate time t of the particle, the 1st to the θ coordinates and the 2nd to the φ one. To maintain the model's self-consistency, we ought to ask for the particle trajectories to be time-like[2]. This in turn translates into

$$-1 = \overline{g}_{\mu\nu}\overline{V}^\mu \overline{V}^\nu, \text{ for all } s \in \mathbb{R}_+. \tag{4.9}$$

[1] one that satisfies $x^\mu(s_0 + \tau) = x^\mu(s_0), \forall s_0 \in \mathbb{R}$ which holds for the spacial slices (given by $\mathbb{S}^2$) we have.

[2] As discussed on Section B, classical particles are always time-like, since their velocity in time are never zero and their spacial velocity can never exceed the speed of light.

In particular, if $s = 0$ we get that our initial conditions have to obey such a relation too, so that

$$(V_0^0)^2 = 1 + (V_0^1)^2 + \sin^2(\theta_0)(V_0^2)^2. \tag{4.10}$$

If we write (V^0, V^1, V^2) as $(\dot{t}, \dot{\theta}, \dot{\varphi})$, the geodesic equations for t, θ and φ assume the following form

$$\ddot{t} + \overline{\Gamma}^0_{00}\dot{t}^2 + \overline{\Gamma}^0_{11}\dot{\theta}^2 + \overline{\Gamma}^0_{22}\dot{\varphi}^2 + 2\overline{\Gamma}^0_{01}\dot{t}\dot{\theta} = 0 \tag{4.11a}$$
$$\ddot{\theta} + \overline{\Gamma}^1_{00}\dot{t}^2 + \overline{\Gamma}^1_{11}\dot{\theta}^2 + \overline{\Gamma}^1_{22}\dot{\varphi}^2 + 2\overline{\Gamma}^1_{01}\dot{\theta}\dot{t} = 0 \tag{4.11b}$$
$$\ddot{\varphi} + 2\dot{\varphi}\left(\overline{\Gamma}^2_{02}\dot{t} + \overline{\Gamma}^2_{12}\dot{\theta}\right) = 0 \tag{4.11c}$$

From the above we see the existence of an *invariant subspace* of the dynamics. By ***invariant subspace***, more formally given by

Definition 4.0.1 (Invariant subspace)**.** Given a space M and a Hamiltonian system $\mathcal{S}$ on M whose Hamiltonian flow is described by the map $\phi : \mathbb{R} \times M \to M$. Then, an **invariant subspace** for the dynamics of $\mathcal{S}$ is a set $\mathcal{A}$ which satisfies

$$\phi(t, \mathcal{A}) \subseteq \mathcal{A}, \forall t \in \mathbb{R}. \tag{4.12}$$

In our case this corresponds to the set

$$\mathcal{A} = \{(t, \theta, \varphi, \dot{t}, \dot{\theta}, \dot{\varphi} = 0) |\, t \in \mathbb{R}, (\theta, \varphi) \in \mathbb{S}^2\}. \tag{4.13}$$

Upon choosing an initial configuration from $\mathcal{A}$, Eqs.(4.11) can be first written as

$$\ddot{t} + \overline{\Gamma}^0_{00}\dot{t}^2 + \overline{\Gamma}^0_{11}\dot{\theta}^2 + 2\overline{\Gamma}^0_{01}\dot{t}\dot{\theta} = 0 \tag{4.14a}$$
$$\ddot{\theta} + \overline{\Gamma}^1_{00}\dot{t}^2 + \overline{\Gamma}^1_{11}\dot{\theta}^2 + 2\overline{\Gamma}^1_{01}\dot{\theta}\dot{t} = 0 \tag{4.14b}$$
$$\ddot{\varphi} = 0 \tag{4.14c}$$

From the last equation we see that $\dot{\varphi}(s)$ is constant in time. Since its starting value was zero, we shall have that $\dot{\varphi}(s) = 0, \forall s \in \mathbb{R}_+$. This means that once we start over $\mathcal{A}$, we shall remain on $\mathcal{A}$.

4.1 A Hamiltonian approach to the equations of motion

To study the relativistic single test particle dynamics, we appeal to the *Hamiltonian formalism* (see Section B.1.2). The equations of motion will no longer be second order, but instead will be a system of coupled first order equations known as *Hamilton's equations of motion*. In our coordinates they can be expressed as

$$\dot{t} = \frac{\partial H}{\partial p_t}, \quad \dot{p}_t = -\frac{\partial H}{\partial t}, \tag{4.15a}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta}, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta}, \tag{4.15b}$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi}, \quad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi}, \tag{4.15c}$$

with our single particle Hamiltonian being given by

$$H = \frac{1}{2m} g^{\alpha\beta} p_\alpha p_\beta. \tag{4.16}$$

Note that for solutions of Eq.(4.15) the above Hamiltonian actually has to be constant precisely because of Eq.(4.9). Indeed, as discussed on Section B.1.2, to find the associated momenta p_μ to the positions x^μ we simply take

$$p_\mu = m g_{\mu\nu} \dot{x}^\nu, \tag{4.17}$$

with $\dot{()} = \frac{d()}{ds}$. This in turn means that the momentum version of Eq.(4.9) is[3]

$$-m^2 = g^{\mu\nu} p_\mu p_\nu, \tag{4.18}$$

thus making Eq.(4.16) constant on the trajectories. More specifically, we have that on each trajectory the Hamiltonian will be

$$H = -\frac{m}{2}, \tag{4.19}$$

[3]such a relation clearly also holds for the vector case, that is

$$-m^2 = g_{\mu\nu} p^\mu p^\nu.$$

which should hold independent of whether or not the metric perturbations are on or off.

In what follows, we will divide our study in essentially two cases: i) the static metric case, for surfaces of revolution, ii) the oscillating metric case, focusing on the metric oscillations of $\mathbb{R} \times \mathbb{S}^2$.

4.2 Surfaces of Revolution: Static metric case

By the term *static metric* we mean a metric tensor whose components do not depend on coordinate time explicitly. Now, when we consider a surface of revolution Σ^2, we deal with the following set of parametric equations

$$x = \rho(\theta)\cos(\varphi), \quad y = \rho(\theta)\sin(\varphi), \quad z = \zeta(\theta). \tag{4.20}$$

Since we consider the dynamics to be happening over the 3−dimensional space-time manifold $\mathbb{R} \times \Sigma^2$, to calculate the metric tensor from Eq.(4.20) we have to take pullback of the Minkowski metric by the inclusion $\iota : \mathbb{R} \times \Sigma^2 \to \mathbb{R}^{1,3}$, leaving us with

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & (\zeta'(\theta))^2 + (\rho'(\theta))^2 & 0 \\ 0 & 0 & (\rho(\theta))^2 \end{pmatrix}, \tag{4.21}$$

whose inverse can be trivially found just by considering the multiplicative inverse of each element on the main diagonal. The Hamiltonian for the system is given by

$$H = \frac{1}{2m}\left(-p_t^2 + \frac{p_\theta^2}{(\zeta'(\theta))^2 + (\rho'(\theta))^2} + \frac{p_\varphi^2}{(\rho(\theta))^2}\right). \tag{4.22}$$

We can thus write the Hamiltonian Eqs.(4.15) in coordinates as

$$\dot{t} = \frac{\partial H}{\partial p_t} = -\frac{p_t}{m}, \qquad \dot{p}_t = -\frac{\partial H}{\partial t} = 0, \tag{4.23a}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m((\zeta')^2 + (\rho')^2)}, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{1}{m}\left(\frac{(\zeta'\zeta'' + \rho'\rho'')}{((\zeta')^2 + (\rho')^2)^2}p_\theta^2 + \frac{\rho'}{\rho^3}p_\varphi^2\right), \tag{4.23b}$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m(\rho)^2}, \qquad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0, \tag{4.23c}$$

where prime denotes differentiation with respect to the variable θ. By Eqs.(4.23a, 4.23c), we see that $p_t \in \mathbb{R}^-$(since the coordinate time t must move in the positive direction) and $p_\varphi = L \in \mathbb{R}$.

The time-like momentum condition of Eq.(4.18) with the inverse metric to Eq.(4.21) is now written as

$$-m^2 = -p_t^2 + \frac{p_\theta^2}{(\zeta'(\theta))^2 + (\rho'(\theta))^2} + \frac{p_\varphi^2}{(\rho(\theta))^2} \tag{4.24}$$

Other calculations and examples are further developed in the preprint by Boatto et al. [SB25], on which the following subsections are based.

4.2.1 An analytic example: The (relativistic) ellipsoid of revolution

We consider an example for the static surface of revolution case, finding its set of equilibrium points and its invariant submanifolds (Definition 4.2.2). More specifically, we focus on the *ellipsoid of revolution*, whose parametrization according to Eq.(4.20) becomes

$$x = R\cos(\varphi)\sin(\theta), \quad y = R\sin(\varphi)\sin(\theta), \quad z = b\cos(\theta) \tag{4.25}$$

where R, b are positive constants. From there, the metric tensor is found to be

$$g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & R^2\cos^2(\theta) + b^2\sin^2(\theta) & 0 \\ 0 & 0 & R^2\sin^2(\theta) \end{pmatrix}. \tag{4.26}$$

Therefore, the Hamiltonian on Eq.(4.22) is given by

$$H = \frac{1}{2m}\left(-p_t^2 + \frac{p_\theta^2}{R^2\cos^2(\theta) + b^2\sin^2(\theta)} + \frac{p_\varphi^2}{R^2\sin^2(\theta)}\right). \tag{4.27}$$

The equations of motion Eqs.(4.23) then become

$$\dot{t} = -\frac{p_t}{m}, \qquad \dot{p}_t = 0 \tag{4.28a}$$

$$\dot{\theta} = \frac{p_\theta}{m(R^2\cos^2(\theta) + b^2\sin^2(\theta))}, \; \dot{p}_\theta = \frac{p_\theta^2(b^2 - R^2)\sin(2\theta)}{2m(R^2\cos^2(\theta) + b^2\sin^2(\theta))^2} + \frac{p_\varphi^2\cos(\theta)}{mR^2\sin^3(\theta)} \tag{4.28b}$$

$$\dot{\varphi} = \frac{p_\varphi}{m\sin^2(\theta)}, \qquad \dot{p}_\varphi = 0 \tag{4.28c}$$

Equilibrium points for the ellipsoid of revolution

The definition of an equilibrium configuration or an equilibrium point for a Hamiltonian system goes as follows

Definition 4.2.1 (Equilibrium configuration - General version)**.** Given a manifold M and a Hamiltonian system defined on T^*M with Hamiltonian $H \in C^\infty(T^*M)$, whose equations of motion read

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}. \tag{4.29}$$

Then, an **equilibrium configuration** for the above is a point $X^* \in T^*M$ such that

$$\frac{\partial H}{\partial p_i}(X^*) = \frac{\partial H}{\partial x_i}(X^*) = 0. \tag{4.30}$$

In other words, an equilibrium configuration is one in which the derivative of our coordinates vanish, meaning that the point we started on is *fixed* under the dynamics.

Due to physical constrains, in our case we are restricted to looking for equilibrium points within the *spacial sub-system* of our dynamics, generated by the coordinates $(\theta, \varphi, p_\theta, p_\varphi)$. In this case, the equilibrium points $X^* = (t_0, \theta_0, \varphi_0, p_{t,0}, p_{\theta,0}, p_{\varphi,0})$ for Eqs.(4.28b, 4.28c) will satisfy

$$\frac{\partial H}{\partial p_\theta}(X^*) = \frac{\partial H}{\partial p_\varphi}(X^*) = \frac{\partial H}{\partial \theta}(X^*) = \frac{\partial H}{\partial \varphi}(X^*) = 0 \tag{4.31}$$

As already studied by Boatto, Dritschel and Schaefer [BDS16] in the non-relativistic case, the fixed points for this dynamics are the poles and the equator. In our framework however, it does not matter the initial position on the ellipsoid, as long as we have zero initial space momentum, the particle will remain at its starting point, as guaranteed by the following

Proposition 4.2.1. *The equilibrium points $X^* = (t_0, \theta_0, \varphi_0, p_{t,0}, p_{\theta,0}, p_{\varphi,0})$ for the dynamics governed by Eqs.(4.28) are of the form*

$$X^* = \{(t_0, \theta_0, \varphi_0, p_{t,0} = -m, p_{\theta,0} = 0, p_{\varphi,0} = 0)\,,\ \forall t_0 \in \mathbb{R}, (\theta_0, \varphi_0) \in \mathbb{S}^2\}. \tag{4.32}$$

Proof: The proof follows trivially by inserting $p_{\theta,0} = p_{\varphi,0} = 0$ on Eq.(4.28). In doing so, we see that Eq.(4.31) is automatically satisfied by every possible value of $t_0 \in \mathbb{R}$ and $(\theta_0, \varphi_0) \in \mathbb{S}^2$. The value for $p_{t,0}$ comes from the relativistic momentum norm condition expressed by Eq.(4.18). □

Invariant submanifolds for the ellipsoid of revolution

We start with the following

Definition 4.2.2 (Invariant submanifold)**.** Given a manifold M and a Hamiltonian system defined on its cotangent bundle T^*M, an **invariant submanifold** for the system is a submanifold $\mathcal{N} \subseteq T^*M$ which gets mapped to itself under the system's Hamiltonian flow.

From a physical point of view, this means that once a particle starts at the submanifold $\mathcal{N}$, its trajectory will lie entirely within $\mathcal{N}$.

Based on the form of the equations of motion, plus Proposition 4.2.1, we may try to observe what happens to the particle dynamics once *one* of the space momenta (say p_θ) is set zero and the other one (p_φ is this case) is not.

On setting $p_{\theta,0} = 0$, we observe that if $\theta_0 = \frac{\pi}{2}$ then $\dot{p}_\theta = 0$. This, together with the fact that our $p_{\theta,0}$ choice makes $\dot{\theta} = 0$ enables us to choose $p_\theta(s) = 0$ and $\theta(s) = \frac{\pi}{2}$, implying that the dynamics is solely happening on the $(t, \varphi, p_t, p_\varphi)$ subspace. A similar reasoning is true for the case where $p_{\varphi,0} = 0$ and we let p_θ run freely instead. More formally speaking, we have the following results

Proposition 4.2.2. *The manifold*

$$\mathcal{A}_{eq} = \left\{ \Big(t(s;t_0), \theta_0 = \frac{\pi}{2}, \varphi(s;\varphi_0), p_t(s;p_{\varphi,0}), p_{\theta,0} = 0, p_\varphi(s;p_{\varphi,0}) \Big) \,|\, t_0 \in \mathbb{R}, \varphi_0 \in \mathbb{S} \right\}$$

$$p_{t,0} = -\sqrt{m^2 + \frac{p_{\varphi,0}^2}{R^2 \sin^2(\theta_0)}} \tag{4.33}$$

is invariant under the Hamiltonian flow generated by Eq.(4.28)*.*

Proposition 4.2.3. *The manifold*

$$\mathcal{A}_{mer} = \left\{ (t(s;t_0), \theta(s;\theta_0), \varphi_0, p_t(s;p_{t,0}), p_\theta(s;p_{\theta,0}), p_{\varphi,0} = 0) \,|\, t_0 \in \mathbb{R}, (\theta_0, \varphi_0) \in \mathbb{S}^2 \right\},$$

$$p_{t,0} = -\sqrt{m^2 + \frac{p_{\theta,0}^2}{R^2 \cos^2(\theta_0) + b^2 \sin^2(\theta_0)}} \tag{4.34}$$

is invariant under the Hamiltonian flow generated by Eq.(4.28)*.*

Their proof is immediate and follow from a direct substitution of the coordinates of each points in either $\mathcal{A}_{\text{mer}}$ or $\mathcal{A}_{\text{eq}}$ into Eqs.(4.23).

As a final comment before departing to the oscillating case, notice that if $b = R$ we are at a sphere and the dynamics of points located in the vicinity of these invariant submanifolds is quite simple and will actually just yield great circles that either pass or not through the north and south poles, depending on the value of $p_{\varphi,0}$. For the general case of $b \neq R$ however, we are at an ellipsoid of revolution whose dynamics around these invariant submanifolds is much richer. For instance, some of the curves traced by the particle in this case may be dense on the subset of allowable motions [Kli95]. The question of stability of such invariant submanifolds will be further explored on the final version of [SB25].

4.3 Surfaces of Revolution: Oscillatory case

We now investigate the dynamical consequences of the metric oscillations on the movement of a single particle. Recall from Section 3.3 that the parameterization we use for our oscillating surface of revolution is

$$x = a(t;\varepsilon)\sin(\theta)\cos(\varphi), \quad y = a(t;\varepsilon)\sin(\theta)\sin(\varphi), \quad z = R\cos(\theta), \tag{4.35}$$

which modeled the more simpler class of isotropic equatorial oscillations for the sphere. The explicit form taken for the perturbation function was

$$a(t,\varepsilon) = R\left(1 + \frac{\varepsilon}{2}\sin^2(\omega t)\right), \tag{4.36}$$

as it never vanishes for all $t \in \mathbb{R}$ and has R as its minimum value. The parameter ω represents the frequency of oscillation and its value will be showed later on once we get into the numerical calculations (Section 4.3.3). The time derivative for $a(t;\varepsilon)$ is given by

$$a' = \partial_t a = \frac{R\omega\varepsilon}{2}\sin(2\omega t), \tag{4.37}$$

with R being the radius of the perturbed sphere and ω the *frequency of the perturbation.*

Notice that, since the associated metric tensor is independent of φ, we still have $\dot{p}_\varphi = 0$ and so, following the same notation as in the previous subsection, we put $p_\varphi = L \in \mathbb{R}$.

Due to such perturbations the metric tensor now changes to that of Eq.(3.21) and its matrix form is given by

$$\overline{g} = \begin{pmatrix} \overline{g}_{tt}(t,\theta;\varepsilon) & \overline{g}_{t\theta}(t,\theta;\varepsilon) & 0 \\ \overline{g}_{t\theta}(t,\theta;\varepsilon) & \overline{g}_{\theta\theta}(t,\theta;\varepsilon) & 0 \\ 0 & 0 & \overline{g}_{\varphi\varphi}(t,\theta;\varepsilon) \end{pmatrix}. \tag{4.38}$$

Based on its inverse, we obtain the following Hamiltonian

$$H = \frac{1}{2m}(\overline{g}^{tt}p_t^2 + \overline{g}^{\theta\theta}p_\theta^2 + \overline{g}^{\varphi\varphi}p_\varphi^2) + \frac{1}{m}\overline{g}^{t\theta}p_t p_\theta. \tag{4.39}$$

Evidently, the equations of motion will be of Hamiltonian form

$$\dot{t} = \frac{\partial H}{\partial p_t} = \frac{1}{m}(\overline{g}^{tt}p_t + \overline{g}^{\theta t}p_\theta),\ \dot{p}_t = -\frac{\partial H}{\partial t} = \frac{-1}{2m}(p_t^2\partial_t\overline{g}^{tt} + p_\theta^2\partial_t\overline{g}^{\theta\theta} + p_\varphi^2\partial_t\overline{g}^{\varphi\varphi}) - \frac{1}{m}p_t p_\theta \partial_t \overline{g}^{\theta t} \tag{4.40a}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{m}(\overline{g}^{\theta t}p_t + \overline{g}^{\theta\theta}p_\theta), \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{-1}{2m}(p_t^2\partial_\theta\overline{g}^{tt} + p_\theta^2\partial_\theta\overline{g}^{\theta\theta} + p_\varphi^2\partial_\theta\overline{g}^{\varphi\varphi}) - \frac{1}{m}p_\theta p_t \partial_\theta \overline{g}^{\theta t}, \tag{4.40b}$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{1}{m}\overline{g}^{\varphi\varphi}p_\varphi, \qquad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0, \tag{4.40c}$$

where we have the following formulae

$$\begin{gathered} \overline{g}^{tt}(t,\theta) = -\frac{(\cos^2(\theta)a(t,\epsilon)^2 + R^2\sin^2(\theta))}{DA}, \quad \overline{g}^{\varphi\varphi}(t,\theta) = \frac{1}{a^2(t)\sin^2\theta}, \\ \overline{g}^{\theta\theta}(t,\theta) = \frac{1-\sin^2(\theta)a'(t,\epsilon)^2}{DA}, \quad \overline{g}^{\theta t}(t,\theta) = \frac{\sin(\theta)\cos(\theta)a(t,\epsilon)a'(t,\epsilon)}{DA}, \\ DA(t,\theta) = \cos^2(\theta)a(t,\epsilon)^2 + R^2\sin^2(\theta)\left(1 - \sin^2(\theta)a'(t,\epsilon)^2\right). \end{gathered} \tag{4.41}$$

4.3.1 Equilibrium solutions

Much like in the static case, when talking about equilibrium solutions we will be specifically referring to those sets of points which make the spacial derivatives of Eq.(4.40) vanish. That is, we look for points $X = (t_0, \theta_0, \varphi_0, p_{t,0}, p_{\theta,0}, p_{\varphi,0})$ which satisfy

$$\frac{\partial H}{\partial p_\theta}(X^*) = \frac{\partial H}{\partial p_\varphi}(X^*) = \frac{\partial H}{\partial \theta}(X^*) = \frac{\partial H}{\partial \varphi}(X^*) = 0. \tag{4.42}$$

Our goal is to show the following

Proposition 4.3.1. *The only possible equilibrium points* $X^* = (t_0, \theta_0, \varphi_0, p_{t,0}, p_{\theta,0}, p_{\varphi,0})$ *are of the form*

$$X^* = \left(t_0, \frac{\pi}{2}, \varphi_0, -m, 0, 0\right), (t_0, 0, \varphi_0, -m, 0, 0), (t_0, \pi, \varphi_0, -m, 0, 0) \tag{4.43}$$

for any $t_0 \in \mathbb{R}, \varphi_0 \in \mathbb{S}$.

Proof: The proof follows from direct calculations. By Eq.(4.41) we readily see that

$$\overline{g}^{\theta t}(X^*) = 0, \tag{4.44}$$

for any of the X^* above. We hence conclude that $\dot{\theta} = \dot{\varphi} = 0$ once we have $p_{\theta,0} = p_{\varphi,0} = 0$. To conclude the same for $\dot{p}_\theta$, notice that

$$\partial_\theta \overline{g}^{tt} = -\frac{1}{(DA)^2}\left(a'^2 R^2 \sin^2(\theta)\cos(\theta)(a^2(3+\cos(2\theta)) + 2R^2\sin^2(\theta))\right) \tag{4.45}$$

By thus setting $\theta_0 = 0, \pi/2$ or π, we shall find $\dot{p}_{\theta,0} = 0$, as we wanted to show. Moreover, these are the only solutions for the above for all $t \in \mathbb{R}$. Indeed, say we had instead

$$a^2(3+\cos(2\theta)) + 2R^2\sin^2(\theta)) = 0.$$

Some simple manipulations using the basic trigonometric identities let us rewrite this equality as

$$\sin^2(\theta)(a(t;\varepsilon)^2 - R^2) - 2a(t;\varepsilon)^2 = 0. \tag{4.46}$$

Notably, any pair of points (t, θ) which satisfy Eq.(4.46) will also have so satisfy

$$(a(t;\varepsilon)^2 - R^2) - 2a(t;\varepsilon)^2 \geq 0, \tag{4.47}$$

since $1 \geq \sin^2(\theta)$. But Eq.(4.47) is equivalent to

$$-(a^2 + R^2) \geq 0, \tag{4.48}$$

which is an absurd. This hence shows that the only possible equilibrium points for the dynamics considered are of the form stated in the proposition. □

4.3.2 Invariant submanifolds for the oscillating case

We already know that any point of the form Eq.(4.43) will be stationary. In particular, we can conclude that *any point in the equator will remain at the equator* as stated in the following

Corollary 4.3.2. *The manifold*

$$\mathcal{V}_{eq} = \left\{ \left(t(s;t_0), \theta_0 = \frac{\pi}{2}, \varphi(s;\varphi_0), p_t(s;p_{t,0}), p_{\theta,0} = 0, p_{\varphi,0} = L \right) \mid t_0, s, L \in \mathbb{R}, \varphi_0 \in \mathbb{S} \right\}$$
$$p_{t,0} = -\sqrt{\frac{-m^2 - g^{\varphi\varphi}(t_0, \theta_0 = \pi/2)L^2}{g^{tt}(t_0, \theta_0 = \pi/2)}} \tag{4.49}$$

is invariant under the Hamiltonian flow of the oscillating metric dynamics described by Eq.(4.40).

Proof: Thanks to Proposition 4.3.1, all we need to show now is that, for any other $p_{\varphi,0} = L$ value $\dot{p}_\theta$ remains equal to zero in the equator (provided $p_{\theta,0} = 0$). This has to still be the case precisely because

$$\partial_\theta \overline{g}^{\varphi\varphi} = -\frac{8\cot(\theta)\csc^2(\theta)}{R^2\left(\epsilon \sin^2(t\omega) + 2\right)^2}, \tag{4.50}$$

so that, at $\theta_0 = \frac{\pi}{2}$, the above derivative vanishes, rendering Eq.(4.40b) identically zero. □

The above result than tells us that the system's dynamics is restricted to the $(t, \varphi, p_t, p_\varphi)$ subspace. Now, much like in the static case however, we have yet another invariant submanifold to our dynamics, as guaranteed by the following

Proposition 4.3.3. *The manifold*

$$\mathcal{V}_{mer} = \{ (t(s;t_0), \theta(s;\theta_0), \varphi_0, p_t(s;p_{t,0}), p_\theta(s;p_{\theta,0}), p_{\varphi,0} = 0) \mid t_0, s, p_{\theta,0} \in \mathbb{R}, (\theta_0, \varphi_0) \in \mathbb{S}^2 \}$$
$$p_{t,0} = \frac{-g^{t\theta}(t_0,\theta_0)p_{\theta,0} - \sqrt{(g^{t\theta}(t_0,\theta_0))^2 p_{\theta,0}^2 - g^{tt}(t_0,\theta_0)(g^{\theta\theta}(t_0,\theta_0)p_\theta^2 + m^2)}}{g^{tt}(t_0,\theta_0)} \tag{4.51}$$

is invariant under the Hamiltonian flow of the oscillating metric dynamics described by Eq.(4.40).

Proof: The proof is immediate and follows directly by observing that upon setting $p_{\varphi,0} = L = 0$ on Eq.(4.40c) we have $\dot{\varphi} = \dot{p}_\varphi = 0$, meanwhile $\dot{\theta}$ and $\dot{p}_\theta$ do not identically vanish. This means that the dynamics happens only on the $(t, \theta, p_t, p_\theta)$ subspace, thus making $\mathcal{V}_{\text{mer}}$ invariant under the Hamiltonian flow. $\square$

Naturally, we shall refer to $\mathcal{V}_{\text{eq}}$ and $\mathcal{V}_{\text{med}}$ as the equatorial and meridian manifolds, respectively. Notably, they possess a non-trivial intersection, given by the set of equilibrium points

$$\mathcal{F} = \mathcal{V}_{\text{eq}} \cap \mathcal{V}_{\text{mer}} = \left\{ \left(t(s;t_0), \frac{\pi}{2}, \varphi_0, -m, 0, 0 \right) | s, t_0 \in \mathbb{R}, \varphi_0 \in \mathbb{S} \right\}, \tag{4.52}$$

discussed on Proposition 4.3.1.

Further notice that the difference between the whole set of equilibrium points and $\mathcal{F}$ is just the north and south poles. This is because if a particle starts at any position (θ_0, φ_0) with $\theta_0 \neq \frac{\pi}{2}$ and $p_{\theta,0} = p_\varphi = 0$, the derivatives of $p_\theta(s)$ and $\theta(s)$ will ***not*** be identically zero, which in turn means that the particle will be set into motion[4], a behavior surely not present in the static case where no matter the initial position, provided with zero initial spatial momenta, the particle will remain fixed at that same place. In this regard, ε is a *dynamics inducing parameter*.

Let's now analyze what happens to the single particle dynamics in a vicinity of each of these manifolds.

4.3.3 Numerical stability in the (θ, p_θ) plane

In this subsection we will be concerned with the stability properties portrayed by the points lying on the invariant submanifold $\mathcal{V}_{\text{eq}}$, focusing on the stability in the (θ, p_θ) subspace. We start by analyzing the poles and then pass to the equatorial manifold $\mathcal{V}_{\text{eq}}$. The technique used to perform such a stability study will be to randomly choose

[4] Notice that the motion of such a particle will happen within $\mathcal{V}_{\text{mer}}$, since $\dot{p}_\varphi = 0$ always, so that $p_\varphi \equiv 0$ due to its initial condition.

a set of initial conditions around our fixed points and see how they evolve under the Hamiltonian flow of Eq.(4.40). That being so, we can not analyze the stability of the meridian submanifold $\mathcal{V}_{\text{mer}}$, as its points will generically not be fixed ones. A thorougher analysis regarding such a set's stability properties is hence left for a future work.

Since our model draws inspiration in the lunar tides we will, in the numerical computations, set

$$\omega = \frac{2\pi}{T}, \tag{4.53}$$

with T being the period of each tide (i.e 12 hours).

Dynamics around the poles

We already know that the north and south poles belong to the set of equilibrium points of the dynamics. That is, if we set them with zero initial space momenta ($p_{\theta,0} = p_{\varphi,0} = 0$), they will remain fixed. When considering the stability of such a set of points, we are interested in knowing whether once we start close to them will for instance *always* remain close, or will potentially "fly off" to other parts of the sphere (and hence, of phase space). Notice however that the problem has an intrinsic $\mathbb{Z}_2$ symmetry. Since, the perturbations are symmetric with respect to the equator, the upper half dynamics that happens on the sphere will be the same as the lower half one. Therefore we can restrict our study of the poles to just one of them. We hence pick the north pole to analyze.

To begin our numerical investigation we randomly choose between 100 to 200 points in the vicinity of the north pole with zero spacial momenta, i.e $p_{\theta,0} = p_{\varphi,0} = 0$. Due to the conservation of angular momentum p_φ the system is invariant under rotations in the φ direction, meaning that if two particle start at the same latitude θ_0 but with different longitudes φ_1 and φ_2, their dynamics will turn out to be completely equivalent, only being shifted by a longitudinal angle of $|\varphi_2 - \varphi_1|$ modulo 2π, meaning that we can choose all particles in the same longitude (in this case $\varphi = 0$).

By examining the trajectories of Figure 4.2 we see the presence of 2 behaviors, namely

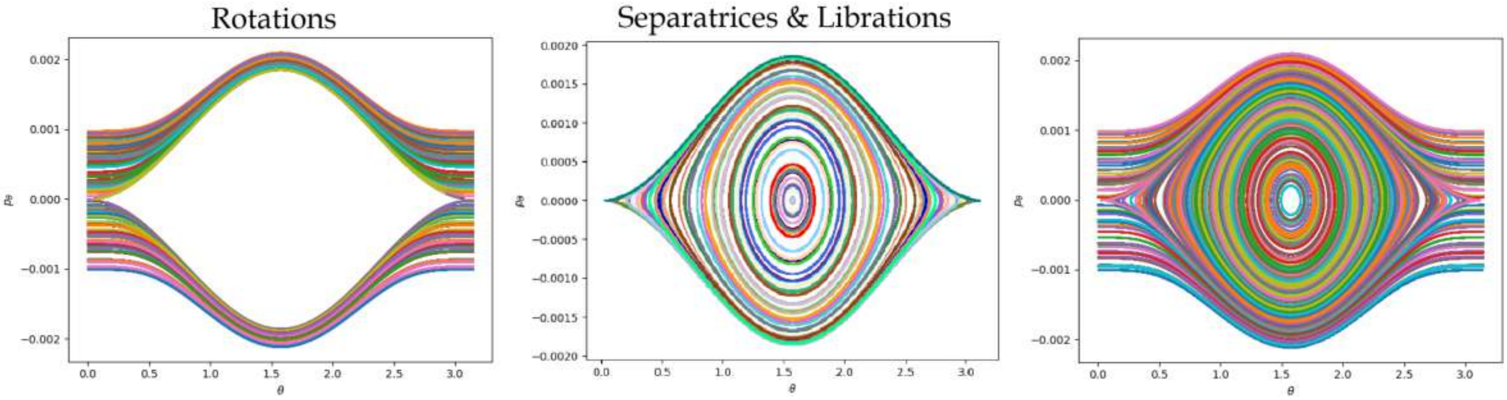


Figure 4.2: (θ, p_θ) plane $\boldsymbol{m = 1, R = 1, \varepsilon = 0.01, \varphi_0 = 0}$. We integrate the orbits from $s_{\min} = 0$ until $s_{\max} = 5 \times 10^5$. ***Left image:*** Single particle dynamics with 150 randomly chosen $\theta_0 \in [10^{-5}, 10^{-4}]$ and $p_{\theta,0} \in [-10^{-3}, 10^{-3}]$. ***Middle image:*** Single particle dynamics with $p_{\theta,0} = p_{\varphi,0} = 0$ for 100 particles with randomly chosen $\theta_0 \in [\frac{\pi}{100}, \frac{\pi}{2}]$. ***Right image:*** Single particle dynamics with 200 random chosen $\theta_0 \in [10^{-5}, \frac{\pi}{2}]$ and $p_{\theta,0} \in [-10^{-3}, 10^{-3}]$.

libration and *rotation*, which are in turn separated by a pair of *separatrices*, a behavior analogous to that of the simple pendulum [Gio22].

The libration motion happens in the interior region to the separatrix (middle image from Figure 4.2) and it represents an oscillation of the angle θ about the equatorial line $\theta = \frac{\pi}{2}$. An example of a trajectory in this regime is given on image $\boldsymbol{(c)}$ of Figure 4.3, where the particle goes from its initial position to its final one and back in a finite amount of time.

In regards to the rotation motion, by exclusion, it happens in the exterior region to the separatrix (left image from Figure 4.2) and it represents a continuous motion of the angle θ around its $[0, \pi]$ range. An example of a trajectory in this regime is given by image $\boldsymbol{(a)}$ of Figure 4.3, where the particle starts from its initial position and moves without bound in the θ direction of phase space, signifying that it goes around the sphere passing through the north and south poles in the configuration space.

Now, exactly in the separatrix however, the particle's motion is quite different. Indeed, if it starts on any one of its points (except the poles), it will take an *infinite* amount of

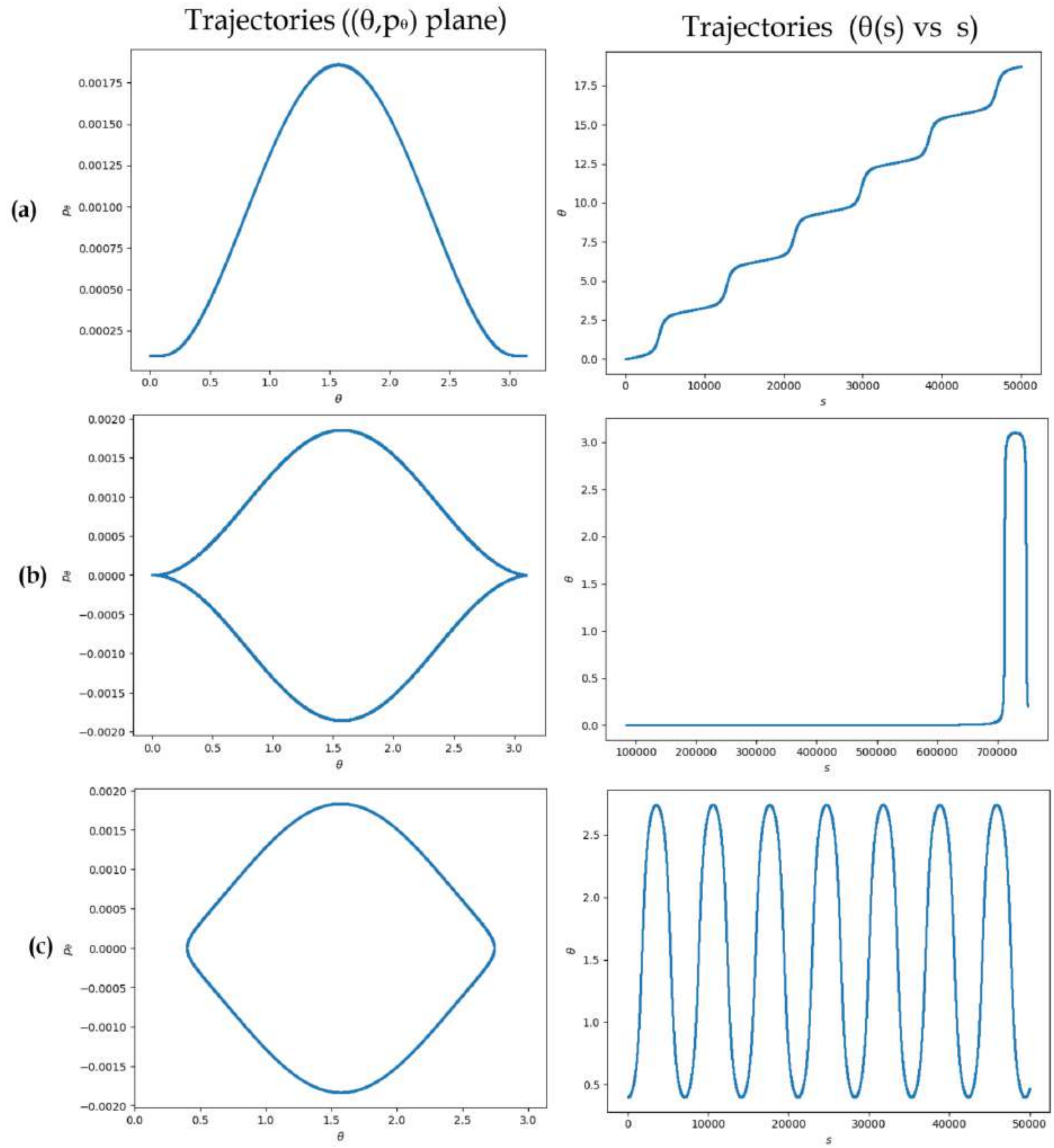


Figure 4.3: Throughout the plots we fixed $m = 1, \varepsilon = 0.01, \varphi_0 = 0, p_{\varphi,0} = 0$ **(a)** Single particle orbit **outside** the separatrix region with $\theta_0 = \pi \times 10^{-4}, p_{\theta,0} = 1e - 5$ **(b)** Single particle orbit **close to** the separatrix with $\theta_0 = 0.001, p_{\theta,0} = 0$ **(c)** Single particle orbit **inside** the separatrix region with $\theta_0 = 0.001, p_{\theta,0} = 10^{-4}$,

time to arrive at either the north pole or the south pole, depending on its initial $p_{\theta,0}$ value. Evidently, if the particle starts at the north or south pole with zero space momenta, it shall remain there forever. An approximate example for such a trajectory is given by image ***(b)*** of Figure 4.3 in which we tried to find a particle that was located precisely at the separatrix (in this case very close to the pole). We observe that the θ angle remains essentially fixed for quite a long time until the particle manages actually starts moving in the libration region, indicating that our approximation was indeed a bit of the separatrix curve.

By also changing the perturbative parameter ε we see how the dynamics around the pole evolves. As $p_{\varphi,0} = 0$ always, by looking at Figure 4.4 we see that for ε values closer to zero, the curves traced are mostly of rotation types. This is expected as when $\varepsilon \to 0$ we recover the manifold $\mathbb{R} \times \mathbb{S}^2$, whose spacial part is simply the (fixed) round sphere. On the flip side if we make ε bigger, more librations seem to appear, that is until the dynamics becomes apparently chaotic, though as we shall see on subsection 4.3.4, this also might be due mostly to numerical errors within our code.

To finish this first discussion, notice that the images on Figure 4.3 repeat themselves infinitely many time, just like in the simple harmonic oscillator case. By thus making use of the same dynamical classification as in the classical theory, we have obtained some numerical evidence indicating that the north pole $X_* = (t(s), \theta(s) = 0, \varphi(s) = 0, p_t(s) = -m, p_\theta(s) = 0, p_\varphi(s) = 0)$ is a saddle fixed point in the dynamics of the subspace defined by

$$\begin{aligned} \mathcal{S}_1 = \{(t, \theta, \varphi, p_t, p_\theta, p_\varphi) | t, p_\theta, p_\varphi \in \mathbb{R}, (\theta, \varphi) \in \mathbb{S}^2, p_\varphi = \varphi = 0\} \\ p_t = \frac{-2g^{t\theta}p_\theta - \sqrt{(2g^{t\theta}p_\theta)^2 - 4g^{tt}m^2}}{2g^{tt}} \end{aligned} \tag{4.54}$$

where the dependence of $g^{\mu\nu}$ on t and θ was left implicit.

Dynamics around the equatorial manifold $\mathcal{V}_{\mathbf{eq}}$

The point-wise stability of the fixed points in the equatorial submanifold is very similar to that of the poles and, as a matter of fact, is more straight forward.

By looking at Figure 4.5, we notice the exclusive presence of librations around the fixed point. Indeed, since now we are analyzing the dynamics with respect to $Y_* = (t(s), \frac{\pi}{2}, \varphi_0 =$

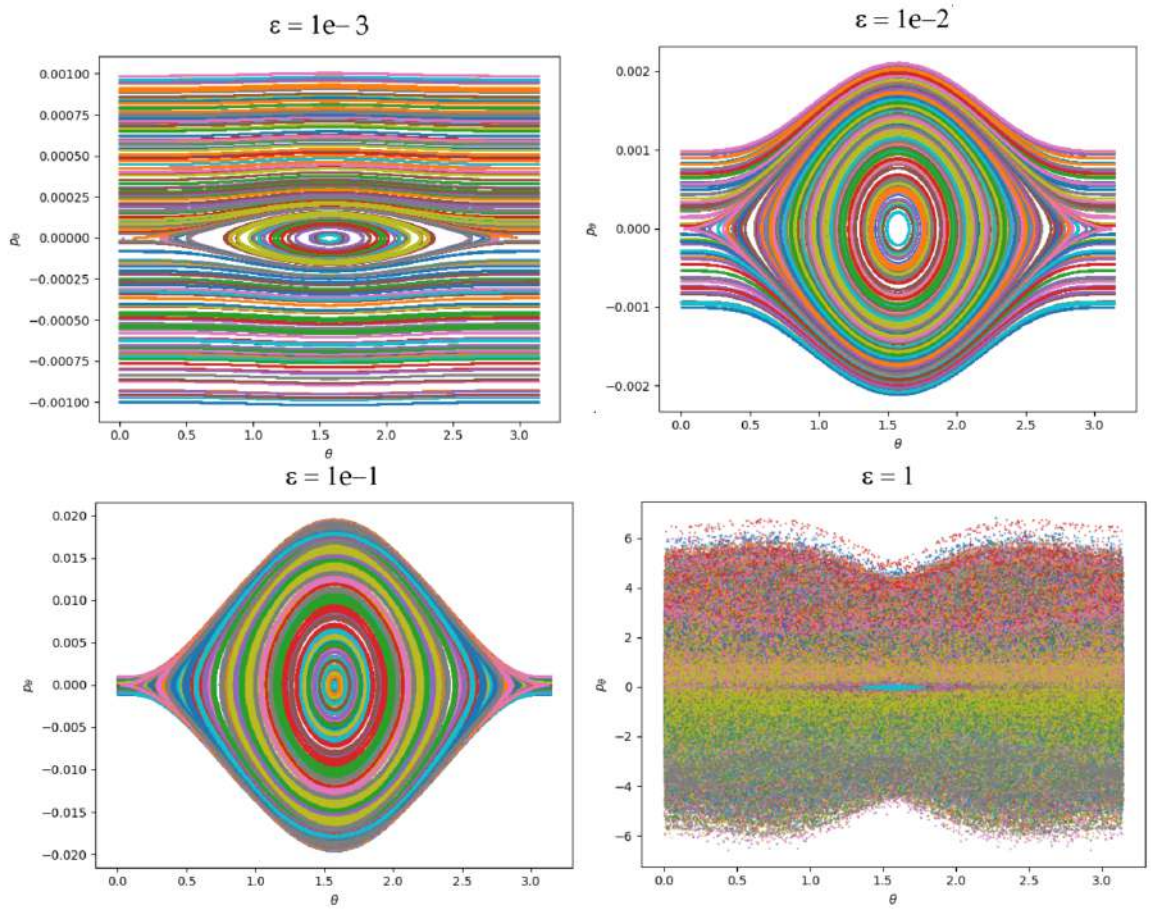


Figure 4.4: Single particle orbits around the north pole with 200 randomly chosen $\theta_0 \in [10^{-5}, \frac{\pi}{2}]$ and $p_{\theta,0} \in [-10^{-3}, 10^{-3}]$ and fixed $m = 1, R = 1, \varphi_0 = 0$. For better visualization the maximum integration time $s_{\max}$ varied between 5×10^4 and 5×10^5. The python solver was LSODA.

$0, p_t(s), p_{\theta,0} = 0, p_{\varphi,0} = 0)$, the important point in such an image is the one on its center, representing the (θ, p_θ) components of Y_*. With respect to it, we see that all sufficiently close initial conditions go around in a periodic fashion (librations), with none of them going to far off in the rotation region.

This thus serves as a first numerical evidence for the fact that Y_* is a center equilibrium point over the subspace $\mathcal{S}_1$ defined on Eq.(4.54).

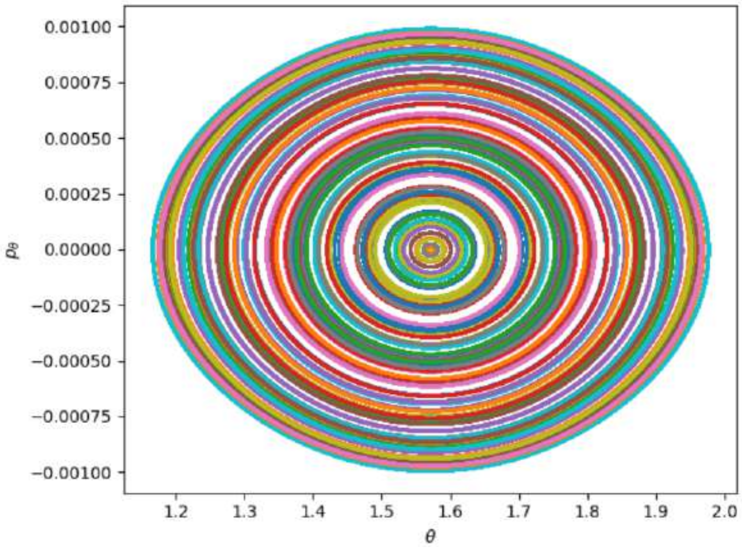


Figure 4.5: We fixed $m = 1, R = 1, \varepsilon = 0.01, \varphi_0 = 0, p_{\varphi,0} = 0$. Single particle dynamics for 100 randomly chosen $\theta_0 \in [\frac{\pi}{2} - 10^{-4}, \frac{\pi}{2} + 10^{-4}]$ and $p_{\theta,0} \in [-10^{-3}, 10^{-3}]$ with $p_{\varphi,0} = 0$.

4.3.4 Details & results on the numerical integration

Throughout the following calculations for this and the following chapter we focused on using the following Python packages: `numpy & numba` for the numerical integration, focusing for now on the LSODA function of the latter and `Scipy` package for the symbolic writing of the metric.

By the above obtained numerical results, we can also give a bound in ε for the overall accuracy of the chosen integration method by computing the relative error for the Hamiltonian of a particle.

Indeed, by looking at Figure 4.6, we observe that for ε values much smaller than 1 we can still guarantee some level of regularity in the integration method, meanwhile greater values imply in higher numerical errors. This result ends up serving a further numerical evidence for the fact that the results obtained through LSODA integrator are more trust worthy precisely in the regime proposed by our Assumption 1, in which we stated that ε should be much smaller than R.

As a final remark, Figure 4.7 tells us that the overall error seems to be of smaller magnitude if the set of initial points is taken closer to the equator which, in a way, is expected as its vicinity is much more regular in the analytical sense than that of the poles.

In either case however the overall error quite large for the expected accuracy requested by the program, which was of order 10^{-10}. A further investigation for a better integration routine will take place in a future work.

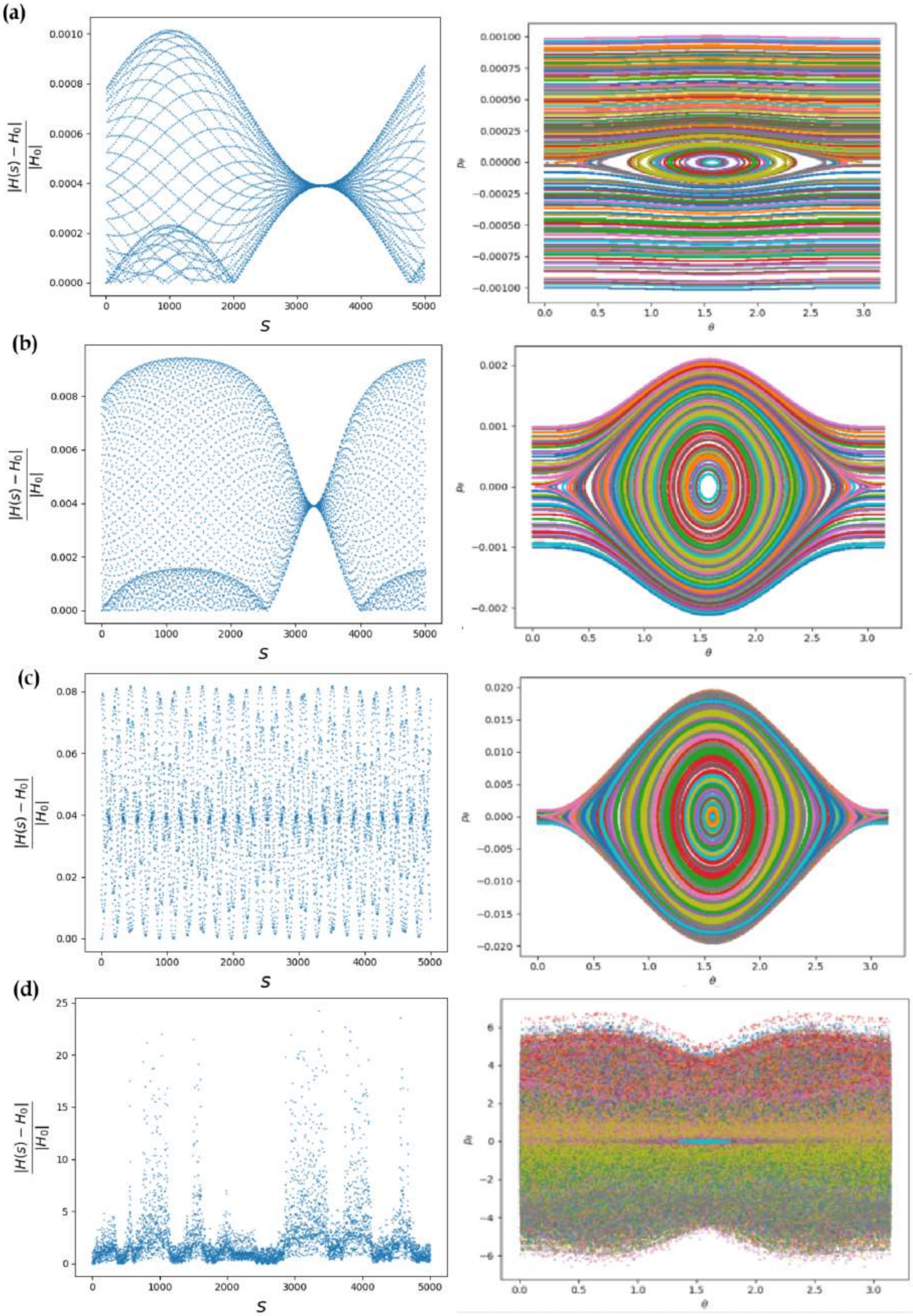


Figure 4.6: (θp_θ plane) with LSODA integration routine on python. Relative Hamiltonian error for a randomly chosen particle for each of the trajectories from Figure 4.4 in which ***(a)*** had $\varepsilon = 10^{-3}$, ***(b)*** had $\varepsilon = 10^{-2}$, ***(c)*** had $\varepsilon = 10^{-1}$, ***(d)*** had $\varepsilon = 1$.

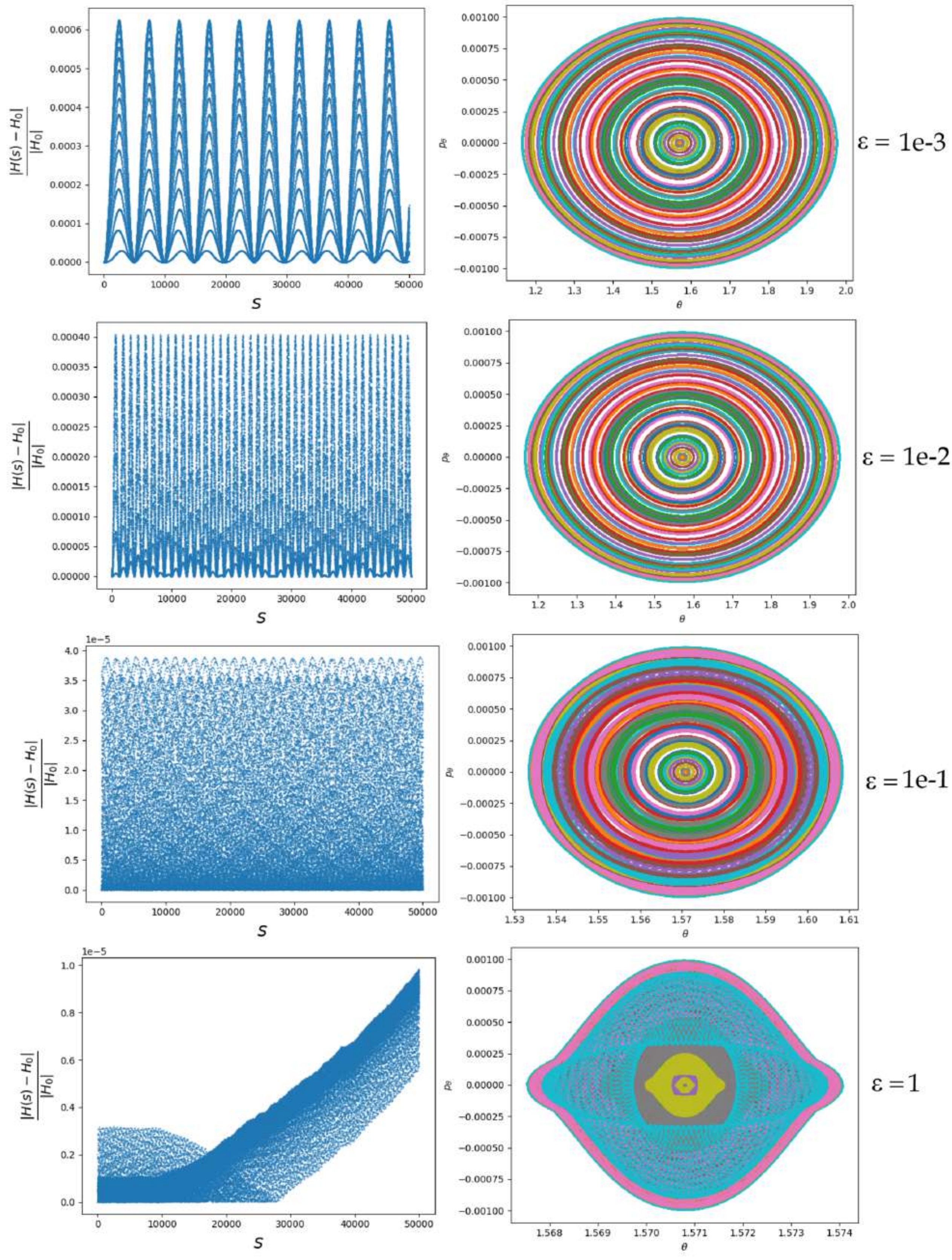


Figure 4.7: (θp_θ plane) with LSODA integration routine on python for the single particle dynamics around the equator. We fixed $\varphi_0 = 0, m = 1, R = 1, p_{\varphi,0} = 0$ and randomly chose 100 points with $\theta_0 \in [\frac{\pi}{2} - 10^{-4}, \frac{\pi}{2} + 10^{-4}]$ and $p_{\theta,0} \in [-10^{-3}, 10^{-3}]$.

Chapter 5

Interacting particle dynamics 1: The Kepler Problem

In the last chapter we saw the motion of a single particle on the perturbed sphere. Now we want to study the motion of two interacting particles. As already mentioned, they do ***not*** alter the space's geometry, which is described by an already given metric - reflecting the Earth and Moon's influences on the atmosphere. To model the particle gravitational interaction, upon considering the extension of gravity to our geometry as being that of a central force field, thus basing ourselves on the Hodge decomposition, we work with the *gravitational Maxwell equations* summarized as

$$dF = 0, \tag{5.1a}$$

$$\delta F = -\gamma j. \tag{5.1b}$$

where j is the current $1-$form possessing the information about the mass distribution and current density, and the $2-$form $F = \frac{1}{2} F_{\mu\nu} e^{\mu} \wedge e^{\nu}$ can be written in matrix form as (see Appendix E)

$$F_{\mu\nu} = \begin{pmatrix} 0 & -\mathcal{E}^x_{\text{grav}} & -\mathcal{E}^y_{\text{grav}} \\ \mathcal{E}^x_{\text{grav}} & 0 & -H \\ \mathcal{E}^y_{\text{grav}} & H & 0 \end{pmatrix} \tag{5.2}$$

We left to Appendix E the calculations relating the electric and magnetic parts of Maxwell's electromagnetic theory to the gravitational acceleration $\mathcal{E}_{\text{grav}}$ and our gravito-magnetic term H, respectively. Moreover, in the case of $\varepsilon-$small amplitude oscillations of the met-

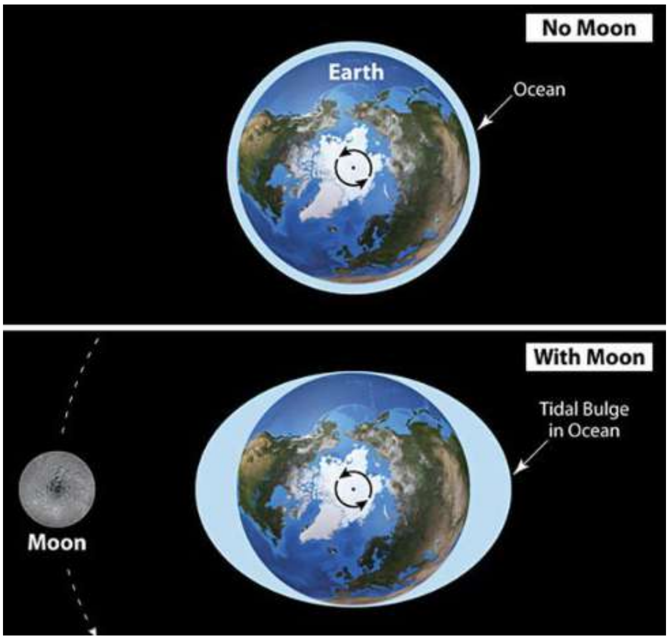


Figure 5.1: Figure taken from the website [Bud] depicting the effect of the Moon on Earth's sea. As shown on Figure 2 however this effect also happens for the atmosphere.

ric, such an H term emerges, being directly proportional to the parameter ε and vanishing in the limit of $\varepsilon \to 0$ (see Section 5.3).

It is important to stress that previous authors like G. Ellis, Roy Maartens and H. van Elst [EvE99, EMM12] have already introduced relativistic gravito-magnetic equations in a different context. In their case, Einstein's field equations are rewritten in a gravito-magnetic way through the use of the *Weyl tensor*, a formalism only possible for *space* dimensions $n \geq 3$ (see subsection B.1.1).

We start our analysis by considering the Kepler problem on the *unperturbed* manifold $\mathbb{R} \times \mathbb{S}^2$, where one interacting particle (view as the source field) is held at the north pole and the other moves on the sphere under the former's influence. The equations of motion in this case are given by the sourced geodesic equations

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = \frac{q}{m}g_{\alpha\beta}F^{\mu\alpha}\frac{dx^\beta}{ds} \tag{5.3}$$

where $F^{\mu\nu}$ is as in Eq.(3.6) and q represents the *gravitational charge* of the particle. As in the previous chapter we can also formulate the equations of motion in terms of a

Hamiltonian (see Appendix D)

$$H = \frac{g^{\mu\nu}}{2m}(p_\mu - qA_\mu)(p_\nu - qA_\nu), \tag{5.4}$$

Observation 5.1. Notice that the above will imply in a change on the relativistic momentum norm [Eq.(4.18)] to

$$-m^2 = g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) \tag{5.5}$$

A further comment for the case of $\mathbb{R} \times \mathbb{S}^2$ is given bellow on Observation 5.2.

Based Eq.(5.26a), we get the following equations of motion

$$\dot{x}^\mu = \frac{1}{m}\left(g^{\mu\alpha}p_\alpha - qg^{\mu\alpha}A_\alpha\right) \tag{5.6a}$$

$$\dot{p}_\mu = -\frac{1}{2m}\partial_\mu g^{\alpha\beta}(p_\alpha - qA_\alpha)(p_\beta - qA_\beta) + \frac{q}{m}g^{\alpha\beta}(p_\alpha - qA_\beta)\partial_\mu A_\beta \tag{5.6b}$$

5.1 The general equations

The gravitational field equations will be given by the following

$$\delta F = -\gamma j, \tag{5.7a}$$

$$dF = 0. \tag{5.7b}$$

which, together with Eq.(5.6) (or Eq.(3.3)), have to be integrated in order for the motion of the particles to be determined (the case of two moving particles, i.e the 2−body problem, will be left to the following chapter). Although difficult, we theoretically solve the former. Seeing F an element of $\mathrm{H}^2(\mathbb{R} \times \mathbb{S}^2)$, we can say that there is some $A \in \Lambda^1(\mathbb{R} \times \mathbb{S}^2)$ and $\gamma \in \mathrm{Ker}(d : \Lambda^2 \to \Lambda^3)$ such that

$$F = -dA + \gamma. \tag{5.8}$$

In such a case Eq.(5.7b) is automatically satisfied and Eq.(5.7a) gets to be written as

$$\delta dA = \Delta_{\mathrm{dR}}A - d\delta A = \gamma j, \tag{5.9}$$

where Δ_{dR} denotes the Hodge-de Rham Laplace operator, a generalization of the Laplace operator for spaces whose metric has arbitrary signature and curvature.

Note that if we could somehow set $\delta A = 0$ in the above, then Eq.(5.9) could be simplified to a Poisson equation, which we know is solvable by Green's functions. Indeed, such a trick is possible once we are provided with the following *gauge transformation*

$$A' = A + d\psi. \tag{5.10}$$

It is a gauge transformation in the sense that the force field F is kept unchanged when we perform it. Indeed

$$F' = -dA' + \gamma = -d(A + d\psi) + \gamma = -dA + \gamma + d^2\psi = -dA + \gamma + 0 = F,$$

so that $F' = F$ and thus, the result we get from solving the field equation for A' is precisely the same we would obtain from solving the field equation for A (at least in terms of the 2−tensor F).

With this in mind, we can simplify Eq.(5.9) by imposing the gauge field ψ to be a solution for the following *gauge condition*

$$\delta A' = 0 \Leftrightarrow \delta d\psi = -\delta A. \tag{5.11}$$

Since $\psi \in \Lambda^0(\mathbb{R} \times \mathbb{S}^2)$ (and so $\delta\psi = 0$) the above gauge condition can be expressed in a potentially more familiar fashion

$$\Delta_{\mathrm{dR}}\psi = -\delta A. \tag{5.12}$$

Since a solution to Eq.(5.12) can always be found[1], the gauge condition for A' expressed on Eq.(5.11) simplifies Eq.(5.9) to the form we wanted, i.e

$$\Delta_{\mathrm{dR}}A' = \gamma j, \tag{5.13}$$

[1]this assertion is due to the fact that for space-time manifolds the Laplace operator becomes of hyperbolic type. Given then a suitable set of initial conditions, expressed as a Cauchy surface for our manifold, the problem is well-posed and a solution can be found [CC03, Car]

the *Poisson equation* with source j. We can write the above in coordinates as

$$(\Delta_{\mathrm{dR}} A')^{\mu} = \nabla_{\nu}\nabla^{\nu} A'^{\mu} + R^{\mu}{}_{\nu} A'^{\nu} = \gamma j^{\mu}, \tag{5.14}$$

and find a solution for it in terms of a Green function $G(x, x') \in \Lambda^0(\mathbb{R} \times \mathbb{S}^2)$ for the Hodge-de Rham Laplacian, satisfying

$$\Delta_{\mathrm{dR}} G(x, x') = \delta^{(3)}(x - x'), \tag{5.15}$$

with $\delta^{(3)}(x - x') = \delta(t - t')\delta(x^1 - x'^1)\delta(x^2 - x'^2)$. From here it follows that $A'^{\mu}(x)$ can be written as [PPV11]

$$A'^{\mu}(x) = \gamma \int_{\mathbb{R}\times\mathbb{S}^2} G(x, x') j^{\mu}(x') d^3x', \tag{5.16}$$

where $d^3x' = \sqrt{-\det(g)(t', \theta')} dt' d\theta' d\varphi'$ is the volume element of $\mathbb{R} \times \mathbb{S}^2$.

Given the solution A'^{μ} of Eq.(5.14) to find the particle's orbit, it remains to plug it into the sourced geodesic equation after computing the $F^{\mu\nu}$ term, or simply calculate its derivatives and solve the set of Eqs.(5.6).

The case of surfaces of revolution

Previously, on Section 4.2, we briefly considered the metric tensor for a surface of revolution on Minkowski space-time which had the equator as an invariant submanifold. For this to be the case we argued that the metric could only have this form

$$g = \begin{pmatrix} g_{tt}(\theta) & 0 & 0 \\ 0 & g_{\theta\theta}(\theta) & 0 \\ 0 & 0 & g_{\varphi\varphi}(\theta) \end{pmatrix}. \tag{5.17}$$

By then using the definition of the Ricci tensor, we can compute what Eq.(5.14) becomes based on the above metric tensor. The only non-zero components will be $R^0{}_0, R^1{}_1$ and

$R^2{}_2$, whose expressions are

$$\begin{aligned}R^0{}_0 =& -\frac{d^2 g_{tt}}{d\theta^2}\frac{1}{(2g_{\theta\theta}(\theta)g_{tt}(\theta))} + \left(\frac{dg_{tt}}{d\theta}\right)^2\frac{1}{4g_{\theta\theta}(\theta)g_{tt}(\theta)^2}\\ &- \frac{dg_{\varphi\varphi}}{d\theta}\frac{dg_{tt}}{d\theta}\frac{1}{4g_{\theta\theta}(\theta)g_{\varphi\varphi}(\theta)g_{tt}(\theta)} + \frac{dg_{\theta\theta}}{d\theta}\frac{dg_{tt}}{d\theta}\frac{1}{4g_{\theta\theta}(\theta)^2 g_{tt}(\theta)},\end{aligned} \tag{5.18a}$$

$$\begin{aligned}R^1{}_1 =& -\frac{d^2 g_{tt}}{d\theta^2}\frac{1}{(2g_{\theta\theta}(\theta)g_{tt}(\theta))} + \left(\frac{dg_{tt}}{d\theta}\right)^2\frac{1}{4g_{\theta\theta}(\theta)g_{tt}(\theta)^2}\\ &- \frac{d^2 g_{\varphi\varphi}}{d\theta^2}\frac{1}{2g_{\theta\theta}(\theta)g_{\varphi\varphi}(\theta)} + \left(\frac{dg_{\varphi\varphi}}{d\theta}\right)^2\frac{1}{4g_{\theta\theta}(\theta)g_{\varphi\varphi}(\theta)^2}\\ &+ \frac{dg_{\theta\theta}}{d\theta}\frac{dg_{tt}}{d\theta}\frac{1}{4g_{\theta\theta}(\theta)^2 g_{tt}(\theta)} + \frac{dg_{\theta\theta}}{d\theta}\frac{dg_{\varphi\varphi}}{d\theta}\frac{1}{4g_{\theta\theta}(\theta)^2 g_{\varphi\varphi}(\theta)},\end{aligned} \tag{5.18b}$$

$$\begin{aligned}R^2{}_2 =& -\frac{d^2 g_{\varphi\varphi}}{d\theta^2}\frac{1}{(2g_{\theta\theta}(\theta)g_{\varphi\varphi}(\theta))} - \frac{dg_{\varphi\varphi}}{d\theta}\frac{dg_{tt}}{d\theta}\frac{1}{4g_{\theta\theta}(\theta)g_{\varphi\varphi}(\theta)g_{tt}(\theta)}\\ &+ \left(\frac{dg_{\varphi\varphi}}{d\theta}\right)^2\frac{1}{4g_{\theta\theta}(\theta)g_{\varphi\varphi}(\theta)^2} + \frac{dg_{\theta\theta}}{d\theta}\frac{dg_{\varphi\varphi}}{d\theta}\frac{1}{4g_{\theta\theta}(\theta)^2 g_{\varphi\varphi}(\theta)}.\end{aligned} \tag{5.18c}$$

From the above, we see that the field equations can be rewritten, in this specific case, as[2]

$$\nabla_\nu \nabla^\nu A^0 + R^0{}_0 A^0 = \gamma j^0, \tag{5.19a}$$
$$\nabla_\nu \nabla^\nu A^1 + R^1{}_1 A^1 = \gamma j^1, \tag{5.19b}$$
$$\nabla_\nu \nabla^\nu A^2 + R^2{}_2 A^2 = \gamma j^2. \tag{5.19c}$$

Recall however that, since A^μ is a short hand notation for the vector $A = A^\mu e_\mu$, each of these covariant derivative are actually dependent on the Christoffel symbols in quite an intricate way

$$\begin{aligned}\nabla_\nu \nabla^\nu A^\mu =& \partial_\nu(\partial^\nu A^\mu g^{\nu\sigma} A^\beta \Gamma^\mu_{\beta\sigma}) + \Gamma^\nu_{\alpha\nu}(\partial^\alpha A^\mu g^{\alpha\sigma} A^\beta \Gamma^\mu_{\beta\sigma})\\ &+ \Gamma^\mu_{\alpha\nu}(\partial^\nu A^\alpha g^{\nu\sigma} A^\beta \Gamma^\alpha_{\beta\sigma}),\end{aligned} \tag{5.20}$$

based on which Eqs.(5.19) can be computed.

5.2 The Kepler Problem over the 2−sphere

The Kepler Problem is, broadly speaking, the scenario where we deal with a point source that is fixed at some place on our manifold and consider the movement a test particle

[2]we drop the prime since the gauge condition lets us interchange between A^μ and A'^μ without losing the physical significance of the model

would go when under the action of said source. In what follows we will study some of the dynamical properties of this problem, proving the existence and unicity of an *equilibrium configuration* when $p_{\theta,0} = 0$, numerically analyzing its stability in the θ variable and also studying the solution's behavior close to such an equilibrium configuration when $p_{\theta,0} \neq 0$.

In all cases, the particle is going to be fixed at the north pole and the fact that it shall remain fixed is attained by setting $j^k = 0$, for $k = 1, 2$ (recall that Roman letters are designated to space indices, whereas Greek ones are for space-time). For the EM theory, this fact represents the *electrostatic regime* of the theory which, in our gravitational case, implies that no magnetic like terms should appear. Indeed, we have the following

Corollary 5.2.1. *The field equations for the unperturbed manifold* $\mathbb{R} \times \mathbb{S}^2$ *can be written as*[3]

$$\Delta_{\mathbb{S}^2}\phi = \gamma\rho, \tag{5.21a}$$

$$\nabla_0\nabla^k\phi = \gamma j^k = 0. \tag{5.21b}$$

Proof: There are two ways to show this statement. The longer one involves directly computing the Christoffel symbols and the Ricci tensor for the manifold $\mathbb{R} \times \mathbb{S}^2$ with metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2(\theta) \end{pmatrix}. \tag{5.22}$$

The only non-zero Christoffel symbols will be

$$\Gamma^1_{22} = -\sin(\theta)\cos(\theta), \quad \Gamma^2_{12} = \frac{\cos(\theta)}{\sin(\theta)}, \tag{5.23}$$

plus the Ricci tensor will be given by

$$R^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.24}$$

Plugging all of these back into Eqs.(5.19) will give us Eqs.(5.21).

The second way of doing it is by noticing that $A^k = 0$ is a consistent solution to the field equations [Eq.(5.19)] in the sense that we get to no contradictions if such an equality is imposed. Based on it, we can conclude that $H = 0$ [Eq.(E.0.9)], as expected. By using the formula for $\mathcal{E}_{grav}$ [Eq.(E.0.5)] on Eqs.(E.0.7, E.0.10), the above result follows immediately. □

More details for this second proving method will be given in [SB25]. Practically speaking, the above means that *our gravitational field is curl free and time independent*. The equation of interest here is just Eq.(5.21a), which was already studied for the case of surfaces of revolution in [DB15a].

For a sphere of unit radius, a mass M located at the north pole generates the following potential [BDS16]

$$\phi(\theta) = \frac{\mathrm{M}}{2\pi}\log(1-\cos(\theta)) = \frac{M}{2\pi}\tilde{\phi}(\theta), \tag{5.25}$$

based on which Eq.(5.4) can be written as

$$H = \frac{1}{2m}\left(-\left(p_t + q\frac{M}{2\pi}\tilde{\phi}(\theta)\right)^2 + p_\theta^2 + \frac{1}{\sin^2(\theta)}p_\varphi^2\right), \tag{5.26a}$$

$$-m^2 = -\left(p_t + q\frac{M}{2\pi}\tilde{\phi}(\theta)\right)^2 + p_\theta^2 + \frac{1}{\sin^2(\theta)}p_\varphi^2 \tag{5.26b}$$

Observation 5.2. Notice that the relativistic condition (originally given by Eq.(4.18)) in this case has to be altered accordingly. This is because if we omit the potential term $\phi(\theta)$ from the time momentum, the two equalities above shall end up yielding contradictory results. This becomes more evident once we write out the equations of motion

$$\dot{t} = \frac{\partial H}{\partial p_t} = \frac{-1}{m}\left(p_t + q\frac{M}{2\pi}\tilde{\phi}(\theta)\right), \quad \dot{p}_t = -\frac{\partial H}{\partial t} = 0 \tag{5.27a}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m}, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{1}{m}\left(q\frac{M}{2\pi}\left(p_t + q\frac{M}{2\pi}\tilde{\phi}(\theta)\right)\partial_\theta\tilde{\phi}(\theta) + \frac{\cos(\theta)}{\sin^3(\theta)}p_\varphi^2\right) \tag{5.27b}$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m\sin^2(\theta)}, \quad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \tag{5.27c}$$

Note that once more we have $p_\varphi = L$ and p_t as our constants of motion. Also note now that, p_t need ***not*** be strictly less than zero, but instead the whole expression $p_t + q\phi(\theta)$ does.

Had we then forgotten the potential term on Eq.(5.26b), one would hence be able to erroneously conclude (after substituting such an equation on Eq.(5.26a)) that the Hamiltonian is *not* a constant of motion on the orbits generated by itself, because it would be dependent on $\phi(\theta)$ which evidently varies as (proper) time passes.

5.2.1 Particle dynamics & periodic motions

We now numerically solve the set of Eqs.(5.27) and plot the dynamics of the moving particle on the sphere accordingly. We can clearly conclude that the system has to be integrable since the Hamiltonian H and the momenta p_t and p_φ are first integrals of motion in involution, meaning that the system should exhibit some type of regular behavior. As mentioned previously though, we will divide the analysis for $p_{\theta,0} = 0$ case.

On the existence of periodic motions

Let's begin by drawing some attention to the fact that *the motion is still bounded by latitudinal circles* like in the oscillating single particle case discussed at the end of the last chapter.

As expected for this problem, the test particle revolves around the source mass, located at the north pole, in a bounded region. As we vary the initial latitude value the lower circle which bounds the dynamics from below also varies accordingly and, although hard to tell just from Figure 5.2, the upper circle varies too.

Much like before, we ought to make use of Eq.(5.26b). By sticking to the above notation for $p_\varphi = L$, notice that once the initial positions $(t_0, \theta_0, \varphi_0)$ and momenta $(p_{\theta,0}, L)$ are given, we can compute p_t to be

$$p_t = -\sqrt{m^2 + p_{\theta,0}^2 + \frac{L^2}{\sin^2(\theta_0)}} - q\frac{M}{2\pi}\tilde{\phi}(\theta_0) \tag{5.28}$$

Recall that the extreme circles are found when the angle $\theta(s)$ achieves its maximum or minimum values. Being it a differentiable function over $\mathbb{R}$, this is tantamount to

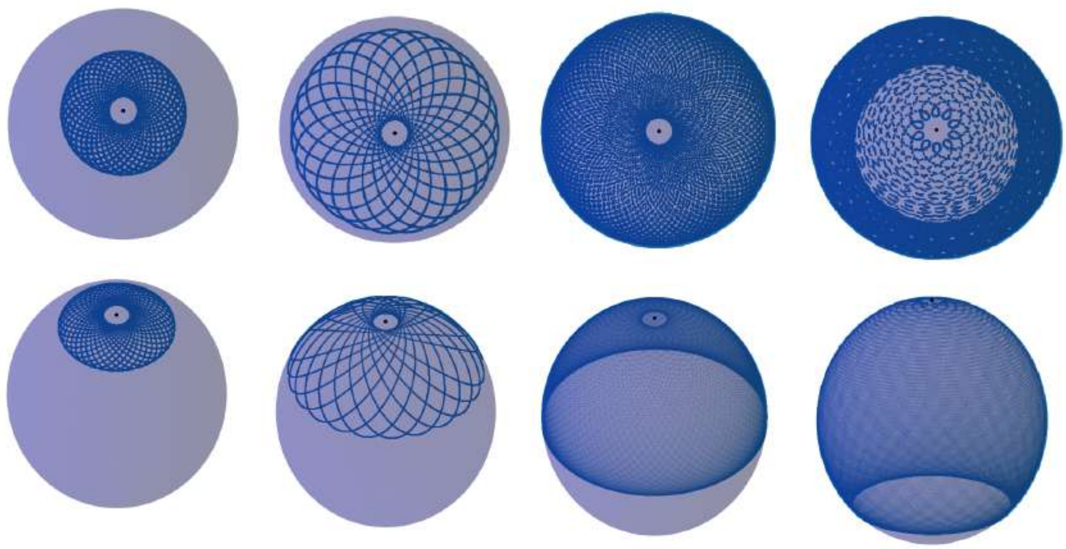


Figure 5.2: Motion over $\mathbb{S}^2$. The parameters were $m = 1, M = 100, p_{\theta,0} = 0, \varphi_0 = 0, p_{\varphi,0} = 10$. Figure depicting the particle motion of the moving particle in the Kepler problem considered. The upper row is a top view (from the north pole) of the solution, meanwhile the botton is an inclined view of the respective trajectory. The black dot represents the fixed particle located at the north pole while the blue curve is the solution to Eqs.(5.27) for the moving particle. The initial latitude values *from left to right* were given by $\theta_0 = \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{4}$

demanding $\dot{\theta} = 0$, which in turn implies that by setting $p_\theta = 0$ we can find such circles, as argued before. After some minor algebraic manipulations we arrive at

$$\sin^2(\theta)\left(\left(p_t + \frac{qM}{2\pi}\ln(1-\cos(\theta))\right)^2 - m^2\right) - L^2 = 0, \tag{5.29}$$

whose solution, although not that trivial, can nonetheless be numerically approximated, with each circle being given by the two *smallest positive* solutions to the above.

We hence see a dependence of these limiting circles with the north pole mass M. Indeed, an interesting behavior emerges when we change M's values within a certain range. What happens is, *what once was the upper limit circle, suddenly becomes the lower limit one*, and vice-versa. This process shows that **there exists a value for M which yields a periodic (circular) orbit** for the system. This orbit will be given by the latitudinal circle the particle started on, which we illustrate as a red curve on Figure 5.3. The same result is to be expect as we vary the moving particle mass m and let M remain fixed.

For, what matters most in this situation is predominantly their ration, rather then each of their single values. More formally, we have

Proposition 5.2.2. *Given an initial condition* $(t_0, \theta_0, \varphi_0, p_t, p_{\theta,0} = 0, L \neq 0)$ *for any* $t_0 \in \mathbb{R}$, $\theta_0 \in (0, \pi)$ *and* $\varphi_0 \in [0, 2\pi)$, *there exists a unique mass value* M *for which the orbit is a latitudinal circle with angular velocity* $\omega_\varphi = \frac{L}{m \sin^2(\theta_0)}$.

Proof of Proposition 5.2.2: We can show unicity and existence directly by a handful of simple calculations. The only way the orbit can be a latitudinal circle is if $\dot{\theta} = \dot{p}_\theta = 0$, that is, if the initial condition is an equilibrium point in (θ, p_θ) subspace.

Given that $p_{\theta,0} = 0$ we already have that $\dot{\theta} = 0$. Now, we want to argue that we can find an M value for which $\dot{p}_\theta = 0$ too. To do so we go back to Eq.(5.27b) and Eq.(5.28). By setting the former to zero and rewriting the later as

$$p_t + q\frac{M}{2\pi}\tilde{\phi}(\theta_0) = -\sqrt{m^2 + \frac{L^2}{\sin^2(\theta_0)}}. \tag{5.30}$$

From the above, we see that $\dot{p}_\theta = 0$ boils down to

$$-q\frac{M}{2\pi}\sqrt{m^2 + \frac{L^2}{\sin^2(\theta_0)}}\partial_\theta\tilde{\phi} + \frac{\cos(\theta_0)}{\sin^3(\theta_0)}L^2 = 0 \tag{5.31}$$

from which we have the single M value

$$M = \frac{\cos(\theta_0)}{\sin^3(\theta_0)} \frac{2\pi L^2}{q\partial_\theta\tilde{\phi}(\theta_0)\sqrt{m^2 + \frac{L^2}{\sin^2(\theta_0)}}} \tag{5.32}$$

which shows the first part. The angular velocity's computation follows immediately from the fact that (by setting $p_\varphi = L$)

$$\dot{\varphi} = \frac{L}{m \sin^2(\theta_0)} =: \omega_\varphi \tag{5.33}$$

which then functions as the frequency of rotation/angular velocity for the integrable subsystem $(t, \varphi, p_t, p_\varphi)$. $\square$

Note in particular that if $\theta_0 > \frac{\pi}{2}$ and $q > 0$ *no* M value can be found which will yield a circular orbit since in this case $M < 0$. If $q < 0$ however, clearly the opposite happens.

Going back to Figure 5.3, after plugging $\theta_0 = \frac{\pi}{3}, L = 10, \varphi_0 = 0$ and setting $p_{\theta,0} = 0$ on Eq.(5.32) we get the approximate value of $M = 24.0938085184$, which seems to fit with the overall behavior.

A more difficult question to ask now is if such an equilibrium configuration exists for *all* values of $M \in \mathbb{R}$. That is, given some $M \in \mathbb{R}$, can we find a *unique* θ_0 for which the initial condition $(t_0, \theta_0, \varphi_0, p_t, p_{\theta,0} = 0, L \neq 0)$ yields and equilibrium configuration? A partial affirmative answer to that question is given in the following

Proposition 5.2.3. *Given a value for the mass M, we can always find a latitude angle θ_0 for which the initial condition $(t_0, \theta_0, \varphi_0, p_t, 0, L)$, where $\varphi_0 \in \mathbb{S}, L \neq 0$ and $t_0 \in \mathbb{R}$, gives a periodic circular orbit for the Kepler problem on $\mathbb{R} \times \mathbb{S}^2$. Furthermore*

PO.1 *if $q > 0$, then $\theta_0 \in (0, \pi/2)$*

PO.2 *if $q < 0$, then $\theta_0 \in (\pi/2, \pi)$*

Proof: We start from Eq.(5.31). It is the equation that actually gives us the condition necessary for $(t_0, \theta_0, \varphi_0, p_t, 0, L)$ to yield a circular orbit for the system. We showed that given θ_0, L, q and m we could always find an M that solved it. Now let's show that given M, L, q and m, we can always find a θ_0 that solves it too.

The first thing to do is rewrite such an equation in a way that gets us ride of the poles at $\theta = 0, \pi$. We can do so by noting that

$$\partial_\theta \tilde{\phi} = \frac{\sin(\theta)}{1 - \cos(\theta)} = \frac{\sin(\theta)}{\sin^2\left(\frac{\theta}{2}\right)}, \tag{5.34}$$

From which Eq.(5.31) can be brought to the form

$$-q\frac{M}{2\pi}\sqrt{m^2 + \frac{L^2}{\sin^2(\theta)}}\frac{\sin(\theta)}{\sin^2\left(\frac{\theta}{2}\right)} + \frac{\cos(\theta)}{\sin^3(\theta)}L^2 = 0. \tag{5.35}$$

We can multiply both sides by sine cubed and further simplify the term under the square root, thus getting to

$$-q\frac{M}{2\pi}\frac{\sin^3(\theta)}{\sin^2\left(\frac{\theta}{2}\right)}\sqrt{L^2+m^2\sin^2(\theta)}+\cos(\theta)L^2=0. \tag{5.36}$$

By using the fact that

$$\sin(\theta)=\sin\left(2\frac{\theta}{2}\right)=2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right), \tag{5.37}$$

we can finally transform Eq.(5.31) into the function

$$f(\theta;q):=-\frac{2qM}{\pi}\sin\left(\frac{\theta}{2}\right)\cos^3\left(\frac{\theta}{2}\right)\sqrt{L^2+m^2\sin^2(\theta)}+\cos(\theta)L^2=0. \tag{5.38}$$

The idea now is to use the Intermediate Value Theorems (IVT) on $f(\theta;q)$. It is evidently continuous in the closed interval $[0,\pi]$ and is such that

$$f\left(0;q\right)=L^2, \tag{5.39a}$$

$$f\left(\frac{\pi}{2};q\right)=-\frac{qM}{2\pi}\sqrt{L^2+m^2}, \tag{5.39b}$$

$$f\left(\pi;q\right)=-L^2. \tag{5.39c}$$

And so, by the IVT, for $q>0$ we can guarantee the existence of at least one root within $(0,\pi/2)$, meanwhile for $q<0$ we assure so in the interval $(\pi/2,\pi)$, thus proving assertions **PO.1** and **PO.2**. □

As a final remark, notice that we can also ask ourselves what effect will it have on the orbit if we change q. Its physical significance was that of a *gravitational charge* which mathematically speaking is allowed to assume positive or negative values, similar to our usual concept of electromagnetic charge. It is important to emphasize the mathematical overtone of this procedure, provided that the equivalence principle implies among other things in the equality between gravitational and rest charges (or masses here) of a particle. In going against such a postulate for the present moment we are assuming that gravity *can* be repulsive and, by the form of the equations of motion, this would be phenomenologically equivalent to the presence of negative (rest) masses within our space. In light of this, we would expect an overall similar though *reversed* behavior to the solution curves of the

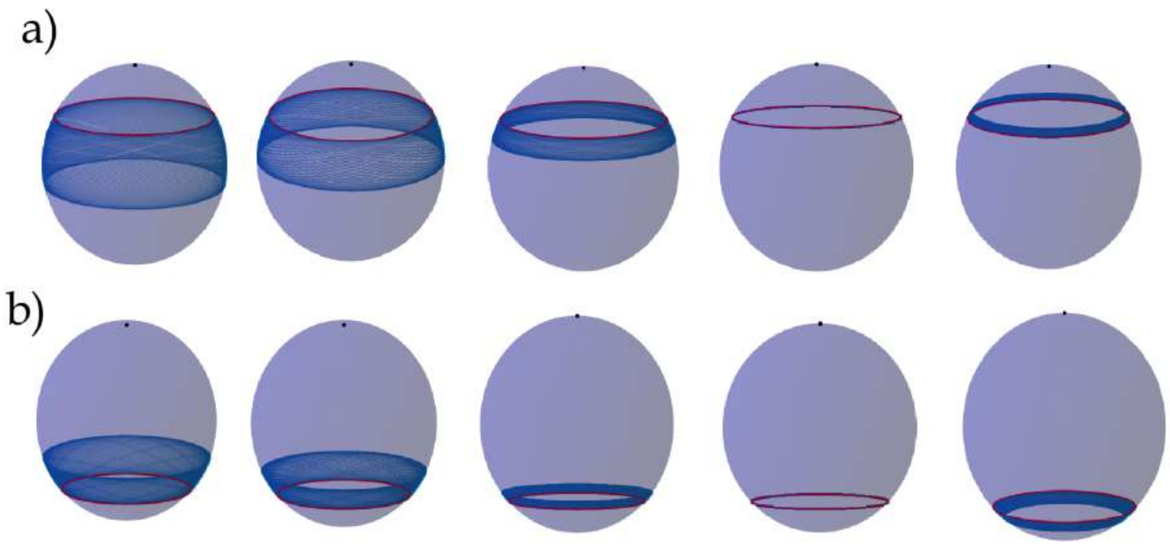


Figure 5.3: *In both cases the red curve is the latitudinal circle on which the particle began its motion.* Picture **a)** depicts the change in the limiting circles for the Kepler Problem over $\mathbb{R} \times \mathbb{S}^2$ with $m = 1, q = m, \theta_0 = \frac{\pi}{3}, p_{\theta,0} = 0, L = 10, \varphi_0 = 0$ and M value given by $M = 10, 15, 20, 24, 26$, respectively. Picture **b)** depicts the change in the limiting circles for the Kepler Problem over $\mathbb{R} \times \mathbb{S}^2$ with $m = 1, q = -m, \theta_0 = \frac{3\pi}{4}, p_{\theta,0} = 0, L = 10, \varphi_0 = 0$ and M values give by $M = 100, 120, 170, 210, 300$, respectively.

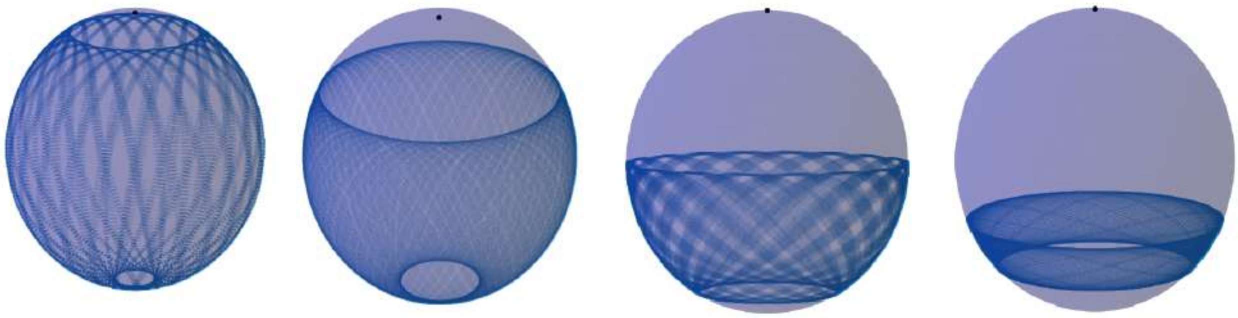


Figure 5.4: Figure depicting the particle motion of the moving particle in the Kepler problem on $\mathbb{R} \times \mathbb{S}^2$. The fixed values were $M = 100, m = 1, q = -m, p_{\theta,0} = 0, \varphi_0 = 0, p_{\varphi,0} = 10$. The initial latitude values *from left to right* were given by $\theta_0 = \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{4}$

Kepler problem when $q = -m$, at least when compared to the results from the previous figures.

As we can check from Figure 5.4, this does indeed seem to be the case. As we get closer to the pole mass M, the gravitational *repulsion* increases, meaning that the moving particle should go farther away. When the starting position is close to the south pole however, repulsion diminishes so that the particle will be bound to move within a smaller strip then previously. As guaranteed by Proposition 5.2.3, for negative q values the equilibrium configuration will sit beneath the equator, as can be verified on Figure 5.3 b).

To find the M value which corresponds to this equilibrium configuration we put $q = -m, m = 1, \theta_0 = \frac{3\pi}{4}, L = 10$ and set $p_{\theta,0} = 0$ on Eq.(5.32), thus obtaining $M = 213.987064289$.

The question of stability of such circular orbits will be further explored in the final version of the preprint [SB25].

5.3 The magnetic component for the oscillating case

Our goal here will be to show the following

Proposition 5.3.1. *The magnetic component of the gravitational field $F_{\mu\nu}$ vanishes as $\varepsilon \to 0$.*

Such a limit is important since it portrays our model's self-consistency, given that the magnetic component was meant to only exists when the $\varepsilon-$small perturbation was on.

Proof of Proposition 5.3.1: Going back to Eqs.(5.21), recall that upon setting $j^k = 0$, then $A^k = 0$ was a valid solution to the field equations used, which led us to conclude that the field ϕ was static and curl free.

This last consideration is crucial because, for a point particle with mass m located at position $x_0(s)$ (for some proper time s), its density is given by

$$\rho(x; s) = m\delta(x - x_0(s)), \tag{5.40}$$

whose associated current density components are

$$j^\mu(x; s) = \rho(x; s)V^\mu(x; s) \tag{5.41}$$

with V^μ being the particle's velocity field. We thus see that if we introduce a metric perturbation $h_{\mu\nu}(t, \theta)$, the first order approximation we made on Section 4 saying that

$\overline{V}^k = V^k + \varepsilon C^k + \mathrm{O}(\varepsilon^2)$ will yield us that $j^k = 0$ and $A^k = 0$ is ***no longer*** a consistent solution to the equations of motion. Indeed, the above tell us that

$$\bar{j}^k = j^k + \varepsilon\rho C^k + \mathrm{O}(\varepsilon^2) = j^k_{(0)} + \varepsilon j^k_{(1)} + \mathrm{O}(\varepsilon^2), \tag{5.42}$$

and so, ρC^k shall act as a *background induced effective current.*

In this way, under the effect of the first order perturbation for the velocity field, upon setting $j^k_{(0)} = 0$, solving the differential equation for $C^k(x; s)$ [Eq.(4.4)] and expanding the Green function $G(x, x')$ (the solution to Eq.(5.15)) in term of ε as

$$G(x, x') = G_{(0)}(x, x') + \varepsilon G_{(1)}(x, x') + \mathrm{O}(\varepsilon^2), \tag{5.43}$$

by Eq.(5.16), $A'^k(x)$ becomes

$$\begin{aligned} A'^k(x) &= \varepsilon\gamma m \int_{\mathbb{R}\times\mathbb{S}^2} (G_{(0)}(x, x') + \varepsilon G_{(1)}(x, x'))j^k_{(1)}(x')d^3x' + \mathrm{O}(\varepsilon^2) \\ &= \varepsilon\gamma \int_{\mathbb{R}\times\mathbb{S}^2} G_{(0)}(x, x')j^k_{(1)}(x')d^3x' + \mathrm{O}(\varepsilon^2) \end{aligned} \tag{5.44}$$

It hence becomes easy to see that as $\varepsilon \to 0$ we have $A^k \to 0$. By Eq.(E.0.9), we know $B \propto \nabla^1 A^2 - \nabla^2 A^1$, from which it follows that $B \propto \varepsilon$, and thus

$$\lim_{\varepsilon\to 0} B(\varepsilon) = 0, \tag{5.45}$$

as we wanted to prove. $\square$

Chapter 6

Interacting particle dynamics 2: The 2−body problem

6.1 Computational Algorithm

The process of solving the field equations for 2 interacting particles and finding their relative motion is actually quite intricate. In what follows we give a short sketch illustrating how one could solve the equations of motion (in geodesic form) computationally.

Recall that for a single particle with mass m located at position $r_{(0)}$ with velocity $v_{(0)}$, the density current is given by

$$j_{(0)} = (\rho, j^1, j^2) = (m\delta(r - r_{(o)}), mv^1_{(o)}, mv^2_{(o)}). \tag{6.1}$$

When considering however the 2 body interactions we need to have in mind that particle 1 is, for a brief instant, a *test particle* is the field of particle 2, and vice-versa. This means that the actual set of equations to be considered for a system composed of 2 *real, interacting* particles is

$$dF_{(1)} = 0, \quad \delta F_{(1)} = -\gamma j_{(1)}, \tag{6.2a}$$
$$dF_{(2)} = 0, \quad \delta F_{(2)} = -\gamma j_{(2)}. \tag{6.2b}$$

where $F_{(j)}$ stands for the gravitational field generated by particle $j = 1, 2$. This in turn means that the sourced geodesic equations to be considered are

$$\frac{d^2x^\mu_{(1)}}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha_{(1)}}{ds}\frac{dx^\beta_{(1)}}{ds} = \frac{q}{m}g_{\alpha\beta}F^{\mu\alpha}_{(2)}\frac{dx^\beta_{(1)}}{ds}, \tag{6.3a}$$

$$\frac{d^2x^\mu_{(2)}}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha_{(2)}}{ds}\frac{dx^\beta_{(2)}}{ds} = \frac{q}{m}g_{\alpha\beta}F^{\mu\alpha}_{(1)}\frac{dx^\beta_{(2)}}{ds}. \tag{6.3b}$$

where $x^\mu_{(j)}$ is the position of particle $j = 1, 2$. From here notice that, once the initial values of position and momenta of each particle are specified, the first thing we ought to do numerically is to solve Eqs.(6.2a, 6.2b) in a small proper time interval Δs. This gives us the "next time step" (since the system is discretized) $F^{\mu\nu}$. With their values in hands, we then plug in such results at Eqs.(6.3a, 6.3b) integrating only *one time step* once more in order to know the updated positions of the particles, based on which we ought to recalculate the new $F^{\mu\nu}$ components and from there repeat the process until the final integration time is hit [Nä22].

A thorougher (numerical) exploration of these sets of equations will be done in future works.

Part II

A study on Braid Theory with applications to Integrable systems

Chapter 7

A brief overview on Integrable Systems

As outlined in the Introduction, the study o integrable system is an old matter, dating back to the times of Jacob Hamilton and Joseph Liouville, being them one of the main figures in the study of such type of systems. Out of all the possible systems one may be interested in studying, here we will restrict our attention to a particular one, called a *Hamiltonian system*. To explain what is a Hamiltonian system, first we begin considering a system of n particles whose configuration space is the manifold $\mathcal{Q}$. Then, let $\mathcal{M} = T^*\mathcal{Q}$ be the cotangent bundle[1] of $\mathcal{Q}$, seen as a symplectic manifold (Definition A.4.1) with canonical $1-$form θ and symplectic form $\omega = -d\theta$, be our phase space. If we write the positions and momenta of each particle as $\mathbf{q}_i$ and $\mathbf{p}_i$ respectively, also denoting the local coordinates of $\mathcal{M}$ as the tuple $(q^1, \ldots, q^n, p_1, \ldots, p_n)$, we will locally have that

$$\omega = \sum_j dq^j \wedge dp_j. \tag{7.1}$$

It is common in the literature to refer to the local coordinates (q_i, p_i) as the ***conjugate variables*** of the system, or the ***conjugate pair of variables***, equivalently.

In this fashion, the system will be called **Hamiltonian** if we can find a function $H \in C^\infty(\mathcal{M})$, such that the time evolution of the phase space coordinates can be described

[1] see Appendix A.1

by the following differential equation

$$\iota_{X_H}\omega = dH, \tag{7.2}$$

or equivalently in coordinates

$$\dot{\mathbf{q}}_i = \frac{\partial H}{\partial \mathbf{p}_i}, \tag{7.3a}$$

$$\dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{q}_i}, \tag{7.3b}$$

where, notably $(\dot{\mathbf{q}}, \dot{\mathbf{p}}) = X_H \in \Gamma(T\mathcal{M})$. Moreover, as laid out on Appendix A.4 one can construct on a symplectic manifold an elementary Poisson structure $\{\cdot, \cdot\}$ (Definition A.4.3) defined over the set $C^\infty(\mathcal{M})$ as

$$\{f, g\} := \sum_j \left(\frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j} \right). \tag{7.4}$$

From it, once $H(q,p)$ is given, we can reframe Eqs.(7.3) to assume the following form:

$$\dot{\mathbf{q}}_i = \{\mathbf{q}, H\} \tag{7.5a}$$

$$\dot{\mathbf{p}}_i = \{\mathbf{p}, H\} \tag{7.5b}$$

So that the time evolution of the system can equivalently be described by Eqs.(7.5). Now, having clarified this matter, we can move on to the key concept of this chapter, namely

Definition 7.0.1 (Liouville Integrability). Let $(\mathcal{M}, \omega)$ be a $2n-$dimension symplectic manifold. A Hamiltonian system over $\mathcal{M}$ is called **Liouville integrable** if one can find n linearly independent first integrals $\{\Phi_j\}_{j=1,\dots,n}$ that are in involution[2].

It is worth mentioning that, there is quite a difference between the notions of *integrability* and *solvability* of physical system. The former has to do with internal properties of the system itself, such as the existence of periodic behaviors and conserved quantities. Whereas the latter has more to do with one's computational power and ability. [Tor16]

For example, we can clearly solve the $1-$dimensional problem of a falling body subject to a constant gravitational force and drag proportional to its velocity. This same system

[2] see Definition A.4.4

however is **not** Liouville integrable (since energy, the only possible integral of motion in this case, is not a conserved quantity anymore). Conversely, the classical two body problem is famously Liouville integrable as much as hard to solve, due to the appearance of elliptic integrals.

In a practical sense, a Hamiltonian system being integrable means that the equations of motion can be (numerically) integrated and the system reduced to the computation of these various integrals, usually referred to as the *quadratures* of the system.

Notice further that the above concept of integrability only works for the case of a Hamiltonian system. The fact that such a symplectic structure exists, and that the *involution* requirement is meat greatly simplifies the work needed. For instance, if we are dealing with a $2n$ degree of freedom system which is *not* Hamiltonian, to fully integrate it we would a priori need precisely $2n-1$ first integrals, instead of just n.

Getting back to our case of interest, a particular set of integrable Hamiltonian systems are the autonomous[3] 1 degree of freedom (d.o.f) ones. To give a proof for this fact we ought to make use of the following

Definition 7.0.2 ($y-$Regularity)**.** Let $f : \mathcal{D}_f \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an at least $\mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^n)$ function. A point $(x_0, y_0) \in \mathcal{D}_f$ will be called ***y−regular*** if

$$\det\left[\frac{\partial f}{\partial y}(x_0, y_0)\right] \neq 0 \tag{7.6}$$

A similar definition holds for $x-$regularity at a point.

With the above definition in mind we can show the following

Theorem 7.0.1. *Let H be a Hamiltonian for an autonomous* 1 *d.o.f system whose conjugate pair of variables is given by (x, y), so that the equations of motion read*

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \tag{7.7a}$$

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}. \tag{7.7b}$$

[3] one in which the Hamiltonian is time independent

Let (x_0, y_0) be a $y-$regular starting point of the dynamics on the phase space. Then, there exists a neighbourhood $\mathcal{U}_{x_0}$ on the $x-$space over which the system is Liouville integrable.

Proof: Since the Hamiltonian is time independent, it is a conserved quantity. By Definition 7.0.1 the system with $n = 1$ that has 1 integral of motion is Liouville integrable, which is the case since $\{H, H\} = 0$. Now, let $H(x(t), y(t)) = E$ be a level set for H and take $(x(0), y(0)) = (x_0, y_0)$. By the Implicit Function Theorem, given the $y-$regularity of the point (x_0, y_0), we can find a neighbourhood $\mathcal{U}_{x_0} \subseteq \mathbb{R}$ of x_0 and a differentiable function g such that $y_0 = g(x_0)$ and $H(x, g(x)) = E$ on $\mathcal{U}_{x_0}$. Putting this back at Eq.(7.7a) we have

$$\frac{dx}{dt} = \left.\frac{\partial H}{\partial y}\right|_{(x,g(x))} := f(x, E) \Rightarrow \int_{x_0}^{x_1} \frac{dx}{f(x, E)} = t_1 - t_0$$

and so the system is deduced by a quadrature. □

Action-angle Variables

An important characteristic of Liouville integrable systems is that, under certain conditions, they can be described by a special set of conjugate pairs named the *action-angle variables*. In order to arrive at such variables however, we ought to perform a series of steps which we briefly lay out below, though whose thorough derivation can be found in most Classical/Hamiltonian Mechanics textbooks or notes alike [Arn89, Gio22]. The first concept we ought to have in mind is that of a *canonical transformation*

Definition 7.0.3 (Canonical Transformation). Given a simplectic manifold (M, ω) a ***canonical transformation*** is a map $f : M \to M$ such that $f^*\omega = \omega$.

In mathematical terms, a canonical transformation is a symplectomorphism and thus, it is, *in particular*, a transformation which preserves the form of the equations of motion (given that under f the form of Eq.(7.2) is kept unchanged). Note that the definition of a canonical transformation is not truly unique. One could also argue for instance that a transformation is canonical *if, and only if*, it preserves the form of Hamilton's equations of motion, a condition less restrictive then ours.

Given a system $\mathcal{S}$ with Hamiltonian $H(q,p)$, the idea behind the construction of the action-angle variables is to use the first integrals in involution $\{\Phi_j\}_{j=1,\dots,n}$ to build a canonical transformations on the phase space M which takes the set of coordinates (q^i, p_i) to a set of conjugate coordinates (θ^i, I_i), with each $\theta^i \in \mathbb{S}^1$ and such that

$$\begin{gathered} H = H(I_1, \dots, I_n) \\ \dot{\theta}^i = \frac{\partial H}{\partial I_i} =: \omega_i, \quad \dot{I}_i = 0 \end{gathered} \tag{7.8}$$

That is, upon performing such a transformation, we are able to write the Hamiltonian of the system solely in terms of the action variables I_i, and the angular variables are trivially integrable, for each ω_i (the frequency of coordinate θ^i) is constant on the curves generated by the Hamiltonian.

The steps one must follows to construct such a set of variables is the following: consider a constant entry vector $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ such that

$$\{(q,p) \in M | \Phi_1(q,p) = c_1, \dots, \Phi_n(q,p) = c_n\} \subseteq M, \tag{7.9}$$

contains a compact, connected submanifold M_c. Then, one can show that M_c is diffeomorphic to an $n-$torus [Gio22]. Moreover, without loss of generality, we can say

$$\det\left(\frac{\partial(\Phi_1, \dots, \Phi_n)}{\partial(p_1, \dots, p_n)}\right) \neq 0 \tag{7.10}$$

based on which we conclude that we can locally invert the $\Phi_j(q,p) = c_j$ equations to obtain $p_i = p_i(\Phi, q)$. Then, on each of the n cycles of M_c, named γ_j for $j = 1, \dots, n$, we can construct the following

$$I_j(\Phi) = \frac{1}{2\pi} \oint_{\gamma_j} \sum_k p_k(\Phi, q) dq_k, \tag{7.11}$$

Based on which, the action functional

$$S(\Phi) = \int \sum_k p_k dq_k, \tag{7.12}$$

can now be written in terms of the I_j, i.e $S = S(I_j)$, in such a way as to enable the following definition for the conjugate pairs to each I_j

$$\theta^j := \frac{\partial S}{\partial I_j}, \tag{7.13}$$

from where Eq.(7.8) follow.

The whole process outlined above is the content of the following

Theorem 7.0.2 (**Arnold-Jost Theorem**). *Let $H(q,p)$ be the Hamiltonian of a system $\mathcal{S}$ whose phase space is given by the symplectic manifold (M, ω). Say $\mathcal{S}$ possess a set of n first integrals in involution $\{\Phi_j\}$ and let $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ be such that the manifold*

$$\{(q,p) \in M | \Phi_1(q,p) = c_1, \dots, \Phi_n(q,p) = c_n\} \tag{7.14}$$

*has a compact connected component M_c. Then, in a neighborhood of M_c we can define a set of **action-angle variables** (θ^i, I_i), in such a way as to make $H = H(I_1, \dots, I_n)$, so that the equations of motion will now be given by*

$$\dot{\theta}^i = \omega^i, \quad \dot{I}_i = 0, \tag{7.15}$$

where $\omega^i = \frac{\partial H}{\partial I_i}$.

Observation 7.1. Notice that the definition of the action-angle variable is always *local* and only exists for certain regimes of the system $\mathcal{S}$. On this same token, the periodic condition present in the θ^i variables leads one into concluding that not all problems admit a *global* canonical transformation which takes us to the action-angle variables (take for instance the free particle problem on $\mathbb{R}^n$, which possess no periodic motions). The only way this could be so, is if for any allowed energy value E the curve

$$\gamma_E := \{(q,p) \in M | H(q,p) = E\}, \tag{7.16}$$

was diffeomorphic to a circle, in which case the dynamics would be globally bounded, making the above construction also globally definable.

Chapter 8

A first study on Braid Theory

8.1 Basic definitions and results

The study of these objects called *braids* dates back to about 200 years ago, with the first mathematicians to seem to have an interest in it being Friedrich Gauss, as some sketches can be found on some of his notes [Lam09], and Adolf Hurwitz, on the 19th century. As we will see next, we can define these braids in three equivalent ways, namely

(1) gemetrically;

(2) physically;

(3) algebraically.

Prior to this objects called *knots* were already being studied, though it was James W. Alexander in his 1923 paper [Ale23] who explicitly layed out a connection between these two by proving that every such knot can be built by the closure of an appropriate braid. Formally speaking we have the following

Definition 8.1.1. A **knot** is an embedding of $\mathbb{S}^1$ into $\mathbb{R}^3$

Definition 8.1.2 (Geometric)**.** A **geometric braid** (or dancing particle braid) is one composed by the time evolution of a system of non-colliding particles.

Definition 8.1.3 (Physical)**.** Consider the parallel planes P_0 and P_1. Let $\{a_1, \ldots, a_n\} \in P_0$ and $b_1, \ldots, b_n \in P_1$, be two sets of points joint by non-intersecting smooth curves. This set of curves is called a **braid on n strings**.

To consider the algebraic definition of a braid, we need to bear in mind the notion of an *equivalence* between braids. Two physical braids α and β will be called **equivalent**[1] (or isotopic) if it is possible to find a continuous deformation of the strings of α to those of β. Furthermore, the construction of algebraic braids will be done through projections on a given plane. Visualizing the physical braid on $\mathbb{R}^3$, we can deform it to an isotopic braid whose crossings occur separately and project it, without loss of generality, on the $xz-$plane.

After having done the projection, while keeping track of which string crosses over which, to the crossing of the ith string *over* the $(i+1)$st one we assign the symbol σ_i. If the opposite happens, we label it σ_i^{-1} instead. For a better visualization, see Figure 8.1a.

Not surprisingly, these symbols will denote the generators of the so called **Artin Braid Group**, introduced be Emil Artin in his 1926 paper. It was from that point on that the study of braids became more rigorous, thanks to the tools coming from Algebra. We are now in position to state

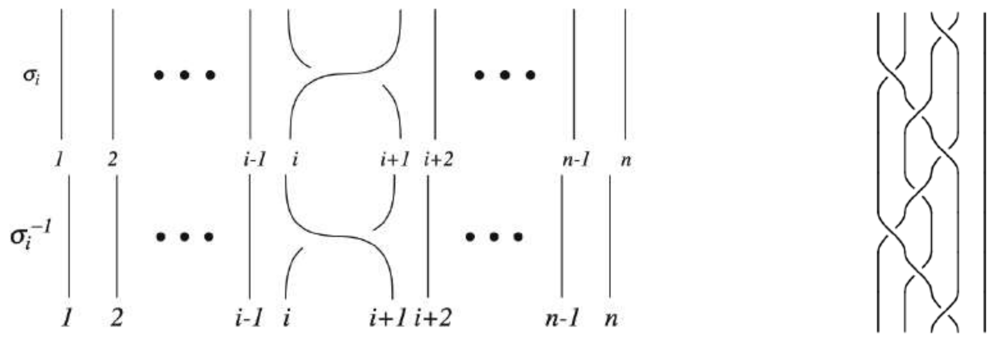
Definition 8.1.4 (Algebraic)**.** An **algebraic n-braid** is an element of the Artin braid group on n strands B_n, thought of as a word on its generators $\{\sigma_i\}_{i=1,\ldots,n-1}$ which themselves obey the following relations

BD.1 $\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| > 1$

BD.2 $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad i = 1, \ldots, n-2$

Moreover, the way in which these objects form a group is by concatenation. That is, the end points of one braid will be the starting points of the other. In this fashion we construct

[1] Represented by $\alpha = \beta$ or $\alpha \simeq \beta$



(a) Generator σ_i and its inverse (b) An element of B_5

Figure 8.1: Algebraic representation of geometric/physical braids

a "tower" of crossings with $n-$strands and such a resulting object will thus be considered an element of B_n (see Figure 8.1b, for example). By the above discussion on these braid elements as projections, it is clear that an $\alpha \in B_n$ is not just one braid but actually represents an *equivalence class* of braids that are isotopic (from a physical/geometrical perspective) to this particular representative we chose to refer to.

Naturally, we call **BD.1** and **BD.2** the **braid relations**. They in turn not only give necessary, but also sufficient conditions for us to determine the braid group, i.e, any other possible relation between the generators can be brought down to just these two. Algebraically speaking, we can say that the above give a valid presentation of the Braid group [GM11].

An important normal subgroup of B_n is P_n the ***pure braid group*** whose generators can be built from the σ_i and are given by

$$\alpha_{ij} := \sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}. \tag{8.1}$$

These α_{ij}, much like the σ_i, also satisfy a set of relations which themselves characterize P_n [AdRJ21]. Intuitively, while and element of B_n permutes the base and final sets of points, an element of P_n does **not** make any permutations on these. For example $\sigma_1^2 \in P_2$ because it doesn't permute its base and end points, i.e it takes 1 to 1 and 2 to 2.

One can also formulate the above reasoning in a more algebraic term. Given the group

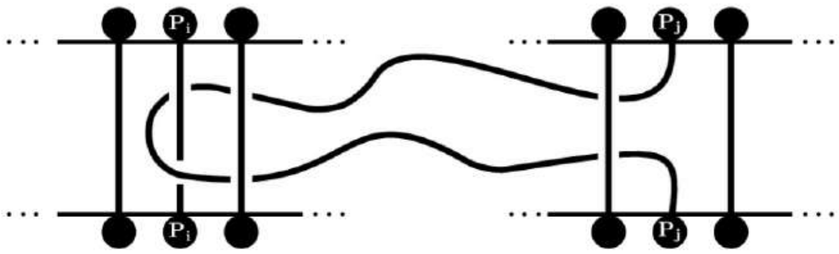


Figure 8.2: Geometric representation of the braid generator α_{ij} for the pure braid group P_n. The image was taken from page 32 of [AdRJ21].

homomorphism

$$\varphi : B_n \longrightarrow \mathrm{S}_n, \tag{8.2}$$

between the braid and permutation groups, which takes a braid generator σ_i and maps it to $\varphi(\sigma_i) = (i\, i+1)$, i.e the element in S_n which exchanges elements i and $i+1$, we can notice that the kernel of such an operator is precisely the group P_n of pure braids. That is

$$\mathrm{Ker}(\varphi) = P_n \tag{8.3}$$

From there, we can conclude by the Isomorphism Theorem that

$$B_n/P_n \simeq \mathrm{S}_n, \tag{8.4}$$

8.1.1 Some fundamental theorems

To be able to refer to the two most classical results on Braid theory, namely Alexander's and Markov's Theorems, we need the following

Definition 8.1.5. The **closure** of a braid $\alpha \in B_n$, denoted $\overline{\alpha}$, is the knot generated by an ordered matching of α's starting and ending points.

Appealing back to the Definition 8.1.3 describing what a physical braid is, the way in which we would define the closure of $\alpha \in B_n$ would simply be to connect the point a_i to b_i without making any new crossings midway, as depicted on Figure 8.3. With this in mind we have the following

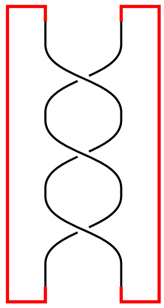


Figure 8.3: Figure depicting the closure of the braid (thick red lines). We joint the starting and final points of the braid as in the above fashion, thus creating a loop in $\mathbb{R}^3$ which can be seen as a knot.

Theorem 8.1.1 (Alexander, 1923). *Every knot can be represented as a closed braid.*

Proof: See [Ale23] and [BZ85].

A possibly intriguing fact about the statement of the above theorem is the lack of the word "unique". Indeed it turns out that the braid that closes to a given knot isn't unique at all. Though, two braids closing to the same knot can't be arbitrarily different either. They have to be related by what we now call **Markov Moves**, described below.

Definition 8.1.6 (Stabilization). Let $\alpha \in B_n \hookrightarrow B_{n+1}$. A **stabilization move** on α is given by the operation $\alpha \mapsto \alpha\sigma_n^{\pm 1}$

Where the inclusion above, namely $B_n \hookrightarrow B_{n+1}$, is just given by the addition of another string to an arbitrary braid of B_n. That is, you add an $(n+1)$st point to the set of starting and ending points of said braid, just to then connect those trivially. The next operation we are interested in is given by

Definition 8.1.7 (Conjugation). Given $\alpha, \gamma \in B_n$, a **conjugation move** on α by γ is defined to be the operation $\alpha \mapsto \gamma\alpha\gamma^{-1}$

Definitions 8.1.6 & 8.1.7 form what we called the Markov moves above. With this at hands we can state

Theorem 8.1.2 (Markov, 1935)**.** *Two braids close to the same knot if, and only if, they are related by Markov moves.*

Proof: The original one can be found here [Mar35]. A more comprehensible one is due to Birman [Bir74].

8.1.2 The center of B_n

Before we move on to discuss how braids can be built in other $2-$manifolds other than the Euclidean plane, let's talk about yet another important subgroup of B_n, namely, its *center*.

On a more general context, given a group G, we define the center of G to be the subgroup

$$Z(G) = \{x \in G|\, xg = gx, \forall g \in G\} \tag{8.5}$$

In other words, the center of a group G is the set of all elements that commute with any other element of G. For a general group, the center need not be finite (in order), nor finitely generated (i.e, generated by a finite set of elements).

In our case, where $G = B_n$, a very nice characterization of the center subgroup happens. To start, consider the element $\Delta \in B_n$, called ***half twist***, given by

$$\Delta = b_{n-1}b_{n-2}\dots b_2b_1, \text{ where } b_k = \sigma_1\sigma_2\dots\sigma_k \tag{8.6}$$

As can be seen on Figure 8.4, the element Δ performs, as the name suggests, a half twist on the strands of the braid by taking, in B_4 for instance, the starting point 1 and mapping it to the ending point 4 and, on the same token, starting point 2 goes to 3, 3 goes to 2 and 4 goes to 1. On the general case of B_n, the half twist takes the initial point i and maps it to $n + 1 - i$.

Upon composing this element with itself we end up with the ***full twist*** Δ^2, which resets the configuration of the initial and final points, though in the process, tangling the

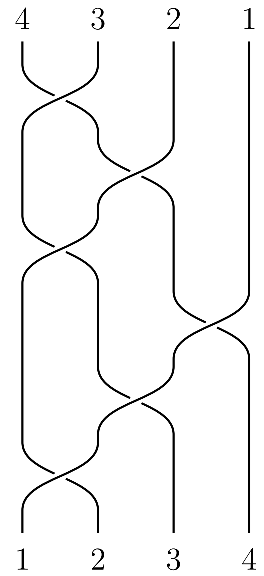


Figure 8.4: Picture depicting the half twist on B_4. In this case, the element can be written as $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$.

strands even more. With this in mind we have the following

Theorem 8.1.3 (Garside, 1969)**.** *Consider the braid group* B_n *and in it the half twist* Δ*. Then, it follows that*

- $Z(B_2) = \langle\Delta\rangle$,
- $Z(B_n) = \langle\Delta^2\rangle$, *for* $n > 2$

That is, Δ *is the generator for the center of* B_2*, meanwhile* Δ^2 *is the generator for the center of* B_n *for* $n > 2$.

Proof: See [Gar69]. □

8.2 Braid group representations

We now briefly lay out the construction of the braid groups in other types of 2−manifolds M other than the plane. Note that we can think of M as the configuration space of n point masses located over it by defining the set

$$\mathcal{C}_n(M) = \{(z_1, \dots, z_n)|\, z_i \in M\} \simeq M^n, \tag{8.7}$$

of all possible positions those particles could occupy on M.

Since we're in general interested in the case where no two distinct particles can occupy the same position simultaneously, we are lead into working with the set

$$\widetilde{\mathcal{C}}_n(M) = \{(z_1, \dots, z_n) \in M^n | \, z_i \neq z_j, \forall i \neq j\}, \tag{8.8}$$

thought of as the collision-less version of $\mathcal{C}_n(M)$. Upon considering the fundamental group of such a space we end up with the set of *Pure braids over* M, that is

$$P_n(M) := \pi_1(\widetilde{\mathcal{C}}_n(M)). \tag{8.9}$$

The geometric reason for this is quite simple: if we take a point $z \in \widetilde{\mathcal{C}}_n(M)$, then z is a collection of n point $(z_1, \dots, z_n)$, *in this particular ordering*, over M. An element of $\pi_1(\widetilde{\mathcal{C}}_n(M), z)$ is just a loop that starts on z and ends on z. The "intermediate point" of this loop can be though of as the movement of the initial set of z_i's. Since the initial set of such points is the same as the final set (by definition), and the points revolve around one another without making any collisions (since they all sit inside $\widetilde{\mathcal{C}}_n(M)$), we end up precisely with a pure braid (if this motion were to be represented by the Artin braid group).

To get to the intrinsic notion of the full braid group B_n, note that there is a natural action of S_n (the permutation group) on $\widetilde{\mathcal{C}}_n(M)$, by the simple permutation of the elements of its points. That is, the group action $S_n \circlearrowright \widetilde{\mathcal{C}}_n(M)$ is given by

$$g.z = (z_{g(1)}, \dots, z_{g(n)}), \text{ for } g \in S_n. \tag{8.10}$$

The action is clearly free, because if $g.z = z$ then $g = Id$ the identity permutation. This in turn enables us to consider orbits of each of the elements of $\widetilde{\mathcal{C}}_n(M)$ and, in particular, also let's us consider the quotient space

$$\mathcal{O}_n(M) = \widetilde{\mathcal{C}}_n(M)/S_n, \tag{8.11}$$

formed by the equivalence classes of each $z \in \widetilde{\mathcal{C}}_n(M)$ under the above natural S_n action. Based on it, we are able to define the *full braid group* intrinsically, as

$$B_n(M) = \pi_1(\mathcal{O}_n(M)), \tag{8.12}$$

which has a similar geometrical interpretation as $P_n(M)$. In this case, instead of just a point z we have an *equivalence class* of points, denoted $[z]$ on which our loop starts and finishes. Since the elements of the class we decide to start our loop on do ***not*** have to exactly agree with the ones we decide to end it on, we have that the initial points we started with may be permuted once we complete the loop. Hence, the Artin representation of such a braid coincides with the ones for the full braid group B_n we saw in the previous subsection.

An important fact when dealing with this fundamental group representation of the braids is that *topological braids* may appear. Their existence is related to the fact that, if $\pi_1(M^n) \neq 0$, then there will be extra strands that we'll need to attach to the Artin representation of our braid.

For instance, take M to be a closed $2-$manifold (i.e compact and without boundary). We can then make the following argument: consider the trivial inclusion

$$\iota : \widetilde{\mathcal{C}}_n(M) \hookrightarrow M^n. \tag{8.13}$$

By the *trivializing theorem* [AdRJ21], we have a naturally induced **surjective homomorphism**

$$\iota_* : P_n(M) := \pi_1(\widetilde{\mathcal{C}}_n(M)) \to \pi_1(M^n). \tag{8.14}$$

The surjectivity of the above map let's us deduce the important

Observation 8.1. Loops not homotopic to a point in M^n are in direct relation with the presence of non trivial pure braids in the Artin representation of $B_n(M)$. More specifically, to each generator, a new strand has to be added in such a representation.

For example, when $n = 1$, since $\tilde{\mathcal{C}}_1(M) \simeq M$, we shall have that $P_1(M) \equiv \pi_1(M)$ by definition (and so, this specific case also holds for open manifolds[2]). The base case of $M = \mathbb{R}^2$ will give us no new strand whatsoever, as expected by construction. Indeed, the Artin representation of $P_1(\mathbb{R}^2)$ is just a line connecting two dots, for $P_n(\mathbb{R}^2)$, with $n > 1$ we will have the generators α_{ij} given by Eq.(8.1).

For the less elementary case of $M = \mathbb{S}\times\mathbb{R}$ however, due to the fact that $\pi_1(\mathbb{S}\times\mathbb{R}) = \mathbb{Z}$, we have to add one extra strand to $P_1(\mathbb{S}\times\mathbb{R})$ in the Artin representation, making such pure braids become $\sigma_1^{2n}, n \in \mathbb{Z}$. In this regard we can say that $P_1(\mathbb{S}\times\mathbb{R}) \simeq P_2(\mathbb{R}^2)$. For $P_n(\mathbb{S}\times\mathbb{R})$, with $n > 1$, we ought to use the fact that

$$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y), \tag{8.15}$$

for any path-connected manifolds X and Y, i.e manifolds which admit a path connecting any two points. In doing so, we conclude that $P_n(\mathbb{S}\times\mathbb{R})$ can be surjectively mapped onto $\pi_1((\mathbb{S}\times\mathbb{R})^n) = \pi_1(\mathbb{S}\times\mathbb{R})^n \simeq \mathbb{Z}^n$, meaning that besides the already existing n strand (associated to each of the n particles), we have to add yet another n more strands associated to each generator of the fundamental group of $(\mathbb{S}\times\mathbb{R})^n$, meaning that $P_n(\mathbb{S}\times\mathbb{R}) \simeq P_{2n}(\mathbb{R}^2)$. Based on this we have the following

Definition 8.2.1 (Topological strand). Given a $2-$manifold M and its associated (pure) braid group $(P_n(M))$ $B_n(M)$, a ***topological strand*** is a strand we add to the Artin representation of $(P_n(M))$ $B_n(M)$ due the non-trivial elements of $\pi_1(M^n)$.

As illustrated above, different generators of $\pi_1(M^n)$ shall correspond to different strands being added. In this general case, for surfaces of large genus we can form some very complicated braids by simply going around the surface in a loop. As we said beforehand, these braids shall be called *topological braids*, because they arise due to the surface's non-trivial topological information captured, to first order, by its fundamental group.

[2]formally, one takes a compact subset of M in which the braid dynamics happens

For further details on other braid representations and their correlations with other areas of algebra and dynamics, see [BCH+09, BZ85, AdRJ21, d'A20b].

Chapter 9

Vortices on the cylinder

In this chapter we focus on the particular case of a system of N point vortices, also denominated a $N-$*vortex system* (see the **Topological dynamics and Vortex motions** subsection of the Introduction).

We will concentrate on the work of Boyland, Stremler and Aref [BSA03], in which the dynamics of an advected particle in the flow of 3 vortices with zero net circulation is discussed under the framework of Braid Theory. We shall first briefly introduce the reader to the Hamiltonian formalism behind vortex dynamics over 2−dimensional configuration space M, though focusing on the plane and the cylinder only. One important property of the $N-$vortex systems is the fact that its configurations space M itself already has a "weighted" symplectic structure

$$\Omega = \sum_{i=1}^{N} dq_i \wedge dp_i = \sum_{i=1}^{N} \Gamma_i dx_i \wedge dy_i, \tag{9.1}$$

with Γ_i being the vortex strength of each point vortex (see Figure 9.1) and (x_i, y_i) a general notation for the position of the $i-$th vortex over M.

Then, based on the computations of [BSA03] we will conclude that a 3 vortex system on the cylinder $\mathbb{R} \times \mathbb{S}$ *with vanishing net circulation* is integrable but nonetheless is not entirely free of chaos, as the topological dynamics of its advected particles have a so-called *pseudo-Anosov* regime of motion.

Lastly, we give various integrability notion based on the braids formed by a system of

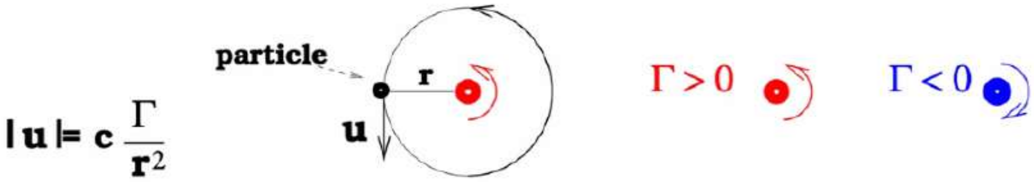


Figure 9.1: *Advected by the velocity field of one point vortex, a test particle follows a circular orbit, with a speed proportional to the absolute value of the vortex circulation and inversely proportional to the square of its distance from the vortex.* With courtesy of Stefanella Boatto and Darren Crowdy from their paper - S. Boatto and D. Crowdy, Point Vortex Dynamics, Enc. of Math. Phys., pag. $66-79$, Academic Press (2006) [BC06]

point particles. Our definitions are general, however we restrict to formulating those solely with the Artin braid group B_n, as we can always think of the dynamics in terms of it as long as topological strands (Definition 8.2.1) are taken into account. We finish by giving a first prove relating to Liouville integrability (LI) of a bounded system of interacting particles and one of the braid integrability definition considered, thus presenting a braid theoretical obstruction to the (LI) of such type of system.

9.1 Interaction Hamiltonian on the plane

Unlike with the massive particles we dealt with throughout Part I of the thesis, when talking about vortex dynamics we can see their configuration space $\mathcal{Q}$ as already being the symplectic manifold we need to describe their dynamical behaviour. This happens because when dealing with vortices the notion of "conjugate momentum" vanishes and, all we need to know to in order to trace out the dynamics is their velocities which, of course, is nothing more then the time derivative of their positions.

In order for this association to be valid, it is clear that our configuration space over which the flow happens has to be even dimensional. For this section, we will focus on $M = \mathbb{R}^2$.

In what follows, talking about *real particles* we shall be refering to the *vortices* within the fluid. Meanwhile by *test particles* we will mean the *fluid particles* that get transported or *advected* by the flow generated by the vortices seen as the sources for the advected dynamics.

Now, given a vorticity field $\omega(r)$, we can characterize the velocity field generated by it still in terms of the aforementioned Hodge decomposition (HD). Now however, in the study of fluids, we are concerned the *divergence free* part of the decomposition instead of the curl free one as in the gravitational case. The conditions used to describe the fluid flow are given by the formulae

$$\operatorname{curl}(u) = \omega, \tag{9.2a}$$
$$\operatorname{div}(u) = 0. \tag{9.2b}$$

Physically, the condition expressed on Eq.(9.2b) asserts that our fluid flow is *incompressible* and so, any deformation in one direction shall be compensated in another.

Furthermore, Eq.(9.2a) let's us write the fluid velocity u in terms of an auxiliary potential $\psi : \mathbb{R}^2 \to \mathbb{R}$ as

$$u = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right). \tag{9.3}$$

The usefulness of the above expression is that is highlights the *Hamiltonian character* of the dynamics. We can also rewrite in the following way

$$u = J\nabla\psi, \text{ with } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{9.4}$$

Now, putting together Eq.(9.3) and Eq.(9.2a) we get that

$$\Delta\psi(r, r'(t)) = -\omega(r, r'(t)) \tag{9.5}$$

Which can be solved by *Green functions*, i.e functions $G(r, r')$ that satisfy

$$\Delta G(r, r') = \delta(r - r') \tag{9.6}$$

with the Laplacian derivation being taken with regards to the r variable. This distinction is important since, it is namely because of it that we are able to express the solution ψ of Eq.(9.5) as

$$\psi(r; t) = -\int G(r, r'(t))\omega(r'(t))dr' \tag{9.7}$$

Notably, the domain of integration is the same over which Eq.(9.5) is valid.

Observation 9.1 (***Emphasizing the test particle perspective***)**.** As written above, the function ψ, known as *stream function*, is the one that describes the behaviour of the *passive, test* particles in the fluid dynamics, ***not*** the one from the vortices themselves.

Indeed $r'(t) = (x'(t), y'(t))$ is the position of a given vortex that we took as a given. A natural question to ask then is, *how do we find* $r'(t)$? In the case of a point-like vortex system, described by the following vorticity field

$$\omega = \sum_i \Gamma_i \delta(r - r_i(t)), \tag{9.8}$$

with r_i being the position of vortex i, we shall find by Eq.(9.7) that the stream function reduces to the sum

$$\psi(r, r_i; t) = -\sum_i \Gamma_i G(r, r_i(t)). \tag{9.9}$$

In order to obtain the vortex interaction Hamiltonian, we can reason that ***real particles are test particles in the field of the other real particles***. That is to say, each vortex (real particle) is a test particles (passively advected particle) in the field of the other vortices. A formal derivation of this procedure can be found in the paper [DB15b]. In the plane, this culminates in the following Hamiltonian

$$H_{\text{vort}} = -\sum_{i \neq j} \Gamma_i \Gamma_j G(r_i, r_j). \tag{9.10}$$

Using the symplectic form from Eq.(9.1), we find the following equations of motions describing each vortex's position.

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{1}{\Gamma_i} \frac{\partial H}{\partial y_i}, \\ \frac{dy_i}{dt} &= -\frac{1}{\Gamma_i} \frac{\partial H}{\partial x_i} \end{aligned} \tag{9.11}$$

From which we get $r_1(t)$ and $r_2(t)$ to substitute back into the Hamiltonian from Eq.(9.9) and solve similar equations to the above (just without the Γ_i's on the denominator).

It becomes clear from Eq.(9.9) that, from a mathematical perspective, the function ψ represents some sort of "interaction potential" of the fluid. In this regard, what Eq.(9.9) tells us then would be the "flow strength" felt by a *test particle* at position r, given a set of point vortices at positions $r_i(t)$.

Following [DB15a], we can equivalently write the vortex interaction Hamiltonian in the following for

$$H = -\frac{1}{4\pi}\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma_i\Gamma_j G(r_i, r_j), \tag{9.12}$$

with the planar Green function being given by

$$G(r, r_0) = \frac{1}{2\pi}\log|r - r_0|\,. \tag{9.13}$$

9.1.1 The Cylinder case

As performed on [ABC$^+$20b, MST03], for the case of the cylinder the Poisson equation is precisely the same, with the difference being in the expression for the Laplace operator.

To find the green function we have to make use of the covering space of the cylinder, given by the plane $\mathbb{R}^2$. This let's us make use of the planar Green function [Eq.(9.13)] to calculate the cylindrical one, however with the catch that an infinite (though regularizable sum) has to be made as an infinite number of copies for each cylinder will appear (Figure 9.2)

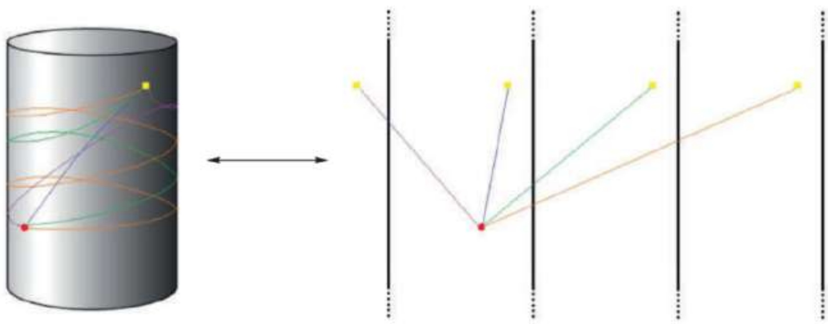


Figure 9.2: Illustration taken from the paper [ABC$^+$20b] showing the plane as the covering space for the cylinder $\mathbb{R}\times\mathbb{S}$. For a radius L cylinder, as it has a $2\pi L$ periodic coordinate, the covering space will contain an infinite number of vortices on each patch that covers $\mathbb{R}\times\mathbb{S}$.

For the case of a cylinder of circumference L the Green function is given by [BSA03]

$$G(x_i, y_i) = \frac{\pi}{2L} \log \left(\frac{1}{2} \left(\cosh \left(\frac{2\pi y_i}{L} \right) - \cos \left(\frac{2\pi x_i}{L} \right) \right) \right), \tag{9.14}$$

with (x_i, y_i) being the coordinates of each vortex. We can now move on to the particular case of interest, namely, when $n = 3$ and total vorticity is zero.

9.2 On the integrability of the 3 vortex system with vanishing net vorticity

We now reprove the integrability of 3 vortices on the cylinder with vanishing net circulation ($\sum_\alpha \Gamma_\alpha = 0$) by following the computations done in [BSA03].

First of all, let's start by parameterizing our configuration space. As the reader is aware, a *cylinder* is nothing more than the manifold $\mathbb{S}^1 \times \mathbb{R}$. As a submanifold of $\mathbb{R}^3$ we have that

$$X = cos(\varphi), \quad Y = sin(\varphi), \quad Z = Z \tag{9.15}$$

However, we can consider a pair of intrinsic normalized coordinates to describe the positions of our vortices. Those are (x_i, y_i) given by

$$x_i = \phi_i, \quad y_i = Z_i, \tag{9.16}$$

in order to later on identify the covering space of the cylinder with the $xy-$plane. Furthermore, to describe the dynamics from a Hamiltonian point of view, we introduce the pair of conjugate variables (q_i, p_i) where [DB15a, BS19]

$$q_i = x_i, \quad p_i = \Gamma_i y_i \tag{9.17}$$

with Γ_i being the *circulation* of each vortex. In these coordinates the symplectic form will be given by[1]:

$$\Omega := \sum_i \Gamma_i dx_i \wedge dy_i = \sum_i dq_i \wedge dp_i. \tag{9.18}$$

[1]we are here assuming that the circulations, once fixed, are not dynamical variable. That is $d\Gamma_i = 0$

The dynamical equations are naturally given by

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{9.19}$$

where the interaction Hamiltonian [Eq.(9.12)] assumes the following form

$$H = -\frac{1}{2\pi}\sum_{\substack{i=1\\ j>i}}^{3} \Gamma_i\Gamma_j G\left(q_i - q_j, \frac{p_i}{\Gamma_i} - \frac{p_j}{\Gamma_j}\right) \tag{9.20}$$

Based on Eq.(9.20), we can see right away the Hamiltonian's invariance with respect to spacial translations. Similar to the gravitational case (Example 2.3.2), this implies in the conservation of the linear momenta of the particles (see Example A.4.2), whose expressions (in the (x_i, y_i) variables) are given by

$$Q = \sum_i \Gamma_i x_i, \quad P = \sum_i \Gamma_i y_i. \tag{9.21}$$

Sticking to the (x_i, y_i) notation, consider the positions of the vortices on the cylinder as given by z_1, z_2 and z_3 with each $z_i = x_i + iy_i$. Since the Hamiltonian only depends on the difference of the positions of the vortices, we can use the quantity $J = \sum \Gamma_\alpha z_\alpha$ together with $Z := z_1 - z_2$ and the fact that $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$ to express the other differences in the following manner:

$$z_1 - z_3 = \frac{-J - \Gamma_2 Z}{\Gamma_3}, \quad z_2 - z_3 = \frac{-J + \Gamma_1 Z}{\Gamma_3}. \tag{9.22}$$

Now, to further reduce the system and prove its integrability, note that for consistency we got to right the Hamiltonian equations of motion in complex form. This can be done in the following way: consider the symplectic form $\omega = \sum_\alpha \Gamma_\alpha dx_\alpha \wedge dy_\alpha$, and a Hamiltonian function $H(x_\alpha, y_\alpha)$ so that

$$\frac{dx_\alpha}{dt} = \frac{1}{\Gamma_\alpha}\frac{\partial H}{\partial y_\alpha}, \tag{9.23a}$$

$$\frac{dy_\alpha}{dt} = -\frac{1}{\Gamma_\alpha}\frac{\partial H}{\partial x_\alpha}. \tag{9.23b}$$

Using the fact that $z = x + iy$, we can express Eqs.(9.23) using complex coordinates in the following way

$$\frac{d\bar{z}_\alpha}{dt} = \frac{2i}{\Gamma_\alpha}\partial_z H(z). \tag{9.24}$$

Setting $\alpha = 1, 2, 3$ and taking the appropriate differences we can get the equations of motion for Z, and thus for $z_1 - z_3$ and $z_2 - z_3$. The Green function of Eq.(9.14), after a very long string of calculations [Dua16], can be written as

$$G(z) = \frac{\pi}{L} \log\left(\left|\sin\left(\frac{\pi}{L}z\right)\right|\right), \tag{9.25}$$

with its real analog just being given by its real part. The equations satisfied by Z is

$$\frac{d\overline{Z}}{dt} = \frac{-\Gamma_3}{2\pi i}\left(\phi(Z) + \phi\left(\frac{-J + \Gamma_1 Z}{\Gamma_3}\right) + \phi\left(\frac{J + \Gamma_2 Z}{\Gamma_3}\right)\right), \quad \phi(Z) := \partial_Z G(z), \tag{9.26}$$

so that we do have a Hamiltonian for this system, namely

$$\mathcal{K}(Z) = \frac{\Gamma_3}{2\pi}\left(G(Z) + \frac{\Gamma_3}{\Gamma_1} G\left(\frac{-J + \Gamma_1 Z}{\Gamma_3}\right) + \frac{\Gamma_3}{\Gamma_2} G\left(\frac{J + \Gamma_2 Z}{\Gamma_3}\right)\right), \tag{9.27}$$

with the canonical symplectic form $\tilde{\omega} = -\frac{1}{2i} dZ \wedge d\overline{Z}$ over the reduced space. By Theorem 7.0.1, we conclude that the above 1 d.o.f system is integrable so that Z, and hence the z_i, can be found by quadratures.

9.2.1 Liouville integrability and topological chaos

On Section 4 of [BSA03] Boyland et al. made use of Braid Theory to classify the dynamics of advected particles under the flow of 3 vortices moving on the cylinder. The main tool used was a result from Nielsen and Thurston classifying the homeomorphism class of a function $f : S \to S$, with S being a compact[2] oriented surface (TN classification) [Nie44, Thu88]. The precise statement of the theorem is the following

Theorem 9.2.1 (Thurston-Nielsen Classification). *If f is a homeomorphosm of a compact surface S, then f is isotopic[3] to a homeomorphism g, of one of the following types:*

TN. 1 *Finite order*

TN. 2 *Pseudo-Anosov*

TN. 3 *Reducible*

[2] the compactness condition here is meat when one considers bounded motions of the vortices. These in turn can only happen on a closed and bounded domain of $\mathbb{R} \times \mathbb{R}^2$

The **TN 1** case means that the map g when composed with itself enough times is the identity over S. **TN 2** means that we can find a pair of transverse, measured foliations which are preserved by the action of g. Typically, one such foliation is stretched while the other is contracted by the map g by a factor of λ and $1/\lambda$, respectively. The complex dynamics of such a pA map generates in the dynamics what is usually understood as *topological chaos*, for such a behavior is preserved by isotopic deformations of g [BSA00]. Finally, by **TN 3** we mean that there is a family of disjoint simple loops which are permuted by the action of g on S. On the complement of such loops, the dynamics of g is either reducible or pseudo-Anosov.

An interesting property of such a vortex system is worth mentioning. As already noted by Aref [AS96], the dynamics for 3 vortices on the cylinder has a wide variety of regimes of motion and as done on [BSA03], we can map such a system from the cylinder to the plane by considering the transformation

$$T(z) = e^{2\pi i z}, \text{ with } z = x + iy \in \mathbb{S} \times \mathbb{R}, \tag{9.28}$$

and once there, we can analyze the dynamics according to the TN classification of the braids formed by these vortices. What one then sees is that, *for certain vorticity ratios, pA regimes emerge in the dynamics, even though the system is completly integrable*. This in turn means that Liouville integrability of the vortex system does ***not*** usually imply in the absence of chaos, at least from a topological perspective. This is mostly due to the fact that topological chaos has to do with the *dynamics of the **advected** particle*, which is differs from that of the sources, as discussed on Observation 9.1.

The way in which this is verified is by considering the Artin representation of the braids from the cylinder to the plane (via Eq.(9.28)) and computing the TN type of each generated braid using a computer software [BH95]. As noted on Subsection 8.2, when considering such a braid representation coming from another manifold, we need to introduce topological strands to our dynamics. Since $\pi_1(\mathbb{R} \times \mathbb{S}) \simeq \mathbb{Z}$ only one extra strand will be needed and, by Eq.(9.28), it shall remain at the origin of $\mathbb{R}^2$. Such a strand has the dynamical interpretation of a *null vorticity vortex* so that, together with the previous ones, we may then say that the 3 vortex problem on the cylinder can me mapped to a 4 vortex problem on the plane, with however only 3 "active" or interacting vortices.

9.3 Integrability notions coming from Braid Theory

A first thing we ought to have in mind is the broad notion of what an *integrable system* is. The technical definition (Definition 7.0.1) says that in order to precisely describe the motion on a Hamiltonian system with $2n$ degrees of freedom, we need to find n independent functions that are kept fixed by the dynamical equations responsible for the evolution of such a system. In doing so, we saw that under certain hypothesis a pair of action-angle variables (I_i, θ_i) can be found (Theorem 7.0.2), in terms of which the dynamical equations read

$$\dot{I}_i = 0, \quad \dot{\theta}_i = \omega_i, \text{ for } \omega_i = \frac{\partial H}{\partial I_i} \tag{9.29}$$

meaning that the dynamics becomes trivial and is in fact described by a set of straight horizontal lines in (I, θ) (or action-angle) phase space .

Typically, one can perform the process of *reduction* to get to Eq.(9.29). In each step, we compute a first integral and use it to reduce the dimension of the phase space until no more of such integrals can be found and the dynamics becomes completely predictable. Meanwhile, focusing on the configuration space now, if we then restrict our attention to the case of a point particle (or vortex) interaction happening over a $2-$manifold M, we can consider the *braids* formed by the motion of these objects and ask ourselves the following: *what happens to the braids generated by this particle dynamics as we perform the reduction of the system*?

Naively, one may think that we simply remove all the braid crossings, getting to the identity braid. This however, is quite far from what goes on. For instance, take a 2 particle system on $\mathbb{R}^2$ that we know to be integrable. Then the solutions to Eq.(9.29) will read

$$\begin{aligned} I_1 &= I_{1,0}, \quad \theta_1(t) = \omega_{1,0}t + \theta_{1,0} \\ I_2 &= I_{2,0}, \quad \theta_2(t) = \omega_{2,0}t + \theta_{2,0} \end{aligned} \tag{9.30}$$

By plotting the angular variables together one thus sees that a priori there is no reason we can't have $\theta_1(t) = \theta_2(t)$ for some t value. Furthermore, since these variable are usually periodic, these braid crossings occur an *infinite* number of times! And so, it's not necessarily true that the resulting braid will have no or even less crossing than the original one.

The question we are lead into thinking now is: *Could we define a sensible notion of (braid) integrablity which bases itself on the* ***untangling*** *of the braids generated by a given physical system?*

The answer is *yes*, and below we outline some possible ways in which we could approach this problem.

The first of them relates to the elements in the center of the braid group. As written on Theorem 8.1.3, the center of the braid group B_n is generated by the full twist for $n > 2$ and by the half twist for $n = 2$. As the name suggest, we can think of the formation of the crossing of these central elements dynamically by imagining that they are created by the anti-clockwise twisting of the end points of the braid, leaving the starting points fixed or vice-versa. In a sense, such a movement is trivially reversible, for we could get free of the crossings by performing this same twist in the clockwise direction and get back to the identity braid.

From here on, all systems $\mathcal{S}$ we shall be referring to are Hamiltonian. With this reasoning in mind, we state

Definition 9.3.1 (Central Braid Integrability). Given a smooth $2-$manifold M, let B_n be the Artin representation of the braid group generated by a point particle/vortex system $\mathcal{S}$ over M. We shall say $\mathcal{S}$ is ***central braid integrable*** (CBI) if, on the regimes associated to bound states of motion, the braids α generated by the system lie in the center of some braid group B_i for some $2 \leq i \leq n$, i.e $\alpha \in \mathrm{Z}(B_i)$ for some $i = 2, \ldots, n$.

Observation 9.2 (**Integrability and Topological Chaos**)**.** Notice that typically CBI systems still exhibit topological chaos. For instance, if we take the full twist on B_3, given by $\Delta_2^2 = (\sigma_1\sigma_2\sigma_1)^2$, following the matrix representation of Boyland et al. [BSA00] one can compute the eigenvalues of Δ_2^2 to be $\lambda_1 = 7 + 4\sqrt{3}$ and $\lambda_2 = 1/\lambda_1$, characterizing a pA regime.

Another notion of braid integrability comes from the dynamics of distinguishable particles. As portrayed on Figure 9.3, if we act twice with some generator σ_i on two given particles in the system, its overall configuration doesn't change, and so we could consider a notion of integrability which takes this discrete symmetry into account.

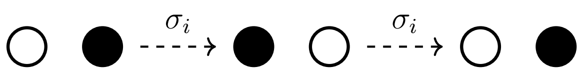


Figure 9.3: Take two distinguishable particles. If we act on it with some σ_i, the organization of the system changes, whereas if we apply the same σ_i once more, the system goes back to its original configuration.

Definition 9.3.2 (**Square Braid Integrability**)**.** Given a smooth 2−manifold M, let B_n be the Artin representation of the braid group generated by a point particle/vortex system $\mathcal{S}$ over M. We shall say $\mathcal{S}$ is ***square braid integrable*** (SBI) if, on the regimes associated to bound states of motion, the braids α generated by the system are such that $\alpha \in \langle \sigma_1^2, \ldots, \sigma_{n-1}^2 \rangle$.

We can suitably define the subgroup

$$\mathfrak{B}_n = \left\langle \sigma_1^2, \ldots, \sigma_{n-1}^2 \right\rangle, \tag{9.31}$$

so that $\alpha \in B_n$ in SBI if, and only if $\alpha \in \mathfrak{B}_n$.

Despite being similar, the above definition are not equivalent, nor do they imply each other. Indeed the braid $\alpha = \sigma_1^2\sigma_2^2$ is SBI but not CBI. Analogously the braid $\beta = \sigma_1$ is

CBI but not SBI. We shall not discuss however which integrability notion is best suited for a given problem.

Although good enough these definitions are all indirectly time dependent. This is because when considering the braid of a physical system, we will in general only take into account the crossings that happened until some final time t_f, specially if the system is known to be aperiodic. This means that for times different from t_f, we will in general have different braids being formed by the system which could in turn change its integrability, possibly even making it become braid non-integrable.

To overcome this issue, first notice that we can always deform the braid crossings in such a way that the positive integer i is always located between two consecutive crossing times $t_{c,i}$ and $t_{c,i+1}$ for $i \geq 1$. This means that until time $t = i$ a total of i braid crossings occurred. Secondly, let's use the notation α_t to denote the braid α generated until the time t for a given point particle system $\mathcal{S}$. With this in mind, consider the sets

$$\mathfrak{A}^{\Delta} = \{t \in \mathbb{N} |\, \alpha_t \text{ is CBI}\}, \tag{9.32a}$$
$$\mathfrak{A} = \{t \in \mathbb{N} |\, \alpha_t \in \mathfrak{B}_n\} \tag{9.32b}$$

based on which the following long time integrability definitions come

Definition 9.3.3 (Long Time Center Braid Integrable). Given a smooth 2−manifold M, let B_n be the Artin representation of the braid group generated by a point particle/vortex system $\mathcal{S}$ over M. Define the set $\mathfrak{A}^{\Delta}_m = \mathfrak{A}^{\Delta} \cap [1, \ldots, m] \subset \mathbb{N}$ and consider

$$a^{\Delta}_m = \left|\mathfrak{A}^{\Delta}_m\right|, \tag{9.33}$$

representing the cardinality of such a set. By defining the quantity

$$\mathrm{d}(\mathfrak{A}^{\Delta}) = \limsup_{m\to\infty} \frac{a^{\Delta}_m}{m}, \tag{9.34}$$

we shall say that $\mathcal{S}$ is ***long time center braid integrable*** (LTCBI) if $\mathrm{d}(\mathfrak{A}^{\Delta}) > 0$. Otherwise, if $\mathrm{d}(\mathfrak{A}^{\Delta}) = 0$ and $\mathfrak{A}^{\Delta} \neq \emptyset$, the system will be called *shor time center braid integrable* (STCBI), where as if $\mathfrak{A}^{\Delta} = \emptyset$ the system is said to be *long time center braid non-integrable* (LTCBnI).

Definition 9.3.4 (Long Time Square Braid Integrable). Given a smooth $2-$manifold M, let B_n be the Artin representation of the braid group generated by a point particle/vortex system $\mathcal{S}$ over M and let $\mathfrak{B}_n$ be as in Eq.(9.31). Define the set $\mathfrak{A}_m = \mathfrak{A} \cap [1, \ldots, m] \subset \mathbb{N}$ and consider

$$a_m = |\mathfrak{A}_m| \,, \tag{9.35}$$

representing the cardinality of such a set. By defining the quantity

$$\mathrm{d}(\mathfrak{A}) = \limsup_{m\to\infty} \frac{a_m}{m}, \tag{9.36}$$

we shall say that $\mathcal{S}$ is ***long time square braid integrable*** (LTSBI) if $\mathrm{d}(\mathfrak{A}) > 0$. Otherwise, if $\mathrm{d}(\mathfrak{A}) = 0$ and $\mathfrak{A} \neq \emptyset$, the system will be called *short time square braid integrable* (STSBI), where as if $\mathfrak{A} = \emptyset$ the system is said to be *long time square braid non-integrable* (LTSBnI).

The above Eqs.(9.34, 9.36) are commonly known as the *natural density* of the sets $\mathfrak{A}^{\triangle}$ and $\mathfrak{A}$. The fact that such a density is positive can be intuitively thought of as implying that the amount of times t for which their defining properties hold is relatively large, as a subset of $\mathbb{N}$ at least. For this reason it is appropriate to say that the system is "long time integrable", given that no matter how much you advance in time, you'll always eventually find some t value for which either property is valid. In doing so, we are hence able to take out the implicit time dependence of the generated braid, thus making the above integrability definitions less arbitrary.

As mentioned above, we can always deform the system's generated braid α in such a way as to make the time t braid α_t possess precisely t crossings (for some $t \in \mathbb{N}$). This is particularly useful when dealing with periodic systems, for we can make the following

Definition 9.3.5 (Natural $\boldsymbol{\tau}$ period). Let $\mathcal{S}$ be a $\tau-$periodic system of point particle over a $2-$manifold M. The ***natural $\boldsymbol{\tau}$ period*** of $\mathcal{S}$ is the smallest natural number τ' for which $\alpha_\tau \simeq \alpha_{\tau'}$, with $\simeq$ representing equality under continuous deformations.

The above definition simply makes use of the fact that, since the system is periodic, at time τ no crossings can be happening, by construction. This means that the braid α_τ is well defined and can thus be deformed to a braid $\alpha_{\tau'}$ for $\tau' \in \mathbb{N}$ possessing precisely τ' crossings and still capturing the system's periodicity. With this in mind we have the following

Proposition 9.3.1. *Given a $\tau-$periodic system $\mathcal{S}$, let τ' be the natural τ period of $\mathcal{S}$. If $\alpha_{\tau'}$ is CBI, then $\mathcal{S}$ is LTCBI.*

Proof: Notice first that the braid group composition operation let's us say that for any pair of times $t, s \in \mathbb{N}$ the following equality holds

$$\alpha_{t+s} = \alpha_t \alpha_{(t+s)-t}, \tag{9.37}$$

that is, the braid generated at time $t + s$ is equal to the braid generated at time t concatenated with the braid generated between the times t and $t + s$, which we denote by $\alpha_{(t+s)-t}$. Hence, by using the fact that the braid is now $\tau'-$periodic and that $\mathfrak{B}_n^\Delta$ is a group, we can say that

$$\alpha_{2\tau'} = \alpha_{\tau'}\alpha_{2\tau'-\tau'} = \alpha_{\tau'}\alpha_{\tau'} = (\alpha_{\tau'})^2 \text{ is CBI,} \tag{9.38}$$

so that, by induction, we are able to conclude that

$$\alpha_{k\tau'} = (\alpha_{\tau'})^k \text{ is CBI,} \tag{9.39}$$

for any $k \in \mathbb{N}$. This then means that the set $\mathfrak{A}^\Delta$ can be written as

$$\mathfrak{A}^\Delta = \{k\tau', \text{ for } k \in \mathbb{N}\} \tag{9.40}$$

so that, by definition, we have that $\mathrm{d}(\mathfrak{A}^\Delta) = \frac{1}{\tau'} > 0$. $\square$

By the same proof, a similar result for $\mathfrak{B}_n$ can also be found. It is important to note that the specific braid $\alpha_{\tau'}$ belongs to $\mathfrak{B}_n^\Delta$. If it were instead some other $\alpha_j \in \mathfrak{B}_n^\Delta$ for

Regime	Braid	TN-type	Expansion constant
I	$(\bar{\sigma}_1\bar{\sigma}_2)^3\bar{\sigma}_1^2(\bar{\sigma}_2\bar{\sigma}_1)^3$	Reducible, all f.o.	
II	$\sigma_3\sigma_2(\sigma_1\sigma_2\sigma_3)^2\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1(\sigma_1\sigma_2\sigma_3)^2\sigma_2\sigma_1\sigma_3\sigma_1\sigma_2\sigma_1\sigma_3$	Reducible, all f.o.	
III	σ_3^2	Reducible, all f.o.	
IV	σ_3^2	Reducible, all f.o.	
V	σ_1^2	Reducible, all f.o.	
VI	$\bar{\sigma}_2^2$	Reducible, all f.o.	
VII	σ_3^2	Reducible, all f.o.	
VIII	$\bar{\sigma}_3^2$	Reducible, all f.o.	
IX	$\sigma_3^2\sigma_2^2$	Reducible, all f.o.	
X	$(\sigma_3\sigma_2)^3$	Reducible, all f.o.	
XI	$(\sigma_3\sigma_2)^2\sigma_3\sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1\sigma_2\sigma_3\sigma_1\sigma_3(\bar{\sigma}_1\bar{\sigma}_2)^3\bar{\sigma}_1$	pA	13.93
XII	$(\sigma_3\sigma_2)^2\sigma_3\sigma_1\sigma_2\sigma_1^2(\sigma_2\sigma_3\sigma_1\sigma_3)^2(\bar{\sigma}_1\bar{\sigma}_2)^3\bar{\sigma}_1$	Reducible, one pA	13.93
XIII	$(\bar{\sigma}_1\bar{\sigma}_2)^2\bar{\sigma}_3^2(\bar{\sigma}_2\bar{\sigma}_1)^5$	pA	9.90

Figure 9.4: Figure taken from [BSA03] illustrating the Artin representation of the braids generated by the vortex system considered. The right-hand-side contain the classification of the braid as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ under the TN classification. The expansion constant refers to the eigenvalue of the pA map.

$j < \tau'$, then the above argument wouldn't work. In particular all we'd be able to say is that

$$\alpha_{\tau'+j} = \alpha_{\tau'}\alpha_{(\tau'+j)-\tau'} = \alpha_{\tau'}\alpha_j$$

which belongs to some coset of the $Z(B_i)$ the braid α_j pertains to. Nonetheless, the above result can also be extended to quasi-periodic systems. By *quasi-periodic* system here we mean a system $\mathcal{S}$ satisfying the property that each particle $\mathfrak{p}_k$ returns after some period τ_k to a *neighborhood* of its starting position. Provided such a neighborhood is fixed, we can consider the system's overall period $\tau = \text{lcm}\{\tau_k\}$. From there we define the natural τ period τ' and if the braid $\alpha_{\tau'}$ generated by the system is CBI, then the system will also be LTCBI.

Even though the above concepts are better, they are still quite hard to be applied to most systems. For instance, the very 3 vortex system which inspired this study of ours only has Regime V as the one generating a CBI dynamics, with a handful of other generating SBI ones. We can however further increment our integrability concepts and consider the following

Definition 9.3.6 (Braid Integrability). Given a smooth 2−manifold M, let B_n be the Artin representation of the braid group generated by a point particle/vortex system $\mathcal{S}$ over M. Consider the quotient group

$$\mathcal{B}_n = B_2/\left\langle \Delta_1^2 \right\rangle \times \cdots \times B_n/\left\langle \Delta_{n-1}^2 \right\rangle \tag{9.41}$$

We will say $\mathcal{S}$ is ***braid integrable*** (BI) if, on the regimes associated to bound states of motion, the braids α generated by the system are such that $\alpha = 1$ in $\mathcal{B}_n$.

The above definition is a bit more workable, mainly due to the following

Proposition 9.3.2. *Let B_n be the braid group on n strands and $\mathcal{B}_n$ the braid quotient group defined as in Eq.(9.41). Then, $\sigma_j^2 = 1$ in $\mathcal{B}_n$ for all $j = 1, \ldots, n-1$.*

Proof: The proof follows by induction. The base case of $j = 1$ is satisfied since $1 = \Delta_1^2 = \sigma_1^2$ in $\mathcal{B}_n$. Assume now that the result holds until some σ_{i-1}, and let's show it is also true for σ_i. Start by noticing that

$$\Delta_i^2 = (b_i b_{i-1} \ldots b_2 b_1)(b_i b_{i-1} \ldots b_2 b_1) = (\sigma_1\sigma_2 \ldots \sigma_i \ldots \sigma_1\sigma_2\sigma_1)(\sigma_1\sigma_2 \ldots \sigma_i \ldots \sigma_1\sigma_2\sigma_1)$$

Now, a string of cancellations will occur. The first cancellation happens between the σ_1 coming from b_1 with the σ_1 coming from b_i (since they merge into a σ_1^2). This in turn let's us write

$$\Delta_i^2 = (\sigma_1\sigma_2 \ldots \sigma_i \ldots \sigma_1\sigma_2)(\sigma_2 \ldots \sigma_i \ldots \sigma_1\sigma_2\sigma_1),$$

from which one sees that the σ_2 coming from b_2 cancels with the σ_2 still from the b_i. To make it explicit one more, notice now that we shall have

$$\Delta_i^2 = (\sigma_1\sigma_2 \ldots \sigma_i \ldots \sigma_1\sigma_2\sigma_3\sigma_1)(\sigma_3 \ldots \sigma_i \ldots \sigma_1\sigma_2\sigma_1),$$

from which we see that σ_3 from b_3 cancels with the σ_3 from b_i. This process then continues until we hit the σ_i from b_i, which can be dragged along the string of generators, since the

σ_{i-1} from b_{i-1} no longer appears. We then get to write

$$\Delta_i^2 = (\sigma_1 \dots \sigma_{i-1}\sigma_i^2\sigma_1 \dots \sigma_{i-2} \dots \sigma_1\sigma_2\sigma_1)(\sigma_1 \dots \sigma_{i-2}\sigma_{i-1} \dots \sigma_1)$$

The above cancellation process then repeats itself once more. It stops once we get to the following braid

$$\Delta_i^2 = \sigma_1\sigma_2 \dots \sigma_{i-1}\sigma_i^2\sigma_{i-1} \dots \sigma_2\sigma_1 \tag{9.42}$$

On $\mathcal{B}_n$ however we now that $\Delta_i^2 = 1$, by definition. From this fact we can conclude that

$$\sigma_1\sigma_2 \dots \sigma_{i-1}\sigma_i^2\sigma_{i-1} \dots \sigma_2\sigma_1 = 1 \;\therefore\; \sigma_i^2 = (\sigma_1\sigma_2 \dots \sigma_{i-1})^{-1}(\sigma_{i-1} \dots \sigma_2\sigma_1)^{-1}$$
$$\Rightarrow \sigma_i^2 = 1$$

which finishes the proof. □

Much like before, we define the set

$$\mathcal{A} = \{t \in \mathbb{N} |\, \alpha_t \in \mathcal{B}_n \text{ is equal to } 1\} \tag{9.43}$$

based on which the following definition and proposition hold

Definition 9.3.7 (**Long Time Braid Integrable**)**.** Given a smooth 2−manifold M, let B_n be the Artin representation of the braid group generated by a point particle/vortex system $\mathcal{S}$ over M and let $\mathcal{B}_n$ be as in Eq.(9.41). Define the set $\mathcal{A}_m = \mathcal{A} \cap [1, \dots, m] \subset \mathbb{N}$ and consider

$$\mathfrak{a}_m = |\mathcal{A}_m|\,, \tag{9.44}$$

representing the cardinality of such a set. By defining the quantity

$$\mathrm{d}(\mathcal{A}) = \limsup_{m\to\infty} \frac{\mathfrak{a}_m}{m}, \tag{9.45}$$

we shall say that $\mathcal{S}$ is ***long time braid integrable*** (LTBI) if $\mathrm{d}(\mathcal{A}) > 0$. Otherwise, if $\mathrm{d}(\mathcal{A}) = 0$ and $\mathcal{A} \neq \emptyset$, the system will be called *short time braid integrable* (STBI), where as if $\mathcal{A} = \emptyset$ the system is said to be *long time braid non-integrable* (LTBnI).

Proposition 9.3.3. *Given a $\tau-$periodic system $\mathcal{S}$, let τ' be the natural τ period of $\mathcal{S}$. If $\alpha_{\tau'} = 1$ in $\mathcal{B}_n$, then $\mathcal{S}$ is LTBI.*

Proof: The proof is the same as in Proposition 9.3.1. Indeed, if $\alpha_{\tau'} = 1$, then $\alpha_{k\tau'} = 1$ for all $k \in \mathbb{N}$, so that $\mathrm{d}(\mathcal{A}) = \frac{1}{\tau'} > 0$. □

As shown on Appendix D, with the exception of Regime II on Figure 9.4, all the other regimes of motion yield BI dynamics. This however just illustrates the fact the notion of braid integrability per se *is* time dependent, as argued above, which is the precise motivation behind the definition of its "long time" version. As a matter of fact, the biggest advantage of the latter integrability definition is that, *for bounded states of motion*, it serves as an obstruction to Liouville integrability, as shown in the following

Theorem 9.3.4. *Let $\mathcal{S}$ be a point particle/vortex system over a $2-$manifold M. Every Liouville integrable bounded regime of $\mathcal{S}$ is LTBI.*

Proof: Say $\mathcal{S}$ is composed of n particles/vortices. The boundedness of the motion asserts to us that the system if found within a potential well, so that the conditions of the Arnold-Jost theorem are verified and we can find a set of action-angle variables (I_i, θ_i) that describe the system (locally). The angle variable are periodic and the equation

$$\boldsymbol{\theta}(t) = (\theta_1(t), \ldots, \theta_n(t)) = (\theta_{1,0} + t\omega_{1,0}, \ldots, \theta_{n,0} + t\omega_{n,0}), \tag{9.46}$$

defines a curve on the torus $\mathbb{T}^n$ which describes the dynamics. Notice moreover that now the braid group for the integrated system is given by B_{n+1}, since $\pi_1(\mathbb{T}^n)$ is given by the free abelian product of $\mathbb{Z}$ with itself n times (thus generating n topological strands), plus the strand related to the path traced out by $\boldsymbol{\theta}(t)$.

In regards to the frequencies $\omega_{i,0}$, notice that if all of them were zero the system would be trivially BI since in this case we would only have the trivial braid being formed. Indeed, by Eq.(9.46), the integrated dynamics would be stationary on the reduced phase space and, on the total space, would possibly correspond to a translation of the whole system.

If however at least one $\omega_{i,0}$ is non-zero, then the path on $\mathbb{T}^n$ traced out by the system will be (almost) periodic. As a matter of fact, each coordinate will have period

$$\tau_i = \frac{2\pi}{\omega_{i,0}}, \tag{9.47}$$

so that the path's period is given by

$$\tau = \text{lcm}(\{\tau_i | \; \tau_i < \infty\}) \tag{9.48}$$

This in turn means that, if we consider τ' as the system's natural τ period, the braid $\alpha_{\tau'}$ generated by the integrated system will repeat itself infinitely many times. If we then see $\alpha_{\tau'} \in \mathcal{B}_{n+1}$, by Lemma D.0.1, we know that

$$\exists k \in \mathbb{N} \text{ such that } (\alpha_{\tau'})^k = 1 \tag{9.49}$$

However, due to the periodicity of the system, we know that $\alpha_{k\tau'} = (\alpha_{\tau'})^k = 1$, which implies that

$$\mathcal{A} = \{(k\tau')n, \text{ for all } n \in \mathbb{N}\} \tag{9.50}$$

From there we have that $\text{d}(A) = \frac{1}{k\tau'} > 0$, as we wished to prove. □

The above theorem further asserts the fact that we can only properly define a time independent braid integrability for *bounded regimes of motion*. In the unbounded case, no motions will be long time integrable[4], and so we are left to consider the time dependent definitions as our guiding ones.

Furthermore, although the word "period" was excessively used, we know that integrable hamiltonian system can be quasi-periodic in the sense that they will eventually get back to a *neighborhood* of the starting point, and not necessarily *the* point per se. We can then use the word *period* in this context to refer to the (average) time this returning process takes, so that the above given proof is equally valid.

[4]this is only true if there is some time t_{sep} at which the particles of the system separate entirely, so that for later times no new braid crossings are formed. In case only one particle "decouples", we reduce in one strand the braid group generated by the system's motion, and so, its long time behavior is still meaningful.

Appendices

Appendix A

Geometry

A.1 Riemannian Geometry

We begin this appendix by defining the important concepts that shall be used later on throughout the thesis.

Definition A.1.1 (Differentiable Manifold)**.** A **differentiable manifold** M of dimension n, written dim $M = n$, is a geometric object that fulfills the following conditions:

DM.1 M is **Hausdorff**

DM.2 M is **second-countable**

DM.3 For each point $x \in M$, there exists a neighbourhood U_x and a homeomorphism φ_U such that $\varphi_U : U_x \to \mathbb{R}^n$ such that on non-empty intersections $U \cap V$ the map $\varphi_U \circ \varphi_V^{-1} : \varphi_V(U \cap V) \to \varphi_U(U \cap V)$ is a diffeomorphism.

In particular we have that DM.1 asserts that points can be separated over M; DM.2 tells us that over each point we get only a finite collection of neighbourhoods and DM.3 is what gives meaning to the word "differentiable" in the definition. Note that throughout the thesis the word *differentiable* will be used to refer to C^∞ maps, i.e, those who admit derivatives of arbitrarily large order.

Formally we can also define the notion of **local charts** on a manifold, which will be nothing more than the pair (U, φ) such that $\varphi : U \to \mathbb{R}^n$ (for an $n-$dimensional manifold), based on which we get the notion of an **atlas** $\mathcal{A}$ given by the collection of all the local charts $\{(U, \varphi)\}$. We can even define a "weaker" version of a differentiable manifold by asking that the *transition maps* $\varphi_U \circ \varphi_V^{-1}$ of **DM.3** instead be $\mathbb{C}^r(\mathbb{R}^n)$, that is, $r-$times continuously differentiable. The case $r = 0$ is that of a **topological manifold**.

Next, we've got the following important

Definition A.1.2 (Tangent Space)**.** The **tangent space** to a point $p \in M$, denoted T_pM, is thought of the collection of all the tangent vectors v to curves $\alpha : (-\varepsilon, \varepsilon) \to M$ so that $\alpha(0) = p$ and $\alpha'(0) = v$.

In general, since we have local description of our manifold as $\mathbb{R}^n$ by means of the local charts, we can take a set of curves $\alpha_i(t)$ and do the following operation

$$(-\varepsilon, \varepsilon) \xrightarrow{\alpha} U \xrightarrow{\varphi} \varphi(U) \in \mathbb{R}^n \tag{A.1.1}$$

so that the constructed function $\varphi(\alpha_i(t)) = x + te_i$ maps $(-\varepsilon, \varepsilon)$ to $\mathbb{R}^n$, and satisfies $d\varphi(\alpha_i'(0)) = e_i$, with $\alpha_i(0) = p$. Moreover, the collection $\{\alpha_i'(0)\}$ of tangent vectors to each curve these α_i generate over M actually forms a basis for the tangent space T_xM (Proposition 3.15 of [Lee13]). This basis is usually written as $\{\partial/\partial x_i\}$ and the reason for such a notation is quite straight forward.

Suppose you are given a function $f \in C^\infty(\mathbb{R}^n)$, i.e, a function $f : \mathbb{R}^n \to \mathbb{R}$. Now, say you'd like to take the derivative of such a function in a certain direction. Locally this can be done in the following manner

$$\left.\frac{d}{dt}\right|_{t=0} f(\alpha_i(t)) = grad(f)(x) \cdot \alpha'(0) = grad(f)(x) \cdot e_i = \frac{\partial f}{\partial x_i}$$

Analogously, based on Eq.(A.1.1), if we are given an $f \in C^\infty(M)$ we construct its **coordinate representation** $\hat{f} \in C^\infty(V), V = \varphi(U) \in \mathbb{R}^n$ by defining $\hat{f} = f \circ \varphi^{-1}$, with (U, φ)

being a coordinate chart. In this way the above set of equalities becomes

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} f(\alpha_i(t)) &= \frac{d}{dt}\bigg|_{t=0} \hat{f}(\varphi(\alpha_i(t))) = grad(\hat{f})(x) \cdot d\varphi(\alpha_i'(0)) \\ &= grad(\hat{f})(p) \cdot e_i = \frac{\partial}{\partial x_i}\bigg|_{x} \hat{f}(p) = \frac{\partial}{\partial x_i}\bigg|_{\varphi(p)} (f \circ \varphi^{-1})(p) \\ &= \frac{\partial}{\partial x_i}\bigg|_{p} (f \circ \varphi^{-1})(\varphi(p)) = \frac{\partial}{\partial x_i}\bigg|_{p} f(p) =: \alpha_i'(0) f(p)\end{aligned}$$

so that the vector $\alpha_i'(0) = \partial/\partial x_i$ acts as a *derivation* on $C^\infty(M)$. For further details on this, the reader is invited to have a look at Lemma 3.1 and Proposition 3.2 of [Lee13]. Now, with Definition A.1.2 at hands we can move on to

Definition A.1.3 (Tangent Bundle)**.** The **tangent bundle** TM over a manifold M is given by

$$TM = \bigsqcup_{p \in M} T_p M \tag{A.1.2}$$

More generically a **vector bundle** over a manifold is a triple (π, E, M) (usually denoted $\pi : E \to M$, or $E \xrightarrow{\pi} M$) of a projection map, a so called total space and a base space, respectively. Both the total and base spaces have manifold structures on them but, there exists a condition the pre-images of π satisfy, namely, that of *local trivialization*, that is a homeomorphism ϕ satisfying

$$\phi : \pi^{-1}(U) \to U \times \mathbb{F}^k \tag{A.1.3}$$

for some open set $U \subset M$ and a field $\mathbb{F}$ (see Definition C.1.4). If $\mathbb{F} = \mathbb{R}$ then (π, E, M) is a **real** $k-$**vector bundle** over M. If it's $\mathbb{C}$ instead then we call it a complex $k-$vector bundle. Moreover, on the case in which $k = 1$ we call it a (real or complex) **line bundle**.

Observation A.1. These line bundles are surprisingly useful. They for instance pop up on the theory of Geometric Quantization. It is a fascinating topic and, although way beyond the scope of this thesis, the reader is invited to have a look at [Woo91] for an introduction, if interested.

Moreover we also ask that at nonempty intersections $U \cap V$ the transition map $\phi_{UV} = \phi_U \circ \phi_V^{-1}$ acts on points $(x, p) \in (U \cap V) \times \mathbb{F}^k$ as

$$\phi_{UV}(x, p) = (x, gp), \quad g \in GL(\mathbb{F}, k) \tag{A.1.4}$$

with $GL(\mathbb{F}, k)$ being the group of $k \times k$ matrices with non-vanishing determinant and whose entries lie in $\mathbb{F}$. Its action over $\mathbb{F}^k$ is a mere matrix multiplication.

Well, getting back to Definitions A.1.3, and A.1.2 we can also define a dual analog of these, namely the **cotangent space** T_p^*M and **cotangent bundle** T^*M (whose definition exactly matches that of Eq.(A.1.2), but with each term now being T_p^*M instead). An element $\alpha \in T_p^*M$ is represented as $\alpha = \alpha_i e^i$, with $\{e^i\}$ being the **dual bases** to $\{\partial/\partial x_i\}$, defined by the relation

$$e^j\left(\frac{\partial}{\partial x_i}\right) = \delta_i^j := \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Furthermore, the action of the e^j on vectors is linear, so that

$$e^j(v) = e^j(v^k e_k) = v^k e^j(e_k) = v^k \delta_k^j = v^j$$

And well, more generically we have that

$$\alpha(v) = \alpha_j e^j(v^k e_k) = \alpha_j v^j \tag{A.1.5}$$

where we've been using Einstein summation convention of repeated indices. A common jargon to refer to the elements of the cotangent bundle is to call them $1-$**forms**. More precisely we have

Definition A.1.4 ($k-$form)**.** Given a vector space V over a field $\mathbb{F}$, a $k-$**form** β over V is an element of $\bigotimes_{i=1}^k V^*$[1] that is $\mathbb{F}$ linear on each entry.

Example A.1.1. Consider the $2-$form $e^1 \otimes e^2$. It acts on $V \times V$ as

$$(e^1 \otimes e^2)(u, v) = e^1(u)e^2(v) = e^1(u^k e_k)e^2(v^j e_j) = u^1 v^2$$

[1]Tensor product of V^* with itself k times.

In general, given a basis $\{e_i\}$ for V, the set $\{e^{i_1}\otimes\cdots\otimes e^{i_k}\}$ for all possible combinations of $i_1,\ldots,i_k = 1,\ldots,k$, gives us a basis for the elements of $\bigotimes_{i=1}^{k} V^*$. A usual notation for such a space of $k-$forms over a space V is $\Omega^k(V)$. Based on this we can further consider the set $\Lambda^k(V)$ of *alternating $k-$forms* over M. An element $\alpha \in \Lambda^k(V)$ has the following property when operating on vectors:

$$\alpha(v_1,\ldots,v_i,\ldots,v_j,\ldots,v_k) = -\alpha(v_1,\ldots,v_j,\ldots,v_i,\ldots,v_k) \tag{A.1.6}$$

which is the reason why we call it alternating in the first place. More generally, given an $\alpha \in \Omega^k(TM)$ we define its associated **alternating form** $Alt(\alpha)$ by:

$$Alt(\alpha)(v_1,\ldots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k} sgn(\sigma)\alpha(v_{\sigma(1)}\ldots v_{\sigma(k)}) \tag{A.1.7}$$

With S_k being the group of permutations of k elements, and $sgn(\sigma)$ the sign of the permutation σ, being $+1$ in case it's even and -1 in case it's odd. Note in particular that, from the very definition $Alt(\alpha)$ is alternating. Indeed if we switch the places of two vectors, what we'll actually be doing is performing an *odd* permutation to the list $(v_1,\ldots,v_k)$. Since said list "enters" the right hand side of Eq.(A.1.7) as it is, when we perform yet another permutation to it, the effect compounds. So that, if the permutation is even, since we already came in with an odd permutation beforehand, the sign will change to a negative. If it's odd, then it'll change to a positive. Either way, no matter which permutation we perform and evaluate α on the new list, it will come out with the opposite sign it did prior to change we made. This in turn enables us to say that

$$Alt(\alpha)(v_1,\ldots,v_i,\ldots,v_j,\ldots,v_k) = -Alt(\alpha)(v_1,\ldots,v_j,\ldots,v_i,\ldots,v_k) \tag{A.1.8}$$

So, the way we can express in coordinates the alternating version of a given $k-$form is by means of the **wedge product**, defined as:

$$e^{i_1}\wedge\cdots\wedge e^{i_k} = Alt(e^{i_1}\otimes\cdots\otimes e^{i_k}) \tag{A.1.9}$$

In such a way that an $\alpha \in \Lambda^k(V)$ shall be given by $\alpha = \frac{1}{k!}\alpha_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k}$ (where once again we use Einstein summation convention for repeated indices).

Getting back on track, the particular case of $V = TM$ is of utter most importance, since it is then and there that we are able to define the crucial structure of this appendix.

Definition A.1.5 (Riemmanian Metric)**.** A **Riemannian Metric** g is an element of $\Omega^2(TM)$ that is:

RM.1 Everywhere non-vanishing

RM.2 Positive definite

From this, we have the important

Definition A.1.6 (Riemannian Manifold)**.** A **Riemannian manifold** is a pair (M, g) where M is a differentiable manifold and g is a Riemannian metric.

Now, getting back to the bundles again, another way to think of these concepts of vectors and forms is by the lens of **sections**.

Definition A.1.7 (Section)**.** Given a vector bundle $\pi : E \to M$ a **section** s is a map $s : M \to E$ such that $\pi \circ s = Id_M$.

Usually the space of sections is denoted $\Gamma(E)$. In particular, a vector field X over M is a section of the tangent bundle TM, i.e $X \in \Gamma(TM)$, because for every point $p \in M$, $X_p \in T_pM \subset TM$ and $\pi(X_p) = p$ (once we note that $X_p = (p, v^1(p), \dots, v^n(p))$ locally!). Analogously we can define form fields over M, that is, sections of T^*M. In particular a Riemannian metric g is an element of $\Gamma(T^*M \otimes T^*M)$, because for every point $p \in M$, $g_p \in T^*M \otimes T^*M$, i.e g_p takes two vectors and returns a real number (interpreted as the inner product of these two vectors), besides also satisfying that $\pi(g_p) = p$.

Another highly important concept in Riemannian Geometry is that of a connection, formally given by

Definition A.1.8 (Connection)**.** Given a vector bundle $E \to M$, a **connection** ∇ over M is a map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ that, given $f \in C^\infty(M)$, satisfies the following conditions:

CO.1 $\nabla_{fX} s = f \nabla_X s$

CO.2 $\nabla_X(fs) = (Xf)s + f\nabla_X s$

The definition is quite broad and, as a matter of fact, one can usually define various connections on the same manifold. In any case, if we are given an open neighbourhood $U \subset M$, a basis $\{s_i\}$ for $\Gamma(U, E)$ (the sections of E over U) and a basis $\{e_i\}$ for $\Gamma(U, TM)$, then we can say that:

$$\nabla_{e_i} s_j = \Gamma^k_{ij} s_k \tag{A.1.10}$$

Though the term *connection coefficients* can be given to the Γ^k_{ij}'s, it is also customary to call them as the *Christoffel symbols* of the connection ∇, after the German mathematical-physicist Elwin Bruno Christoffel (1829-1900). They will naturally depend on the chosen coordinate basis and can surprisingly be interpreted as a set of "E valued matrix $1-$forms". This is because in the general setting (but still over the neighbourhood U) one has that a section s is written as $s = a^j s_j$ (for $a^j \in C^\infty(U)$), so that by condition **CO.2** we have:

$$\nabla_{e_k} s = (\partial_k a^j + a^l \Gamma^j_{kl}) s_j \tag{A.1.11}$$

The E-valuedness part of the above statement is a bit trickier to understand but, basically it refers to the fact that, if we let $\Gamma^j_k = \Gamma^j_{lk} e^k$ (with e^j being the dual of e_j, like described above), then we can build the matrix $\Gamma = (\Gamma^j_k)_{j,k}$ and define its (linear) action on $\Gamma(U, E)$ by[2]:

$$\Gamma s = \Gamma(a^k s_k) = a^k \Gamma s_k = a^k \Gamma^j_k s_j = a^k \Gamma^j_{lk} e^l \otimes s_j = a^l \Gamma^j_{kl} e^k \otimes s_j$$

[2]the tensor product is just to keep things more formal. As long as you know how to compute the desired quantity, you can omit it without any issues!

so that the Christoffel symbols are E-valued $1-$forms because they can be seen as functions that take a vector and return an element of E (due to the way in which the tensor product operates)! The space of such functions will be denoted $\Omega(U, E)$, in accordance with the above notation for $1-$form spaces over M. Moreover, by defining the *derivation* d of a section $s = a^j s_j$ as

$$ds = \frac{\partial a^j}{\partial x^i} e^i \otimes s_j$$

we can simply rewrite Eq.(A.1.11) as:

$$\nabla s(e_k) = (ds + \Gamma s)e_k \tag{A.1.12}$$

Which (despite potentially not looking much like it) makes a lot of sense, given that $\nabla s : \Gamma(TM) \to \Gamma(E)$, according to Definition A.1.8. It is thus based on Eq.(A.1.12) that we usually write

$$\nabla = d + \Gamma, \quad \Gamma \in \Omega(U, E) \tag{A.1.13}$$

Another very important concepts in Riemannian geometry are that of **metric compatibility condition**, expressed as:

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad X, Y, Z \in \Gamma(TM), \tag{A.1.14}$$

and of **torsion-free connection**, namely:

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \tag{A.1.15}$$

By playing around locally with Eq.(A.1.14), one is able to conclude that the Christoffel symbols assume the following form:

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} (\partial_i g_{mj} + \partial_j g_{mi} - \partial_k g_{ij}), \tag{A.1.16}$$

and actually satisfy Eq.(A.1.15). We call (A.1.16) the **Levi-Civita** connection and by the Fundamental Theorem of Riemannian Geometry (check Theorem I.5.1 of [Cha06] for a proof), it is the unique metric compatible, torsion-free connection over our manifold M.

Now, taking a couple of steps back to Definition A.1.8, we define the following

Definition A.1.9 (Parallel Transport)**.** Given a section $s \in \Gamma(E)$ and a vector field $X \in \mathfrak{X}(M)$, we say that X **parallel transports** s if

$$\nabla_X s = 0, \tag{A.1.17}$$

Or equivalently s is parallel transported in the X direction. This is a crucial notion because it is based on such that we have

Definition A.1.10 (Geodesic)**.** Given a curve $\gamma : I \to M$, we say that γ is a **geodesic curve** over M if

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0, \tag{A.1.18}$$

As you can tell, Eq.(A.1.18) is just a particular version of Eq.(A.1.17) and, indeed one could say that a geodesic curve is one whose tangent vector is parallel transported along itself. In coordinates $(x^1, \dots, x^n)$ we can say (by slight abuse of notation) that $\gamma(t) = (x^1(t), \dots, x^n(t))$. So that, if $\{\partial_j\}$ generate the tangent space at a point, say $\gamma(0) = p \in M$, then we can say that

$$\dot{\gamma} = \dot{x}^j \partial_j,$$

based on which Eq.(A.1.18) assumes the form[3]

$$\ddot{x}^j + \Gamma^j_{kl}\dot{x}^k\dot{x}^l = 0, \tag{A.1.19}$$

commonly referred to as the *(affine) geodesic equation.* Physically, this equation represents the trajectory a test particle of unit mass would follow if to be put within the manifold M, under the action of no other forces! Indeed, Newton's famous second law $F = ma$ is more generally written as

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \boldsymbol{f}, \tag{A.1.20}$$

which in coordinates gets to be

$$\ddot{x}^j + \Gamma^j_{kl}\dot{x}^k\dot{x}^l = f^j. \tag{A.1.21}$$

[3]check [Cha06] for a more in depth derivation.

Note however that Eq.(A.1.19) tells us that on a curved space, we expect free moving particles to follow curvy, non-linear trajectories as view from the outside (unlike with $\mathbb{R}^n$). Other objects that enable us to measure curvature are the **Riemann** and **Ricci** tensors. Given $X, Y, Z \in \mathfrak{X}(M)$, the former is define by:

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \tag{A.1.22}$$

with $[\cdot,\cdot]$ being your usual vector field commutator (i.e, $[X,Y] = XY - YZ$). In coordinates we have that the Riemann tensor is

$$R^l{}_{ijk} = \partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^r_{ij}\Gamma^l_{rk} - \Gamma^r_{ik}\Gamma^l_{rj}, \tag{A.1.23}$$

Upon contraction of the first and third indices we find the Ricci tensor mentioned above, so that:

$$R_{ik} = \partial_k \Gamma^l_{il} - \partial_l \Gamma^l_{ik} + \Gamma^r_{il}\Gamma^l_{rk} - \Gamma^r_{ik}\Gamma^l_{rl}, \tag{A.1.24}$$

By construction it follows that the Ricci tensor is symmetric. For relations on the indices of the Riemann tensor the reader is invited to have a look at [Cha06]. We nonetheless explicit here one particularly useful formula know as the **second (differential) Bianchi identity** , that is satisfied by the Riemann tensor, namely[4]:

$$\nabla_{(\mu} R_{\alpha\beta)\rho\sigma} = 0, \ \ \forall \rho, \sigma \leq n \tag{A.1.25}$$

Furthermore, note that we can create a particular scalar quantity that encodes the curvature of the manifold known as the **Ricci scalar** given by:

$$R = g^{ij} R_{ij}, \tag{A.1.26}$$

And, it is with these last two quantities in mind that we are able to construct the **Einstein Tensor**, which is yet another measure of curvature and it is given by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{A.1.27}$$

[4]as usual, the parenthesis on the indices denote a symmetric permutation of them.

having the property that it's divergence is zero, that is $\nabla_\nu G^{\mu\nu} = 0$.

To finish this part we invite the reader to have a look at some illuminating applications to physics of this subject [Poi04, Car14, Mag08, RD22].

A.1.1 A bit more on curvature forms

Above, we defined the Riemann and Ricci tensors in a rush, you could say. We now explore a bit more of their relationship. First note the following: given any finite dimensional vector space V with a metric g, we can consider over its dual a set of rank-4 tensors denoted $\mathscr{R}(V)$, being such that $\forall\, R \in \mathscr{R}(V)$ we have[5]:

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}, \tag{A.1.28a}$$
$$R_{a(bcd)} = R_{abcd} + R_{acdb} + R_{adbc} = 0, \tag{A.1.28b}$$

The set $\mathscr{R}(V)$ is suitably called the **space of algebraic curvature tensors on** V, because the set of Eqs.(A.1.28) is composed of the algebraic relations the Riemann curvature tensor satisfies [Cha06]. Along side it we can also consider the set $S^2(V)$ of **symmetric 2-tensors on** V[6] and define the *trace function* that relates the two in the following manner:

$$\begin{aligned} \mathrm{Tr} : \mathscr{R}(V) &\longrightarrow S^2(V) \\ R &\longmapsto \mathrm{Tr}(R)_{ac} = g^{bd} R_{abcd} \end{aligned}$$

That is, as we described on the previous section, we perform a contraction of the second and last (or equivalently, first and third) indices to obtain a "Ricci-like" tensor.

On the basis of this trace map, we can build the so called **Weyl space** $\mathscr{W}(V) = Ker(\mathrm{Tr}) \cap \mathscr{R}(V)$, and in particular consider it over a manifold M by making $V = TM$.

[5]There is a purposefully missing factor of $1/3$ on the first equality of Eq.(A.1.28b) because it equals zero so, such a factor wouldn't contribute to anything.

[6]defined on a similar way as to $\Lambda^2(V)$ though this time, instead of the alternating we've got the symmetric product $\alpha \circ \beta$ for $\alpha \in \Omega^k(V)$ and $\beta \in \Omega^l(V)$

It's possible to show that, if dim $M = n$, the following decomposition holds:

$$R_{\lambda\mu\nu\kappa} = \frac{1}{n-2}(g_{\lambda\nu}R_{\mu\kappa} + g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} - g_{\mu\kappa}R_{\lambda\nu}) - \frac{R}{(n-1)(n-2)}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}) + C_{\lambda\mu\nu\kappa}, \quad (A.1.29)$$

with $C_{\lambda\mu\nu\kappa} \in \mathscr{W}(V)$ being the *Weyl tensor* associated to $R_{\lambda\mu\nu\kappa}$.

There is very interesting argument that asserts the triviality of the Weyl tensor in dimensions less than or equal to 3. In essence, it follows from the following: take an operator b defined over the tensor product space $\bigoplus^4 V$ such that $\text{Ker}(b) := \mathscr{R}(V)$. That is, $\forall R \in \bigoplus^4 V$ define

$$b(R)(e_a, e_b, e_c, e_d) = \frac{1}{3}R_{a(bcd)} = R_{abcd} + R_{acdb} + R_{adbc}), \quad (A.1.30)$$

for e_i elements of the base of V. By noticing that $S^2\Lambda^2 V \subset \bigoplus^4 V$[7] we can consider $b|_{S^2\Lambda^2 V}$ (which, by abuse of notation, we just keep referring to as b as well), since $b(S^2\Lambda^2 V) = S^2\Lambda^2 V$, and $b^2 = b$ (as can be checked manually), b acts as some type of projection over $S^2\Lambda^2 V$, and so we have the following decomposition

$$S^2\Lambda^2 V = \text{Ker}(b) \oplus \text{Im}(b). \quad (A.1.31)$$

More over, it's not hard to show that $\text{Im}(b) = \Lambda^4 V$. Indeed, take $x, y, z, t \in V$ arbitrary vectors and consider the more general version of Eq.(A.1.30)

$$T(t, x, y, z) := b(R)(t, x, y, z) = \frac{1}{3}(R(t, x, y, z) + R(t, y, z, x) + R(t, z, x, y)). \quad (A.1.32)$$

To show the above assertion is to show that an exchange of any two coordinates of the above defined T will be equivalent to a mere sign change on T's value. Indeed, for any

[7] An element $R \in S^2\Lambda^2 V$ is, in coordinates, written as

$$R = R_{ijkl}(e^i \wedge e^j) \circ (e^k \wedge e^l)$$

$R \in S^2\Lambda^2 V$ we have:

$$\begin{aligned} T(x,t,y,z) &= \frac{1}{3}(R(x,t,y,z) + R(x,y,z,t) + R(x,z,t,y)) \\ &= \frac{1}{3}(-R(t,x,y,z) + R(z,t,x,y) + R(t,y,x,z)) \\ &= \frac{1}{3}(-R(t,x,y,z) - R(t,z,x,y) - R(t,y,z,x)) \\ &= -T(t,x,y,z). \end{aligned}$$

A similar argument shows that the other exchanges such as $T(t,y,x,z)$ and $T(t,x,z,y)$ yield the same result.

This in particular shows that, for the cases where $n = 2,3$, we have that $b \equiv 0$ (because $\dim\Lambda^4 V = \binom{n}{4}$) so that $\mathscr{R}(E) = S^2\Lambda^2 V$. Moreover, by counting the dimensions of $S^2\Lambda^2 V$ and comparing it to the dimension of $S^2 V$ in these cases, we see that the trace map considered above is injective and so its kernel is trivial, containing only the zero tensor. By the definition of the Weyl space, we have that *no non-trivial Weyl tensor exists for dimensions* 2 *and* 3.

To conclude this fact for the $n = 3$ Proposition 7.1 of [Via11] must be used. Overall, the reader is invited to check [Bes87, Via11] for more on this matter.

A.2 Derivations of forms

One very important derivation one encounters in geometry is the famous **exterior derivative**, given by:

$$d : \Omega^k(M) \to \Omega^{k+1}(M), \tag{A.2.1}$$

The way it acts on a given $k-$form $\alpha = \alpha_{i_1 \dots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k}$ is simple, for instance:

$$d\alpha := \frac{\partial \alpha_{i_1 \dots i_k}}{\partial x^\beta} e^\beta \otimes e^{i_1} \otimes \cdots \otimes e^{i_k}, \tag{A.2.2}$$

And it doesn't stop there! We can keep going further and define on the same manner the action of d on higher order forms. Indeed we actually have a *chain*

$$\Omega^0(M) \xrightarrow{d} \Omega(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{\mathrm{d}} \dots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

with $\Omega^0(M) = \mathcal{C}^\infty(M)$ the space of infinitely differentiable functions over M. Similarly we can define the action over the space of alternating forms instead so that we also obtain the chain

$$\Lambda^0(M) \xrightarrow{d} \Lambda(M) \xrightarrow{d} \Lambda^2(M) \xrightarrow{d} \dots \xrightarrow{d} \Lambda^n(M) \xrightarrow{d} 0$$

Though in this case, the definition is a bit more subtle. For instance, on the previous section we defined $\alpha \in \Lambda^k(M)$ to have the form $\alpha = \frac{1}{k!}\alpha_{i_1\dots i_k}e^{i_1} \wedge \dots \wedge e^{i_k}$. Though, if we were to write it down more explicitly, note that we could also have:

$$\alpha = \frac{1}{k!}\sum_{i_1,\dots,i_k} \alpha_{i_1\dots i_k}e^{i_1} \wedge \dots \wedge e^{i_k} = \sum_{i_1<\dots<i_k} \alpha_{i_1\dots i_k}e^{i_1} \wedge \dots \wedge e^{i_k}$$

due to the fact that the coefficients are anti-symmetric themselves. Thus, we can define the differential of α to be:

$$d\alpha = \sum_{i_1<\dots<i_k}\sum_j \partial_j \alpha_{i_1\dots i_k}e^j \wedge e^{i_1} \wedge \dots \wedge e^{i_k} \tag{A.2.3}$$

If we want to get back to our previous notation though, all we have to do is to consider the *anti-symmetrized* version of the former defined expression, namely:

$$d\alpha = \frac{1}{k!}\sum_{i_1,\dots,i_{k+1}} \partial_{[i_1}\alpha_{i_2\dots i_{k+1}]}e^{i_1} \wedge \dots \wedge e^{i_{k+1}} \tag{A.2.4}$$

Example A.2.1. Let $A \in \Lambda^1(M)$, then $A = A_\mu e^\mu$. To maintain the Einstein summation convention as described above, we'd need to consider $dA = \partial_{[\alpha}A_{\mu]}e^\alpha \wedge e^\mu$ or more explicitly, we'd have that

$$dA = \frac{1}{2}(\partial_\alpha A_\mu - \partial_\mu A_\alpha)e^\alpha \wedge e^\mu$$

Example A.2.2. Let $F \in \Lambda^2(M)$, so that $F = \frac{1}{2}F_{\mu\nu}e^\mu \wedge e^\nu$. Now, using the fact that $F_{\mu\nu} = -F_{\nu\mu}$, we can write its differential as:

$$dF = \frac{1}{6}(\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu})e^\alpha \wedge e^\mu \wedge e^\nu$$

To finish off this part, note that given a metric structure $g \in \Gamma(T^*M \otimes T^*M)$, we can define the inner product on the space of forms in the following fashion:

$$g_x(\alpha, \beta) = \langle \alpha, \beta \rangle_x = \frac{1}{k!} g^{i_1 j_1} \dots g^{i_k j_k} \alpha_{i_1 \dots i_k} \beta_{j_1 \dots j_k} \tag{A.2.5}$$

where $x \in M$ and $\alpha, \beta \in \Lambda^k(M)$. In the case where M is *compact* we can further define the quantity

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle \, \mathrm{dVol}_M \tag{A.2.6}$$

known as the $\boldsymbol{L^2}$**−metric** over $\Lambda^k(M)$. Differential forms that have finite L^2-norm are suitably called $\boldsymbol{L^2}$***−integrable forms***. For compact manifolds any smooth form is L^2-integrable, whereas for open manifolds such a condition essentially demands that the functions of each component of your form go "fast enough" to zero at infinity.

To finish this first part, note that based on Eq.(A.2.6) we are able to define the dual operator to d, namely the **codifferential operator** δ satisfying

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle \,, \quad \alpha \in \Lambda^k(M), \beta \in \Lambda^{k+1}(M) \tag{A.2.7}$$

On the end of the next section we shall see how to compute this codifferential. We end this part by defining the ***k*th de Rham cohomology class** (over the real numbers) as:

$$\mathrm{H}^k(M, \mathbb{R}) = \frac{\mathrm{Ker}(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M))} \tag{A.2.8}$$

The elements of this group are equivalence classes of differential forms. We say that $\alpha \sim \beta$ in $\mathrm{H}^k(M, \mathbb{R})$ if there exists some $\phi \in \Lambda^{k-1}(M)$ such that

$$\alpha - \beta = d\phi, \tag{A.2.9}$$

that is, α and β belong to the same cohomology class if they differ by an exact form. More on the properties of this space can be found at Chapters 17 and 18 of [Lee13].

The Hodge Star

Given the above construction of differential $k-$forms on a manifold, we can define yet another very useful operation that takes us from one class of forms to another. As the title suggests, this transformation is called the **Hodge star operator**, and it's defined in the following fashion.

Suppose you are given an oriented (pseudo-)Riemannian Manifold (M, g), and over it you consider the space of alternating $k-$forms $\Lambda^k(M)$. Then, the *Hodge dual* $\star\alpha$, of some $\alpha \in \Lambda^k(M)$, is going to be the $(n-k)-$form defined by the following:

$$\beta \wedge \star\alpha = \langle \beta, \alpha \rangle \, \mathrm{dVol}_M, \forall \beta \in \Lambda^k(M) \tag{A.2.10}$$

where naturally dVol_M is the volume form of M. To explicitly compute the exact form of the Hodge dual in a given coordinate basis is a tricky task (to say the least!). Nonetheless, we shall proceed with the calculations since they will come in handy on Appendix A.3.

To begin, lets say we are given the $k-$form $\beta = \alpha = \frac{1}{k!} e^{i_1} \wedge \cdots \wedge e^{i_k}$ (summing over i_1 through i_k), and we wish to compute its dual. First, one should note that[8]:

$$\|\alpha\|^2 = \frac{1}{k!} g^{i_1 j_1} \dots g^{i_k j_k}, \text{ and } \mathrm{dVol}_M = \sqrt{|g|} e^1 \wedge \cdots \wedge e^n$$

where we decide to keep the volume form in terms of the ordered summation since it simplifies the computations a bit. Thus, if we go back to Eq.(A.2.10), we find that:

$$e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge \star(e^{i_1} \wedge \cdots \wedge e^{i_k}) = g^{i_1 j_1} \dots g^{i_k j_k} \sqrt{|g|} e^1 \wedge \cdots \wedge e^n$$

Note that it is then reasonable to let $\star(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \frac{f^{i_1 \dots i_k}_{j_{k+1} \dots j_n}}{(n-k)!} e^{j_{k+1}} \wedge \cdots \wedge e^{j_n}$, in such a way that:

$$g^{i_1 j_1} \dots g^{i_k j_k} \sqrt{|g|} e^1 \wedge \cdots \wedge e^n = \frac{f^{i_1 \dots i_k}_{j_{k+1} \dots j_n}}{(n-k)!} e^{j_1} \wedge \cdots \wedge e^{j_k} \wedge e^{j_{k+1}} \wedge \cdots \wedge e^{j_n} \tag{A.2.11}$$

Where we have just switched the indices i_p for j_p on the right hand side above, so that notation looks less messy and more uniform. Now note the following fact. Defining the *Levi-Civita symbols* $\varepsilon_{\mu_1 \dots \mu_k}$ as a totally anti-symmetric quantity whose values are defined

[8]the summations over the i_m's, j_m's and l_m's are left implicit, just for the sake of having a cleaner expression!

by:

$$\begin{cases} 1, & \text{if } (\mu_1, \ldots, \mu_k) \text{ is an } \textbf{even} \text{ permutation of } (1, \ldots, k) \\ -1, & \text{if } (\mu_1, \ldots, \mu_k) \text{ is an } \textbf{odd} \text{ permutation of } (1, \ldots, k) \\ 0, & \text{otherwise} \end{cases}$$

we can relate the formula of a given alternating $k-$form $e^{i_1} \wedge \cdots \wedge e^{i_k}$ to $e^1 \wedge \cdots \wedge e^k$ by:

$$e^{i_1} \wedge \cdots \wedge e^{i_k} = e^1 \wedge \cdots \wedge e^k \varepsilon^{i_1 \ldots i_k} \tag{A.2.12}$$

Now, using Eq.(A.2.12) on the above formula, we find that:

$$\frac{f^{i_1 \ldots i_k}_{j_{k+1} \ldots j_n}}{(n-k)!} e^{j_1} \wedge \cdots \wedge e^{j_k} \wedge e^{j_{k+1}} \wedge \cdots \wedge e^{j_n} = \frac{f^{i_1 \ldots i_k}_{j_{k+1} \ldots j_n}}{(n-k)!} \varepsilon^{j_1 \ldots j_k j_{k+1} \ldots j_n} e^1 \wedge \cdots \wedge e^n \tag{A.2.13}$$

To be able to solve for the f's above, we shall also need the following relation on the ε's, namely:

$$\varepsilon^{j_1 \cdots j_k l_{k+1} \ldots l_n} \varepsilon_{j_1 \ldots j_k p_{k+1} \ldots p_n} = k! \delta^{l_{k+1} \ldots l_n}_{p_{k+1} \ldots p_n} \tag{A.2.14}$$

with $\delta^{l_{k+1} \ldots l_n}_{p_{k+1} \ldots p_n}$ being the **generalized Krönecker delta**, defined as:

$$\delta^{l_{k+1} \ldots l_n}_{p_{k+1} \ldots p_n} = \begin{vmatrix} \delta^{l_{k+1}}_{p_{k+1}} & \ldots & \delta^{l_{k+1}}_{p_n} \\ \vdots & & \vdots \\ \delta^{l_n}_{p_{k+1}} & \ldots & \delta^{l_n}_{p_n} \end{vmatrix}$$

and each δ^ν_μ is such that it values 1 if $\mu = \nu$ and zero otherwise. Now, using Eq.(A.2.13) together with Eq.(A.2.14) we see that Eq.(A.2.11), once multiply by $\varepsilon_{j_1 \ldots j_k \mu_{k+1} \ldots \mu_n}$ on both sides, yields the following:

$$\begin{aligned} \sqrt{g} g^{i_1 j_1} \ldots g^{i_k j_k} \varepsilon_{j_1 \ldots j_k \mu_{k+1} \ldots \mu_n} &= \frac{f^{i_1 \ldots i_k}_{j_{k+1} \ldots j_n}}{(n-k)!} k! \delta^{j_{k+1} \ldots j_n}_{\mu_{k+1} \ldots \mu_n} \\ &= f^{i_1 \ldots i_k}_{[\mu_{k+1} \ldots \mu_n]} k! \\ &= k! f^{i_1 \ldots i_k}_{\mu_{k+1} \ldots \mu_n} \end{aligned}$$

where on the second line we've used the anti-symmetrization identity of the generalized Krönecker delta. Given that the f's are already anti-symmetric, we obtain the above result. Now, switching the dummy variables μ_j by i_j we finally have that:

$$f^{i_1 \ldots i_k}_{i_{k+1} \ldots i_n} = \frac{\sqrt{g} g^{i_1 j_1} \ldots g^{i_k j_k}}{k!} \varepsilon_{j_1 \ldots j_k i_{k+1} \ldots i_n} \tag{A.2.15}$$

Hence, with Eq.(A.2.15) in hands we conclude that:

$$\star(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \frac{\sqrt{g} g^{i_1 j_1} \dots g^{i_k j_k}}{(n-k)!k!} \varepsilon_{j_1 \dots j_k i_{k+1} \dots i_n} e^{i_{k+1}} \wedge \cdots \wedge e^{i_n}$$

and more generically for a $k-$form $\alpha = \alpha_{i_1 \dots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}$, after another change of dummy variables, it follows that:

$$\star\alpha = \frac{\sqrt{g} \alpha^{i_1 \dots i_k} \varepsilon_{i_1 \dots i_k i_{k+1} \dots i_n}}{(n-k)!k!} e^{i_{k+1}} \wedge \cdots \wedge e^{i_n} \tag{A.2.16}$$

where $\alpha^{i_1 \dots i_k} = g^{i_1 j_1} \dots g^{i_k j_k} \alpha_{j_1 \dots j_k}$. Note also that by considering the *Levi-Civita tensor* $E_{i_1 \dots i_n} = \sqrt{|g|} \varepsilon_{i_1 \dots i_n}$, we can rewrite Eq.(A.2.16) with the α indices lowered and the ε indices raised instead (which follows because, as the name suggests, $E_{i_1 \dots i_n}$ transforms like a tensor and hence has its indices lowered/raised by the metric $g_{\mu\nu}$ of M).

Now, being clear that $\star : \Lambda^k(M) \to \Lambda^{n-k}(M)$, we can notice the following short chain:

$$\Lambda^k(M) \xrightarrow{\star} \Lambda^{n-k}(M) \xrightarrow{\star} \Lambda^k(M)$$

and so it becomes natural to ask what happens to a $k-$form once we apply the Hodge dual twice to it. Well, we have the following:

$$\begin{aligned}
\star\star\alpha &= \star\left(\frac{\sqrt{g}\alpha_{i_1 \dots i_k} \varepsilon^{i_1 \dots i_k}{}_{i_{k+1} \dots i_n}}{(n-k)!k!} e^{i_{k+1}} \wedge \cdots \wedge e^{i_n}\right) \\
&= \frac{|g|\, \alpha_{i_1 \dots i_k} \varepsilon^{i_1 \dots i_k}{}_{i_{k+1} \dots i_n} \varepsilon^{i_{k+1} \dots i_n}{}_{j_1 \dots i_k}}{(n-k)!k!^2} e^{j_1} \wedge \cdots \wedge e^{j_k} \\
&= (-1)^{k(n-k)} \frac{|g|\, \alpha_{i_1 \dots i_k} \varepsilon^{i_1 \dots i_k i_{k+1} \dots i_n} \varepsilon_{i_{k+1} \dots i_n j_1 \dots i_k}}{(n-k)!k!^2} e^{j_1} \wedge \cdots \wedge e^{j_k} \\
&= (-1)^{k(n-k)} \frac{|g|\, \alpha_{i_1 \dots i_k} \delta^{i_1 \dots i_k}_{j_1 \dots i_k}}{k!^2} e^{j_1} \wedge \cdots \wedge e^{j_k} \\
&= (-1)^{k(n-k)} \frac{|g|\, \alpha_{j_1 \dots j_k}}{k!} e^{j_1} \wedge \cdots \wedge e^{j_k} = (-1)^{k(n-k)} |g|\, \alpha
\end{aligned}$$

Now, connecting the ends we have:

$$\star\star\alpha = (-1)^{k(n-k)} |g|\, \alpha, \ \ \forall \alpha \in \Lambda^k(M) \tag{A.2.17}$$

That is, the Hodge dual of the Hodge dual of a $k-$form is necessarily going to be a multiple of this $k-$form. In particular we can deduce a formula for the inverse Hodge in terms of the Hodge itself, namely:

$$\star^{-1} = \frac{(-1)^{k(n-k)}}{|g|}\star \tag{A.2.18}$$

Or even more explicitly we have that:

$$\star^{-1}\alpha = \frac{(-1)^{k(n-k)}}{\sqrt{|g|}(n-k)!k!}\alpha^{i_1\ldots i_k}\varepsilon_{i_1\ldots i_n}e^{i_{k+1}}\wedge\cdots\wedge e^{i_n}, \forall\alpha\in\Lambda^k(M) \tag{A.2.19}$$

To finish off this section we compute δ, as mentioned at the end of Section A.1. For this we suppose M compact (so that is particular $\partial M = \emptyset$) and note that by the definition of the Hodge dual [Eq.(A.2.10)] we have that the L^2-metric defined on Eq.(A.2.6) can be rewritten as

$$(\alpha, \beta) = \int_M \alpha\wedge\star\beta \tag{A.2.20}$$

from this, assuming $\alpha\in\Lambda^{k-1}(M), \beta\in\Lambda^k(M)$ and, by making use of Green's theorem for $k-$forms we have the following equalities:

$$\begin{aligned}(d\alpha,\beta) &= \int_M d\alpha\wedge\star\beta = \int_M d(\alpha\wedge\star\beta) - (-1)^{k-1}\int_M \alpha\wedge d(\star\beta)\\ &= \int_{\partial M}\alpha\wedge\star\beta + (-1)^k\int_M \alpha\wedge d(\star\beta) = (-1)^k\int_M \alpha\wedge d(\star\beta)\\ &= \int_M \alpha\wedge\star\delta\beta = (\alpha,\delta\beta)\end{aligned}$$

since this equality holds for all $\alpha\in\Lambda^{k-1}(M)$, it follows that

$$\delta\beta = (-1)^k\star^{-1}d\star\beta \tag{A.2.21}$$

So that in general we'd have that $\delta\gamma = (-1)^{\deg(\gamma)}\star^{-1}d\star\gamma$, form some form γ of degree $\deg(\gamma)\in\mathbb{Z}_{\geq 0}$.

Note A.2.1. The computations done above though assume a **Riemannian** metric over our manifold and thus, do not directly apply to the physical, relativistic case that interests us in this work. The catch here is that we can *choose* in the pseudo-Riemannian case to have the following identity

$$\varepsilon^{i_1 \dots i_n} := \varepsilon_{i_1 \dots i_n} \tag{A.2.22}$$

from which we'd have that the Levi-Civita tensor

$$E_{i_1 \dots i_n} := \sqrt{|g|}\varepsilon_{i_1 \dots i_n} \tag{A.2.23}$$

would have inverse given by

$$E^{i_1 \dots i_n} = \frac{\mathrm{sgn(g)}}{\sqrt{|g|}}\varepsilon^{i_1 \dots i_n} \tag{A.2.24}$$

with $\mathrm{sgn}(g)$ being the signature of the metric (check Definition A.5.1). This choice enables us to proceed with the calculations in the same way we did above. So that for instance the operators defined on Section A.3 make sense for both the Riemannian and pseudo-Riemannian cases. The other possible choice we could have made would have been to consider $\varepsilon^{i_1 \dots i_n} := \mathrm{sgn}(g)\varepsilon_{i_1 \dots i_n}$ which would in turn drop off the signature term of the inverse Levi-Civita tensor, but would make some minus signs appear in relations such as those of Eq.(A.2.14), which is not really that desirable for us.

Hodge's Theorem and Decomposition

Before moving on to some specify derivations that can be constructed with the exterior derivative and the Hodge dual, we'd like to explicitly mention the famous *Hodge decomposition Theorem* on forms. We first enunciate it in its classical, already know form, and just give references on which the reader can find the proof. We then state and prove its pseudo-Riemannian analog for L^2-integrable forms.

The classical Hodge Decomposition Theorem comes as a corollary to the following[9]:

Theorem A.2.1 (Hodge Theorem)**.** *Let M be a compact, connected Riemannian manifold. Naming $\Lambda_2^k(M)$ as the space of L^2-integrable $k-$forms over M, it follows that*

[9]This result is actually quite more general but, since it goes beyond the scope of the thesis to talk about it properly, we refer the reader to Theorem 5.1 of [Voi02].

there is an orthonormal basis of $\Lambda_2^k(M)$ consisting of eigenvalues of the Laplace-Beltrami operator $\Delta : \Lambda_2^k(M) \to \Lambda_2^k(M)$. That in turn means that Δ is diagonalizable and hence that:

$$\Lambda_2^k(M) = \mathcal{H}^k(M) \oplus Im(\Delta) \tag{A.2.25}$$

where $\mathcal{H}^k(M) = Ker(\Delta)$ is the space of harmonic $k-$forms.

Proof: Check Section 1.3 of [Ros97]. □

Corollary A.2.2 (Hodge Decomposition)**.** *Let M be a compact Riemannian manifold, and consider $\Lambda_2^k(M)$ the space of L^2-integrable $k-$forms over M. Then, the following decomposition holds:*

$$\Lambda_2^k(M) = \mathcal{H}^k(M) \oplus Im(d) \oplus Im(\delta) \tag{A.2.26}$$

Proof: Let's start by showing the following

Lemma A.2.1. Given the Hodge Laplacian $\Delta = \delta d + d\delta$, it follows that

$$\alpha \in \text{Ker}(\Delta) \Leftrightarrow d\alpha = \delta\alpha = 0 \tag{A.2.27}$$

Proof of lemma: ($\Leftarrow$) Follows trivially from the definition of the Hodge-Laplacian

($\Rightarrow$) If $\alpha \in \text{Ker}(\Delta)$, then we have that

$$0 = (\alpha, \Delta\alpha) = (\alpha, \delta d\alpha) + (\alpha, d\delta\alpha) = \|d\alpha\|_{L^2}^2 + \|\delta\alpha\|_{L^2}^2 .$$

Since both of the above quantities are non-negative, for their sum to be zero each one of the has to be zero. By the property of norms we have that $d\alpha = \delta\alpha = 0$. ■

Getting back to the corollary now, note that all we need to show (by Theorem A.2.1) is that

$$\text{Im}(\Delta) = \text{Im}(d) \oplus \text{Im}(\delta). \tag{A.2.28}$$

To do so we can simply prove that one set is contained in the other.

(⊆) If $\beta \in \text{Im}(\Delta)$, then we have a $k-$form α such that $\beta = \Delta\alpha$. This hence implies that

$$\beta = \delta d\alpha + d\delta\alpha = \delta\alpha_1 + d\alpha_2 \in \text{Im}(d) \oplus \text{Im}(\delta).$$

(⊇) Given now $\beta \in \text{Im}(d) \oplus \text{Im}(\delta)$, if we show that $\beta \perp \text{Ker}(\Delta)$, then by Theorem A.2.1, we will have shown that $\beta \in \text{Im}(\Delta)$, as desired. Indeed, take any $\gamma \in \text{Ker}(\Delta)$. Then, by Lemma A.2.1, it follows that

$$(\gamma, \beta) = (\gamma, \delta d\alpha) + (\gamma, d\delta\alpha) = (d\gamma, d\alpha) + (\delta\gamma, \delta\alpha) = 0,$$

which finishes the proof. □

The above corollary then states that, for any L^2-integrable $k-$form α, we can find $\phi \in \Lambda_2^{k-1}(M), \beta \in \Lambda_2^{k+1}(M), \gamma \in \mathcal{H}^k(M)$, such that

$$\alpha = d\phi + \delta\beta + \gamma, \tag{A.2.29}$$

which is formally know as the *Hodge decomposition* of the $k-$form α.

Furthermore, with Theorem A.2.1 at our disposal, it is possible to perform the following identification:

Theorem A.2.3. *The natural map*

$$\mathcal{H}^k(M) \to H^k_{dR}(M, \mathbb{R})$$

that takes α and maps it to its de Rham class $[\alpha]$ is a vector space isomorphism.

Proof: Check Theorem 5.23 of [Voi02]. □

And so, by having knowledge about the kernel of Δ (on $k-$forms), we can also obtain information about the cohomology groups of our space (and vice-versa!).[10]

[10] This not at all a trivial information to have on other types of spaces. For instance, have a look at [Phi90].

An analog for Corollary A.2.2 also holds true for the case of pseudo-Riemannian manifolds, more specifically we have[11]

Theorem A.2.4 (Relativistic Hodge decomposition)**.** *Let* (M, g) *be an* $n-$*dimensional globally hyperbolic manifold and consider* $V \in \Lambda^k_{(2)}(M)$*. Then, there exists unique* $\phi \in \Lambda^{k-1}_{(2)}(M)$ *and* $F \in \Lambda^{k+1}_{(2)}(M)$ *such that:*

$$V = d\phi + \delta F \tag{A.2.30}$$

Proof: Consider $W \in \Lambda^k_{(2)}(M)$ such that

$$\Delta W = V$$

by Lemma 1 of [Gem00] we guarantee the uniqueness and existence of W and by the definition of the Laplacian, let $\phi := \delta W$ and $F := dW$ so that the result follows. □

A good further reading to be done concerning this higher order, metric signature counting generalization of the Hodge decomposition is [SBE06].

A.3 Some specific derivations

Inspired on some of Manfredo's classical Differential Geometry book's exercises [dC14], we will derive the general form of the well known operators Curl, Divergence and Laplacian, on general coordinates, over a manifold M^n with (pseudo-)Riemannian metric g_{ij}. Denoting $\{e_i\}$ for the vectors spanning the tangent space to some point on our manifold, we have that:

1. $g(e_i, e_j) = g_{ij}$
2. $e^j = g^{jk} e_k$
3. $e^i(e_j) = g^i_j = \delta^i_j$

[11] check Section A.5 for more details.

Where on the second item e^j is the *dual vector* to e_j and it acts on it as shown on the third item above, being δ^i_j the Krönecker delta, which evaluates to 1 if $i = j$ and to zero otherwise.

Divergence

Let $v \in T_pM^n$, for some $p \in M^n$, and define ω_v as its associated 1−form given by $\omega_v = g(v, \cdot) = g_{ij}v^je^i$. We thus define the *divergence* of v, namely $div(v)$, as the element of $\Lambda^0(M^n)$ which satisfies:

$$\star div(v) = d \star \omega_v$$

Or equivalently, using the inverse Hodge dual [Eq.(A.2.19)], we can say that:

$$div(v) = \star^{-1} d \star \omega_v = -\delta\omega_v \tag{A.3.1}$$

according to Eq.(A.2.21) and its proceeding observation. To calculate the above expression, remember that by Eq.(A.2.16) with $k = 1$ it follows that:

$$\star\omega_v = \frac{\sqrt{g}}{(n-1)!}v^{i_1}\varepsilon_{i_1\dots i_n}e^{i_2} \wedge \cdots \wedge e^{i_n}$$

And now, from Eq.(A.2.19), Eq.(A.2.14) and the definition of the exterior derivative, it easily follows that:

$$\begin{aligned}
-\delta\omega_v &= \star^{-1} d \left(\frac{\sqrt{g}}{(n-1)!}v^{i_1}\varepsilon_{i_1\dots i_n}e^{i_2} \wedge \cdots \wedge e^{i_n} \right) \\
&= \star^{-1} \left(\frac{1}{(n-1)!}\partial_{[\mu}(\sqrt{g}v^{i_1})\varepsilon_{i_1\dots i_n]}e^{\mu} \wedge e^{i_2} \wedge \cdots \wedge e^{i_n} \right) \\
&= \star^{-1} \left(\frac{1}{(n-1)!n!}\delta^{j_1 j_2 \dots j_n}_{\mu i_2 \dots i_n}\partial_{j_1}(\sqrt{g}v^{i_1})\varepsilon_{i_1 j_2 \dots j_n}e^{i_2} \wedge \cdots \wedge e^{i_n} \right) \\
&= \frac{1}{\sqrt{g}}\partial_{j_1}(\sqrt{g}v^{i_1})\frac{\varepsilon_{i_1 j_2 \dots j_n}\varepsilon^{\mu i_2 \dots i_n}\delta^{j_1 j_2 \dots j_n}_{\mu i_2 \dots i_n}}{(n-1)!n!} = \frac{1}{\sqrt{g}}\partial_{j_1}(\sqrt{g}v^{i_1})\frac{\varepsilon_{i_1 j_2 \dots j_n}\varepsilon^{j_1 j_2 \dots j_n}}{(n-1)!} \\
&= \frac{1}{\sqrt{g}}\partial_{j_1}(\sqrt{g}v^{i_1})\delta^{j_1}_{i_1} = \frac{1}{\sqrt{g}}\partial_{j_1}(\sqrt{g}v^{j_1})
\end{aligned}$$

Hence, we recover the Voss-Weyl formula for the divergence, which is indeed given by:

$$div(v) = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}v^{\mu})}{\partial x^{\mu}} \tag{A.3.2}$$

Moreover, Eq.(A.3.2) can be covariantly written as:

$$div(v) = \nabla_\mu v^\mu, \quad v = v^\mu e_\mu \tag{A.3.3}$$

with ∇_μ being the covariant derivative in the e_μ direction as defined on Appendix A.1.

Example A.3.1. A trivial but nonetheless instructive example is given by the following. Let $\mathbf{F} = F^i e_i \in \mathbb{R}^3$, then $\omega_F = \delta_{ij} F^j e^i$, but since $\delta_{ij} = diag(1, 1, 1)$, it follows that $F^i = F_i \equiv \omega_i$. In this way we have:

$$\star\omega_F = F^{j_1}\epsilon_{j_1 j_2 j_3} e^{j_2} \wedge e^{j_3} = F^1 \epsilon_{1 j_2 j_3} e^{j_2} \wedge e^{j_3} + F^2 \epsilon_{2 j_2 j_3} e^{j_2} \wedge e^{j_3} + F^3 \epsilon_{3 j_2 j_3} e^{j_2} \wedge e^{j_3}$$

since $j_2 < j_3$ our only options for their values are $(j_2, j_3) = (2, 3), (1, 3), (1, 2)$. Hence:

$$\star\omega_F = F^1 \epsilon_{123} e^2 \wedge e^3 + F^2 \epsilon_{213} e^1 \wedge e^3 + F^3 \epsilon_{312} e^1 \wedge e^2 = F^1 e^2 \wedge e^3 - F^2 e^1 \wedge e^3 + F^3 e^1 \wedge e^2$$

$$\therefore d \star \omega_F = \frac{\partial F^1}{\partial x^1} e^1 \wedge e^2 \wedge e^3 + \frac{\partial F^1}{\partial x^2} e^1 \wedge e^2 \wedge e^3 + \frac{\partial F^1}{\partial x^3} e^1 \wedge e^2 \wedge e^3$$

From where it follows that $div\mathbf{F} = \frac{\partial F^i}{\partial x^i}$ (with a sum over i), as expected.

Example A.3.2. Consider now $\mathbb{R}^2$ expressed in polar coordinates with basis vectors given by $\{\partial_r, \partial_\theta\}$. Let $\mathbf{F} = F^r \partial_r + F^\theta \partial_\theta$. Then $\omega_F = F^r dr + r^2 F^\theta d\theta$, where the metric is given by $g_{ij} = diag(1, r^2)$. In this way we have that:

$$\star\omega_F = rF^{j_1}\epsilon_{j_1 j_2} e^{j_2} = rF^1 \epsilon_{1 j_2} e^{j_2} + rF^2 \epsilon_{2 j_2} e^{j_2}$$

$$\Rightarrow \star\omega_F = rF^1 \epsilon_{12} e^2 + rF^2 \epsilon_{21} e^1 = rF^r d\theta - rF^\theta dr$$

From there it follows that:

$$d \star \omega_F = \frac{\partial(rF^r)}{\partial r} dr \wedge d\theta - \frac{\partial(rF^\theta)}{\partial \theta} d\theta \wedge dr = \left(\frac{\partial(rF^r)}{\partial r} + \frac{\partial(rF^\theta)}{\partial \theta} \right) dr \wedge d\theta$$

$$\Rightarrow div\mathbf{F} = \frac{1}{r}\frac{\partial(rF^r)}{\partial r} + \frac{\partial(F^\theta)}{\partial \theta}$$

as it's known to be the case.

Before moving to the next operation, note that by Eq.(A.3.1) plus the use of the musical isomorphism known as *sharp operator*, we can extend the concept of "divergence" to any $k-$form α over M^n, be defining

$$div(\alpha^\sharp) = (-1)^k \delta\alpha, \tag{A.3.4}$$

with $\alpha^\sharp \in \Gamma(\Lambda^k(TM^n))$ being given by $\alpha^\sharp = g^{i_1 j_1} \dots g^{i_k j_k} \alpha_{i_1 \dots i_k} e_{j_1} \wedge \dots \wedge e_{j_k}$.

Curl

On the same note as in the previous subsection, let $v \in \Gamma(TM^n)$, so that we can define a particular $(n-2)$-form by the following procedure

$$v \mapsto \omega_v \mapsto d\omega_v \mapsto \star^{-1} d\omega_v \tag{A.3.5}$$

Again make use of the the musical isomorphism, we define

$$curl(v) = (\star^{-1} d\omega_v)^\sharp \tag{A.3.6}$$

Thus observe that[12] $(curl(v))^\flat = \star^{-1} d\omega_v$. Well, in coordinates Eq.(A.3.6) looks like this:

$$\star^{-1} d\omega_v = \frac{1}{(n-2)!\sqrt{g}} \partial_{\mu_1}(v_{\mu_2}) \varepsilon^{\mu_1 \mu_2}{}_{\mu_3 \dots \mu_n} e^{\mu_3} \wedge \dots \wedge e^{\mu_n}$$

If we consider the multivector version of the above, we arrive at:

$$curl(v) = \sum_{\mu_3 < \dots < \mu_n} \frac{1}{\sqrt{g}} \partial_{\mu_1}(v_{\mu_2}) \varepsilon^{\mu_1 \mu_2 \mu_2 \dots \mu_n} e_{\mu_3} \wedge \dots \wedge e_{\mu_n} \tag{A.3.7}$$

Example A.3.3. Applying formula from Eq.(A.3.7) to the particular case of a vector field $v = v^i e_i, i = 1, 2$ in $\mathbb{R}^2$ with metric $g_{ij} = \delta_{ij}$ we have that:

$$curl(v) = \varepsilon^{\gamma\beta} \partial_\gamma(\delta_{\mu\beta} v^\mu) = \partial_1 v^2 - \partial_2 v^1$$

as expected.

[12] above the curl is what we call the *flat operator*. Contrary to the sharp operator, it take *vector* and map them to *forms*. As the reader might have noted $(X^\flat)^\sharp = X$ and $(\alpha^\sharp)^\flat = \alpha$

Example A.3.4. Let's verify the well know relation asserting that $div(curl(v)) = 0$ or, more specifically, that $div((curl(v))^\flat) = 0, \forall v \in \Gamma(TM^n)$ smooth. Indeed we have that:

$$div((curl(v))^\flat) = \star^{-1} d \star (\star^{-1} d\omega_v) = \star^{-1} d \star^{-1} \star d\omega_v = \star^{-1} d^2 \omega_v = 0$$

Where the last equality holds from the fact that $d^2 = 0$, as observed on Appendix A.2.

Laplacian

To be able to define the Laplace operator, we need first to introduce the gradient of a function $f : M^n \to \mathbb{R}$. Indeed, the way in ehich one does it is by first considering the vector $u \in T_pM^n$ and defining the *directional derivative* of f in the u direction by

$$D_{\mathbf{u}} f = \nabla f \cdot u \tag{A.3.8}$$

What we do here is to then generalize this notions of directional derivation to manifolds admitting a metric. Now, once we note that:

$$f : M^n \to \mathbb{R}$$

$$(x^1, \ldots, x^n) \mapsto f(x^1, \ldots, x^n)$$

We naturally have, by Appendix A.1, that $df : TM^n \to \mathbb{R}$ is given by:

$$df = \frac{\partial f}{\partial x^1} e^1 + \ldots + \frac{\partial f}{\partial x^n} e^n$$

so that $df(u)$ with $u = u^j e_j$ is equal to:

$$df(u) = \frac{\partial f}{\partial x^i} e^i (u^j e_j) = \frac{\partial f}{\partial x^i} u^j e^i (e_j) = \frac{\partial f}{\partial x^i} u^j g^i_j = \frac{\partial f}{\partial x^i} u^i = uf$$

In this way, we again conclude that vector field over TM^n can be seen as derivation of $\mathcal{C}^\infty(M^n)$. Now, based on Eq.(A.3.8), we define the *gradient* of f as the $1-$form ∇f that satisfies:

$$df(u) = g(\nabla f, u) \tag{A.3.9}$$

In local coordinates, if $u = e_j$, we have that:

$$g(\nabla f, e_j) = g_{ij}(\nabla f)^i = \frac{\partial f}{\partial x^j} \therefore \delta^k_j (\nabla f)^j = g^{kj}\frac{\partial f}{\partial x^j}$$

That is to say:

$$(\nabla f)^k = g^{kj}\frac{\partial f}{\partial x^j}$$

And thus $\nabla f = g^{kj}\frac{\partial f}{\partial x^j}e_k$ which in turn can be thought of as a $1-$form by the musical isomorphism given by the metric, as discussed on Appendix A.1, $\nabla f = e^j \partial_j f$. This in turn lets us express the gradient of a function in terms of its exterior derivative, that it

$$\nabla f = df$$

as a $1-$form. From this one defines:

$$\Delta f = div(\nabla f) = \delta df \tag{A.3.10}$$

By the computation of Appendix A.3, we get to the following expression for the Laplacian of a function:

$$\Delta f = \frac{1}{\sqrt{g}}\partial_j(\sqrt{g} g^{ji}\partial_i f) \tag{A.3.11}$$

For a general $k-$form though, the definition is altered a bit, leading us to the **Laplace - de Rham operator** $\Delta : \Lambda^k(M^n) \to \Lambda^k(M^n)$, given by:

$$\Delta\alpha = d\delta\alpha + \delta d\alpha \tag{A.3.12}$$

One can show that in coordinates Eq.(A.3.12) assumes the following form

$$\Delta\alpha = \left(\frac{1}{\sqrt{g}}\partial_\beta(\sqrt{g}g^{\xi a_0}g^{i_1 a_1}\dots g^{i_k a_k}\partial_{a_0}\alpha_{a_1\dots a_k})\frac{\delta^{\gamma_1\dots\gamma_k\beta}_{i_1\dots i_k\xi}}{(k+1)!}g_{\gamma_1\rho_1}\dots g_{\gamma_k\rho_k} + \partial_{\rho_1}\left(\frac{1}{\sqrt{g}}\partial_{\gamma_k}(\sqrt{g}\alpha^{i_1\dots i_k})\right)\frac{\delta^{\gamma_1\dots\gamma_k}_{i_1\dots i_k}}{k!}g_{\gamma_1\rho_2}\dots g_{\gamma_{k-1}\rho_k}\right)e^{\rho_1}\wedge\dots\wedge e^{\rho_k} \tag{A.3.13}$$

which can in turn be covariantly written in the following form:

$$\Delta\alpha = \left(\nabla_\beta\nabla^\beta\alpha_{\rho_1\dots\rho_k} + (-1)^k\nabla_{\rho_1}\nabla^\beta\alpha_{\beta\rho_2\dots\rho_k}\right)e^{\rho_1}\wedge\dots\wedge e^{\rho_k} \tag{A.3.14}$$

Example A.3.5. Now, for the sake of consistency, we end up this section with a trivial example. Indeed, let $g_{ij} = \delta_{ij}$ be the $\mathbb{R}^n$ metric, so that $\sqrt{g} = 1$ and such that its entries are non-vanishing if, and only if, $i = j$. Plugging this right back at Eq.(A.3.11), we have that:

$$\Delta_{\mathbb{R}^n} f = \partial_k(\delta^{ki}\partial_i f) = \partial_i^2 f$$

as expected.

A.4 Symplectic Geometry

In this appendix we are going to define some objects needed to mathematically formalize (and even extend) Classical Mechanics, if you will. Assuming that the reader already had a look at Appendix A.1 and thus knows what manifolds and forms are, we can jump right into the following

Definition A.4.1 (Symplectic Manifold)**.** A **symplectic manifold** is a pair $(\mathcal{M}, \omega)$, such that $dim(\mathcal{M}) = 2n$, for some $n \in \mathbb{N}$ and ω is a non-degenerate $2-$form defined over $\mathcal{M}$.

We can consider yet another very interesting and rich structure over any symplectic manifold, namely a **Poisson Structure**. Formally we have

Definition A.4.2 (Poisson Structure)**.** A map $\{\cdot, \cdot\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ is called a **Poisson Structure** if it satisfies the following

PS.1 $\{f, g\} = -\{g, f\}$ (Anti-commutative)

PS.2 $\{f + ag, h\} = \{f, h\} + a\{g, h\}, \forall a \in \mathbb{R}$ ($\mathbb{R}-$linear)

PS.3 $\{fg, h\} = f\{g, h\} + \{f, h\}g$ (Leibniz's Rule)

PS.4 $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ (Jacobi identity)

The way you build this object on the symplectic case is very simple. Consider $f, g \in C^\infty(\mathcal{M})$. Find their Hamiltonian vector fields X_f and X_g, by solving

$$\iota_{X_f}\omega = df, \quad \iota_{X_g}\omega = dg. \tag{A.4.1}$$

From there follows

Definition A.4.3 (Poisson Bracket)**.** Based on the vector fields coming from Eq.(A.4.1), we define the **Poisson bracket** of two function f and g in $C^\infty(\mathcal{M})$ by

$$\{f, g\} = \mathcal{L}_{X_f}(g) = \omega(X_g, X_f) \tag{A.4.2}$$

with $\mathcal{L}_{X_f}$ being the Lie derivative in the X_f direction.

Hence, if we are given a system of local coordinates (q, p) such that $\omega = \sum_i dq_i \wedge dp_i$, Eq.(A.4.2) is actually written as:

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{A.4.3}$$

A very important property that follows from Definition A.4.3 is

Definition A.4.4 (Involution)**.** Based on the above structure, we say two functions f and g are in **involution** when $\{f, g\} = 0$.

Note in particular that, by **PS.3**, we can get a vector field out of a Poisson structure by fixing its first argument. That is, given a function $f \in C^\infty(\mathcal{M})$, consider the derivation $X_f = \{f, \cdot\} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$. As mentioned on Appendix A.1, such derivations can be understood as elements of $\mathcal{X}(\mathcal{M}) = \Gamma(T\mathcal{M})$, hence turning X_f into a vector field over $\mathcal{M}$.

The condition expressed on Definition A.4.4 says that the quantity g is constant along the flow of f and vice-versa due to **PS.1** and Eq.(A.4.2).

One important geometric quantity we deal with in mechanics and symplectic geometry is the *tautological* 1*−form*. Given some smooth manifold M, its cotangent bundle T^*M

with local coordinates (x, ξ) and canonical projection $\pi : T^*M \to M$ is a symplectic manifold. Over $\Lambda^1(T^*M)$, we can consider a form θ being given by

$$\theta_{(x,\xi)}(v) = \langle \xi, d\pi(v) \rangle \,, \tag{A.4.4}$$

with $(x, \xi) \in T^*M$ and $v \in T_{(x,\xi)}T^*M$. With this in mind we state the very important

Proposition A.4.1 (Canonical Symplectomorphism)**.** *Given two manifolds M_1 and M_2 whose cotangent bundles $(\pi_1, T^*M_1, M_1), (\pi_2, T^*M_2, M_2)$ have tautological $1-$forms θ_1 and θ_2 respectively, for every $\phi \in$ Diff(M_1, M_2), we can construct a bundle map $\hat{\phi} : T^*M_1 \to T^*M_2$ defined by*

$$\hat{\phi}(x, p) = (\phi(x), ([D\phi]^{-1})^T p) \tag{A.4.5}$$

that satisfies $\hat{\phi}^*\theta_2 = \theta_1$.

Proof: We prove this by performing direct computation. Take $(x, \xi) \in T^*M_1$ and some $v \in T_{(x,\xi)}T^*M_1$. Using Eq.(A.4.4), it follows that

$$\begin{aligned}
(\hat{\phi}^*\theta_2)_{(x,\xi)}(v) &= \theta_{2,\hat{\phi}(x,\xi)}(D\hat{\phi}(v)) = \left\langle \hat{\phi}(x,\xi), (D\pi_2 \circ D\hat{\phi})(v) \right\rangle \\
&= \left\langle ([D\phi]^{-1})^T(\xi), D(\pi_2 \circ \hat{\phi})(v) \right\rangle = \left\langle ([D\phi]^{-1})^T(\xi), D(\phi \circ \pi_1)(v) \right\rangle \\
&= \left\langle \xi, [D\phi]^{-1} \circ D\phi \circ D\pi_1)(v) \right\rangle = \langle \xi, D\pi_1(v) \rangle = \theta_{1,(x,\xi)}(v),
\end{aligned}$$

as we wished to prove. $\square$

The map $\hat{\phi}$ is called the **cotangent lift** of the map ϕ. The naming becomes clearer once we look at the following diagram

$$\begin{array}{ccc}
T^*M_1 & \xrightarrow{\hat{\phi}} & T^*M_2 \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\
M_1 & \xrightarrow{\phi} & M_2
\end{array}$$

where π_i are the projections of the bundles down to their respective base spaces. Such type of map is particularly important when one considers the case $M_1 = M_2 := M$. In it, certain maps ϕ can be used in the process of *reduction* (see subsection 2.3.1), thus making it easier to solve the equations of motion governing the dynamical evolution of our system over T^*M.

Group Actions

Say we have a Lie group G (a group with a manifold structure) with Lie Algebra (the tangent space to the identity) Lie(G) $:= \mathfrak{g}$. Given an element $u \in \mathfrak{g}$, the ***infinitesimal generator*** of the action $\psi : G \times M \to M$ is given by

$$\mathfrak{u} = \left.\frac{d}{dt}\right|_{t=0} \psi(exp(tu), x). \tag{A.4.6}$$

Much like with the Hamiltonian vector field we mentioned above, we can consider

Definition A.4.5 (***Hamiltonian action***)**.** Given a group G, with Lie algebra $\mathfrak{g}$, acting on a symplectic manifold (M, ω) by $\psi : G \times M \to M$, we call the group action *Hamiltonian*, if

HGA.1 There exists a map $\mu : M \to \mathfrak{g}^*$ called the ***momentum map***.

HGA.2 The infinitesimal generator is Hamiltonian, with

$$\iota_{\mathfrak{u}}\omega = d\left\langle \mu, u \right\rangle, \tag{A.4.7}$$

where $\langle \mu, u \rangle (x) = \langle \mu(x), u \rangle$ and $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the canonical evaluation pairing between forms and vectors.

HGA.3 The function μ is $G-$equivariant, that is

$$\mu \circ \psi_g = \mathrm{Ad}^T_{g^{-1}} \circ \mu \tag{A.4.8}$$

Moreover, it is usual to denote $\mu^u = \langle \mu, u \rangle$, so that Eq.(A.4.7) can be written as

$$\iota_{\mathrm{u}}\omega = d\mu^u. \tag{A.4.9}$$

The function μ^u is called the ***co-momentum map***, and its clear from its definition, together with **HGA.2**, that $\mu^u \in C^\infty(M)$

Based on the cotangent bundle case we can define an **exact manifold** to be a pair (M, ω) such that there is a $1-$form $\theta \in \Omega^1(M)$ satisfying $\omega = -d\theta$. We shall be calling θ the **symplectic potential**. With this in mind we have the following

Proposition A.4.2. *Given an exact manifold (M, ω) with symplectic potential θ and a group action $G \circlearrowright M$ such that $\psi_g^*\theta = \theta$, then the action is Hamiltonian. Moreover, it follows that the moment map $\mu^u = \iota_{\mathrm{u}}\theta$.*

Proof: We can show the result in two straightforward steps. First we prove that the moment map μ is of the above mentioned form. Then we show that it's $G-$equivariant. For the first part let's consider Cartan's magic formula

$$\mathcal{L}_{\mathrm{u}}\theta = d\iota_{\mathrm{u}}\theta + \iota_{\mathrm{u}}d\theta = d\iota_{\mathrm{u}}\theta - \iota_{\mathrm{u}}\omega \tag{A.4.10}$$

By the definition of Lie derivative, since $\psi_g^*\theta = \theta$, it follows that $\mathcal{L}_{\mathrm{u}}\theta = 0$ so that Eq.(A.4.10) yields

$$0 = d\iota_{\mathrm{u}}\theta - \iota_{\mathrm{u}}\omega \Rightarrow d\iota_{\mathrm{u}}\theta = \iota_{\mathrm{u}}\omega = d\mu^u$$

as we said above. Now we check the $G-$equivariancy.

$$\begin{aligned}\langle \mu \circ \psi_g, u \rangle = \psi_g^*\mu^u = \psi^*\iota_{\mathrm{u}}\theta = \iota_{(\psi_{g^{-1}})_*\mathrm{u}}\psi_g^*\theta = \iota_{Ad_{g^{-1}}(u)}\theta \\ = \langle \mu, Ad_{g^{-1}}(u) \rangle = \left\langle Ad_{g^{-1}}^T \circ \mu, u \right\rangle,\end{aligned}$$

which finishes the proof. □

We finish off this part with some examples that will be useful throughout the main text

Example A.4.1 ($SO(3)$ action)**.** Consider the group action $SO(3) \circlearrowright \mathbb{R}^3$, which is just matrix multiplication, and its natural symplectic extension to the manifold $\mathbb{R}^3 \times \mathbb{R}^3$. In this example, our aim is to show that the momentum map of such a group action is precisely the angular momentum

$$L = \boldsymbol{x} \times \boldsymbol{p}, \tag{A.4.11}$$

for $(\boldsymbol{x}, \boldsymbol{p}) \in \mathbb{R}^3 \times \mathbb{R}^3$. To prove this, first notice that, much like with the group $SO(3)$ acting on $\mathbb{R}^3$ by matrix multiplication, we can also consider the Lie algebra $\mathfrak{so}(3)$ acting like-wise on the same space. There is however another thing that happens with the Lie algebra action that we can use to our advantage. Formally we have that

$$\mathfrak{so}(3) = \{A \in \mathfrak{gl}(3, \mathbb{R}) | \, A + A^T = 0\}, \tag{A.4.12}$$

so that a generic matrix $A \in \mathfrak{so}(3)$ can actually be written as

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}, \text{ for } a, b, c \in \mathbb{R}. \tag{A.4.13}$$

If we thus take some $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and consider the new vector $A\boldsymbol{x}$, we notice that

$$A\boldsymbol{x} = \begin{pmatrix} x_3 b - x_2 c \\ x_1 c - x_3 a \\ x_2 a - x_1 b \end{pmatrix} = \xi_A \times \boldsymbol{x}, \text{ with } \xi_A = (a, b, c), \tag{A.4.14}$$

And so, we can bring back an element of $\mathfrak{so}(3)$ that acts on a vector on $\mathbb{R}^3$ to a cross product of two $\mathbb{R}^3$ vectors in a consistent manner.

Now, getting back to our goal and following the prior notation, take an element $u \in \mathfrak{so}(3)$ and consider its infinitesimal generator

$$\mathfrak{u} = \left.\frac{d}{dt}\right|_{t=0} \psi(e^{tu}, (\boldsymbol{x}, \boldsymbol{p})) = \left.\frac{d}{dt}\right|_{t=0} (e^{tu}\boldsymbol{x}, e^{tu}\boldsymbol{p}) = (u\boldsymbol{x}, u\boldsymbol{p}). \tag{A.4.15}$$

Now, using a slight abuse of notation to denote the simplectic form of $\mathbb{R}^3 \times \mathbb{R}^3$ as

$$\omega = d\boldsymbol{x} \wedge d\boldsymbol{p}, \tag{A.4.16}$$

to compute the momentum map we proceed as laid out above, that is

$$\iota_u\omega = (u\boldsymbol{x})\cdot d\boldsymbol{p} - (u\boldsymbol{p})\cdot d\boldsymbol{x} = (\xi_u\times\boldsymbol{x})\cdot d\boldsymbol{p} - (\xi_u\times\boldsymbol{p})\cdot d\boldsymbol{x} \text{ (by Eq.(A.4.14))}$$
$$= (\boldsymbol{x}\times d\boldsymbol{p})\cdot\xi_u + (d\boldsymbol{x}\times\boldsymbol{p})\cdot\xi_u = d(\xi_u\cdot(\boldsymbol{x}\times\boldsymbol{p})).$$

Since ξ_u can be thought of as an element of the Lie algebra $\mathfrak{so}(3)$, the result follows from identifying $\mu = \boldsymbol{x}\times\boldsymbol{p}$ at Eq.(A.4.7), as we wished to prove.

Example A.4.2 (Momentum conservation - Vortex case)**.** Close to the end of Section 9.2 we mentioned that the Hamiltonian for the interaction of 3 vortices will, *in* (q_i, p_i) *coordinates*, be given by

$$H = -\frac{1}{2\pi}\sum_{i<j}\Gamma_i\Gamma_j G\left(q_i - q_j, \frac{p_i}{\Gamma_i} - \frac{p_j}{\Gamma_j}\right). \tag{A.4.17}$$

The translation invariance it possess can be expressed in the form of a group action, with the acting group being our manifold itself. That is, we can define the action $\mathbb{S}\times\mathbb{R}\circlearrowright\mathbb{S}\times\mathbb{R}$ by setting

$$(q_0, p_0)\cdot(q,p) = (q_0 + q, p_0 + p). \tag{A.4.18}$$

Note however that the coordinates which the Hamiltonian is dependent are *not* the same as the ones in which our symplectic form is written. Indeed, meanwhile the form is written with respect to (q,p), the Hamiltonian depends on $(q, p/\Gamma)$. This difference, although simple, is actually quite significant in the process of finding the integrals of motion, for it was only defined when both the Hamiltonian and the symplectic form depended on the same variables. We can easily fix this by defining the scaled momentum

$$\tilde{p}_i = \frac{p_i}{\Gamma_i}. \tag{A.4.19}$$

With respect to it, our symplectic form gets to be written as

$$\omega = \sum_i \Gamma_i dq_i \wedge d\tilde{p}_i, \tag{A.4.20}$$

and the Hamiltonian is rewritten to be

$$H = -\frac{1}{2\pi}\sum_{i<j}\Gamma_i\Gamma_j G(q_i - q_j, \tilde{p}_i - \tilde{p}_j). \tag{A.4.21}$$

The infinitesimal generator for the action of Eq.(A.4.18) (now in the $(q_i, \tilde{p}_i)$ coordinates) is easily found to be

$$\mathfrak{u} = (q_0, \tilde{p}_0). \tag{A.4.22}$$

To find its associated momentum map, we repeat the usual procedure

$$\begin{aligned}
\iota_{\mathfrak{u}}\omega &= \sum_i \Gamma_i dq_i((q_0, \tilde{p}_0))d\tilde{p}_i - \sum_i \Gamma_i d\tilde{p}_i((q_0, \tilde{p}_0))dq_i \\
&= \left(\sum_i \Gamma_i d\tilde{p}_i\right) q_0 - \left(\sum_i \Gamma_i dq_i\right) \tilde{p}_0 \\
&= d\left\langle \left(\sum_i \Gamma_i \tilde{p}_i, -\sum_i \Gamma_i q_i\right), (q_0, \tilde{p}_0)\right\rangle,
\end{aligned}$$

from which it follows that our momentum map is given by

$$\mu = \left(\sum_i \Gamma_i \tilde{p}_i, -\sum_i \Gamma_i q_i\right). \tag{A.4.23}$$

We if furthermore make use of the fact that

$$q_i = x_i, \quad \tilde{p}_i = \frac{p_i}{\Gamma_i} = \frac{1}{\Gamma_i}(\Gamma_i y_i) = y_i, \tag{A.4.24}$$

we recover the quantities of Eq.(9.21).

Symplectic Reduction

Theorem A.4.3 (Nöether's Theorem)**.** *Let $\mathcal{M}$ be a symplectic manifold, G a connected group whose action over $\mathcal{M}$ is Hamiltonian with moment map μ, and $H \in C^\infty(\mathcal{M})$. It then follows that*

H is $G-$invariant	$\Longleftrightarrow$	*μ^u is preserved by the flow generated by H*

Proof: (⇒) What we need to show here is that given $\mathfrak{u} \in \mathfrak{g} = Lie(G)$ its associated moment map $\mu^{\mathfrak{u}}$ is preserved by the symplectic vector field X_H generated by the Hamiltonian H. In order to do so we once again make use of the Lie derivative in the following fashion

$$\mathcal{L}_{X_H}\mu^{\mathfrak{u}} = d\mu^{\mathfrak{u}}(X_H) = \iota_{\mathfrak{u}}\omega(X_H) = \omega(\mathfrak{u}, X_H) = -\omega(X_H, \mathfrak{u}) = -dH(\mathfrak{u})$$
$$= -\mathcal{L}_{\mathfrak{u}}H = 0$$

thanks to the $G-$invariance of H, i.e, $\psi_g^* H = H$.

(⇐) To prove this other direction we use the following nice Lie group results:

Lemma A.4.1. Let G be a connected Lie Group and U a neighbourhood of e. Then

$$G = \bigcup_{n=1}^{\infty} U^n \tag{A.4.25}$$

where U^n are all the $n-$fold products of elements of U.

Lemma A.4.2. The map $exp : \mathfrak{g} \to G$ is C^∞ and $dexp : \mathfrak{g}_o \to G_e$ is the identity. That is, exp gives a diffeomorphism of a neighbourhood of $0 \in \mathfrak{g}$ to one of $e \in G$.

Proof: Check [War83] Proposition 3.18 and Theorem 3.21 (d). ■

Now, to prove the theorem note that since μ is preserved by the flow of H, by definition it follows that

$$0 = \mathcal{L}_{X_H}\mu^{\mathfrak{u}} = \mathcal{L}_{\mathfrak{u}}H$$

as can be checked on the above computations in the other direction of the proof. Now notice that $\forall t \in \mathbb{R}$ we have:

$$\frac{d}{dt}H(exp(t\mathfrak{u})x) = (\mathcal{L}_{\mathfrak{u}}H)(exp(t\mathfrak{u})x) = 0$$
$$\Rightarrow H(exp(t\mathfrak{u})x) = H(x) \tag{A.4.26}$$

Now, to prove this for any $g \in G$ we use Lemmas A.4.1 and A.4.2 to express $g = h_1 \dots h_n$ with each $h_i = exp(u_i), u_i \in \mathfrak{g}$. By iterating the result of Eq.(A.4.26) we find that

$$H(gx) = H(x), \; \forall g \in G \tag{A.4.27}$$

which let's us conclude that $\psi_g^* H = H$, i.e H is $G-$invariant. □

What Nöether's theorem tells us is that given a continuous symmetry of the system, *that leaves the Hamiltonian invariant*, we can find a quantity that remains unchanged throughout the orbit of the system. In physics jargon we call such a **conserved charge** and for higher tier physical theories the presence of these charges is of uttermost importance. So much so that one can even build a theory[13] based purely on symmetry arguments, as well as by thinking about which quantities are physically reasonable to be conserved charges.

A.5 Pseudo-Riemannian Geometry

In this section of the appendix we define the mathematical objects we shall explicitly make use throughout the thesis. The main goal is to define Lorentzian and globally hyperbolic manifolds, the former being the definition of a space-time manifold. In order to do so, we will need the concept of causality and that of a pseudo-Riemannian metric. To talk about the latter we first introduce the following concept

Definition A.5.1 (Signature)**.** Given a metric g on a manifold M, we will denote the signature of g as a pair of numbers (p, q) where p (resp. q) stands for the number of positive (resp. negative) signs of the metric.

Notably, we determine p and q by looking at an orthogonal basis for the tangent space of our manifold (which can always be found at least in a vicinity of a point by the famous

[13]actually its Lagrangian!

Gram-Schmidt process). In the Riemannian case for instance, positive definiteness means that the signature of the metric is $(n, 0)$ with n being the dimension of the manifold at play. More generally, a **pseudo-Riemannian manifold** is a differentiable manifold with a non-positive definite metric, i.e, one whose negative part of the signature is non-null. One very important example of such a manifold is the following

Definition A.5.2 (Lorentzian Manifold)**.** A differentiable manifold M of dimension n is called **Lorentzian** when over it we have a metric tensor g whose signature is $(n-1, 1)$.

Example A.5.1. In the case (M, g) is Lorentzian and $g = \text{diag}(-1, 1, \ldots, 1)$ we call M a *Minkowskian manifold*. Note in particular that a Lorentzian manifold is locally always Minkowskian.

On a pseudo-Riemannian manifold, we can classify vectors v at each tangent space in three different ways[14]:

1. *Time-like*

2. *Light-like*

3. *Space-like*

From this we say that a curve $\gamma \in M$ is *time-like* when $\gamma'(t)$ is time-like for all t over which γ is defined. Similar definitions hold for null and space-like curves. Note though that what might differ between authors is the characterization of time and space like vector/curves since it depends on whether $p > q$ or not (For instance, check the caption of the above figure). Anyhow, the geometrical scenarios are clear, if the vector sits inside the light-cone we call it time-like. If it belongs to the the cone itself, we say it's null, and if its on the outer part, space-like.

[14]the terminology *null* instead of light-like is more usual in the literature so, we shall use both interchangeably throughout the text.

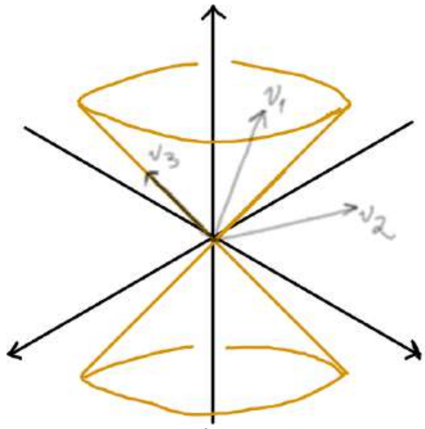


Figure A.1: Example of the tangent space of a Lorentzian manifold, as given on Definition A.5.2. Here v_1 is time-like, v_2 space-like and v_3 null. This is because for a signature $(n-1,1)$ metric, we have that $g(v,v)<0$ implies that the vector v sits inside the light-cone, in turn defined by the set of vectors such that $g(v,v)=0$. Lastly, $g(v,v)>0$ in this case represents space-like vectors, since those sit outside the region enclosured by the light-cone.

Notice that on Fig.A.1, it's clear that we have two possible directions on our tangent space, provided by the presence of the light-cone itself. Indeed, when a vector v sits inside the "upper sheaf" (resp. "lower sheaf") we call it **future directed** (resp. **past directed**), though the notion of upper and lower is determined by the presence of a unit time-like vector e_0. In the Lorentzian scenario for instance, we've got *future directed* vectors v being represented by $g(e_0,v)<0$, while *past directed* ones by $g(e_0,v)>0$. We this in mind, we have

Definition A.5.3 (Time Orientability)**.** A pseudo-Riemannian Manifold will be **time orientable** if $\exists X \in \Gamma(TM)$ that is everywhere time-like. Moreover we have that

TO.1 The choice of such a vector field is called a **time orientation**.

TO.2 With respect to a time-orientation, a time-like vector $v_p \in T_pM$ is called **future directed** if v_p is in the same connected component as $X(p)$. It is called **past directed** if $-v_p$ is future directed.

Definition A.5.4 (Causal curve)**.** A curve $\gamma \in M$ is **causal** if it's either null or time-like.

That is to say, causal curves physically speaking are the ones that connect points that can have some type of influence over one another. Indeed, we shall refer to *points* in our pseudo-Riemannian manifolds as *events* since, at least on the Lorentzian case, that's what they physically represent. Hence causally connected points are to be understood as events that influence one another, depending on which one is sitting where in the inner light-cone that contains the two. More explicitly, we have

Definition A.5.5 (Causality)**.** We say $x, y \in M$ are **causally connected** if there is a causal curve that connects the two.

By putting together Definition A.5.3 and Definition A.5.4, we are able to think of **future directed** (or future oriented) curves on M. Naturally, we define those as the curves whose tangent vectors are always future directed. A similar definition holds for past oriented curves and, we see in particular that causal time-like curves can be either future or past directed, depending on parameteization. With this concept in mind we have

Definition A.5.6. Let (M, g) be a time-oriented pseudo-Riemannian manifold. Then we say

1. $x \ll y$ if there is a future oriented, time-like curve connecting x to y.

2. $x \leq y$ if $x = y$ or, there is a causal smooth curve connecting the two.

3. $x < y$ if $x \leq y$ and $x \neq y$.

Based on this we can define two important sets, namely

$$J^+_M(x) = \{y |\, x \leq y\} \tag{A.5.1a}$$
$$J^-_M(x) = \{y |\, y \leq x\} \tag{A.5.1b}$$

Eq.(A.5.1a) is called the **causal future** of x, while Eq.(A.5.1b) the **causal past** of such point. We can moreover define a **diamond** as the set $J_M(x, y) = J^+_M(x) \cap J^-_M(y)$. With this in mind we move to the following

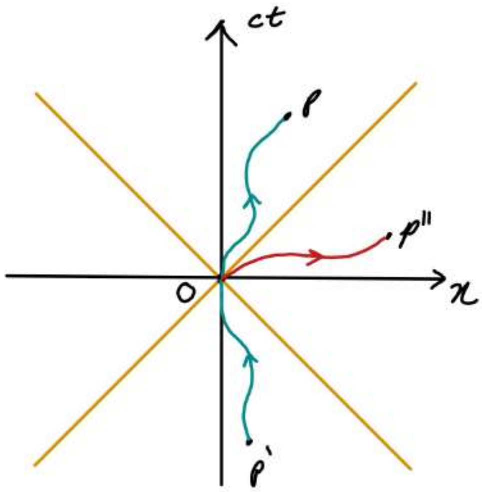


Figure A.2: 2−dimensional Minkowski space-time (Example A.5.1). We see that the curves connecting p' to O and O to p are time-like causal and future oriented. The one joining O to p'' is space-like instead.

Definition A.5.7 (Future and Past Compactness)**.** Let (M, g) be a time-oriented pseudo-Riemannian manifold. Then a subset $A \subseteq M$ is called **future compact** (resp. **past compact**) if $J^+_M(p) \cap A$ (resp. $J^-_M(p) \cap A$) is compact for all $p \in M$.

Finally, note that a time oriented, pseudo-Riemannian manifold is **causal**, if there are no closed causal curves on M. In particular that is to say that one can **not** get to its past by just traveling to the future! With this in mind we have

Definition A.5.8 (Global Hyperbolicity)**.** Given a time oriented, pseudo-Riemannian manifold (M, g) we say that M is **globally hyperbolic** if

GH.1 (M, g) is causal

GH.2 $\forall\, x, y \in M$, the diamonds $J_M(x, y)$ are compact.

Based on this definition, we call globally hyperbolic spaces **space-time manifolds**, since we have a direction of time and a notion of causality, being globally well defined by **GH.1**. Condition **GH.2** though, has more to do with the geometry/topology of the manifold, prohibiting for instance certain types of holes to exist.

Example A.5.2 (Non-globally hyperbolic plane)**.** Take $M = \mathbb{R}^2 \setminus \mathbb{E}$, with the set $\mathbb{E}$ being given by[15]

$$\mathbb{E} = \{(x_1, x_2) \in \mathbb{R}^- \times \mathbb{R} | \, x_1 \le x_2, \, x_1 \le -x_2\}, \tag{A.5.2}$$

and with the Lorentizian metric

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.5.3}$$

The null curves of the metric through a point $x \in M$ are just $45°$ angle straight lines centered at such a point, and so, time-like curves are straight lines that sit inside such a light cone.

The openness of M, in this case, guarantees us its causality so that **GH.1** is satisfied. For **GH.2** however, note that given a point $x_0 \in M$ located on the third quadrant and $y_0 \in M$ on the second one, the diamond $J_M(x_0, y_0)$, although non-empty, is not compact since there are causal curves emanating from x_0 and others incoming into y_0 that hit the boundary of $\mathbb{E}$. Such a diamond is thus neither closed nor open and since M is finite dimensional the diamond can't be compact, so that M is not globally hyperbolic.

A.5.1 Distributions

Intuitively speaking, a *distribution* over some space is a function on its dual. Indeed, based on [Wal12] and [Gem00] we shall work with the following notion

Definition A.5.9 (Distribution on a Manifold)**.** On defining $T^k M := TM \times \cdots \times TM$, k times, we have that a **tensor distribution** $u^{\mathcal{D}}$ over M is a continuous linear functional

$$u^{\mathcal{D}} : \Gamma(\Lambda^k T^* M \otimes T^k M) \to \mathbb{C}$$

explicitly given by the formula

$$u^D(T, (U_1, \dots, U_k)) = \int_M \langle T, (U_1, \dots, U_k) \rangle \, \mathrm{dVol}_M \tag{A.5.4}$$

[15]we may think of $\mathbb{E}$ as the causal wedge of $(-\infty, 0]$ [HE23].

With $\langle T, (U_1, \ldots, U_k)\rangle = T(U_1, \ldots, U_k)$. The number k will be the **order** of the tensor distribution u and, the space of such distributions will be called $\Lambda^k_{\mathcal{D}}(M)$.

We are going to make a slight abuse of notation and instead of referring to the tensor distribution $u^{\mathcal{D}}$ taking an element of $\Gamma(\Lambda^k T^*M \otimes T^k M)$ and giving back a number, let $T^{\mathcal{D}}$ be the distribution that takes k vectors, pairs them with the tensor T, and then integrates this pairing over M. With this notation, we can make use of Definition A.5.9 to talk about the uniqueness property of Lemma 1 of [Gem00] for vector/tensor fields instead of distributions too. Indeed, note that by definition we have that for a tensor $T \in \Lambda^k(M)$

$$(\Delta T)^{\mathcal{D}}(U_1, \ldots, U_k) = \int_M \langle \Delta T, (U_1, \ldots, U_k)\rangle \, \mathrm{dVol}_M$$

so that if we have the distributional equality $(\Delta T)^{\mathcal{D}} = V^{\mathcal{D}}$ then we must have $\Delta T = V$ a.e (almost everywhere, with respect to the volume measure dVol_M) since the former would imply that

$$\int_M \langle \Delta T - V, (U_1, \ldots, U_k)\rangle \, \mathrm{dVol}_M = 0, \; \forall U_1, \ldots, U_k \in \mathfrak{X}(M).$$

from which $\Delta T = V$ holds a.e. Notice however that, when we are performing the above integrals we are bluntly *assuming* the integrability of our $k-$forms over M. This is a bit "slackish" and so, to properly consider distributions in this regard we ought to take certain spaces into account. First, let's take a step back and work over a *Riemannian* manifold M for now. We can then define the space $L^p(M)$ as

$$L^p(M) := \left\{ f \in \mathcal{C}^\infty(M) | \; \|f\|_p^p = \int_M |f(x)|^p \, \mathrm{dVol}_M(x) < \infty \right\}. \tag{A.5.5}$$

For values of $p \geq 1$, the norm $\|\cdot\|_p$ makes the pair $(L^p(M), \|\cdot\|_p)$ a Banach space (by the Fischer-Riesz Theorem). This isn't enough though. To talk about integrability of forms we need to consider another norm. To define it, we can rely on the above L^p norm and consider the following: when given a Riemannian $m-$manifold M, we can have associated to it a Riemannian metric g and a metric compatible connection ∇. If we take a $k-$form $T \in$

$\Lambda^k TM$ and consider the *multi index covariant derivative* $D^\alpha T_{i_1\ldots i_k} = \nabla^{\alpha_1}\ldots\nabla^{\alpha_l}T_{i_1\ldots i_k}$ for some real entry multi index $\alpha = (\alpha_1,\ldots,\alpha_l)$ we can make the following

Definition A.5.10 (*Weak derivative*)**.** Given a $k-$form T we say v is its ***weak derivative of order*** $\boldsymbol{\alpha}$ if, for every test function with compact support $\phi \in \mathcal{C}_c^\infty(M)$ we have that

$$\sum_{i_1,\ldots,i_k}\int_M T_{i_1\ldots i_k}D^\alpha\phi \mathrm{dVol}_M = (-1)^{|\alpha|}\sum_{i_1,\ldots,i_k}\int_M v^{\alpha_1\ldots\alpha_l}_{i_1\ldots i_k}\phi \mathrm{dVol}_M, \tag{A.5.6}$$

where $|\alpha| = \sum_{j\leq l}\alpha_j$ is the *norm* of α.

From this we see that on this general setting, the weak derivative of a $k-$form is an $(l,k)-$tensor distribution, with l being the *length* of the multi index α, that is integrable (at least) on every compact set of M. Thanks to this notion of weak derivation we are able to construct the following

$$\|T\|^p_{\mathcal{W}^{j,p}(\Lambda^k TM)} := \sum_{i_1,\ldots i_k}\sum_{|\alpha|\leq j}\|D^\alpha T_{i_1\ldots i_k}(x)\|^p_p\,, \tag{A.5.7}$$

where in the above, since M is Riemannian $m-$dimensional, we choose its coordinates to be enumerated from 1 to m, and so $D^0T_{i_1\ldots i_k}(x) \equiv T_{i_1\ldots i_k}(x)$. With this in mind, we can call $\|\cdot\|_{\mathcal{W}^{j,p}(\Lambda^k TM)}$ a norm. For, if $\|T\|_{\mathcal{W}^{j,p}(\Lambda^k TM)} = 0$, since we are summing over positive terms only, each one of them has to be zero. By our above comment we then have that $T_{i_1\ldots i_k} = 0$ for all $i_1,\ldots,i_k = 1,\ldots k$, hence $T\equiv 0$. The $\mathbb{R}-$linearity follows trivially and by using the linearity of the derivatives ∇^l together with Minkowski's inequality on each summand, we conclude the triangle inequality. It's thus based on Eq.(A.5.7) (and hence Definition A.5.10) that we make

Definition A.5.11 (*Sobolev Space*)**.** The ***Sobolev space*** $\mathcal{W}^{j,p}(\Lambda^k TM)$ over a Riemannian manifold M is the space of differential $k-$forms whose weak derivatives of order α and norm less than or equal to j exist and lie in $L^P(M)$. More explicitly, we have

$$\mathcal{W}^{j,p}(\Lambda^k TM) := \{T \in \Lambda^k(TM) |\, \|T\|_{\mathcal{W}^{j,p}(\Lambda^k TM)} < \infty\} \tag{A.5.8}$$

As shown in Section 2.2 these Sobolev spaces play a big role in mathematics and physics. For, they end up being the spaces over which one should solve partial differential equations. As mentioned in the body of text, in this same cited section, we can in certain situations be more interested in working with the $p = 2$ case of Sobolev spaces instead. These will in turn get a name of their own, and shall be referred to as

$$\mathcal{H}^j(\Lambda^k TM) = \mathcal{W}^{j,2}(\Lambda^k TM). \tag{A.5.9}$$

For each $j \geq 0$ the spaces $\mathcal{H}^j$ (for short) are ***Hilbert spaces***, i.e, complete spaces (in the sense of Cauchy sequences) endowed with an inner product and a norm, naturally induced by the latter. It's over these spaces that we are able to properly define inner products of functions and $k-$forms now too. They're specially important in classical Quantum Mechanics (cQM) and are a key component of Spectral Theory which can be very broadly thought of as the mathematical theory which formalizes the calculations done in cQM.

For the pseudo-Riemannian case, the above constructions can be carried out presumably on the same fashion since, no particular mention to the signature of the metric was made.

For further reading, have a look at [Gem00, Wal12, KM19]. A good introductory book on the matter of Banach and Hilbert spaces, along with the fundamentals of Spectral Theory is due to John Conway [Con19].

A.5.2 Some topological considerations for $\mathcal{L}_M$

Throughout the thesis we mostly dealt with a collection of pseudo-Riemannian metrics defined over the sphere (or more specifically the 3−cylinder $\mathbb{R} \times \mathbb{S}^2$) which were parameterized by some time t and induced by a certain perturbative parameter ε in order to account for the interpretation of our model, as explained on Chapter 1.

Here we wish to formalize a bit this notion of "perturbation" that we used so much throughout the work. To do so, given a Riemannain Manifold M, we consider the space

$\mathcal{L}_M$ of ***Lorentzian structures*** over M. The general existence of a Lorentzian metric over a generic manifold M can not be usually guaranteed, but some interesting results can be found either way. Indeed, any non-compact, connected smooth manifold admits a Lorentz metric [O'n83] and, as studied on [Ler73], when provided with a suitable topology we can have some very nice regularity and stability results regarding the metrics of $\mathcal{L}_M$.

On the matter of topology, in this thesis we shall be dealing with the $\boldsymbol{C^\infty}$**−topology** over $\mathcal{L}_M$ since its the one that better suits our perturbations. To define it, we first need the following

Definition A.5.12 (C^∞−norm)**.** Let $g' \in \mathfrak{Riem}(M)$ be a Riemmanian metric over M. Consider the norm $\|\cdot\|_{g'}$ induced by g' on the space of k−forms $\Omega^k(M)$ given by

$$\|T(x)\|^2_{g'} = (g')^{i_1 j_1}_x \dots (g')^{i_k j_k}_x T(x)_{i_1 \dots i_k} T(x)_{j_1 \dots j_k}, \text{ for some } x \in M, \tag{A.5.10}$$

where, $T(x)$ and g^{ij}_x represent the form T and inverse metric $(g')^{ij}$ evaluated at $x \in M$, respectively. The $\boldsymbol{C^\infty}$***−norm*** over $\mathcal{L}_M$ will be given by

$$\|T\|_\infty = \sup_{x \in M} \|T(x)\|_{g'} . \tag{A.5.11}$$

With Eq.(A.5.11) at our disposal, it's natural to consider

Definition A.5.13 (C^∞−topology)**.** The $\boldsymbol{C^\infty}$**−topology** over $\mathcal{L}_M$ is simply the topology induced by the C^∞−norm.

The open balls of radius R, centered at some metric g, in this topology will be given by the sets

$$\mathcal{B}_R(g) = \{h \in \mathcal{L}_M | \ \|h - g\|_\infty < R\}, \tag{A.5.12}$$

with the difference $h - g$ being point-wise well defined and indeed being given by the component-wise difference of each quantity (evaluated at the specific point).

Definition A.5.14 ($\varepsilon-$proximity)**.** Given two pseudo-Riemannian forms $g_1, g_2 \in (\mathcal{L}_M, \|\cdot\|_\infty)$ and an $\varepsilon \in \mathbb{R}$, we shall say that g_1 is ε close to g_2 if

$$\|g_1 - g_2\|_\infty < \varepsilon, \tag{A.5.13}$$

that is, if $g_1 \in \mathcal{B}_\varepsilon(g_2)$.

It is through Definition A.5.14 that we are able to give some formalization to the approach we took in the body of the thesis, namely, of considering a "small" metric perturbation $\overline{g}$ with respect to some base metric $g \in \mathcal{L}_M$. Indeed we see that

$$\|\overline{g} - g\|_\infty = \sup_{x\in M} \|g + \varepsilon h - g\|_{g'} = \varepsilon \sup_{x\in M} \|h\|_{g'} \tag{A.5.14}$$

Upon taking $M = \mathbb{R} \times \mathbb{S}^2$ with $g' = \mathrm{diag}(1, 1, \sin^2(\theta))$ and $g = \mathrm{diag}(-1, 1, \sin^2(\theta))$, the metric perturbation h we choose at Eq.(3.24) is such that

$$\sup_{x\in M} \|h\|_{g'} = \sqrt{1 + R^2} < 1 + R, \tag{A.5.15}$$

for $R > 0$. Putting the above equations together, we conclude that

$$\|\overline{g} - g\|_\infty < \varepsilon(1 + R), \tag{A.5.16}$$

so that $\overline{g} \in \mathcal{B}_{\varepsilon(1+R)}(g)$.

We can further use the $C^\infty-$norm as a guarantee for how good the approximation we are doing really is. For instance, by demanding $\varepsilon \ll \|g\|_\infty \|h\|_\infty^{-1}$, our approximation should be good provided that, in a sense, it naturally extends the Minkowskian metric perturbation $\overline{g} = \eta + \varepsilon h$, where one could demand $\|\varepsilon h\|_\infty \ll \|\eta\|_\infty$ so that $\varepsilon \ll \|h\|_\infty^{-1}$, and hence first order perturbation theory, at least from the $\|\cdot\|_\infty$ perspective, makes sense.

Appendix B

Basics of General Relativity

In this section our aim is to introduce the reader to the computations that are usually done in General Relativity when one is interested in finding so called "perturbative solutions", i.e solutions to the field equations that arise from a perturbation of an already know "base solution". Since to specify a metric is to specify a geometry, the base solutions will here be referred to as **base manifolds**.

The object we shall be primarily interested in here are *pseudo-Riemannian metrics* (see Section A.5) since our manifolds of interest will be *space-time* like.

Definition B.0.1 (Space-Time Manifold)**.** A **space-time manifold** is a pair (M, g) in which M is a differentiable, globally hyperbolic manifold and g is a Lorentzian metric.

In physics language, it is customary to say that M is a $(n+1)-$dimensional manifold, where n refers to its spacial dimensions and 1 for its time dimension. In principle you could consider more time dimensions on your manifold, though the utility of such an approach is not quite clear as of now.

B.1 Setting the grounds of GR

The now almost 110 year old theory of General Relativity (GR) was formally developed by the German theoretical physicist Albert Einstein (1875-1955) in the beginning of the

20th century. Although now quite well understood by physicists and already experimentally verified, back then the theory revolutionized how we should see and think about gravity. The reason being that from that point on, one could picture the action of gravity by means of geometry and, more specifically pseudo-Riemannian geometry[1].

What the theory tells us is that we have an equivalence between *gravity* and *curvature* expressed in the following set of field equations[2]

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{B.1.1}$$

where $T_{\mu\nu}$ is the so called **Energy-Momentum tensor**, $G_{\mu\nu}$ is the **Einstein tensor** and κ is a constant that depends on the dimension of our space-time manifold.

The components of the energy-momentum tensor are such that T^{00} represents the *relativistic energy density of the particle* i.e energy/ (volume$\times c^2$), with c being the speed of light. T^{0i} has to do with the particle's relativistic momentum density ($\boldsymbol{p} = m\boldsymbol{u}, m = m_0\gamma$) where m_0 is the *inertial mass* of the particle, that is, the mass measured by an observer that is co-moving with the particle itself, and γ is called the *Lorentz factor* given by:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{B.1.2}$$

with v being the particle's spacial velocity[3]. Finally, the components T^{ij} represent the **stress tensor** over the particle. The diagonal terms are related to pressures suffered by the particle along the directions of the chosen basis, whereas the off diagonal terms have to do with sheer stress[4] felt by the particle. Meanwhile, the Einstein tensor is given by the following:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{B.1.3}$$

[1] Now a days though this picture is a bit "worn off". We tend to think of the field equations as more of a set of partial differential equations and forget a bit about the geometry behind it. Apparently it seems to be a good view point to have when tackling more modern mathematical problems that emerge in the theory [RD22]

[2] we shall adhere to the physics notation here since it is quite illustrative, more often than not!

[3] Note that for small velocities, i.e when $v \ll c$, the Lorentz factor approximates to 1. This in turn explains why no relativistic effects happen at the classical level.

[4] on rubbing two surfaces together for instance, sheer stress is created between them.

with $R_{\mu\nu}$ being the Ricci tensor [Eq.(A.1.24)] and R the Ricci scalar [Eq.(A.1.26)]. The inspiration behind this definitions comes from the following functional known as the **Einstein-Hilbert action**, given by:

$$S[g] = \frac{1}{2\kappa}\int_M R\sqrt{-g}d^n x \tag{B.1.4}$$

with $\sqrt{-g}d^n x$ being the volume element of our $n-$dimensional pseudo-Riemannian manifold.

Note B.1.1. The motivation for one to consider the Einstein-Hilbert action is simple: it is the simplest action one can think of building when considering "gravitationally interesting" objects. You see, given $p \in M$ we can always find a coordinate system over p such that the metric is, to first order, the Euclidian (or the Minkowski) one and thus whose Christoffel symbols [Eq.(A.1.16)] vanish. Though the same can **not** be said about the derivatives of the latter! Hence, locally, the simplest non-trivial object that gives us sight of curvature (and hence, of gravitational phenomena) are the Riemann/Ricci tensors at p. Since the action functional gets a scalar function over M as an argument, the Ricci scalar is a good guess to be made indeed. Nevertheless, more intricate actions for gravity, involving other possible combination of Ricci, Riemann and metric tensors can be constructed. Indeed $\boldsymbol{f(R)}$**−gravity theories** are those whose underlying action functionals are of the form:

$$S[f,g] = \frac{1}{2\kappa}\int_M f(R)\sqrt{-g}d^n x \tag{B.1.5}$$

with $f(R)$ representing here an arbitrary function of the aforementioned quantities. Such class of theories have been widely studied and are still part of the body of research done in gravity [SF10, JPS12].

Now, getting back on track, by the well known *extreme action principle*, our equations of motion for the metric will be given by the set of metrics such that:

$$\frac{\delta S}{\delta g^{\mu\nu}}[g] = 0 \tag{B.1.6}$$

Upon doing such a variation on Eq.(B.1.4) we end up with:

$$\delta S = \frac{1}{2\kappa}\int_M (\delta(R)\sqrt{-g} + R\delta(\sqrt{-g}))d^n x \tag{B.1.7}$$

which can in turn be written as [Car14, Ton]:

$$\delta S = \frac{1}{2\kappa}\int_M G_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu} d^n x + \frac{1}{2\kappa}\int_M \nabla_\mu X^\mu \sqrt{-g} d^n x$$
$$X^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\Gamma^\beta_{\alpha\beta} \tag{B.1.8}$$

The second integral appearing on Eq.(B.1.8) is zero by Stokes theorem on manifolds. Since we ask Eq.(B.1.6) to hold for every possible variation, we thus end up with Einstein's vacuum field equations:

$$G_{\mu\nu} = 0 \tag{B.1.9}$$

Which notably correspond to the field equations [Eq.(B.1.1)], with $T_{\mu\nu} = 0$. As a matter of fact, to obtain Eq.(B.1.1) on its full form our action of interest actually needs to be:

$$S_{tot} = S_{EH} + S_{matter} \tag{B.1.10}$$

where S_{EH} is as in Eq.(B.1.4) and S_{matter} is the action associated to the matter content present on space-time. Said action is usually written as

$$S_{matter} = \int \mathcal{L}_{matter}\sqrt{|g|}d^n x \tag{B.1.11}$$

where $\mathcal{L}_{matter}$ is the *Lagrangian* for the matter distribution considered. From Eq.(B.1.11) we define

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta(\mathcal{L}_{matter}\sqrt{|g|})}{\delta g^{\mu\nu}} \tag{B.1.12}$$

So that we recover Eq.(B.1.1) as desired.

Until now all we've talked about was the metric tensor and all we have done was to try and show how its dynamical equations look like. If however we are interested in the dynamical equations of the sources of said gravitational fields we ought to solve the following

$$\nabla_\mu T^{\mu\nu} = 0. \tag{B.1.13}$$

By the physical interpretation the components of our stress-energy tensor have it follows that Eq.(B.1.13) functions as some type of $F = ma$ for a *real*, field generating particle. Indeed, by taking $\nu = i = 1, 2, 3$ we have that

$$0 = \nabla_0 T^{0i} + \nabla_k T^{ki} = \nabla_0 p^i + \nabla_k T^{ki}$$
$$\Rightarrow \nabla_0 p^i = -\nabla_k T^{ki},$$

with the last equality representing a generalized version of

$$\frac{dp}{dt} = -\text{grad}(U),$$

if we were to consider some conservative force $F = -\text{grad}(U)$.

Regarding however the equations of motion for test particles the situation changes a bit. Indeed what we have is

Proposition B.1.1 (Geodesic motion). *A test particle moves according to the geodesic equation.*

Proof: This comes from a generalized version of Newton's laws of motion. In particular, a relativistic version of **New.2** would read

$$\nabla_u (m_0 u) = F \tag{B.1.14}$$

for a particle with mass m_0, velocity u and upon which a force F is acting. In coordinates we have that

$$u^\alpha = \frac{dx^\alpha}{ds}, \tag{B.1.15}$$

with ds being the infinitesimal distance element of the world-line[5] of the particle.

From there it follows that, since gravity is to be thought of as the curvature of the space-time the particle sits in, there actually are no other external forces present in our

[5]this is just the path the particle traces out on space-time

system (provided we are considering a *single* test particle, of course). This tantamounts to saying that $F \equiv 0$ and hence, the coordinate version of Eq.(B.1.14) becomes

$$u^{\alpha}\nabla_{\alpha}(m_0 u^{\beta} e_{\beta}) = 0, \tag{B.1.16}$$

for a basis e_{β} of the tangent space. By the chain rule and the definition of covariant derivative of a vector, we fist find that

$$(u^{\alpha}u^{\beta}\partial_{\alpha}m_0 + m_0 u^{\alpha}\partial_{\alpha}u^{\beta} + m_0\Gamma^{\beta}_{\sigma_{\alpha}}u^{\sigma}u^{\alpha})e_{\beta} = 0. \tag{B.1.17}$$

We can further expand on the above equation by using the chain rule on Eq.(B.1.15) as

$$u^{\alpha}\partial_{\alpha} = u^{\alpha}\frac{\partial}{\partial x^{\alpha}} = \frac{dx^{\alpha}}{ds}\frac{\partial}{\partial x^{\alpha}} = \frac{d}{ds}, \tag{B.1.18}$$

thus yielding us the following

$$\frac{dx^{\beta}}{ds}\frac{dm_0}{ds} + m_0\frac{d^2x^{\beta}}{ds^2} + m_0\Gamma^{\beta}_{\sigma\alpha}\frac{dx^{\sigma}}{ds}\frac{dx^{\alpha}}{ds} = 0. \tag{B.1.19}$$

Since we usually deal with test particles with constant mass, the first term vanishes and so we get the usual geodesic equation. □

To end this part we briefly mention some of the key points, from a physical perspective, that at the end lead Einstein to formulate the General Theory of Relativity.

Much like mathematics depends on a set of axioms for us to work with, physical theories also have their axioms commonly referred to as *postulates* or work hypothesis. For special relativity we turn out to have the following

Postulate B.1.1 (Principle of Relativity)**.** The laws of physics are the same for every inertial observer.

In the above, inertial refers to the fact that the observer moves with constant velocity (relative to something else). Any two observers that move with constant velocity with respect to each other for instance will get to the same laws of physics, provided they are able to make identical experiments to test whatever theories they might be after.

Moving further we also have

Postulate B.1.2 (Constancy of c)**.** The speed of light c is constant.

In particular, the need to deal with pseudo-Riemannian spaces comes from the jointing both of these postulates together. Indeed if c is a *universal* speed limit, we need somehow to be able to change between two coordinate systems (referring to inertial observers) in a way such that its value remains unchanged. The geometry provided by Minkowski space for example precisely enables us to make sense of that. Thus, the need to deal with these types of manifolds.

Theories which assume Postulates B.1.2 & B.1.1 are thus called *relativistic*. In particular, General Relativity, as the name suggests, is a relativistic theory of gravity. Newtonian gravity on the other hand is **not** relativistic because no assumption about any maximal speed limit is made. That's why we consider it to be a *classical* theory of gravity, in this regard.

For the construction of GR however, the reasoning becomes more intricate. To start we have

Postulate B.1.3 (Mach's Principle)**.** The matter distribution determines the geometry of the universe.

Although this is not *the* first formulation of the "original" Mach's principle, with some work one can get to this form of it (see [RD22] and references therein). Based on Postulate B.1.3 we can better appreciate the physical meaning of Eq.(B.1.1), by seeing where it was deduced from.

Postulate B.1.4 (Relativistic equivalence principle)**.** An observer that linearly accelerates with respect to an inertial observer is locally identical to an observer at rest in a gravitational field.

In other words a local, isolated observer sitting inside a space-ship can **not** perform any experiments that will enable him/her to distinguish between being in free fall at acceleration g near a massive body versus moving, in the opposite direction to its free-falling one, at a constant acceleration g.

The last two other principals Einstein used to go from special to general relativity were the principal of general covariance and of general relativity. The former says that physical equations should be able to be expressed in tensor form, meanwhile the latter asserts that all observers are equivalent and thus no preferred coordinate choice exists, aside from the one that might make certain computations easier.

More aspect of the above can be checked at the amazing book by D'Inverno and Vickers [RD22]. We now move on to talk a bit about another interesting aspect of GR that actually culminates in an analogy with electromagnetism.

B.1.1 Aspects of $(3+1)-$dimensional gravity

Here we elucidate some properties of four dimensional Einstein gravity, in particular, a very useful decomposition that happens in this case which itself yields a beautiful parallel between gravity and electromagnetism. We start with a pseudo-Riemannian manifold (M, g) of dimension four with Minkowski signature $(-, +, +, +)$.

Given a particle of mass m, we define its *four velocity* as in Eq.(B.1.15) and normalize it in such a way that

$$u_\mu u^\mu = -1. \tag{B.1.20}$$

This means in particular that even if the particle doesn't move in space, it still does so in time.

Associated to said velocity field we can come up with *projection operators* that we define as follows

$$U^\mu{}_\beta = -u^\mu u_\beta \tag{B.1.21a}$$
$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \tag{B.1.21b}$$

Evidently, $U^{\mu}{}_{\beta}$ projects onto the direction of motion, whereas less so clearly one can conclude that $h_{\mu\nu}$ is a metric over the instantaneous rest-space of observers moving with four velocity u^{α}. This is because, at each point $p \in M$, normal to our 4−vector is a constant time slice hypersurface (which in this case shall be a 3−manifold $\mathcal{M}^3$). We can then choose our local coordinates at p in such a way that $u^{\mu} = (1, 0, 0, 0)$, is the time-like unit vector normal to $\mathcal{M}^3$. In these coordinates Eq.(B.1.21b) reduces to

$$h_{ij} \equiv g_{ij},$$

which is indeed a metric tensor, by the definition of g and its constant time slice restriction.

From there, along with the metric compatible [Eq.(A.1.14)] covariant derivative ∇_{μ}, we also have its **totally orthogonal projection** $\tilde{\nabla}$, particularly given by:

$$\tilde{\nabla}_{\alpha} u_{\beta} = h^{\mu}_{\alpha} h^{\nu}_{\beta} \nabla_{\mu} u_{\nu} \tag{B.1.22}$$

which can in turn be further generalized to higher order tensors as

$$\tilde{\nabla}_{\sigma} T^{\beta_1 \ldots \beta_q}{}_{\alpha_1 \ldots \alpha_p} = h^{\mu_1}_{\alpha_1} \ldots h^{\mu_p}_{\alpha_p} h^{\beta_1}_{\gamma_1} \ldots h^{\beta_q}_{\gamma_q} h^{\nu}_{\sigma} \nabla_{\nu} T^{\gamma_1 \ldots \gamma_q}{}_{\mu_1 \ldots \mu_p} \tag{B.1.23}$$

The expression on Eq.(B.1.22) though can be rewritten in terms of the full connection ∇ by defining the **time covariant derivate** along the world lines of the mass m. Indeed we have that such a formula is given by

$$\dot{T}^{\beta_1 \ldots \beta_q}{}_{\alpha_1 \ldots \alpha_p} = u^{\nu} \nabla_{\nu} T^{\beta_1 \ldots \beta_q}{}_{\alpha_1 \ldots \alpha_p} \tag{B.1.24}$$

And so, by using Eq.(B.1.24) we have that

$$\tilde{\nabla}_{\alpha} u_{\beta} = \nabla_{\alpha} u_{\beta} + u_{\alpha} \dot{u}_{\beta} \tag{B.1.25}$$

or, equivalently

$$\nabla_{\alpha} u_{\beta} = -u_{\alpha} \dot{u}_{\beta} + \tilde{\nabla}_{\alpha} u_{\beta} \tag{B.1.26}$$

Now, since we can regard $\tilde{\nabla}_{\alpha} u_{\beta}$ as a rank−2 tensor, then we have the following decomposition holding

$$\tilde{\nabla}_{\alpha} u_{\beta} = \frac{1}{3} \Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \tag{B.1.27}$$

where $\sigma_{\alpha\beta}$ is a *trace-less symmetric tensor*, $\omega_{\alpha\beta}$ in an *anti-symmetric tensor*, $\Theta = \tilde{\nabla}_\alpha u^\alpha$ (a.k.a the trace of $\tilde{\nabla}_\alpha u_\beta$) and $h_{\alpha\beta}$ is as in Eq.(B.1.21b). More explicitly we have that[6]

$$\sigma_{\alpha\beta} = \nabla_{\langle\mu} u_{\nu\rangle} := \left(h_\alpha^{(\mu} h_\beta^{\nu)} - \frac{1}{3} h^{\mu\nu} h_{\alpha\beta} \right) \nabla_\mu u_\nu \tag{B.1.28a}$$

$$\omega_{\alpha\beta} = h_\alpha^{[\mu} h_\beta^{\nu]} \nabla_\mu u_\nu \tag{B.1.28b}$$

Based on Eq.(B.1.28b) we define yet another kinematical quantity

$$\omega_\alpha = \frac{1}{2} \varepsilon_{\gamma\alpha\beta\rho} u^\gamma \omega^{\beta\rho}, \tag{B.1.29}$$

which can be interpreted as a sort of "3−vorticity", if you will. Advancing a bit further now, thanks to the *Weyl tensor* $C_{\alpha\beta\mu\nu}$ (check Eq.(A.1.29)), we can define the following quantities:

$$E_{\alpha\beta} = C_{\alpha\mu\beta\nu} u^\mu u^\nu \tag{B.1.30a}$$

$$H_{\alpha\beta} = \frac{\sqrt{|g|}}{2} \varepsilon_{\gamma\alpha\rho\mu} C^{\rho\mu}{}_{\beta\nu} u^\gamma u^\nu \tag{B.1.30b}$$

which are the **electric** and **magnetic** parts of the Weyl tensor, respectively. Physically they represent the 'free gravitational field', i.e the one outside matter sources and thus, its expected that their equations of motion reflect how said matter affects the fabric of space-time[7].

To get to said set of equations we need to make use of the *differential Bianchi identity* (see Eq.(A.1.25)) together with the formula for the Weyl tensor in terms of $E_{\alpha\beta}$ and $H_{\alpha\beta}$, given by [MB98][8]:

$$C_{\alpha\beta}{}^{\mu\nu} = 4 \left(u_{[\alpha} u^{[\mu} E_{\beta]}{}^{\nu]} + h_{[\alpha}{}^{[\mu} E_{\beta]}{}^{\nu]} \right) + 2\varepsilon_{\gamma\alpha\beta\rho} u^\gamma u^{[\mu} H^{\nu]\rho} + 2\varepsilon^{\gamma\mu\nu\rho} u_\gamma u_{[\alpha} H_{\beta]\rho} \tag{B.1.31}$$

[6] Remember that $T^{(ab)} = \frac{1}{2}(T^{ab} + T^{ba})$ and $T^{[ab]} = \frac{1}{2}(T^{ab} - T^{ba})$

[7] which by the way is just an effective way of visualizing what goes on. Truly, space-time is not 'made' of anything. There is no 'space-time particle' for instance and, gravitons (the particle mediators of gravity), much like any other fundamental particles are nothing but energetic excitations of the gravitational field, when seen as a quantum field that permeates all of space.

[8] where we have

$$\text{curl } A_{\alpha\beta} = u^\gamma \varepsilon_{\gamma\rho\mu(\alpha} \tilde{\nabla}^\rho A_{\beta)}{}^\mu$$

and $E_{\langle\alpha\beta\rangle}$ as on Eq.(B.1.28a).

From here we use the trace-less part of the contracted Bianchi identities and get to the following set of *vacuum Maxwell-like equations* for $E_{\alpha\beta}$ and $H_{\alpha\beta}$:

$$\tilde{\nabla}^{\beta} E_{\alpha\beta} = -3\omega^{\beta} H_{\alpha\beta} + \varepsilon_{\rho\alpha\mu\nu} u^{\rho} \sigma^{\mu}{}_{\sigma} H^{\nu\sigma} \tag{B.1.32a}$$

$$\tilde{\nabla}^{\beta} H_{\alpha\beta} = 3\omega^{\beta} H_{\alpha\beta} - \varepsilon_{\rho\alpha\mu\nu} u^{\rho} \sigma^{\mu}{}_{\sigma} E^{\nu\sigma} \tag{B.1.32b}$$

$$\begin{aligned}\dot{E}_{\langle\alpha\beta\rangle} - \text{curl } H_{\alpha\beta} = &- \Theta E_{\alpha\beta} + 3\sigma_{\mu\langle\alpha} E_{\beta\rangle}{}^{\mu} \\ &- \omega^{\mu} u^{\gamma} \varepsilon_{\gamma\mu\rho(\alpha} E_{\beta)}{}^{\rho} + 2u^{\gamma} \dot{u}^{\mu} \varepsilon_{\gamma\mu\rho(\alpha} H_{\beta)}{}^{\rho}\end{aligned} \tag{B.1.32c}$$

$$\begin{aligned}\dot{H}_{\langle\alpha\beta\rangle} - \text{curl } E_{\alpha\beta} = &- \Theta H_{\alpha\beta} + 3\sigma_{\mu\langle\alpha} H_{\beta\rangle}{}^{\mu} \\ &- \omega^{\mu} u^{\gamma} \varepsilon_{\gamma\mu\rho(\alpha} H_{\beta)}{}^{\rho} - 2u^{\gamma} \dot{u}^{\mu} \varepsilon_{\gamma\mu\rho(\alpha} E_{\beta)}{}^{\rho}\end{aligned} \tag{B.1.32d}$$

B.1.2 Hamiltonian formulation

Much like Classical Mechanics, General Relativity also admits a Hamiltonian formalism. In order for it to work though, an important assumption has to be made: that 4 *dimensional space-time is decomposable in a* $3+1$ *dimensional fashion*. That is, our space-time admits for each time component a connected, manifold-like spacial slice that in turn is endowed with some Riemannian metric. This formalism is useful for a variety of reasons, such as computational methods to find Einstein's field equation's solution and to perform canonical quantization of such field equations [Ber].

Our focus here will however be much more modest. We just want to assure to the reader that even in GR we can make use of the *free particle Hamiltonian* to talk about particle motion, and furthermore use such a formalism to our advantage in the qualitative description of the particle's behavior (at least in some specific case).

To do so, recall that, formally, to built a Hamiltonian for a particle we ought to consider first the Lagrangian (density) describing its motion. For a free particle of mass m on a pR-manifold (M, g) that would be

$$\mathcal{L} = \frac{m}{2} g_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta}, \tag{B.1.33}$$

where $\dot{()} = \frac{d()}{ds}$ Its corresponding action is then given by

$$S = \frac{m}{2} \int g_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta} ds, \tag{B.1.34}$$

and its minimization yields the geodesic equation [RD22]. From Eq.(B.1.33), we construct the free particle Hamiltonian via the Legendre transform

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\dot{x}^\mu - \mathcal{L}. \tag{B.1.35}$$

As usual we define the quantity

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}, \tag{B.1.36}$$

to be the ***conjugate momentum*** to the variable x^μ. In the free particle case, such a variable equals

$$p_\mu = m g_{\mu\beta}\dot{x}^\beta. \tag{B.1.37}$$

And so, we can invert the formula for $\dot{x}^\mu$ obtaining

$$\dot{x}^\alpha = \frac{1}{m} g^{\alpha\mu} p_\mu, \tag{B.1.38}$$

from which we see that Eq.(B.1.35) assumes the form

$$\mathcal{H} = \frac{1}{2m} g^{\alpha\beta} p_\alpha p_\beta. \tag{B.1.39}$$

From here we have our usual equations of motion

$$\dot{x}^\mu = \frac{\partial \mathcal{H}}{\partial p_\mu}, \tag{B.1.40a}$$

$$\dot{p}_\mu = -\frac{\partial \mathcal{H}}{\partial x^\mu}. \tag{B.1.40b}$$

Clearly, Eq.(B.1.40a) and Eq.(B.1.38) are the same. For Eq.(B.1.40b) though, it can be written as

$$\dot{p}_\mu = -\frac{1}{2m} \partial_\mu g^{\alpha\beta} p_\alpha p_\beta. \tag{B.1.41}$$

We can deduce its connection with the geodesic equation in two ways: first, from Eq.(B.1.37) we have that

$$p^\mu = m\dot{x}^\mu, \tag{B.1.42}$$

and so, when we take its derivative we have that

$$\dot{p}^\mu = m\ddot{x}^\mu = -m\Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\frac{1}{m}\Gamma^\mu_{\alpha\beta}p^\alpha p^\beta$$
$$\Rightarrow \dot{p}^\mu + \frac{1}{m}\Gamma^\mu_{\alpha\beta}p^\alpha p^\beta = 0. \tag{B.1.43}$$

The second way to do it is directly from Eq.(B.1.41). More specifically, the metric compatibility condition gives us that

$$\partial_\mu g^{\alpha\beta}p_\alpha p_\beta = -2g^{\alpha\gamma}\Gamma^\beta_{\gamma\mu}p_\alpha p_\beta. \tag{B.1.44}$$

More over, since our particle is time-like, we have

$$p^\mu p_\mu = -m^2, \tag{B.1.45}$$

being satisfied. From there it follows that

$$p_\mu\dot{p}^\mu = -p^\mu\dot{p}_\mu. \tag{B.1.46}$$

Plugging back Eqs.(B.1.41, B.1.44) on Eq.(B.1.46) and perfoming some index manipulation we end up with

$$p_\mu\left(\dot{p}^\mu + \frac{1}{m}\Gamma^\mu_{\alpha\beta}p_\alpha p_\beta\right) = 0, \tag{B.1.47}$$

so that it is *sufficient* for the geodesic equation in momentum form [Eq.(B.1.43)] to hold for the above equality to be true.

B.2 Perturbations around Minkowski metric

To fully solve the field equations [Eq.(B.1.1)] is a tremendously difficult task in general. Particular solutions assuming certain symmetries of space can be found (Schwarzschild, Kerr, ...) though, we won't be focusing on those here. Instead we take the perturbative approach. Which itself lead Einstein to conclude the existence of so called *gravitational waves* some hundred years ago (and which were only experimentally verified on 2015).

For that we start with the following assumption on the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(t, x^i) \tag{B.2.1}$$

That is, we will assume a prescribed distribution of matter and energy given in the **energy-momentum tensor** $T_{\mu\nu}$. By using the definition of the Christoffel symbols associated to $g_{\mu\nu}$ we find that

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}(\partial_\mu h^{\alpha}_{\nu} + \partial_\nu h^{\alpha}_{\mu} - \partial^{\alpha} h_{\mu\nu}) \tag{B.2.2}$$

Where $\eta^{\alpha\beta}\partial_\beta = \partial^\alpha$ and $\eta^{\alpha\beta}h_{\beta\nu} = h^{\alpha}_{\nu}$. From here we have that

$$R_{\mu\nu} = \frac{1}{2}(\partial_\alpha\partial_\mu h^{\alpha}_{\nu} + \partial_\alpha\partial_\nu h^{\alpha}_{\mu} - \partial_\alpha\partial^\alpha h_{\mu\nu} - \partial_\nu\partial_\mu h^{\alpha}_{\alpha}) \tag{B.2.3}$$

Renaming $h^{\alpha}_{\alpha} = h$ and $\Box = \partial_\alpha\partial^\alpha$, to first order in $h_{\mu\nu}$ the Ricci scalar will be of the form:

$$R = \partial_\alpha\partial_\mu h^{\alpha\mu} - \Box h \tag{B.2.4}$$

We thus have that the Einstein tensor will be of the form

$$G_{\mu\nu} = \frac{1}{2}(\partial_\alpha\partial_\mu h^{\alpha}_{\nu} + \partial_\alpha\partial_\nu h^{\alpha}_{\mu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} + \eta_{\mu\nu}\Box h - \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}) \tag{B.2.5}$$

To further simplify the equation we define the **tranverse-traceless tensor** given by

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \tag{B.2.6}$$

and consider a gauge in which it is a conserved quantity, i.e, $\partial_\mu \tilde{h}^{\mu}_{\nu} = 0$ which in terms of $h_{\mu\nu}$ can be written as

$$\partial_\mu h^{\mu}_{\nu} = \frac{1}{2}\partial_\nu h \tag{B.2.7}$$

After plugging (B.2.7) back into (B.2.5) and using (B.2.6) we are left with

$$G_{\mu\nu} = -\frac{1}{2}\Box \tilde{h}_{\mu\nu} \tag{B.2.8}$$

The field equations then take the form

$$\Box \tilde{h}_{\mu\nu}(t, x^i) = -\frac{16\pi G}{c^4} T_{\mu\nu}(t, x^i). \tag{B.2.9}$$

Hence, if $h_{\mu\nu}$ is given, we can use the above to define a first order accurate energy-momentum tensor. Conversely, if $T_{\mu\nu}$ is given, we can solve the above by means of retarded Green's functions.

B.3 Perturbations around a general metric

Like above, we now consider a metric perturbation of the form

$$\overline{g}_{\mu\nu} = g_{\mu\nu} + \varepsilon L_{\mu\nu} \tag{B.3.1}$$

where now $g_{\mu\nu}$ is a generic (base) space-time tensor and $L_{\mu\nu}$ is a matter induced perturbation over such base space-time. As mentioned previously, the path followed by a particle subject to just the effect that the geometry of its embbeding space has over it[9], is given by a geodesic curve, that is, a solution to:

$$\frac{d^2x^\alpha}{ds^2} + \overline{\Gamma}^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{B.3.2}$$

Where the Christoffel symbols will be given by:

$$\begin{aligned}\overline{\Gamma}^\alpha_{\mu\nu} &= \frac{1}{2}\overline{g}^{\alpha\beta}(\partial_\mu \overline{g}_{\beta\nu} + \partial_\nu \overline{g}_{\beta\mu} - \partial_\beta \overline{g}_{\mu\nu}) = \frac{1}{2}(g^{\alpha\beta} - \varepsilon L^{\alpha\beta})(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) \\ &= \frac{1}{2}(g^{\alpha\beta} - \varepsilon L^{\alpha\beta})(\partial_\mu(g_{\beta\nu} + \varepsilon L_{\beta\nu}) + \partial_\nu(g_{\beta\mu} + \varepsilon L_{\beta\mu}) - \partial_\beta(g_{\mu\nu} + \varepsilon L_{\mu\nu}))\end{aligned}$$

Which to first order in ε becomes:

$$\begin{aligned}\overline{\Gamma}^\alpha_{\mu\nu} &= \Gamma^\alpha_{\mu\nu} + \frac{\varepsilon}{2}g^{\alpha\beta}(\partial_\mu L_{\beta\nu} + \partial_\nu L_{\beta\mu} - \partial_\beta L_{\mu\nu}) - \frac{1}{2}\varepsilon L^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) \\ &= \Gamma^\alpha_{\mu\nu} + \frac{\varepsilon}{2}g^{\alpha\beta}(\partial_\mu L_{\beta\nu} + \partial_\nu L_{\beta\mu} - \partial_\beta L_{\mu\nu}) - \varepsilon L^{\alpha\beta} g_{\sigma\beta}\Gamma^\sigma_{\mu\nu}\end{aligned}$$

Now, using the fact that the Christoffel symbol of the base manifold is Torsion free, we arrive at:

$$\overline{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \frac{\varepsilon}{2}g^{\alpha\beta}(\nabla_\mu L_{\beta\nu} + \nabla_\nu L_{\beta\mu} - \nabla_\beta L_{\mu\nu}) \tag{B.3.3}$$

Moreover, the term containing the $\varepsilon L_{\mu\nu}$'s shall be renamed by now just for the sake of practicality since, later on when doing the actual computation we will naturally have to explicitly calculate every part of the above expression. Hence, by doing this we end up with:

[9] and thus, subject just to the force of gravity

$$\overline{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + \varepsilon\Omega^{\alpha}_{\mu\nu} \tag{B.3.4}$$

Note that since we are only interested in the first order correction of ε, every term involving ε^2 will be considered higher order and thus neglected. With this in hand we are now able to compute the Riemann curvature Tensor. That is:

$$\begin{aligned}\overline{R}^{\alpha}{}_{\mu\beta\nu} &= \partial_\beta\overline{\Gamma}^{\alpha}_{\mu\nu} - \partial_\nu\overline{\Gamma}^{\alpha}_{\mu\beta} + \overline{\Gamma}^{\delta}_{\mu\nu}\overline{\Gamma}^{\alpha}_{\delta\beta} - \overline{\Gamma}^{\delta}_{\mu\beta}\overline{\Gamma}^{\alpha}_{\delta\nu} \\ &= \partial_\beta(\Gamma^{\alpha}_{\mu\nu} + \varepsilon\Omega^{\alpha}_{\mu\nu}) - \partial_\nu(\Gamma^{\alpha}_{\mu\beta} + \varepsilon\Omega^{\alpha}_{\mu\beta}) + (\Gamma^{\delta}_{\mu\nu} + \varepsilon\Omega^{\delta}_{\mu\nu})(\Gamma^{\alpha}_{\delta\beta} + \varepsilon\Omega^{\alpha}_{\delta\beta}) \\ &- (\Gamma^{\delta}_{\mu\beta} + \varepsilon\Omega^{\delta}_{\mu\beta})(\Gamma^{\alpha}_{\delta\nu} + \varepsilon\Omega^{\alpha}_{\delta\nu})\end{aligned}$$

Which then becomes:

$$\overline{R}^{\alpha}{}_{\mu\beta\nu} = R^{\alpha}{}_{\mu\beta\nu} + \partial_\beta\varepsilon\Omega^{\alpha}_{\mu\nu} - \partial_\nu\varepsilon\Omega^{\alpha}_{\mu\beta} + \Gamma^{\delta}_{\mu\nu}\varepsilon\Omega^{\alpha}_{\delta\beta} + \Gamma^{\alpha}_{\delta\beta}\varepsilon\Omega^{\delta}_{\mu\nu} - \Gamma^{\delta}_{\mu\beta}\varepsilon\Omega^{\alpha}_{\delta\nu} - \Gamma^{\alpha}_{\delta\nu}\varepsilon\Omega^{\delta}_{\mu\beta} \tag{B.3.5}$$

Which in terms of the covariant derivatives of the base manifold coordinates can be rewritten as:

$$\overline{R}^{\alpha}{}_{\mu\beta\nu} = R^{\alpha}{}_{\mu\beta\nu} + \varepsilon(\nabla_\beta\Omega^{\alpha}_{\mu\nu} - \nabla_\nu\Omega^{\alpha}_{\mu\beta}) \tag{B.3.6}$$

Which, upon contraction of the indexes α and β yields the modified Ricci tensor, namely:

$$\overline{R}_{\mu\nu} = R_{\mu\nu} + \varepsilon(\nabla_\alpha\Omega^{\alpha}_{\mu\nu} - \nabla_\nu\Omega^{\alpha}_{\mu\alpha}) \tag{B.3.7}$$

Once again, after some convenient renaming of the terms involving $L_{\mu\nu}$ (which appear as factors of $\Omega^{\alpha}_{\mu\nu}$) we are left with:

$$\overline{R}_{\mu\nu} = R_{\mu\nu} + \varepsilon R^{\Omega}_{\mu\nu} \tag{B.3.8}$$

Now with the effective manifold metric $g_{\mu\nu}$ we are able to find the Ricci scalar

$$\overline{R} = \overline{g}^{\mu\nu}\overline{R}_{\mu\nu} = (g^{\mu\nu} - \varepsilon L^{\mu\nu})(R_{\mu\nu} + \varepsilon R^{\Omega}_{\mu\nu}) = R + \varepsilon(g^{\mu\nu}R^{\Omega}_{\mu\nu} - L^{\mu\nu}R_{\mu\nu})$$

$$\overline{R} = R + \varepsilon R^{\Omega} \tag{B.3.9}$$

Finally we are able to see how does the Einstein tensor change when $L_{\mu\nu}$ is taken into account. We are thus left with

$$\begin{aligned}
\overline{G}_{\mu\nu} &= \overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{R} = \overline{R}_{\mu\nu} - \frac{1}{2}(g_{\mu\nu} + \varepsilon L_{\mu\nu})(R + \varepsilon R^{\Omega}) \\
&= R_{\mu\nu} + \varepsilon R^{\Omega}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{\varepsilon}{2}g_{\mu\nu}R^{\Omega} - \frac{\varepsilon}{2}L_{\mu\nu}R \\
&= \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \varepsilon\left(R^{\Omega}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\Omega} - \frac{1}{2}L_{\mu\nu}R\right) \\
&\Rightarrow \overline{G}_{\mu\nu} = G_{\mu\nu} + \varepsilon\left(R^{\Omega}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\Omega} - \frac{1}{2}L_{\mu\nu}R\right)
\end{aligned} \tag{B.3.10}$$

Which, when written a bit more explicitly, assumes the following form:

$$\begin{aligned}
\overline{G}_{\mu\nu} = G_{\mu\nu} + \varepsilon\big(\nabla_\alpha \Omega^{\alpha}_{\mu\nu} - \nabla_\nu \Omega^{\alpha}_{\mu\alpha} - \\
\frac{1}{2}g_{\mu\nu}\left(\nabla_\alpha\nabla_\beta L^{\alpha\beta} - \frac{1}{2}\nabla^\alpha\nabla_\alpha L - L^{\alpha\beta}R_{\alpha\beta}\right) + RL_{\mu\nu}\Big)
\end{aligned} \tag{B.3.11}$$

Appendix C

Algebraic Structures

C.1 Fundamental concepts and definitions

The idea of this appendix is mostly to state the definitions needed for the thesis, as well as give some examples of those. We begin with the most basic one, namely

Definition C.1.1 (Group)**.** A **group** $(G, \cdot)$ is a double composed of a set G and an operation $\cdot$ that satisfies the following conditions:

GP.1 $g \cdot h \in G, \forall g, h \in G$ (Closedness Condition)

GP.2 There is an element $e \in G$ such that $\forall g \in G, e \cdot g = g \cdot e = g$ (Neutral Element)

GP.3 $\forall g \in G$ there is a $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$ (Inverse element)

More specifically, when the operation under which the set G is a group is addition, we call G an **additive group**. If its multiplication instead, we call it a **multiplicative group**.

Example C.1.1. The set of real numbers is a additive group. Indeed **GP.1** is satisfied because the addition of two real numbers is again a real number, by construction. The element $0 \in \mathbb{R}$ is the neutral element since $a + 0 = 0 + a = a, \forall a \in \mathbb{R}$. Moreover $\forall a \in \mathbb{R}$ there is an element $(-a) \in \mathbb{R}$ such that $a + (-a) = (-a) + a = 0$, so that conditions **GP.2** and **GP.3** are also satisfied. Hence $(\mathbb{R}, +)$ is and additive group.

Another much smaller example of an additive group is the set of integers $\mathbb{Z}$, or the set of rationals $\mathbb{Q}$. The set of natural numbers $\mathbb{N}$ however does **not** form an additive group due to failing conditions **GP.3**. Though, it comes close enough! Given that it satisfies the other two. Based on this, we have the following

Definition C.1.2 (Monoid)**.** A **monoid** is a double $(G, \odot)$ that satisfies conditions **GP.1** and **GP.2**.

That is, it is a set that is almost a group, except for the non-existence of inverse elements. For instance $(\mathbb{Z}, \cdot)$ is a multiplicative monoid (what's the inverse of 2 here?) and, as discussed above $(\mathbb{N}, +)$ is an additive one.

We can combine Definitions C.1.1 and C.1.2 into the following concept.

Definition C.1.3 (Ring)**.** A **ring** $(R, +, \cdot)$ is a triple such that R is a set, $(R, +)$ is a group and $(R, \cdot)$ is a monoid.

Definition C.1.4 (Field)**.** A **field** $(K, +, \cdot)$ is a set such that both $(K, +)$ and $(K, \cdot)$ are groups.

Moreover, the above operations are related by the following expression:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

For instance, by the above discussion we see that $(\mathbb{Z}, +, \cdot)$ is the ring of integers and $(\mathbb{Q}, +, \cdot)$ is the field of rational numbers (given that $(Q, \cdot)$ is also a group, as the reader might check!).

Now, for many purposes in Algebra we need the notion of a **closed field** to be able to fully access a certain result we are interested in proving. For this we have the following:

Definition C.1.5 (Algebraic Closedness)**.** A field K is **algebraically closed** if every polynomial over K has a root in K.

Although quite simple, for the sake of completeness we call a *polynomial in one variable* a function $p : K \to K$ that is well defined, i.e if $a = b$ in K then $p(a) = p(b)$ in K too. We usually denote it as $p \in K[x]$ with $K[x]$ being the **ring of functions** in one variable over K formally given by:

$$K[x] = \{a_0 + a_1 x + \cdots + a_n x^n | a_i \in K, n \geq 0\}$$

That is, it's the set of all polynomials in one variable with coefficients in K. To generalize this is quite simple. Just take the ring $K[x_1, \ldots, x_n]$ of polynomials in n variables over K defined on a similar fashion, in which case $p \in K[x_1, \ldots, x_n]$ is now a function $p : K^n \to K$ that is well defined.

C.2 Groups and matrices

Another very important type of groups that appear, be it in physics or math, are the so called *matrix groups*. As the name suggests, those are nothing more than groups whose elements are matrices and, in this short subsection we shall define some of them.

The first and perhaps most important such group for us is the set $\mathrm{GL}(n, F)$, namely *the group of invertible n by n matrices with entries in F* (where F is any field we like). We call $\mathrm{GL}(n, F)$ the *general linear group* and, indeed it is to be though of the set of linear transformations from F^n to itself.

Take now $F = \mathbb{R}$ and consider the vector space $\mathbb{R}^n$ with usual inner product $\langle \cdot, \cdot \rangle$ given by:

$$\langle v, w \rangle = \sum_{i=1}^{n} v^i w^i \tag{C.2.1}$$

Consider now a matrix $A \in \mathrm{GL}(n, \mathbb{R})$ satisfying the following property

$$\langle Av, Aw \rangle = \langle v, w \rangle , \quad \forall v, w \in \mathbb{R}^n \tag{C.2.2}$$

from which we have that $\left\langle v, A^T A w \right\rangle = \langle v, w \rangle$ (by the definition of transpose). Since this is meant to be valid for all vectors in $\mathbb{R}^n$ we have that

$$A^T A = I \tag{C.2.3}$$

Matrices that obey Eq.(C.2.3) (or Eq.(C.2.2) for that matter) are called *orthogonal* and, the set of orthogonal matrices, denoted as $O(n)$, is thus defined in the following manner:

$$O(n) := \{A \in \mathrm{GL}(n, \mathbb{R}) |\; A^T A = I\} \tag{C.2.4}$$

We can see that this set is indeed a group because, $I \in O(n)$ is the neutral element; given $A, B \in O(n)$ then $(AB)^T AB = B^T A^T AB = B^T B = I$, so that $AB \in O(n)$; $\forall A \in O(n)$ it follows that $(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = I^{-1} = I$[1].

The group $O(n)$ has two connected components, namely that formed by the matrices A such that $\det(A) = -1$ which we shall refer to as **reflection matrices**, and that formed by the matrices A such that $\det(A) = 1$ which we will call **rotation matrices**. The set of rotation matrices is more special in a sense because it contains the identity element in it. Not only this but, it is itself a subgroup of $O(n)$ (as the reader may check)! To this group we give the name $SO(n)$, i.e, the group of *special orthogonal matrices*, which can equivalently be defined as the connected component of the identity within the group $O(n)$.

We can make this same construction taking as base field F the set of complex number $\mathbb{C}$ with inner product given by

$$\langle v, w \rangle = \sum_{i=1}^{n} v^i \overline{w}^i \tag{C.2.5}$$

where the over-line represents complex conjugation. The complexified version of $O(n)$ is now denoted $U(n)$ and we call the *the group of unitary matrices* being defined as

$$U(n) := \{A \in \mathrm{GL}(n, \mathbb{C}) |\; A^\dagger A = I\}, \text{ where } A^\dagger = \overline{A^T} \tag{C.2.6}$$

inside which sits $SU(n)$, the *special unitary group* of matrices, defined in an analogous way to $SO(n)$ above.

[1] remember that the inverse of an invertible matrix is unique! So, if $A^T A = I$, then $AA^T = I$ as well.

C.3 Galois and Morales-Ramis Theories

C.3.1 Classical Galois Theory

Before jumping ahead to its differential counterpart, let's talk a bit about algebraic, or classical Galois Theory first. Its name comes from the French mathematician Évariste Galois (1811-1832) who famously ended up dying at young age in a dual, due to political reasons. Nonetheless, despite his short life, Galois managed to make an interesting connection between polynomial roots and groups (which weren't even strictly well defined in his time). In modern terms, the idea is that given a polynomial $p(x)$ of degree n over a field K (Definition C.1.4), we can associate to $p(x)$ something called the **Galois group** of p over K.

This group can usually be denoted as $Gal_K(p(x))$ or $Gal(p(x))$, when the field is clear. To be able to construct this group we need to define a few concepts first.

Definition C.3.1 (Field Extension)**.** Given two fields L and K, we say that L is a **field Extension** of K, expressed as $K \subset L$, when $L = K(\alpha_\lambda)$[2], for $\lambda \in \Lambda$, with Λ a (possibly infinite) set.

Some elementary examples are $\mathbb{R} \subset \mathbb{C}$ with $\mathbb{C} = \mathbb{R}(i)$ or even $\mathbb{R} = \mathbb{Q}(\Lambda)$ with Λ in this case being the set of irrational numbers. Based on these two examples, we call a field extension $K \subset L$ *finite* when $L = K(\alpha_1, \ldots, \alpha_n)$, and infinite otherwise. We can further taper the type of extensions we are dealing with in the following two ways:

Definition C.3.2 (Normal Extension)**.** We say that $K \subset L$ is a **normal extension** when L is the **splitting field** of some polynomial $p \in K[x]$.

To illustrate the above definition, consider the following

[2] see Appendix C for more details on this field

Example C.3.1. Let $K = \mathbb{Q}$ and $p(x) = x^2 - 2$. The set of number that satisfy $p(x) = 0$ is $\{\sqrt{2}, -\sqrt{2}\}$. Calling $\alpha = \sqrt{2}$, we have that the field extension $L = \mathbb{Q}(\alpha) \supset \mathbb{Q}$ is normal because over $L[x]$ it follows that $p(x) = (x - \alpha)(x + \alpha)$, i.e, the polynomial splits.

Though, not only does the polynomial $x^2 - 2$ split over $\mathbb{Q}(\sqrt{2})$. For instance, $(x^2 - 2)^n$ also does so for every $n \in \mathbb{N}$. To avoid this type of "polynomial misbehaviour" we have the following:

Definition C.3.3 (Separable Extension)**.** Consider the field extension $L = K(\alpha_1, \ldots, \alpha_n)$, with α_i the common roots of a polynomial $p \in K[x]$. We call $K \subset L$ a **separable extension** if $\gcd(p(x), p'(x)) = 1$, where $p'(x)$ is the derivative of p.

In particular, Example C.3.1 also illustrates a separable extension of the rationals given that $p'(x) = 2x$ in that case, whose greatest common divisor with $x^2 - 2$ clearly equals one.

More generally, a field extension $K \subset L$ that obeys Definitions C.3.2 and C.3.3 is called a **Galois extension**. We are now in position to state

Definition C.3.4 (Galois Group)**.** Consider the field extension $K \subset L$. The **Galois group** of $K \subset L$, written $Gal(L/K)$ is the group of automorphisms of L that fix K, i.e $Gal(L/K) = Aut_K(L)$.

By construction, we have that an element φ of the Galois group of a field extension, generated by the addition of a set of roots of a polynomial $p \in K[x]$ to K, has to send a root of p to a root of p because, if $p(x) = \sum_i a_i x^i$, for $a_i \in K$ and α_j is a root of $p(x)$ then:

$$0 = \varphi(0) = \varphi(p(\alpha_j)) = \sum_i a_i \varphi(\alpha_j)^i$$

so that $\varphi(\alpha_j)$ is a root of p too. Because of this fact, if $K \subset L$ is finite then so will it be $Gal(L/K)$. Hence, in the algebraic setting, the Galois group of a finite field field extension will necessarily be a subgroup of S_n, for some $n \in \mathbb{N}$.

To make things more concrete, let's look at Example C.3.1 with more attention now.

Example C.3.2. Consider again $K = \mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) = L$. We saw that this extension is Galois since L is obtained by the adjunction of the roots of $p(x) = x^2 - 2$ to $\mathbb{Q}$. Now, to compute its Galois group note that if $\varphi \in Gal(L/K)$, then either $\varphi(\sqrt{2}) = \sqrt{2}$, in which case $\varphi = Id$, or $\varphi(\sqrt{2}) = -\sqrt{2}$, so that $\varphi^2 = Id$, i.e φ is an element of order two.

By abuse of notation we can write $Gal(L/K) = \{Id, \varphi\} \simeq \mathbb{Z}_2$, where the group homomorphism is $\psi : Gal(L/K) \to \mathbb{Z}_2$ given by $\psi(Id) = 0, \psi(\varphi) = 1$ and $\psi(\varphi_1 \circ \varphi_2) = \psi(\varphi_1) + \psi(\varphi_2)$. On the same token, one shows that $\mathbb{Z}_2 \simeq S_2$ so that, as mentioned above $Gal(L/K) \simeq S_2$.

Another highly important concept to the theory is that of *solvability*. In the algebraic case, we are interested in being able to solve polynomial equations by radicals. This simply means that we want to solve equations by taking $n-$th roots, adding/subtracting, multiplying/dividing numbers.

More specifically, we have the following definitions.

Definition C.3.5 (Radical Extension)**.** A field extension $K \subset L$ is **radical** if $L = K(\alpha_1, \ldots, \alpha_n)$ and, defining $L_k = K(\alpha_1, \ldots, \alpha_k)$, for each α_k there is an $n_k \in \mathbb{Z}$ such that $\alpha_k^{n_k} \in L_{k-1}$.

Definition C.3.6 (Solvability by Radicals)**.** A polynomial $p(x) \in K[x]$ will be **solvable by radicals** if there exists a radical extension $K \subset L$ such that p splits over L.

Definition C.3.7 (Group Solvability)**.** Given a group G with identity e, we say that G is **solvable** if, there exists a normal chain of subgroups

$$G = G_n \trianglerighteq G_{n-1} \trianglerighteq \cdots \trianglerighteq G_1 \trianglerighteq G_0 = e$$

such that each quotient G_i/G_{i-1} is abelian.

It's not hard to convince oneself that every abelian group is solvable. Indeed just take $G_1 = G$ so that $G_1/G_0 = G/e = G$ satisfies the above criterion.

As the name suggests, it turns out that for $p \in K[x]$, $p(x)$ is solvable by radicals if, and only if, its Galois group $Gal(L/K)$ is solvable[3]. It also happens that the Galois group of a general quintic or higher order polynomial is non-solvable and thus, no general formula for a root of such polynomial (depending on its coefficients) can be found [Mor96]. To finish this section we invite the interested reader to have a look at other books on the subject such as the already mentioned [Mor96], but also [DF03, MT68, Ste04].

C.3.2 Differential Galois Theory

With the definitions of the previous section at hand, we can now jump into Differential Galois Theory. Our object of study will now be systems of *linear differential equations* that will be defined over a differential field (K, d) (which we'll simply denote by K), i.e, a field with a map $d : K \to K$ that is linear of satisfies Leibniz's rule.

Much like when studying calculus, we also encounter here numbers whose differential is zero. Based on this we construct

$$\mathcal{C}_K = \{a \in K \mid da = 0\}$$

the **field of constants** of K.

The motivation behind the development of the theory also has to do with the quest of determining the integrability of a given system. Indeed, the common usage of Galois Theory is to prove the *non-integrability* of a certain dynamical system [ABC$^+$20a].

To get to know this is done, let's start with the following: given a differential field K, we call α an *exponential element* over K if $d\alpha/\alpha \in K$. It is *integral* if $d\alpha \in K$ and *algebraic* if there is a polynomial $f(x) \in K[x]$ such that $f(\alpha) = 0$. With this in mind we have

[3] This can be proven by the Fundamental Theorem of Galois Theory, also dubbed Galois Correspondence. For further details check [Mor96]

Definition C.3.8 (Generalized Liouville Extension)**.** Consider the differential field extension $K \subset L$. It will be called a **generalized Liouville extension** if we can find a field chain

$$K = K_0 \subset K_1 \subset \cdots \subset K_{n-1} \subset K_n = L$$

such that $K_i = K_{i-1}(\alpha_i)$ with α_i being either **exponential**, **integral** or **algebraic** over the differential field K_{i-1}.

Definition C.3.9 (Differential Galois Group)**.** Given a differential field extension $K \subset L$, its **differential Galois group** $Gal(L/K) := Aut_K(L)$, i.e, it's the group of automorphisms of L that fix K.

Now, much like with its algebraic counterpart, our differential Galois group can be put into another very famous group, though this time of matricial character. To see why, note that if we are given a linear differential equation

$$\mathcal{L}(y) = a_n \frac{d^n y}{dx^n} + \cdots + \frac{dy}{dx} + a_0 y = 0, \tag{C.3.1}$$

and a set of n independent solutions $\{y_j\}$, the whole idea is that through the Galois group should send (in a short had notation) solutions of $\mathcal{L}$ to solutions of $\mathcal{L}$. With this in mind, since the operator is linear we will have

$$\sigma(y_i) = \sum_j a_{ij} y_j, \tag{C.3.2}$$

and so if we act on the "solution vector" $\boldsymbol{y}$ with σ, we will see something like

$$\sigma(\boldsymbol{y}) = \sigma\left(\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right) = \begin{pmatrix} \sigma(y_1) \\ \vdots \\ \sigma(y_n) \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = A_\sigma \boldsymbol{y}.$$

Now, given that by Definition C.3.9, σ is an automorphism, i.e a homomorphic bijection, its representing matrix A_σ shall also be a bijection from the extended field L to itself. Hence, A_σ is invertible and so, by definition $A_\sigma \in \mathrm{GL}(n, K)$. This computation let's us conclude that $\mathrm{Gal}(L/K) \leqslant \mathrm{GL}(n, K)$ if we are considering a degree n extension.

Since this inclusion is general we may end up with disconnected Galois groups (like $\mathrm{O}(n, K)$, for instance). Because of this we usually restrict ourselves to work with the part of the Galois group that contains the identity element. We thus call it the ***identity component of the Galois group*** and usually denote it by $(\mathrm{Gal}(L/K))^0$.

The next important concept we'd like to introduce is that of a **Picard-Vessiot extension**. The way it works is the following: much like in the algebraic case we wish to find an extension field L that contains all the solutions of Eq.(C.3.1) but, in this case, that also satisfies $\mathcal{C}_L = \mathcal{C}_K$, since we want our derivation to be proper, uniquely defined. We see that this field mimics the splitting field of the polynomial case. With this in mind we have

Definition C.3.10. Given a homogeneous ODE of degree n, $\mathcal{L}(y) = 0$ defined over a differential field K, a **Picard-Vessiot extension** $K \subset L$ is a differential field extension such that

PV.1 $L = K\langle y_1, \ldots, y_n \rangle$, with $\{y_j\}$ being a fundamental set of solutions Eq.(C.3.1).

PV.2 $\mathcal{C}_L = \mathcal{C}_K$

Have in mind that, in the notation of **PV.**1, we are leaving implicit the higher order derivatives of each solution. A proper, though more cumbersome notation for it would be $K(y_1, \ldots, y_n, y_1', \ldots, y_n', \ldots, y_1^{(n-1)}, \ldots, y_n^{(n-1)})$. Thus making this addition more explicit.

Notably, it is crucial to know whether the objects we are defining and working with actually make sense and/or even exist. To guarantee this we have following nice

Theorem C.3.1. *Consider a differential field K with algebraically close*[4] *field of constants $\mathcal{C}_K$ and a homogeneous ODE of order n, namely $\mathcal{L}(y) = 0$. Then, there exists a Picard-Vessiot extension $K \subset L$.*

[4]see Definition C.1.5

Proof: Check Theorem 5.6.5 of [CH11].

In essence, the reader should have in mind that Picard-Vessiot theory is the differential analog of classical Galois Theory for linear differential equations [AHJ19b].

Now, based on the generalized Liouville extension (Definition C.3.8), we have that a linear ordinary differential equations is ***Picard-Vessiot integrable*** (or, PV integrable) if we can find a PV extension of the base field whose solutions to the linear differential equation are of generalized Liouvillian type (i.e, exponential, integral or algebraic). With that in mind we can state the following important

Theorem C.3.2 (**Galois integrability - Linear case**)**.** *Given a linear ordinary differential equation* $\mathcal{L}(y)$*, it follows that* $\mathcal{L}(y)$ *is PV integrable if, and only if, the identity component of its Galois group is solvable.*

Proof: Check Chapter 6 of [BM96].

As a final comment, take note of the fact that the distinctions coming from Definition C.3.8 are extremely important, and depending on which one of these our element α ends up being, the differential Galois group of our extension will assume a different form (at least for second order linear ODEs). This fact is an important result due to Kovacic, on the computation of differential Galois groups of second order differential equations. So much so that we name such computational method *Kovacic's algorithm.* For more details, check Section 2 of [RR99].

C.3.3 Tastings of Morales-Ramis Theory

We now introduce a bit of the theory connecting Differential Galois to integrability. This connection is made manifest by considering the Galois group of a linear differential equation, coming from a first order approximation of a Hamiltonian dynamics over one of its solution curves.

Indeed, given a Hamiltonian H and its associate vector field X_H, whose solution curve we denote Γ and is parameteized by $\gamma(t) = (q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$, the linear system we are interested in, named the *first variational equation* (VE) over Γ, is defined by

$$\dot{\xi} = A(\gamma(t))\xi. \tag{C.3.3}$$

The way to arrive at Eq.(C.3.3) is quite simple: note first that the set of Hamilton's equations over the solution curve Γ assume the form

$$\dot{\gamma}(t) = J \left.\frac{\partial H}{\partial \gamma}\right|_{\gamma(t)}, \quad J = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0. \end{pmatrix} \tag{C.3.4}$$

Considering now the variational curve $\gamma'(t) = \gamma(t) + \xi(t)$, so that

$$\begin{aligned} \dot{\gamma}'(t) = \dot{\gamma}(t) + \dot{\xi}(t) = J \left.\frac{\partial H}{\partial \gamma}\right|_{\gamma'(t)} &= J \left.\frac{\partial H}{\partial \gamma}\right|_{\gamma(t)+\xi(t)} \\ = J \left.\frac{\partial H}{\partial \gamma}\right|_{\gamma(t)} &+ J\frac{\partial}{\partial \gamma}\left.\left(\frac{\partial H}{\partial \gamma}\right)\right|_{\gamma(t)} \xi(t) \\ = J \left.\frac{\partial H}{\partial \gamma}\right|_{\gamma(t)} &+ J\mathrm{Hess}(H)|_{\gamma(t)}\xi(t), \end{aligned}$$

from which, by Eq.(C.3.4), we recover Eq.(C.3.3) upon the identification

$$A(\gamma(t)) = J\mathrm{Hess}(H)|_{\gamma(t)}. \tag{C.3.5}$$

To announce the theorem by Morales-Ramis-Simó and others that characterizes differential Galois theory, we need the following definitions.

Definition C.3.11 (Holomorphic function)**.** A function $f : \mathcal{U} \subseteq \mathbb{C} \to \mathbb{C}$ is called ***holomorphic*** over $\mathcal{U}$ when it's complex differentiable at every point of $\mathcal{U}$. Equivalently, writing $f = u + iv$, we may say that f is holomorphic if, and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{C.3.6}$$

hold for every point $z_0 = (x_0, y_0) \in \mathcal{U}$.

Definition C.3.12. A function $f : \mathcal{U} \subseteq \mathbb{C} \to \mathbb{C}$ is ***meromorphic*** if f is holomorphic at almost every point of $\mathcal{U}$.

By "almost every point" we mean that, *except* by a set of points with measure zero, f is holomorphic over $\mathcal{U}$. For more details on complex analysis, the reader is invited to check [BG12].

We will refer to a *meromorphically integrable Hamiltonian system* as one that is Liouville integrable and whose first integrals are meromorphic functions over the phase space (seen as a complex manifold). With this, we are able to announce

Theorem C.3.3 (*Moralis-Ramis*)**.** *Assume that a Hamiltonian system is meromorphically integrable in a connected neighbourhood of a phase curve* Γ*. Then the identity component of the differential Galois group of the VEs along* Γ *is Abelian.*

Proof: See Theorem 3.1 of [Aud02] with [MP09, AHJ19b] as guiding references.

With the above theorem in mind, we could say that the essence behind differential Galois theory when applied to systems of linear differential equations is to be able to give a group theoretic relation between the analytical and algebraic simplicities of our system.

We end this chapter by saying that for further reference on Morales-Ramis and Galois Theories the reader is invited to check the already mentioned [BM96, RR99] as well as [MR, MRR01, BMRZMR16, DW21].

Appendix D

Other computations and results

Hamiltonian formulation for the equations of motion - Kepler Problem

Our goal here is to show that the Hamiltonian for the 3−vector potential A^μ is precisely the one given in Eq.(5.4). To show this, recall that the Lagrangian for a charged particle with mass m and charge q in an electromagnetic field is given by

$$\mathcal{L} = \frac{m}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + q g_{\mu\nu} A^\mu \dot{x}^\nu \tag{D.0.1}$$

From which the conjugate momentum to x^μ is found to be

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = m g_{\mu\nu} \dot{x}^\nu + q g_{\mu\nu} A^\nu. \tag{D.0.2}$$

We can then write $\dot{x}^\mu$ in terms of the p_μ by inverting the above relation, obtaining

$$\dot{x}^\mu = \frac{g^{\mu\nu}}{m} p_\nu - \frac{q}{m} A^\nu, \tag{D.0.3}$$

based on which, upon minor algebraic manipulations with the indices, the following equations are found

$$\frac{m}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{g^{\mu\nu}}{2m} (p_\mu p_\nu - q p_\mu A_\nu - q p_\mu A_\nu + q^2 A_\mu A_\nu), \tag{D.0.4a}$$

$$q g_{\mu\nu} A^\mu \dot{x}^\nu = \frac{q}{m} g^{\mu\nu} p_\mu A_\nu - \frac{q^2}{m} g^{\mu\beta} A_\mu A_\beta. \tag{D.0.4b}$$

Using the definition for the Hamiltonian together with Eqs.(D.0.4) we find that

$$
\begin{aligned}
H &= \dot{x}^\mu p_\mu - \mathcal{L} \\
&= \frac{g^{\mu\nu}}{m} p_\mu p_\nu - \frac{q}{m} g^{\mu\nu} p_\mu A_\nu - \frac{g^{\mu\nu}}{2m} p_\mu p_\nu + \frac{q}{m} g^{\mu\nu} p_\mu A_\nu - \frac{q^2}{2m} g^{\mu\nu} A_\mu A_\nu \\
&- \frac{q}{m} g^{\mu\nu} p_\mu A_\nu + \frac{q^2}{m} g^{\mu\nu} A_\mu A_\nu \\
&= \frac{g^{\mu\nu}}{2m} p_\mu p_\nu - \frac{q}{m} g^{\mu\nu} p_\mu A_\nu + \frac{q^2}{2m} g^{\mu\nu} A_\mu A_\nu = \frac{g^{\mu\nu}}{2m} (p_\mu - qA_\mu)(p_\nu - qA_\nu) \quad \square
\end{aligned}
$$

Classifying the braid integrability of the 3 vortex system on the cylinder with $\Gamma = 0$

With Definition 9.3.6 and Proposition 9.3.2 in our hands, we can much more easily inspect whether the braids from Figure 9.4 are BI or not. Regimes III through IX are trivially BI and so we won't write them down explicitly. In what follows, we show that all the other regimes, *except for Regime II*, are so too. Notice that $\overline{\sigma}_i \equiv \sigma_i^{-1}$.

- Regime I

  From now on, all the calculations will be done over $\mathcal{B}_n$.

$$
(\overline{\sigma}_1 \overline{\sigma}_2)^3 \overline{\sigma}_1^2 (\overline{\sigma}_2 \overline{\sigma}_1)^3 = (\overline{\sigma}_1 \overline{\sigma}_2 \overline{\sigma}_1 \overline{\sigma}_2 \overline{\sigma}_1 \overline{\sigma}_2)(\overline{\sigma}_2 \overline{\sigma}_1 \overline{\sigma}_2 \overline{\sigma}_1 \overline{\sigma}_2 \overline{\sigma}_1) = 1
$$

- Regime II

  First notice that we can write

$$
(\sigma_1 \sigma_2 \sigma_3)^2 = \sigma_1 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_3
$$

  So that, by making use of the fact that $\sigma_1 \sigma_3 = \sigma_3 \sigma_1$ and $\sigma_2 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 \sigma_3$, the following equalities hold

$$
\begin{aligned}
&\sigma_3 \sigma_2 (\sigma_1 \sigma_2 \sigma_3)^2 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1 (\sigma_1 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_3) \sigma_2 \sigma_1 \sigma_3 \sigma_1 \sigma_2 \sigma_1 \sigma_3 \\
&= \sigma_3 \sigma_2 (\sigma_1 \sigma_2 \sigma_3)^2 \sigma_3 \sigma_2 \sigma_3 \sigma_2 \sigma_3 \sigma_2 \sigma_3 \sigma_2 \sigma_1 \sigma_3 = \sigma_3 \sigma_2 (\sigma_1 \sigma_2 \sigma_3)^2 \sigma_3 \sigma_2 \sigma_1 \sigma_3 \\
&= \sigma_3 \sigma_2 (\sigma_1 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_3) \sigma_3 \sigma_2 \sigma_3 \sigma_1 = \sigma_3 \sigma_2 \sigma_1 \sigma_2
\end{aligned}
$$

  which is not equal to 1 on $\mathcal{B}_n$

- Regime X

$$(\sigma_3\sigma_2)^3 = (\sigma_3\sigma_2\sigma_3)(\sigma_2\sigma_3\sigma_2) = (\sigma_2\sigma_3\sigma_2)(\sigma_2\sigma_3\sigma_2) = 1$$

By the same reasoning, we have

$$(\sigma_i\sigma_{i+1})^3 = (\overline{\sigma}_i\overline{\sigma}_{i+1})^3 = 1 \tag{D.0.5}$$

for all $i = 1, \ldots, n-2$

- Regime XI

$$(\sigma_3\sigma_2)^2\sigma_3\sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1\sigma_2\sigma_3\sigma_1\sigma_3(\overline{\sigma}_1\overline{\sigma}_2)^3\overline{\sigma}_1 = (\sigma_3\sigma_2)^2\sigma_3\sigma_2\sigma_3\sigma_1\sigma_3\sigma_1$$
$$= (\sigma_3\sigma_2)^3 = 1$$

- Regime XII

$$(\sigma_3\sigma_2)^2\sigma_3\sigma_1\sigma_2\sigma_1^2(\sigma_2\sigma_3\sigma_1\sigma_3)^2(\overline{\sigma}_1\overline{\sigma}_2)^3\overline{\sigma}_1 = (\sigma_3\sigma_2)^2\sigma_3\sigma_1\sigma_2(\sigma_2\sigma_1)^2\overline{\sigma}_1$$
$$= (\sigma_3\sigma_2)^2\sigma_3\sigma_1\sigma_2(\sigma_2\sigma_1\sigma_2\sigma_1)\overline{\sigma}_1 = (\sigma_3\sigma_2)^3 = 1$$

- Regime XIII

$$(\overline{\sigma}_1\overline{\sigma}_2)^2\overline{\sigma}_3^2(\overline{\sigma}_2\overline{\sigma}_1)^5 = (\overline{\sigma}_1\overline{\sigma}_2)^2(\overline{\sigma}_2\overline{\sigma}_1)^3(\overline{\sigma}_2\overline{\sigma}_1)^2 = (\overline{\sigma}_1\overline{\sigma}_2)^2(\overline{\sigma}_2\overline{\sigma}_1)^2$$
$$= (\overline{\sigma}_1\overline{\sigma}_2\overline{\sigma}_1\overline{\sigma}_2)(\overline{\sigma}_2\overline{\sigma}_1\overline{\sigma}_2\overline{\sigma}_1) = 1$$

Some algebraic results for $\mathcal{B}_n$

Our goal here is to show prove the following

Lemma D.0.1. Let B_n be the Artin braid group and consider Δ_i^2 the full twists of order i each given by

$$\Delta_i^2 = (b_i \dots b_1)^2, \text{ where } b_j = \sigma_1 \dots \sigma_j \tag{D.0.6}$$

Then, the following group isomorphism holds

$$\mathcal{B}_n := B_2 / \left\langle \Delta_1^2 \right\rangle \times \dots \times B_n / \left\langle \Delta_{n-1}^2 \right\rangle \simeq \mathrm{S}_n \tag{D.0.7}$$

Proof: By Proposition 9.3.2, we know that we can write $\mathcal{B}_n$ in the following way

$$\mathcal{B}_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \middle| \sigma_i^2 = 1,\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2 \right\rangle \tag{D.0.8}$$

The above is actually a *Coxeter group*, since we can also say that

$$\mathcal{B}_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \middle| \sigma_i^2 = 1, (\sigma_i \sigma_{i+1})^3 = 1, (\sigma_i \sigma_j)^2 = 1 \text{ if } |i-j| \geq 2 \right\rangle \tag{D.0.9}$$

Such a Coxeter group is isomorphic to the Coxeter representation of the symmetric group S_n [RTV05], given by

$$S_n \simeq \left\langle s_1, \dots, s_{n-1} \middle| s_i^2 = 1, (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1 \text{ if } |i-j| \geq 2 \right\rangle \tag{D.0.10}$$

where one suitably has $s_i \to (i\, i+1)$. By taking the group isomorphism that maps $\sigma_i \to s_i$ we have the desired isomorphism between $\mathcal{B}_n$ and S_n. $\square$

Appendix E

Computations over $\mathbb{R} \times \Sigma$

We dedicate this section for the coordinate calculation of the field equations we used to construct the extended gravitational interaction and show that they produce in a (2+1)−dimension manifold, denoted here by $\mathbb{R}\times\Sigma$, equations similar to the 3−dimensional electromagnetic ones.

Recall that our gravitational field equations, named gravitational Maxwell equations (GMEs), were given in Eq.(1.9). From them, we concluded that $F = -dA + \alpha$ for some harmonic 2−form $\alpha \in \mathrm{H}^2(\mathbb{R}\times\Sigma)$ and $A \in \Lambda^1(R\times\Sigma)$, and hence that $\delta dA = \gamma j$. Expanding this last equation in coordinates will yield us:

$$\nabla_\nu \nabla^\nu A^\mu - \nabla_\nu \nabla^\mu A^\nu = \gamma j^\mu, \tag{E.0.1}$$

where $A^\mu = (\phi, A^1, A^2)$ and $j^\mu = (\rho, j^1, j^2)$. Here ϕ is the gravitational potential and ρ the matter distribution. The quantities A^i relate to the curl part of the gravitational field and j^i to the matter current. By putting $\mu = 0$ we get

$$\nabla_\nu \nabla^\nu \phi - \nabla_\nu \nabla^0 A^\nu = \gamma\rho, \tag{E.0.2}$$

whereas if we set $\mu = k$, we find that:

$$\nabla_\nu \nabla^\nu A^k - \nabla_\nu \nabla^k A^\nu = \gamma j^k. \tag{E.0.3}$$

We can further rewrite Eq.(E.0.2) as

$$\nabla_\nu (\nabla^\nu \phi - \nabla^0 A^\nu) = \gamma\rho. \tag{E.0.4}$$

Following what is done for test particles in electromagnetism [CA, RD22], we introduce the *gravitational field* as

$$\mathcal{E}^{\nu}_{grav} = -\nabla^{\nu}\phi + \nabla^0 A^{\nu}. \tag{E.0.5}$$

Observe also that $\mathcal{E}_{grav}$ is a purely spacial vector, i.e $\mathcal{E}^0_{grav} = 0$, by its very definition. This in turn leaves with the terms $\mathcal{E}^k_{grav}$ only which can be expressed as

$$\mathcal{E}^k_{grav} = -g^{kj}\nabla_j\phi + \nabla^0 A^k \tag{E.0.6}$$

From Eq.(E.0.2) we see that

$$\nabla_k \mathcal{E}^k_{grav} = -\gamma\rho, \tag{E.0.7}$$

which is itself the coordinate version of the equation $\text{div}(\mathcal{E}_{grav}) = -\gamma\rho$.

Observation E.1. Upon comparing with electromagnetism, we see that the only (and most important) difference between Eq.(E.0.7) and the first of Maxwell's equations, namely

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0},$$

is the sign difference on the right hand side. This illustrates the attractive character of gravity and further reassures us that the classical extension we proposed [Eq.(5.1)] is indeed reasonable.

Regarding Eq.(E.0.3), a similar reasoning can be applied to it. Indeed, by separately summing over the time and spacial components we see that

$$\gamma j^k = \nabla_0(\nabla^k\phi - \nabla^0 A^k) - \nabla_j(\nabla^j A^k - \nabla^k A^j), \tag{E.0.8}$$

where the first term is already known from above. To better grasp the second one though, notice the following: if we define a 2−tensor $H^{jk} = \nabla^j A^k - \nabla^k A^j$, then we see that $\delta_\Sigma H = \nabla_j H^{ij}$ is precisely the second term of Eq.(E.0.8). Though, since the *divergence of a tensor is the curl of its dual*[1], for $\delta_\Sigma H = \star_\Sigma^{-1} d_\Sigma \star_\Sigma H = \star_\Sigma^{-1} d_\Sigma(\star_\Sigma H)$, by thus setting

$$B := \star_\Sigma H = \sqrt{g_\Sigma}(\nabla^1 A^2 - \nabla^2 A^1) \tag{E.0.9}$$

[1] times ±1, depending on the degree of the tensor!

it follows that

$$\begin{aligned}\mathrm{curl}_\Sigma(B) := \star_\Sigma^{-1} d_\Sigma B &= \star_\Sigma^{-1} d_\Sigma \left[\frac{\sqrt{g_\Sigma}}{2}\left(\nabla^i A^j - \nabla^j A^i\right)\varepsilon_{ij}\right] \\ &= \star_\Sigma^{-1}\left[\frac{\sqrt{g_\Sigma}}{2}\nabla_k\left(\nabla^i A^j - \nabla^j A^i\right)\varepsilon_{ij} e^k\right] \\ &= -\frac{\varepsilon^{kl}\varepsilon_{ij}}{2}\nabla_k(\nabla^i A^j - \nabla^j A^i) e_l \\ &= (\nabla_k \nabla^l A^k - \nabla_k \nabla^k A^l) e_l,\end{aligned}$$

and hence, Eq.(E.0.8) can be rewritten as

$$-\nabla_0 \mathcal{E}^k_{grav} + \mathrm{curl}_\Sigma(B)^k = -\gamma j^k. \tag{E.0.10}$$

Observation E.2. Once again, a quick comparison with the second of Maxwell's field equations

$$-\frac{1}{c}\partial_t \mathbf{E} + \nabla \times \mathbf{B} = \mu_0 \mathbf{j},$$

revels to us that the major difference here lies in a sign change on the right hand side of the above equation, as expected.

Notice that the other Maxwell field equations also have a place in this framework. Indeed, if we consider the $\Sigma-$Hodge dual of the scalar B

$$\star_\Sigma B = \frac{\sqrt{g_\Sigma}}{2}\varepsilon_{ij} B e^i \wedge e^j, \tag{E.0.11}$$

since it now is a $2-$form on the 2 dimensional manifold Σ, its spacial exterior derivative should vanish identically, that is

$$d_\Sigma \star_\Sigma B = \frac{\sqrt{g_\Sigma}}{2}\varepsilon_{ij}\nabla_k B e^k \wedge e^i \wedge e^j \equiv 0. \tag{E.0.12}$$

Hence, by the definition of the co-differential we get that

$$\delta_\Sigma B := \star_\Sigma^{-1} d_\Sigma \star_\Sigma B = 0, \tag{E.0.13}$$

which in turn resembles the divergence free condition of the magnetic tensor in 3 dimensions

$$\nabla \cdot \mathbf{B} = 0. \tag{E.0.14}$$

The last of Maxwell's equations we need to get is

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{E.0.15}$$

By using the above formulae for $\star_\Sigma^{-1}, d_\Sigma$ and $\mathcal{E}_{\text{grav}}$, together with the definition for the curl, we have that

$$\star_\Sigma^{-1} d_\Sigma \mathcal{E}_{\text{grav}} = \frac{\varepsilon_{ij}}{2\sqrt{g_\Sigma}}([\nabla^k \nabla^j - \nabla^j \nabla^k] A^0 + \nabla^j \nabla^0 A^k - \nabla^k \nabla^0 A^j), \tag{E.0.16}$$

where we recall that $A^0 \equiv \phi$, is the scalar potential of the interaction. We can moreover use the properties and definition of the Riemann tensor to rewrite the above equation in terms of it (see subsection A.1.1). In doing so we get that

$$\star_\Sigma^{-1} d_\Sigma \mathcal{E}_{\text{grav}} = \frac{\varepsilon_{ij}}{2\sqrt{g_\Sigma}}(R_\mu{}^{0jk} + R_\mu{}^{k0j} + R_\mu{}^{jk0}) A^\mu - \frac{\varepsilon_{ij}}{2\sqrt{g_\Sigma}} \nabla^0 H^{kj}.$$

From here, using the definition of B we finally attain the following

$$\star_\Sigma^{-1} d_\Sigma \mathcal{E}_{\text{grav}} = \frac{-1}{\sqrt{g_\Sigma}} \nabla^0 \left(\frac{B}{\sqrt{g_\Sigma}} \right), \tag{E.0.17}$$

which will be equivalent to Eq.(E.0.15), provided that the determinant of the spacial part of the metric equals one.

Bibliography

[ABC$^+$20a] J. Andrade, S. Boatto, T. Combot, et al. N-body dynamics on an infinite cylinder: the topological signature in the dynamics. *Regul. Chaot. Dyn.*, 25:78–110, 2020.

[ABC$^+$20b] Jaime Andrade, Stefanella Boatto, Thierry Combot, Gladston Duarte, and Teresinha J Stuchi. N-body dynamics on an infinite cylinder: the topological signature in the dynamics. *Regular and Chaotic Dynamics*, 25:78–110, 2020.

[AdRJ21] Ruben Aldrovandi and Roldão da Rocha Jr. *A Gentle Introduction to Knots, Links and Braids*. World Scientific, 2021.

[AHJ19a] P. Acosta-Humánez and G. Jiménez. Some tastings in morales-ramis theory. *Journal of Physics: Conference Series*, 1414(1):012011, 2019.

[AHJ19b] P. Acosta-Humánez and G. Jiménez. "some tastings in morales-ramis theory.". *Journal of Physics: Conference Series*, 1414(1):012011, 2019.

[AKM65] Roy L. Adler, Alan G. Konheim, and M. Harry McAndrew. Topological entropy. *Transactions of the American Mathematical Society*, 114(2):309–319, 1965.

[Ale23] J. W. Alexander. A lemma on systems of knotted curves. *Proc. Nat. Acad. Sci. USA*, 9:93–95, 1923.

[Alf42] Hannes Alfvén. "existence of electromagnetic-hydrodynamic waves.". *Nature*, 3805:405–406, 1942.

[Aok89] N. Aoki. "chapter 15 topological dynamics". *Topics in General Topology*, pages 625–740, 1989.

[Arn89] V. I. Arnold. *Mathematical Method of Classical Mechanics*, volume I. Arnold, Springer, 2nd edition, 1989.

[Arn90] Vladimir I. Arnold. *Huygens and Barrow, Newton and Hooke: pioneers in mathematical analysis and catastrophe theory from evolvents to quasicrystals*. Springer Science & Business Media, 1990.

[AS96] H. Aref and M. A. Stremler. On the motion of three point vortices in a periodic strip. *Journal of Fluid Mechanics*, 314:1–25, 1996.

[Aud02] Michèle Audin. Intégrabilité et non-intégrabilité de systèmes hamiltoniens. *Séminaire Bourbaki*, 282:880–893, 2002.

[BC06] S. Boatto and D. Crowdy. Point-vortex dynamics. In Jean-Pierre Françoise, Gregory L. Naber, and Tsou Sheung Tsun, editors, *Encyclopedia of Mathematical Physics*, pages 66–79. Academic Press, Oxford, 2006.

[BCH$^+$09] A. J. Berrick, F. Cohen, E. Hanbury, Y. L. Wong, and J. Wu. *Braids: Introductory lectures on braids, configurations and their applications*, volume 19. World Scientific, 2009.

[BDS16] Stefanella Boatto, David G. Dritschel, and Rodrigo G. Schaefer. N-body dynamics on closed surfaces: The axioms of mechanics. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering SciencesVolume*, 472, 2016.

[Ber] Edmund Bertschinger. Hamiltonian formulation of general relativity (lecture notes). mit. edu/edbert/GR/gr11. pdf (2002).

[Bes87] A. L. Besse. *Einstein Manifolds.* Springer-Verlag, 1987.

[BF59] G. Birkhoff and J. Fisher. Do vortex sheets roll up? *Rend. Circ. Mat. Palermo*, 8:77–90, 1959.

[BG12] Carlos A. Berenstein and Roger Gay. *Complex variables: an introduction*, volume 125. Springer Science & Business Media, 2012.

[BG22] June Barrow-Green. "george birkhoff's forgotten manuscript and his programme for dynamics.". In *Proc. Int. Cong. Math, vol. 7*, pages 5748–5770, 2022.

[BH95] M. Bestvina and M. Handel. Train tracks for surface homeomorphisms, topology. 34:109–140, 1995.

[Bir74] J. S. Birman. *Braids, Links and Mapping Class Groups.* Ann. of Math. Studies 82 (Princeton Univ. Press, Princeton, 1974.

[BM96] Daniel Bertrand and Andy R. Magid. "lectures on differential galois theory.". *Bulletin of the American Mathematical Society*, 33(2):289, 1996.

[BMRZMR16] Alexey Bolsinov, Juan J. Morales-Ruiz, Nguyen Tien Zung, and Juan J. Morales-Ruiz. Integrable systems and dif ierential galois theory. *Geometry and Dynamics of Integrable Systems*, pages 1–33, 2016.

[BNPB12] Harsh Bhatia, Gregory Norgard, Valerio Pascucci, and Peer-Timo Bremer. The helmholtz-hodge decomposition—a survey. *IEEE Transactions on visualization and computer graphics*, 19(8):1386–1404, 2012.

[Boa08] S. Boatto. Curvature perturbation and stability of a ring of vortices. *Discrete and Continuous Dynamical Systems Series B*, 10:349–375, 2008.

[Bog77] V. A. Bogomolov. Dynamics of vorticity at a sphere. *Fluid Dynamics*, (6) pag.: 863 a 870, 1977.

[BP92] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Springer Science & Business Media, 1992.

[BS08] S. Boatto and C. Simó. Thomson's heptagon: A case of bifurcation at infinity. *Physica D*, 237:2051, 2008.

[BS19] S. Boatto and C. Simó. A vortex ring on a sphere: the case of total vorticity equal to zero. *Phil. Trans. R. Soc. A*, 377, 2019. Article 20190019.

[BSA00] P. Boyland, M. Stremler, and H. Aref. Topological fluid mechanics of stirring. *J. Fluid Mech.*, 403:277–304, 2000.

[BSA03] P. Boyland, M. Stremler, and H. Aref. Topological fluid mechanics of point vortex motions. *Physica D*, 175:69–95, 2003.

[Bud] Science Buddies. Moon effect on earth's sea.

[But06] Eugene Butikov. Precession and nutation of a gyroscope. *European journal of physics*, 27(5):1071, 2006.

[BZ85] G. Burde and H. Zieschang. *Knots*. de Gruyter Studies in Mathematics, Berlin and New York, 1985.

[CA] Luca Comisso and Felipe A. Asenjo. Magnetic reconnection as a mechanism for energy extraction from rotating black holes. *Phys. Rev. D*, 103:023014.

[Car] Steven Carlip. "lectures in (2+ 1)-dimensional gravity.". arXiv preprint gr-qc, (1995).

[Car14] Sean Carroll. *Space Time and Geometry: An introduction to General Relativity*. Pearson Education Limited, 2014.

[CC03] Steven Carlip and Steven Jonathan Carlip. *Quantum gravity in 2+ 1 dimensions*, volume 50. Cambridge University Press, 2003.

[CH11] T. Crespo and Z. Hajto. Algebraic groups and differential galois theory. *Graduate studies in Mathematics*, 122, 2011.

[Cha06] Isaac Chavel. *Riemannian Geometry: A modern Introduction*. University Pres, Cambridge, 2nd edition, 2006.

[Con19] John B. Conway. *A course in functional analysis*, volume 96. Springer, 2019.

[CRDB21] Gabriele Gradoni Cameron R. D. Bunney. Electromagnetism in curved spacetimes: Coupling of the doppler and gravitational redshifts. April 2021. Cameron R. D. Bunney, Gabriele Gradoni, [gr-qc].

[d'A20a] Loïc Poulain d'Andecy. Fusion for the yang–baxter equation and the braid group. *Winter Braids Lecture Notes*, 7:1–49, 2020.

[d'A20b] Loïc Poulain d'Andecy. "fusion for the yang–baxter equation and the braid group.". *Winter Braids Lecture Notes 7*, pages 1–49, 2020.

[DB15a] D. G. Dritschel and S. Boatto. The motion of point vortices on closed surfaces. *Proc. R. Soc. A*, 471, 2015. Article 20140890.

[DB15b] David Gerard Dritschel and Stefanella Boatto. The motion of point vortices on closed surfaces. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2176):20140890, 2015.

[dC14] Manfredo Perdigão do Carmo. *Geometria Diferencial de Curvas e Superfícies*. SBM, 2014.

[DF03] David S. Dummit and Richard M. Foote. *Abstract Algebra*. John Wiley & Sons Inc, 3th edition, 2003.

[Dir28] Paul Adrien Maurice Dirac. "the quantum theory of the electron.". *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 778:610–624, 1928.

[Dua16] G. Duarte. "a dinâmica de 2 corpos sobre a superfície de um cilindro infinito". Master's thesis, Universidade Federal do Rio de Janeiro]. Research Gate, 2016.

[DW21] Thomas Dreyfus and Jacques-Arthur Weil. "differential galois theory and integration.". *Anti-Differentiation and the Calculation of Feynman Amplitudes*, pages 145–171, 2021.

[EMM12] George FR Ellis, Roy Maartens, and Malcolm AH MacCallum. *Relativistic cosmology*. Cambridge University Press, 2012.

[EvE99] G. F. R. Ellis and H. van Elst. Cosmological models: Cargese lectures 1998, nato sci. *Ser. C*, 541:1–116, 1999. [gr-qc]].

[FPS15] Masoud Faraki, Maziar Palhang, and Conrad Sanderson. Log-euclidean bag of words for human action recognition. *IET Computer Vision*, 9(3):331–339, 2015.

[Gar69] Frank A Garside. The braid group and other groups. *The Quarterly Journal of Mathematics*, 20(1):235–254, 1969.

[Gem00] Gianluca Gemelli. Second-order covariant tensor decomposition in curved spacetime. *Mathematical Physics, Analysis and Geometry*, 3:195–216, 2000.

[GH55] W. H. Gottschalk and G. A. Hedlund. "topological dynamics.". *American Mathematical Society Colloquium Publications*, 36, 1955.

[GHL04] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry*, volume 3 of *Univesitext*. Springer, Berlin, 2004.

[Gil16] Adrian E. Gill. *Atmosphere—ocean dynamics*. Elsevier, 2016.

[Gio22] Antonio Giorgilli. Notes on hamiltonian dynamical systems. 102, 2022.

[GM11] Juan Gonzáles-Menezes. *Basic results on Braid Groups: Tutorial on the Braid Group*. Juan Gonzáles-Menezes, Annales Mathématiques, 2011.

[GW06] Eli Glasner and Benjamin Weiss. *"On the interplay between measurable and topological dynamics."*. In Handbook of dynamical systems, vol. 1. Elsevier, pp. 597-648. Science, 2006.

[Har21] James B. Hartle. *Gravity: an introduction to Einstein's general relativity*. Cambridge University Press, 2021.

[HE23] Stephen W Hawking and George FR Ellis. *The large scale structure of space-time*. Cambridge university press, 2023.

[Hea56] Thomas L. Heath. *The Thirteen Books of Euclid's Elements*. Dover Publications, Inc, 1956.

[HH52] Alan L. Hodgkin and Andrew F. Huxley. A quantitative description of membrane current and its application to conduction and excitation in nerve. *The Journal of physiology*, 117(4):500, 1952.

[Hub29] Edwin Hubble. "a relation between distance and radial velocity among extra-galactic nebulae.". *Proceedings of the national academy of sciences*, 15(3):168–173, 1929.

[IKMS23] Hitoshi Ikemori, Shinsaku Kitakado, Yoshimitsu Matsui, and Toshiro Sato. "runge–lenz vector as a 3d projection of so(4) moment map in phase space.". *Journal of Physics A: Mathematical and Theoretical*, 56(22):225204, 2023.

[JPS12] Luisa G. Jaime, Leonardo Patino, and Marcelo Salgado. f (r) cosmology revisited. arXiv preprint, 2012.

[Kim99] Y. Kimura. Vortex motion on surfaces with constant curvature. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 455:245–259, 1999.

[Kli95] Wilhelm P. A. Klingenberg. *Riemannian Geometry*. De Gruyter, Berlin, New York, 1995.

[KM19] Arnab Kayal and Bhakti Bhusan Manna. *Sobolev Space On Riemannian Manifolds*. PhD thesis, Indian Institute of Technology Hyderabad, 2019.

[KS69] Thomas R. Kane and M. P. Scher. A dynamical explanation of the falling cat phenomenon. *International journal of solids and structures*, 5(7):663–670, 1969.

[KS04] L. H. Kauffman and A. Sossinsky. Knots: Mathematics with a twist. *The American Mathematical Monthly*, 111(9):830, 2004.

[Lam09] Sophia Lambropoulou. Knots and braids. *Advanced School on Knot Theory and its Applications to Physics and Biology, ICTP*, pages 11–22, 2009.

[Lam15] Jules Lamers. A pedagogical introduction to quantum integrability, with a view towards theoretical high-energy physics. arXiv preprint, 2015.

[LC69] R. S. Lindzen and S. Chapman. Atmospheric tides. *Space Science Reviews*, 10(1):3–188, 1969.

[Lee13] John M. Lee. *Introduction to smooth manifolds.* Springer, 2013.

[Ler73] David E. Lerner. The space of lorentz metrics. *Communications in mathematical Physics*, 32:19–38, 1973.

[Mag08] Michele Maggiore. Gravitational waves. *Oxford University Press*, 2008.

[Mar35] A. A. Markov. Über die freie Äquivalenz geschlossener zöpfe. *Recueil Mathematique Moscou*, 43(1936):73–78, 1935.

[MB98] Roy Maartens and Bruce A. Bassett. Class. *Quantum Grav.*, 15, 1998. [gr-qc]].

[Moo] T. K. Subrahmonian Moothathu. *Lectures notes on Topological Dynamics.* University of Hyderabad, School of Mathematics and Statistics.

[Mor96] Patric Morandi. *Field and Galois Theory.* Springer, 1996.

[MP09] Andrzej J. Maciejewski and Maria Przybylska. "differential galois theory and integrability.". *International Journal of Geometric Methods in Modern Physics*, 08:1357–1390, 2009.

[MPS24] A.-M. Muscaş, D. Paşca, and C. Stoica. "the restricted three-body problem with logarithm potential.". *New Astronomy*, 105:102086, 2024.

[MR] Juan J. Morales-Ruiz. "meromorphic non-integrability of hamiltonian systems i: Theory.". In *this proceedings.*

[MRR01] Juan J. Morales-Ruiz and Jean Pierre Ramis. "galoisian obstructions to integrability of hamiltonian systems.". *Methods and Applications of Analysis*, 8(1):33–96, 2001.

[MST03] James Montaldi, Anik Souliere, and Tadashi Tokieda. Vortex dynamics on a cylinder. *SIAM Journal on Applied Dynamical Systems*, 2(3):417–430, 2003.

[MT68] L. M. Milne-Thomson. *Theoretical Hydrodynamics.* Dover Publications, 5th edition, 1968.

[MT13] J. Montaldi and T. Tokieda. *Deformation of geometry and bifurcations of vortex rings, in Recent Trends in Dynamical Systems.* 335 a 370, vol. **35** of Proceedings in Mathematics and Statistics, Springer, pag, 2013.

[MW21] Dan Margalit and Rebecca R. Winarski. "braid groups and mapping class groups: The birman–hilden theory.". *Bulletin of the London Mathematical Society*, 3:643–659, 2021.

[NAS] NASA. Nasa science - tides.

[NH94] Aoki Nobuo and Kōichi Hiraide. "topological theory of dynamical systems: recent advances.". 1994.

[Nie44] Jakob Nielsen. Surface transformation classes of algebraically finite type. *Mat.-Fys. Medd. Danske Vid. Selsk*, 21:3–89, 1944.

[Nä22] Joonas Nättilä. Runko: Modern multiphysics toolbox for plasma simulations. *Astronomy & Astrophysics*, 664:A68, 2022.

[O'n83] Barrett O'neill. *Semi-Riemannian geometry with applications to relativity.* Academic press, 1983.

[Ped13] Joseph Pedlosky. *Geophysical fluid dynamics.* Springer Science & Business Media, 2013.

[Phi90] Rafe Mazzeo Ralph S. Phillips. Hodge theory on hyperbolic manifolds. *Duke Mathematical Journal*, 60(2), 1990.

[Poi04] Eric Poisson. *A relativist's toolkit: The mathematics of Black Hole mechanics.* Cambridge University Press, 2004.

[PPV11] Eric Poisson, Adam Pound, and Ian Vega. The motion of point particles in curved spacetime. *Living Reviews in Relativity*, 14:1–190, 2011.

[QD90] Wang Qiu-Dong. The global solution of the n-body problem. *Celestial Mechanics and Dynamical Astronomy*, 50:73–88, 1990.

[RD22] James Vickers Ray D'Inverno. *Introducing Einstein's Relativity: A deeper understanding.* Oxford University Press, 2nd edition, 2022.

[Ros97] Steven Rosenberg. *The Laplacian on a Riemannian manifold - An introduction to analysis on manifolds.* Cambridge University Press, 1997.

[RR99] Juan José Morales Ruiz and Juan J. Morales Ruiz. *Differential Galois theory and non-integrability of Hamiltonian systems.* Birkhäuser, Basel, 1999.

[RTV05] Louis Rowen, Mina Teicher, and Uzi Vishne. Coxeter covers of the symmetric groups. *Journal of Group Theory*, 8(2):139–169, 2005.

[SB25] A. Fuzfa V. Pessanha R. G. Schaeffer Arturo Vieiro S. Boatto Luca Comisso ˙ Generalization of newtonian gravity to surfaces of oscillating metrics. 2025.

[SBE06] José M. M. Senovilla S. Brian Edgar. A weighted de rham operator acting on arbitrary tensor fields and their local potentials. *Journal of Geometry and Physics*, 56(10):2135–2162, 2006. ISSN 0393-0440.

[Sch14a] Rodrigo Schaefer. *A dinâmica de N corpos sobre superfcies: um exemplo para 2 corpos*. PhD thesis, Universidade Federal do Rio de Janeiro, 2014.

[Sch14b] Rodrigo Schaeffer. A dinâmica de n corpos em superfícies: Um exemplo para 2 corpos. 2014.

[SF10] Thomas P. Sotiriou and Valerio Faraoni. f (r) theories of gravity. *Reviews of Modern Physics*, 82(1):451, 2010.

[Ste04] Ian N. Stewart. *Galois Theory*. Chapman & Hall/CRC, 3rd edition, 2004.

[Thu88] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bulletin of the American mathematical society*, 19(2):417–431, 1988.

[Ton] David Tong. *Lecture notes on General Relativity*. Cambridge University Press.

[Tor16] Alessandro Torrielli. Lectures on classical integrability. 7 2016.

[Van06] Jack Vanderlinde. *Classical electromagnetic theory*, volume 145. Springer Science & Business Media, 2006.

[Via11] Jeff A. Viaclovsky. *Lecture notes: Topics in Riemannian Geometry*. Fall, 2011.

[Voi02] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry*. Cambridge University Press, 2002.

[Wal12] Stefan Waldman. Lecture notes: Geometric wave equations. 8 2012.

[War83] Frank W. Warner. Foundations of differentiable manifolds and lie groups. *Graduate Texts in Mathematics*, 94, 1983.

[Whi19] Marshall Whittlesey. *Spherical geometry and its applications*. Chapman and Hall, /CRC, 2019.

[Woo91] N. M. J. Woodhouse. *Geometric Quantization*. Clarendon Press ·, Oxford, 2nd edition, 1991.