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Some Properties of ASH Attractors

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Tese de doutorado apresentada ao Programa de Pós-graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Matemática

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"No importa lo despacio que vayas, siempre y cuando no te detengas."
— Confucio

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Abstract

This text refers to a doctoral thesis in dynamical systems, which focuses on continuous-time dynamics. More specifically, it advances the theory of asymptotically sectional-hyperbolic (ASH) flows through several fundamental results. First, we prove that every star ASH attractor of a C^1 vector field with positive topological entropy is necessarily sectional-hyperbolic. Second, we establish that all ASH attractors satisfy the intermediate entropy property. Third, we demonstrate that any ASH attractor in three-dimensional vector fields is entropy-expansive and admits periodic orbits. Finally, we provide a lower bound for the growth rate of periodic orbits in an ASH attractor.

Keywords: Asymptotically sectional-hyperbolicity, seccional hyperbolicity, entropy expansiveness, growth rate of periodic orbits.

Resumo

Este texto se refere a uma tese de doutorado em sistemas dinâmicos, que trata da dinâmica contínua, mais especificamente, avançamos na teoria dos fluxos assimptoticamente seccional-hiperbólicos (ASH) por meio de vários resultados fundamentais. Primeiro, provamos que todo atrator ASH estrela de um campo vetorial C^1 com entropia topológica positiva é necessariamente seccional-hiperbólico. Segundo, estabelecemos que todos os atratores ASH satisfazem a propriedade da entropia intermediária. Terceiro, demonstramos que qualquer atrator ASH em campos vetoriais tridimensionais é entropia-expansivo e admite órbitas periódicas. Por fim, fornecemos um limite inferior para a taxa de crescimento das órbitas periódicas em um atrator ASH.

Palavras-chave: Assintoticamente seccional-hiperbólico, seccional hiperbolco, entropia expansivo, taxa de crescimento de órbitas periódicas.

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1 Introduction

The study of the dynamics of flows, also known as continuous-time dynamics, from both topological and statistical viewpoints, is a significant area of research in mathematics. This field traces its origins back to the work of Poincaré, who utilized these concepts to deepen the understanding of the topology of underlying manifolds.

Over time, the study of continuous-time dynamics evolved into a distinct mathematical discipline. Following the groundbreaking contributions of Anosov and Smale, it emerged as a particularly fruitful area of research. Specifically, the discovery of the hyperbolic nature of flows through Smale's horseshoe [46] as a source of stability, along with Anosov's work on the hyperbolicity of the geodesic flow on negatively curved Riemannian manifolds [1], became celebrated cornerstones of the theory. These developments introduced techniques from both differential topology and ergodic theory, greatly advancing the understanding of the dynamics of a broad and important class of flows.

Subsequently, concepts from information theory and statistical physics, introduced by Kolmogorov, Sinai, Ruelle, Bowen, and others, were incorporated to better understand the complexity of dynamics and the relevant invariant measures of the system, particularly those that maximize entropy and, by the variational principle, achieve topological entropy. This integration led to a fruitful symbiosis between topological dynamics and the ergodic theory of dynamical systems.

One particular source of chaos and positive entropy is the presence of horseshoes. The pursuit of identifying horseshoe-type subdynamics within more general flows became a highly active area of research ([27], [18], [19]). A more challenging question also emerged, which can be termed the *flexibility of entropy*: given any value between zero and the topological entropy, can we find a compact subset (or an ergodic measure) whose entropy matches that value? One of the aims of this work is to address this question for certain flows that present new challenges due to the presence of singularities, as will be discussed in the following chapters.

Although hyperbolic theory is very powerful, it requires that the subspace generated by the vector field is continuous, which implies, among other things, that its dimension must be locally constant. Consequently, no singularity can be approached by regular orbits within the hyperbolic set. In [26], Lorenz discovered a robust attractor (with a dense orbit) that exhibits some properties resembling those of hyperbolic systems, yet includes singularities that are accumulated by regular orbits within the attractor. Inspired by this model, Guckenheimer, Shilnikov and Turaev in an independently way ([20] and [48]) introduced a geometric example that resembles the system studied by Lorenz. Therefore, to develop a comprehensive program for understanding most dynamical systems, it is essential to analyze such open sets.

The search for such a program motivated the search for a systematic theory to describe such dynamics, which was established by Morales and Metzger [29] under the name *sectional hyperbolicity* (see chapter 2). In their work, they proved that the geometric Lorenz attractor is sectional-hyperbolic. Moreover, in a seminal work by Morales, Pacifico, and Pujals [33], it was shown that any robust attractor in a 3-dimensional manifold must be sectional-hyperbolic. In fact, Tucker [47] later proved that the attractor obtained from original Lorenz equations is, in fact, sectional-hyperbolic.

One might assume that sectional-hyperbolic systems share similar properties with hyperbolic ones. While this is sometimes true, it is important to exercise caution. For instance, the problem of finding a periodic orbit can yield different outcomes: any isolated nontrivial hyperbolic set must have a periodic orbit, but there are isolated nontrivial sectional-hyperbolic sets without periodic orbits [30]. We will revisit this issue later. Another example is the following: Every isolated transitive hyperbolic set is robustly transitive, but this is not true in the sectional hyperbolic theory anymore

In his PhD thesis, Rovella constructed another flow, similar to the Lorenz attractor, but with a singularity exhibiting different behavior [39]. He was able to find examples of attractors that, unlike the Lorenz attractor, do not exhibit robustness. However, inspired by several works on the quadratic family, such as those by Benedicks-Carleson [12] and Jakobson [21], as well as on homoclinic bifurcations (see also the work of Palis and Yoccoz [37]), he proved that such attractors persist in a certain sense, likewise in a codimension 2 submanifold. Once again, understanding such examples is crucial for a comprehensive understanding of dynamical systems.

It turns out that the Rovella attractor fits within another theory with a hyperbolic flavor: the *asymptotic sectional-hyperbolic dynamics*, introduced by [34]. In fact, in [41] the authors shown that the Rovella attractor is an asymptotically sectional-hyperbolic attractor. Another example of sets satisfying this weak kind of hyperbolicity is known as the contractive singular horseshoe [34]. It should be noted that these examples are not sectional-hyperbolic.

In this work we continue the study of such dynamics. Let us now precise its definition. Let M be a compact Riemannian manifold endowed with metric d, induced by the Riemannian metric $\|\cdot\|$. We denote by X to a C^1 vector field on M, and we will refer by its *flow* on M to the family of maps $\Phi = \{X_t\}_{t\in\mathbb{R}}$, induced by X. A compact subset Λ of M is called X-invariant if $X_t(\Lambda) = \Lambda$, for every $t \in \mathbb{R}$.

Next, recall that a compact invariant set Λ has a *dominated splitting* if there are a continuous invariant splitting $T_\Lambda M = E \oplus E^c$ (respect to DX_t) and constants $K, \lambda > 0$

satisfying

$$\frac{\|DX_t(x)|_{E_x}\|}{m(DX_t(x)|_{E^c_x})} \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \forall t > 0,$$

where $m(A)$ denotes the conorm of A. In this case, we say that E^c is *dominated* by E. We say that Λ is *partially hyperbolic* if E is a contracting subbundle , i.e.,

$$\|DX_t(x)|_{E_x}\| \leq Ke^{-\lambda t},$$

for every $t > 0$ and $x \in \Lambda$. Finally, denote by $W^s(Sing(X))$ the union of the stable manifolds of the singularities of X.

Definition 1.0.1. Let Λ be a compact invariant partially hyperbolic set of a vector field X. We say that Λ is *asymptotically sectional-hyperbolic* (ASH for short) if the singularities of Λ are hyperbolic and its central subbundle is eventually asymptotically expanding outside the stable manifolds of the singularities, i.e., there exists $C > 0$ such that

$$\limsup_{t\to+\infty} \frac{\log|\det(DX_t(x)|_{L_x})|}{t} \geq C, \tag{1.1}$$

for every $x \in \Lambda' = \Lambda \setminus W^s(Sing(X))$ and every two-dimensional subspace L_x of E^c_x. We say that an ASH set is *non-trivial* if it is not reduced to a singularity.

Remark 1.0.1. ASH sets satisfy the *Hyperbolic lemma* i.e., any compact and invariant set without singularities is hyperbolic. The proof of this result can be found in [41].

It can be easily seen from the definition that the sectional-hyperbolic theory is encompassed within asymptotically sectional-hyperbolic theory, which in turn contains the hyperbolic theory. Moreover, the Rovella's attractor and the contracting singular horseshoes show that those inclusions are proper. An important consideration regarding the difference between sectional hyperbolicity and asymptotic sectional hyperbolicity is that in the case of sectional-hyperbolic dynamics, uniform estimates are often obtainable (which hold for nearby vector fields). However, in the asymptotic scenario, we must exercise more caution, as in the case of the Rovella attractor, where uniform estimates are not expected. Indeed, from ASH property we see that for every point $x \in \Lambda$ outside $W^s(Sing(X))$, and every plane $L_x \in G(2, F)$ (the Grasmannian of two-planes contained in the subbundle F) there is an unbounded increasing sequence of positive numbers $t_k = t_k(x, L_x) > 0$, called *hyperbolic times*, such that

$$|\det DX_{t_k}(x)|_{L_x}| \geq e^{Ct_k}, \quad k \geq 1. \tag{1.2}$$

Any unbounded increasing sequence satisfying the relation (1.2) will be called a sequence *C-hyperbolic times* for x.

A vector field X is *star* if there is a C^1-neighborhood of X formed by vector fields whose all singularities and periodic orbits are hyperbolic. Star vector fields form a key concept for dealing with global dynamics. They were introduced by Liao and Mañé, who showed that if the dynamics cannot bifurcate through non-hyperbolic periodic orbits, then robustly, all periodic orbits must exhibit uniform hyperbolic strength up to their period, along with a dominated splitting arising from the union of their hyperbolic splittings (see [25],[28]). Morales and Pacifico (in dimension 3) and Shi, Gan and Wen (in dimension 4) proved that, generically, the star property is equivalent to sectional hyperbolicity (see [31] and [44]). However, in higher dimensions, this equivalence does not hold, as exemplified by the work of Bonatti and Da Luz [13]. Thus, one can ask when the ASH theory diverges from the star theory. Our first result shows that, in any dimension, when restricted to asymptotic sectional-hyperbolic dynamics with positive topological entropy, the star property is equivalent to sectional hyperbolicity.

Theorem A. Every asymptotically sectional-hyperbolic attractor associated with C^1 vector field X on M, having positive topological entropy and satisfying the star property is sectional-hyperbolic.

The crucial step in proving Theorem A is the existence of a periodic orbit contained in the attractor Λ. Nevertheless, by our next results, one can obtain periodic orbits. Moreover, as we will see later in Theorem D, this hypothesis is satisfied in the three-dimensional scenario.

Next, we delve into the entropy theory of ASH flows. More precisely, we address their entropy flexibility. Here, we denote by $h_{top}(X)$ and $h_\mu(X)$ the topological and metric entropies of X, respectively (see chapter 2 for precise definitions). We say that a subset Λ and vector field X has *intermediate entropy property* if for every $h \in [0, h_{top}(X\mid_\Lambda))$ there exists an ergodic measure μ supported on Λ such that $h = h_\mu(X)$.

In our next result, we address the intermediate entropy property of ASH flows.

Theorem B. Let X be a C^1-vector field on M. Suppose M contains an asymptotically sectional-hyperbolic attractor Λ for X. Then Λ has intermediate entropy property.

As mentioned earlier, due to the variational principle, it is natural to inquire whether a measure of maximal entropy exists. Bowen, in [15], introduced a property called *entropy-expansiveness* to guarantee the upper semi-continuity of the entropy map, thus ensuring the existence of measures with maximal entropy. In our next result, we prove this property for ASH attractors in dimension three.

Theorem C. Every asymptotically sectional-hyperbolic attractor Λ associated with a C^1 vector field X on a three-dimensional manifold M is entropy-expansive.

Another natural question concerns the positivity of entropy. As discussed earlier, this is related to the existence of horseshoe-like subdynamics, particularly the presence of periodic orbits. As showed by [11], it is known that any attracting[1] sectional-hyperbolic set has a periodic orbit. However, there are attracting ASH sets without periodic orbits [41]. Thus, the issue of the existence of periodic orbits remains a subtle one. Our next result addresses this problem in dimension three.

Theorem D. Any asymptotically sectional-hyperbolic attractor Λ associated to three-dimensional vector fields X of class C^1 has a periodic orbit. Actually it contains a nontrivial homoclinic class. Thus its topological entropy is positive. If the periodic orbits are dense on Λ, then it is a homoclinic class.

We suspect that the attractor ASH is generally a homoclinic class. Indeed, this holds true in the sectional-hyperbolic setting, as shown by Arroyo and Pujals [8]. However, if we consider higher regularity, this result extends to any dimension.

Theorem E. If a C^1-vector field X on M contains a asymptotically sectional-hyperbolic attractor with positive topological entropy, then M contains a non-trivial homoclinic class.

All the Theorems C, D and E have consequences about the entropy of the attractor under perturbations.

Corollary F. Let X be a C^1 vector field on M, and let Λ be an asymptotically sectional-hyperbolic attractor. Then, there is a neighborhood U of Λ and a C^1 neighborhood $\mathcal{U}$ of X such that $X|_\Lambda$ is a point of lower semicontinuity for the entropy function on

$$\mathcal{X}^1(M,U) = \left\{ Y|_{\Lambda_Y} : \Lambda_Y = \bigcap_{t\geq 0} Y_t(U) \right\}.$$

In addition, if M is three-dimensional, then $X|_\Lambda$ is a point of continuity for the entropy function on $\mathcal{X}^1(M,U)$.

In our final main result, we put the topological and measure theoretical viewpoint together to relate the entropy with the periodic orbits of a ASH flow. This result is based on the work of Bowen in the begining of seventies [15], where its shown that, for axiom A systems, there is a relationship between the topological entropy and the growth rate of the periodic orbits. Namely, he proved that if f is an Axiom A diffeomorphism, then

$$\limsup_{n\to\infty} \frac{1}{n} \log(\#P_n(f)) = h_{top}(f),$$

where $\#P_n(f)$ denotes the number of periodic orbits for f of period n. Later, Katok in [22] proved for $C^{1+\alpha}$ surface diffeomorphisms that the topological entropy is a lower bound

[1] A set Λ is attracting if it has a neighborhood so that every point in the neighborhood eventually enters and remains within the set under the dynamics, i.e., $\Lambda = \bigcup_{t>0} X_t(U)$ for some open set satisfying $X_1(\overline{U}) \subset U$.

for the growth rate of its periodic orbits. In [50] this result was extended to generic C^1 vector fields X. In this work we obtain this result for ASH attractors associated to C^1 vector fields X. More precisely, if $P_t(X\mid_\Lambda)$ is the set of periodic orbits contained in Λ with period less than or equal to t, we have the following theorem:

Theorem G. Let X be a C^1 vector field on M. Suppose M contains an asymptotically sectional-hyperbolic attractor Λ for X. Then,

$$\limsup_{t\to\infty} \frac{1}{t} \log \#P_t(X|_\Lambda) \geq h_{top}(X|_\Lambda).$$

The organization of this work is as follows: In chapter 2, we introduce the basic concepts and results that will be used throughout this text. In chapter 3, we study the hyperbolicity of ergodic measures for ASH sets and explore the relationship between the Oseledets splitting and the dominated splitting. We also examine the relation between the dominated splitting and the Oseledets splitting for the Poincaré linear flow. Chapter 4 is devoted to studying the entropy theory of ASH attractors and some of its applications. More precisely, we prove Theorem A, Theorem B, Theorem E, Corollary F and Theorem G. In chapter 5 we will study ASH dynamics in the three-dimensional setting and we will provide the proofs for Theorem C and Theorem D.

2 Preliminaries

This chapter is devoted to provide the precise definitions for all the concepts used in this work. We also provide some useful known results from the theory of singular flows.

2.1 Basic Setting

Throughout this work, M denotes a compact Riemannian manifold endowed with a metric d, induced by the Riemannian metric $\|\cdot\|$. We denote by X to a C^r vector field on M. For any $r \geq 1$, denote by $\mathcal{X}^r(M)$ the space of C^r vector fields on M endowed the C^r norm. Denote by $\Phi = \{X_t\}_{t\in\mathbb{R}}$ the *flow* induced by X. As usual, we say that a set Λ is X-invariant if $X_t(\Lambda) = \Lambda$, for every $t \in \mathbb{R}$.

For $x \in M$, the *orbit* of x is the set

$$\mathcal{O}(x) = \{X_t(x) : t \in \mathbb{R}\}.$$

For $a, b \in \mathbb{R}$, the *orbit segment from a to b* of a point x is defined by

$$X_{[a,b]}(x) = \{X_t(x) : t \in [a, b]\}.$$

We say that $\sigma \in M$ is a *singularity* of X if $X(\sigma) = 0$. Denote the set of singularities of the vector field X by $Sing(X)$. A singularity σ is hyperbolic if all eigenvalues of $DX_t(\sigma)$ have nonzero real parts. A point $x \in M$ is *regular* if it is not a singularity. A regular point x is *periodic* if there is $t > 0$ such that $X_t(x) = x$. The *period* $\pi(x)$ a periodic point x is defined as the smallest positive number t satisfying $X_t(x) = x$. Denote by $Per(X)$ the set of periodic orbits of X.

We say that a compact invariant set Λ is *attracting* if there exists a neighborhood U_0 of Λ (called trapping region) such that

$$\overline{X_t(U_0)} \subset U_0, \quad \forall t > 0,$$

and

$$\Lambda = \bigcap_{t\geq 0} X_t(U_0).$$

We say that an attracting set Λ is an *attractor* if it is transitive, i.e., there is $z \in \Lambda$ such that $\overline{\mathcal{O}_+(z)} = \Lambda$, where $\mathcal{O}_+(z) = \{X_t(z) \mid t \geq 0\}$ denotes the positive orbit of z.

2.2 Hyperbolic, Sectional Hyperbolic and Star Flows

In the sequel, we shall recall some concepts that will be widely explored in this work, and were mentioned in the previous section.

Definition 2.2.1 (Dominated splitting)**.** A compact invariant set Λ is said to have a *dominated splitting* if there are a continuous invariant splitting $T_\Lambda M = E \oplus E^c$ and constants $K, \lambda > 0$ satisfying:

$$\frac{\|DX_t(x)|_{E_x}\|}{m(DX_t(x)|_{E^c_x})} \le Ke^{-\lambda t}, \quad \forall x \in \Lambda, \forall t > 0,$$

where $m(A)$ denotes the conorm of a linear transformation A. In this case, we say that E is *dominated* by E^c.

Remark 2.2.1. Note that $T_\Lambda M = E \oplus E^c$ is a dominated splitting for $X_t \mid_\Lambda$ if and only if it is a dominated splitting for the time-one map $X_1 \mid_\Lambda$.

The following lemma from [40] states that any two dominated splittings with the same index i for the subbundle E must coincide; that is, such a splitting is unique.

Lemma 2.2.2. Let $X \in \mathcal{X}^1(M)$ and Λ be an X_t-invariant set Λ admits two dominated splittings $T_\Lambda M = E \oplus E^c$ and $T_\Lambda M = F \oplus F^c$. If $\dim E \le \dim F$, then $E \subseteq F$ and $E^c \supseteq F^c$. In particular, if $\dim E = \dim F$, then $E = F$ and $E^c = F^c$.

Proof. Let $u \in E(x)$ be a unit vector, i.e., $\|u\| = 1$.Then u admits aunique decomposition as $u = u_F + u_{F^c}$, where $u_F \in F(x)$ and $u_{F^c} \in F^c(x)$ with $u_{F^c} \neq 0$. Then,

$$\begin{aligned}
\|DX_t(x)\mid_{E(x)}\| &\ge \|DX_t(x)u\| \\
&\ge \|DX_t(x)u_{F^c}\| - \|DX_t(x)u_F\| \\
&\ge m(DX_t(x)\mid_{F^c(x)})\|u_{F^c}\| - \|DX_t(x)\mid_{F(x)}\|\|u_F\| \\
&= \|DX_t(x)\mid_{F(x)}\|\Big(\frac{m(DX_t(x)\mid_{F^c(x)})}{\|DX_t(x)\mid_{F(x)}\|}\|u_{F^c}\| - \|u_F\|\Big) \\
&\ge \|DX_t(x)\mid_{F(x)}\|\Big(\frac{1}{K}e^{\lambda t}\|u_{F^c}\| - \|u_F\|\Big) \text{ (by domination)},
\end{aligned}$$

thus, $\dfrac{\|DX_t(x)\mid_{E(x)}\|}{\|DX_t(x)\mid_{F(x)}\|} \to \infty$ when $t \to \infty$. We notice that if there exist $v \in F(x)$ such that $v = v_E + v_{E^c}$ with $v_E \in E(x)$, $v_{E^c} \in E^c(x)$ and $v_{E^c} \neq 0$ we can repeat the previous argument and conclude that $\dfrac{\|DX_t(x)\mid_{F(x)}\|}{\|DX_t(x)\mid_{E(x)}\|} \to \infty$ when $t \to \infty$ which is a contradiction. Therefore, by the dimension hypothesis $\dim E(x) \le \dim F(x)$ we have $E(x) \subseteq F(x)$. In a similar way it is shown that $E^c(x) \supseteq F^c(x)$. $\square$

Definition 2.2.3 (Hyperbolic set)**.** A compact and invariant set Λ is said to be *hyperbolic* if it has a dominated splitting $T_\Lambda M = E^s \oplus \langle X \rangle \oplus E^u$ such that, E^s is contracting and E^u is expanding, i.e.,

$$\|DX_t(x)|_{E_x}\| \le Ke^{-\lambda t} \text{ and } \|DX_{-t}(x)|_{E^u_x}\| \le Ke^{-\lambda t},$$

for every $t > 0$ and $x \in \Lambda$.

A *hyperbolic periodic point* is a periodic point whose orbit is a hyperbolic set. A singularity is hyperbolic if it is a hyperbolic set.

Let $\sigma \in M$ be a hyperbolic singularity of X. We call the dimension of its stable space the index of σ, denoted $\text{Ind}(\sigma)$. Denote by $W^s(\sigma)$ and $W^u(\sigma)$ the stable and unstable manifolds for σ. As usual, for any $x \in M$ and $r > 0$, denote by $B(x, r)$ the r-ball centered at x. We say that Λ is a non-trivial transitive set if Λ is not reduced to a singularity or a periodic orbit.

Lemma 2.2.4. Let Λ be a non-trivial transitive set and let $\sigma \in Sing_\Lambda(X)$ hyperbolic. Then, $(W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda \neq \emptyset$ and $(W^u(\sigma) \setminus \{\sigma\}) \cap \Lambda \neq \emptyset$.

Proof. Since $\sigma \in \Lambda$, we have that it is hyperbolic of saddle type. In this way, we can find $\varepsilon > 0$ such that, for every $z \in M$, if $d(X_t(z), \sigma) \leq \varepsilon$ for all $t \geq 0$, then $z \in W^s(\sigma)$. Since Λ is non-trivial and transitive, there is a regular point $y \in \Lambda$ whose orbit is dense in Λ.

Notice that for any neighborhood U of σ, the closer a point $z \in U$ is to σ, the longer its orbit remains in U. Thus, one can find a sequence of points $y_n \in \mathcal{O}(y)$ and an arbitrarily large sequence of positive numbers t_n, $n \geq 1$, such that $y_n \to \sigma$ and

$$d(X_{t-t_n}(y_n), \sigma) \leq \varepsilon, \quad \forall t \in [0, t_n],$$

as well as

$$X_{-t_n}(y_n) \in B(\sigma, \varepsilon) \setminus B\left(\sigma, \frac{\varepsilon}{2}\right), \quad \forall n \geq 1.$$

Therefore, for every accumulation point x of $\{X_{-t_n}(y_n)\}_{n\geq 1}$ one has by continuity of the flow that $\frac{\varepsilon}{2} \leq d(x, \sigma) \leq \varepsilon$ and $d(X_t(x), \sigma) \leq \varepsilon$ for all $t \geq 0$. This shows that $x \in (W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda$. Similar arguments prove that $(W^u(\sigma) \setminus \{\sigma\}) \cap \Lambda \neq \emptyset$. □

Definition 2.2.5 (Partially Hyperbolic set)**.** A compact and invariant set Λ is said to be *partially hyperbolic* if it has a dominated splitting $T_\Lambda M = E \oplus E^c$ such that E is contracting.

Lemma 2.2.6. Let Λ be a compact invariant set with a partially hyperbolic splitting $T_\Lambda M = E \oplus E^c$. Then, $X(x) \in E^c(x)$ for any point $x \in \Lambda$.

Proof. Without loss of generality we can assume that the Riemannian metric left the subfibrates E and E^c orthogonal. We note that for $x \in \Lambda$ we have $X(x) = v_e + v_{ec}$ with $v_e \in E(x)$ and $v_{ec} \in E^c(x)$. Suppose that $X(x) \notin E^c(x)$ for some $x \in \Lambda$, then $v_e \neq 0$. Since E and E^c are DX_t-invariant and orthogonal we have that for $t \in \mathbb{R}$

$$\begin{aligned}\|X(X_t(x))\|^2 =& \|DX_t(x)X(x)\|^2 = \|DX_t(x)(v_e + v_{ec})\|^2 = \|DX_t(x)v_e + DX_t v_{ec}\|^2 \\ =& \|DX_t(x)v_e\|^2 + \|DX_t v_{ec}\|^2, \end{aligned} \tag{2.1}$$

which is a contradiction: The left side of (2.1), $\|X(X_t(x))\|$ is bounded for $t \in \mathbb{R}$ and the right side of (2.1) is not bounded since E is uniformly contracting because $0 \neq v_e \in E$ and

$$\|DX_t(x)v_e\| \geq \frac{1}{K}e^{\lambda t}\|v_e\| \to \infty \text{ when } t \to -\infty.$$

□

The following lemma relates the dimensions of the partially hyperbolic splitting and hyperbolic splitting at singularities, and this lemma will be used in the proof of Theorem A.

Lemma 2.2.7. Let Λ be a non-trivial partially hyperbolic attractor with splitting $T_\Lambda M = E \oplus E^c$. Then, for every hyperbolic singularity $\sigma \in \Lambda$ with hyperbolic splitting $T_\sigma M = \overline{E}^s_\sigma \oplus \overline{E}^u_\sigma$, we have

$$\dim(\overline{E}^s_\sigma) = \dim(E(\sigma)) + l \quad \text{ with } \quad l \in \mathbb{Z}^+.$$

Proof. Suppose the contrary, that is, we have two splittings at σ:

$$\begin{aligned} T_\sigma M =& E(\sigma) \oplus E^c(\sigma) \text{ partially hyperbolic splitting and} \\ T_\sigma M =& \overline{E}^s_\sigma \oplus \overline{E}^u_\sigma \text{ hyperbolic splitting} \end{aligned}$$

with $\dim E(\sigma) \geq \dim(\overline{E}^s_\sigma)$ for some hyperbolic singularity $\sigma \in \Lambda$. We note that the two splittings are dominated and since $\dim(\overline{E}^s_\sigma) \leq \dim E(\sigma)$, by lemma 2.2.2 we have $\overline{E}^s_\sigma \subseteq E(\sigma)$. By lemma 2.2.4, there is $x \in (W^s(\sigma) \cap \Lambda) \setminus \{\sigma\}$. Thus, $X(X_t(x)) \in T_{X_t(x)}W^s(\sigma)$ for $t > 0$. Now, consider the unit vectors $\dfrac{X(X_n(x))}{\|X(X_n(x))\|}$ with $n \in \mathbb{Z}^+$ by the compactness of the unitary tangent bundle we have that there exists a subsequence n_k and $v \in E(\sigma)$ such that

$$\frac{X(X_{n_k}(x))}{\|X(X_{n_k}(x))\|} \to v \text{ when } k \to \infty.$$

We note that $v \neq 0$ since $\|v\| = 1$. This fact contradicts the continuity of dominated splittings since, by lemma 2.2.6 we have $\dfrac{X(X_{n_k}(x))}{\|X(X_{n_k}(x))\|} \in E^c(X_{n_k}(x))$ and by the continuity of the dominated splitting implies $v \in E^c(\sigma)$, thus $0 = v$ because $v \in E(\sigma) \cap E^c(\sigma)$. □

Definition 2.2.8 (Sectional Hyperbolic set)**.** Let Λ be a compact invariant partially hyperbolic set of a vector field X whose singularities are hyperbolic. We say that Λ is *sectional-hyperbolic* (SH for short) if its central subbundle is sectional expanding, i.e., there exists $K, \lambda > 0$ such that for every two-dimensional subspace L_x of E^c_x one has

$$|\det DX_t(x)|_{L_x}| \geq Ke^{\lambda t}, \quad \forall x \in \Lambda, \forall t > 0. \tag{2.2}$$

We now conclude this subsection by providing the definition of star flow that will be considered in Theorem A.

Definition 2.2.9 (Star flow)**.** A vector field X is said to be a *star vector field* (or *star flow*) if there is a C^1-neighborhood $\mathcal{U}$ of X such that if $Y \in \mathcal{U}$, then any critical element(singularity or periodic orbit) of Y is hyperbolic. We say that a compact invariant set Λ for X is *star* if there is a neighborhood U of Λ such that X is a star flow on U.

Remark 2.2.2*.* We have the following remarks

- SH flows are contained in ASH flows. This follows directly from the Definitions 1.0.1 and 2.2.8.
- SH flows are contained in stars flows. Note that the partial hyperbolicity and the property (2.2) are open conditions. Therefore, by the hyperbolic lemma, we have that SH flows are Star flows.
- Theorem A establishes the relationship between ASH flows and Star flows in the case of attractors.

2.3 Stable Manifolds and Homoclinic Classes

Recall that for a hyperbolic periodic point p associated to a C^1 vector field X, the *strong stable* and *strong unstable manifold* of p are defined, respectively, by

$$W^{ss}(p) = \{y \in M : \lim_{t\to\infty} d(X_t(p), X_t(y)) = 0\}$$

and

$$W^{uu}(p) = \{y \in M : \lim_{t\to-\infty} d(X_t(p), X_t(y)) = 0\}.$$

We then define the stable and unstable manifolds of p, respectively, by

$$W^s(p) = \bigcup_{t\in\mathbb{R}} W^{ss}(X_t(p)) \text{ and } W^u(p) = \bigcup_{t\in\mathbb{R}} W^{uu}(X_t(p)).$$

Now, recall that for a hyperbolic periodic orbit p associated to a C^1 vector field X, the *homoclinic class* of p, denoted by $H(p)$, is defined as

$$H(p) := \overline{W^s(p) \pitchfork W^u(p)}.$$

In this way, for any pair of periodic orbits γ_p and γ_q, we say that $\gamma_p \sim \gamma_q$ if $W^s(p) \pitchfork W^u(q)$ and $W^s(q) \pitchfork W^u(p)$. In this case, they belong to the same homoclinic class. We say that a homoclinic class $H(p)$ is *non trivial* if $H(p) \neq \mathcal{O}(p)$.

2.4 Ergodic Theory

Next, we reall some concepts from the ergodic theory of flows. We say that a Borelian probility measure μ in M is Φ-*invariant* if $\mu(X_t(B)) = \mu(B)$, for every Borelian subset $B \subset M$ and every $t \in \mathbb{R}$. An invariant probability measure is *ergodic* if for every X-invariant set $B \subset M$, one has $\mu(B)\mu(B^c) = 0$. Let $GL(d, \mathbb{R})$ denote the set of all invertible $d \times d$ matrices with real entries. A measurable function $A : M \times \mathbb{R} :\to GL(d, \mathbb{R})$ is called a *cocycle* if for all $x \in M$ $A(x, 0) = Id$ and $A(x, t + s) = A(X_t(x), s)A(x, t)$ for $t, s \in \mathbb{R}$. We write $A_t(\cdot) = A(\cdot, t)$. The natural example of a cocycle over a smooth flow X_t on a manifold is the derivative cocycle $A_t(x) = DX_t(x)$ on the tangent bundle TM of a finite-dimensional compact manifold M.

The following theorem, due to Oseledets [35], will be crucial to our purposes.

Theorem 2.4.1 (**Oseledets**)**.** Let $X_t : M \to M$ a measurable flow preserving a probability μ and $A_t : M \to GL(d, \mathbb{R})$ a measurable cocycle. If exist $a > 0$ such that $\sup_{t\in[-a,a]} \log^+ \|A_t(x)\| \in L^1(\mu)$. There is a Φ-invariant full set B such that for every $x \in B$, there are $k(x) > 1$, real numbers $\chi_1(x) < \cdots < \chi_{k(x)}(x)$, and a splitting

$$T_xM = \bigoplus_{i=1}^{k(x)} H_i(x)$$

satisfying the following properties:

- The maps $x \mapsto H_i(x)$, $i = 1, \ldots, k(x)$, are measurable.
- The splitting is DX_t-invariant, i.e., $A_t(x)H_i(x) = H_i(X_t(x))$ for every $t \in \mathbb{R}$.
- For every $v \in H_i(x) \setminus \{0\}$, $i = 1, \ldots, k(x))$, the following limit exists:

$$\chi(x, v, A) = \lim_{t\to\pm\infty} \frac{1}{|t|} \log \|A_t(x)v\| = \chi_i(x).$$

- For $S \subset N := \{1, 2, \ldots, k(x)\}$, let $H_S(x) := \bigoplus_{i\in S} H_i(x)$. Then,

$$\lim_{t\to\infty} \frac{1}{t} \log(\sin |\measuredangle(H_S(X_t(x)), H_{N\setminus S}(X_t(x)))|) = 0, \tag{2.3}$$

  where $\measuredangle(u, v)$ denotes the angle formed by u and v. Moreover, for $u, v \in H_i$ we have

$$\lim_{t\to\infty} \frac{1}{t} \log \sin |\measuredangle(A_t(x)u, A_t(x)v)| = 0.$$

The numbers $\chi_i(x)$, $i = 1, \ldots k(x)$, given in Theorem 2.4.1 are called the *Lyapunov exponents of* x. Next, we provide a list of elementary facts about Lyapunov exponents that will be used in chapter 3.

- $\chi(x, 0) := -\infty$,

- $\chi(x, \alpha v) = \chi(x, v)$ for $\alpha \in \mathbb{R} \setminus \{0\}$,
- $\chi(x, v + w) \leq \max\{\chi(x, v), \chi(x, w)\}$. Furthermore,

$$\text{if } \chi(v) \neq \chi(w), \text{ then } \chi(x, v + w) = \max\{\chi(x, v), \chi(x, w)\}.$$

A very important property of measures is when their Lyapunov exponents are far from zero, as we define next.

Definition 2.4.2. An invariant probability measure μ for the flow of a C^1 vector field X is a *hyperbolic measure* if μ-almost every point has only one zero Lyapunov exponent, which corresponds to the flow direction.

We define the support of a measure in the usual way:

$$supp(\mu) = \{x \in M \mid \mu(V) > 0 \text{ for every neighborhood } V \text{ of } x\}$$

that is the smallest closed set of full measure.

2.5 Linear Poincaré Flow

To achieve our goal, we need to use the technology of the linear Poincaré flow, a concept that we now recall in detail. For $X \in \mathcal{X}^1(M)$, denote the *normal bundle of* X by

$$\mathcal{N} = \bigcup_{x \in M \setminus Sing(X)} \mathcal{N}_x,$$

where

$$\mathcal{N}_x = \{v \in T_x M : v \perp X(x)\}.$$

Denote $O_x : T_x M \to \mathcal{N}_x$ the orthogonal projection on $\mathcal{N}_x$.

Definition 2.5.1. The *Poincaré linear flow* associated to X is the flow $\Psi = \{\psi_t\}_{t \in \mathbb{R}}$ on $\mathcal{N}$, $\psi_t(x) : \mathcal{N}_x \to \mathcal{N}_{X_t(x)}$ defined by

$$\begin{aligned}\psi_t(x)v :=& O_{X_t(x)} DX_t(x)v \\ =& DX_t(x)v - \frac{\langle DX_t(x)v, X(X_t(x))\rangle}{\|X(X_t(x))\|^2} X(X_t(x)).\end{aligned}$$

for $v \in \mathcal{N}_x$ and for all $t \in \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ is the inner product on $T_x M$ given by the Riemannian metric.

It is easy to see that $\Psi = \{\psi_t\}_{t \in \mathbb{R}}$ satisfies the cocycle relation

$$\psi_{t+s}(x) = \psi_t(X_s(x))\psi_s(x) \text{ for every } t, s \in \mathbb{R}.$$

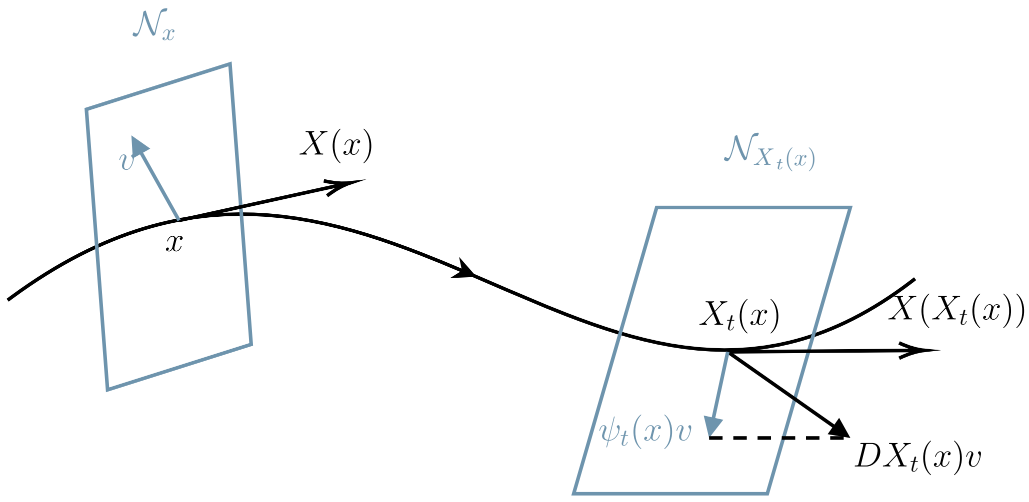

Figure 1 – Poincaré Linear Flow.

Let $T_\Lambda M = E \oplus E^c$ be an splitting partially hyperbolic on Λ. Denote

$$\mathcal{E}_x = O_x(E(x)) \text{ and } \mathcal{E}_x^c = O_x(E^c(x)).$$

Since $X(x) \in E^c(x)$ for $x \in \Lambda$ (Lemma 2.2.6) it is easy to show that $O_x(E^c) = E^c \cap \mathcal{N}$. In this case, one has a splitting of the normal bundle given by $\mathcal{N} = \mathcal{E} \oplus \mathcal{E}^c$.

Notice that if μ is a hyperbolic invariant measure for the flow of X, it is possible to consider the following splitting on $sup(\mu)$: $T_{supp(\mu)}M = E^s(x) \oplus \langle X(x) \rangle \oplus E^u(x)$, where

$$E^s(x) = \bigoplus_{\chi_i(x)<0} H_i(x) \text{ and } E^u(x) = \bigoplus_{\chi_i(x)>0} H_i(x).$$

Denote $\mathcal{E}_x^s = O_x(E^s(x))$ and $\mathcal{E}_x^u = O_x(E^u(x))$ for every $x \in supp(\mu)$.

Lemma 2.5.2. If $T_\Lambda M = E \oplus F$ is a dominated splitting, then $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ is dominated splitting w.r.t. the linear Poincaré flow ψ_t, where $\mathcal{E}_x = O_x(E(x))$ and $\mathcal{F}_x = O_x(F(x))$.

Proof. Fix $v \in E$. Since

$$T_x M = \mathcal{N}_x \oplus \langle X(x) \rangle,$$

we can write $v = O_x(v) + \alpha X(x)$ for some $\alpha \in \mathbb{R}$. Therefore,

$$DX_t(x)O_x(v) = O_{X_t(x)}DX_t(x)O_x(v) + \beta X(X_t(x)),$$

for some $\beta \in \mathbb{R}$. Thus, we obtain

$$\begin{aligned} DX_t(x)v &= DX_t(x)O_x(v) + DX_t(x)(\alpha X(x)) \\ &= O_{X_t(x)}DX_t(x)O_x(v) + \beta X(X_t(x)) + \alpha DX_t(x)X(x) \\ &= O_{X_t(x)}DX_t(x)O_x(v) + (\alpha + \beta)X(X_t(x)). \end{aligned}$$

On the other hand,

$$DX_t(x)v = O_{X_t(x)}DX_t(x)v + DX_t(x)v - O_{X_t(x)}DX_t(x)v.$$

So, since $DX_t(x)v - O_{X_t(x)}DX_t(x)v \in \langle X(x)\rangle$,

$$\psi_t(x)O_x(v) = O_{X_t(x)}DX_t(x)O_x(v) = O_{X_t(x)}DX_t(x)v \in \mathcal{E}_{X_t(x)}.$$

This shows the ψ_t-invariance of $\mathcal{E}$. In addition,

$$(\alpha+\beta)X(X_t(x)) = DX_t(x)v - O_{X_t(x)}DX_t(x)v.$$

Thus, we have

$$\begin{aligned}\|DX_t(x)v\| &= \|O_{X_t(x)}DX_t(x)O_x(v) + DX_t(x)v - O_{X_t(x)}DX_t(x)v\|\\ &= \|\psi_t(x)O_x(v) + (Id - O_{X_t(x)})DX_t(x)v\|\\ &\geq \|\psi_t(x)O_x(v)\| - \|Id - O_{X_t(x)}\| \cdot \|DX_t(x)v\|\\ &\geq \|\psi_t(x)O_x(v)\| - \|DX_t(x)v\|,\end{aligned}$$

where the last inequality comes from the fact that $Id - O_{X_t(x)}$ is a projection and

$$\|Id - O_{X_t(x)}\| \leq 1.$$

We then conclude that

$$\|\psi_t(x)O_x(v)\| \leq 2\|DX_t(x)v\|.$$

By taking the supreme over all unitary vectors in $\mathcal{E}_x$, we have

$$\|\psi_t(x)|_{\mathcal{E}}\| \leq 2\|DX_t(x)|_E\|. \tag{2.4}$$

By an analogous reasoning, we obtain the ψ_t-invariance of $\mathcal{F}$ and

$$\|\psi_{-t}(x)|_{\mathcal{F}}\| \leq 2\|DX_{-t}(x)|_F\|. \tag{2.5}$$

Now, by combining the domination property of the splitting $T_\Lambda M = E \oplus F$ with the relations (2.4) and (2.5), we have

$$\|\psi_t(x)|_{\mathcal{E}}\| \cdot \|\psi_{-t}(x)|_{\mathcal{F}}\| \leq 4\|DX_t(x)|_E\| \cdot \|DX_{-t}(x)|_F\| \leq 4Ce^{-\lambda t},$$

for every $t > 0$. This proves the result. □

Another flow mentioned in this work is the rescaled Poincaré linear flow.

Definition 2.5.3. The *rescaled Poincaré linear flow* associated with X is the flow $\Psi^* = \{\psi_t^*\}$ on $\mathcal{N}$, where $\psi_t^* : \mathcal{N} \to \mathcal{N}$ is defined by

$$\psi_t^*(x)v = \frac{\|X(x)\|}{\|X(X_t(x))\|}\psi_t(x)v = \frac{1}{\|DX_t(x)\mid_{\langle X(x)\rangle}\|}\psi_t(x)v,$$

where $\langle X(x)\rangle$ is the 1-dimensional subspace of T_xM spanned by the vector $X(x) \in T_xM$.

We have that $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ is a dominated splitting with respect to ψ_t if and only if it is a dominated splitting with respect to ψ_t^*.

As with the linear Poincaré flow, it is also not difficult to show that the rescaled Poincaré linear flow satisfies the cocycle condition:

$$\psi_{t+s}^*(x) = \psi_t^*(X_s(x))\psi_s^*(x), \quad \text{for all } t, s \in \mathbb{R}.$$

Given an ergodic measure μ, for μ-almost every $x \in M \setminus \mathrm{Sing}(X)$, the Lyapunov exponent for DX_t along the flow direction is zero, namely,

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|DX_t(x) \mid_{\langle X(x)\rangle} \| = 0.$$

Therefore,

$$\begin{aligned}\lim_{t\to\pm\infty} \frac{1}{t} \log \|\psi_t^*(x)v\| &= \lim_{t\to\pm\infty} \frac{1}{t} \log \|\psi_t(x)v\| - \lim_{t\to\pm\infty} \frac{1}{t} \log \|DX_t(x) \mid_{\langle X(x)\rangle} \| \\ &= \lim_{t\to\pm\infty} \frac{1}{t} \log \|\psi_t(x)v\|.\end{aligned}$$

This means that the Lyapunov exponents of the rescaled Poincaré linear flow and the linear Poincaré flow are the same. Hence, the rescaled Poincaré linear flow and the linear Poincaré flow also share the same Oseledets splitting.

2.6 Topological Entropy

Let $F \subset M$ and fix $\varepsilon, n > 0$. We say that a subset K of F is

- a *n-ε-separated set of F* if for any pair of distinct points $x, y \in K$ there is some $0 \leq n_0 \leq n$ such that $d(f^{n_0}(x), f^{n_0}(y)) > \varepsilon$. Denote $S(n, \varepsilon, F)$ the maximal cardinality of an n-ε-separated subset of F.
- a *n-ε-generator for F* if for every $x \in F$ there exists $y \in K$ such that $d(f^i(x), f^i(y)) \leq \varepsilon$, for every $0 \leq i \leq n$. Denote $R(n, \varepsilon, F)$ the minimum cardinality of the n-ε-generators for F.

Note that $S(n, \varepsilon, F)$ and $R(n, \varepsilon, F)$ are always finite due to the compactness of M. We then define the *topological entropy of f on F* as the number

$$h(f, F) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log(S(n, \varepsilon, F)) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log(R(n, \varepsilon, F)).$$

Observe that when $F = M$, the number $h(f, F) = h(f)$ is the topological entropy of f. The definition of topological entropy for flows is similar. In fact, $h(\Phi, F) = h(X_1, F)$.

Now, for a map $f : M \to M$, the *metric entropy* of an invariant measure μ for f is given by

$$h_\mu(f) = \sup\{h_\mu(f, \mathcal{P}) : \mathcal{P} \text{ is a finite partition}\},$$

where

$$h_\mu(f, \mathcal{P}) = -\lim_{n\to\infty} \frac{1}{n} \sum_{P\in\mathcal{P}_n} \mu(P) \log \mu(P), \quad \mathcal{P}_n = \mathcal{P}_0 \vee f^{-1}(\mathcal{P}_1) \vee \cdots \vee f^{n-1}(\mathcal{P}).$$

and the partition $\mathcal{P} \vee \mathcal{Q}$ is defined as the common refinement of $\mathcal{P}$ and $\mathcal{Q}$, that is, the collection of all sets of the form $P \cap Q$ with $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$.

Denote by $\mathcal{M}(\Phi)$ the set of invariant measures for the flow of X. In this way, the metric entropy of an invariant measure μ for the flow Φ is defined as $h_\mu(\Phi) = h_\mu(X_1)$. Recall $\Phi = \{X_t\}_{t\in\mathbb{R}}$ the *flow* induced by X.

On the other hand, we called *potential* to a continuous function $\phi : M \to \mathbb{R}$. The *topological pressure* of Φ with respect to ϕ is defined by

$$P(\Phi, \phi) = \sup_{\nu\in\mathcal{M}(\Phi)} \left\{h_\nu(\Phi) + \int \phi d\nu\right\}.$$

A measure $\mu \in \mathcal{M}(\Phi)$ is called an *equilibrium state for the potential* ϕ *associated to* Φ if the above supremum is attained at $\nu = \mu$. If $\phi \equiv 0$, the equilibrium state μ associated to ϕ is called *measure of maximal entropy.*

Let (M, d) be a compact metric space and let $f : M \to M$ be a homeomorphism. Recall that the δ-dynamical ball of a point $x \in M$ is the set

$$B_\delta^\infty(x, f) = \{y \in M : d(f^n(x), f^n(y)) \le \delta, \forall n \in \mathbb{Z}\}.$$

Definition 2.6.1. A continuous flow Φ on a compact metric space M is said to be *entropy-expansive* if its time-one map is entropy-expansive, i.e., there exists $\delta > 0$ such that $h(X_1, B_\delta^\infty(x, X_1)) = 0$, for every $x \in M$.

In [36] the authors showed that any SH invariant set for C^1 flows is entropy-expansive in any dimension. Our result the Theorem C states that this property holds for three-dimensional ASH attractors.

Remark 2.6.1. It is worth mentioning that we will prove our results by different approaches. As we will see in chapter 5, this is due to the lack of uniform area expansion in the central bundle. This represents the main challenge when extending results from the sectional-hyperbolic to the ASH setting. This motivates us to develop new tools to deal with these issues.

3 Hyperbolic Measures for ASH Flows.

In this chapter we will show that every ergodic measure is hyperbolic, an essential tool for proving our main results. To present the statement of this result, it is necessary to recall the notion of a hyperbolic measure.

3.1 Hyperbolic and Regular Measure.

Recall that an invariant measure μ for the flow of X is *supported on* Λ is *supp*$(\mu) \subset \Lambda$, where *supp*(μ) denotes the support of μ. Now, recall that an ergodic probability measure for X is *atomic* if it is supported on either a singularity or periodic orbit. We say that a probability measure μ is *regular* if it is not atomic.

The main result of this section is as follows.

Theorem 3.1.1. Let Λ be an asymptotically sectional-hyperbolic set for a C^1 vector field X, and let μ be a regular ergodic invariant measure for the flow of X supported on Λ. Then, μ is a hyperbolic measure.

Before presenting the proof of Theorem 3.1.1, an original contribution of this thesis, we gather some preliminary lemmas. We decided to present their proofs for the sake of completeness.

Let us begin with the following lemma, which says that a regular invariant measure for a C^1 vector field X is not supported on $W^s(Sing(x))$.

Lemma 3.1.2. Let Λ be a compact invariant set for a C^1-vector field X. If μ is a regular ergodic measure for the flow of X, then $\mu(W^s(Sing(X)) = 0$.

Proof. Since $W^s(Sing(X)) \setminus Sing(x)$ is invariant, the ergodicity of μ implies that

$$\mu(W^s(Sing(X)) \setminus Sing(x))$$

has either full or zero measure. However, since no point in $W^s(Sing(X)) \setminus Sing(x)$ can be recurrent, we conclude by Poincaré's recurrence theorem that

$$\mu(W^s(Sing(X)) \setminus Sing(X)) = 0.$$

Finally, since μ is regular and $Sing(X)$ is finite, we have $\mu(Sing(X)) = 0$. Therefore, $\mu(W^s(Sing(X))) = 0$. □

The next lemma separates negative and non-negative exponents within the ASH splitting.

Lemma 3.1.3. Let Λ be an asymptotically sectional-hyperbolic set for a C^1 vector field X, and let μ be a regular ergodic measure for X supported on Λ. Let $T_\Lambda M = E \oplus E^c$ be the ASH splitting of Λ. Then,

$$E(x) = \bigoplus_{\chi_i(x)<0} H_i(x) \qquad \text{and} \qquad E^c(x) = \bigoplus_{\chi_i(x)\geq 0} H_i(x),$$

for μ-almost every point $x \in \Lambda$, where the subspaces $H_i(x)$ are given by Oseledets Theorem (Theorem 2.4.1).

Proof. Let μ be a regular and ergodic measure for X supported on Λ and let $B \subset \Lambda$ be the set given by Oseledets Theorem (Theorem 2.4.1). Denote

$$F^-(x) = \bigoplus_{\chi_i(x)<0} H_i(x) \qquad \text{and} \qquad F^+(x) = \bigoplus_{\chi_i(x)>0} H_i(x), \quad \forall x \in B.$$

We claim that

$$E(x) = F^-(x) \text{ and } E^c(x) = \langle X(x)\rangle \oplus F^+(x) \quad \forall x \in B.$$

To see why the claim holds, recall that μ is a regular measure and, therefore, is not supported on $Sing(X)$. Moreover, by Lemma 3.1.2 we can assume that

$$B \cap W^s(Sing(X)) = \emptyset.$$

Let us first show that $E(x) = F^-(x)$. Indeed, due to the uniform contraction of E, the Lyapunov exponent of x, concerning any $v \in E(x)$, is negative and therefore $E(x) \subset F^-(x)$. Now, suppose there is a non-zero vector $v \in F^-(x) \setminus E(x)$. We have that $v = u + w$ with $u \in E(x)$, $w \in E^c(x)$ and $w \neq 0$. Notice that $w \notin \langle X(x)\rangle$, because

$$\chi(x, w) = \chi(x, v - u) \leq \max\{\chi(x, v), \chi(x, -u)\} \leq \max\{\chi(x, v), \chi(x, u)\} < 0.$$

Moreover, we can assume that w satisfies $w \perp X(x)$ and $\|w\| = 1$.

Recall that $X(x) \in E^c(x)$ by Lemma 2.2.6, and denote $L_x = \langle X(x), w\rangle$ the two-dimensional subspace spanned by the vectors $X(x)$ and w. Note that L_x is contained in $E^c(x)$. Let $\theta_t(x)$ be the angle between $DX_t(x)X(x) = X(X_t(x))$ and $DX_t(x)w$. Then,

$$|\det(DX_t(x)|_L)| = \frac{\sin\theta_t \cdot \|X(X_t(x))\| \cdot \|DX_t(x)w\|}{\|X(x)\|}. \tag{3.1}$$

On one hand, by the choice of B and the asymptotic area expansion of Λ, we have

$$\limsup_{t\to\infty} \frac{1}{t} \log \|\det DX_t(x)|_{L_x}\| > C > 0. \tag{3.2}$$

On the other hand, by definition of the Lyapunov exponents,

$$\limsup_{t\to\infty} \frac{1}{t} \log \|\det DX_t(x)|_{L_x}\| = \chi(x, X(x)) + \chi(x, w) < 0. \tag{3.3}$$

This is a contradiction. Therefore, $E(x) = F^{-}(x)$. In particular,

$$\dim(E^c(x)) = \dim(\langle X(x)\rangle \oplus F^{+}(x)). \tag{3.4}$$

Next, we show that $E^c(x) = \langle X(x)\rangle \oplus F^{+}(x)$. For $u \in E^c(x) \cap \langle X(x)\rangle^{\perp}$, by repeating the above argument for the plane $M = \langle X(x), u\rangle$ we have that $\chi(x, u) > 0$. This implies that $u \in F^{+}(x)$. Therefore,

$$E^c(x) = (E^c(x) \cap \langle X(x)\rangle^{\perp}) \oplus \langle X(x)\rangle \subset F^{+}(x) \oplus \langle X(x)\rangle,$$

and by (3.4) we have $E^c(x) = \langle X(x)\rangle \oplus F^{+}(x)$. □

Finally, we are ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Let μ be a regular ergodic measure for X and let B be the set given by Oseledets theorem. By Lemma 3.1.2, we can assume $B \cap W^s(Sing(X)) = 0$. Next, let $T_\Lambda M = E \oplus E^c$ be the ASH splitting associated to Λ. For $x \in B$, let $H(x)$ be vector subspace, given by Oseledets theorem, associated with the null Lyapunov exponents. Notice that by Lemma 3.1.3 one has $\langle X(x)\rangle \subset H(x) \subset E^c_x = \langle X(x)\rangle \oplus F^{+}(x)$, where $F^{+}(x)$ is the subspace associated to the positive Lyapunov exponents. So, $H(x) = \langle X(x)\rangle$. This proves the result. □

3.2 Hyperbolic Measure and Linear Poincaré Flow.

The following lemma states that if a measure is hyperbolic for the derivative cocycle, it reveals how its dominated decompositions relate to the subspaces given by the Oseledets theorem for the Poincaré linear flow of the cocycle.

Lemma 3.2.1. Let Λ be an asymptotically sectional-hyperbolic set for a C^1 vector field X, and let μ be a regular hyperbolic measure for X supported on Λ. Let $T_\Lambda M = E \oplus E^c$ be the ASH splitting of Λ. Then,

$$\mathcal{E}(x) = \bigoplus_{\chi_i(x,\Psi)<0} H_i^{\Psi}(x) \qquad \text{and} \qquad \mathcal{E}^c(x) = \bigoplus_{\chi_i(x,\Psi)>0} H_i^{\Psi}(x),$$

for μ-almost every point $x \in \Lambda$, where $\chi(\cdot, \Psi)$ and $H_i^{\Psi}(\cdot)$ are the Lyapunov exponent and subspaces given by the Oseledets Theorem (Theorem 2.4.1).

Proof. Let ψ_t the Poincaré linear flow since $\|\psi_t(x)\| \leq \|DX_t(x)\|$ for $x \in M \setminus Sing(X)$ and for all $t \in \mathbb{R}$ we have $\sup_{t\in[-1,1]} \log^{+} \|\psi_t(x)\| \in L^1(\mu)$ because it is continuous and bounded. Applying Oseledets Theorem (Theorem 2.4.1) twice, we obtain B a set of full measure on Λ where the Lyapunov exponents exist for both cocycles. We note $\chi_i(\cdot)$ is the Lyapunov exponent and $H_i(\cdot)$ its associated subspaces given by the oseledecs for cocycle derivate and $\chi_i(\cdot, \Psi)$ is the Lyapunov exponent and and $H_i^{\Psi}(\cdot)$ its associate subspaces given by the oseledecs for cocycle Poincaré linear flow.

We recall that the $T_\Lambda M = E \oplus E^c$ dominated splitting induces a $\mathcal{N} = \mathcal{E} \oplus \mathcal{E}^c$, where $\mathcal{E} = O(E)$ and $\mathcal{E}^c = O(E^c)$.

Fix $x \in B$, since E is uniformly contracting and

$$\|\psi_t(x) O_x(v)\| \leq 2\|DX_t(x)v\| \text{ for } v \in E(x),$$

inequality obtained in the lemma 2.5.2. Then, the Lyapunov exponents along $\mathcal{E}\,(= O(E))$ are negative, thus,

$$\mathcal{E}(x) \subset \bigoplus_{\chi_i(x,\Psi)<0} H_i^{\Psi}(x). \tag{3.5}$$

Since μ is regular hyperbolic by the lemma 3.1.1 we have that

$$E^c(x) = \langle X(x) \rangle \oplus \bigoplus_{\chi_i(x)>0} H_i(x)$$

and that $\mathcal{E}^c = O(E^c) = E^c \cap \mathcal{N} \subset E^c$, then

$$\|\psi_t(x)v\| = \|DX_t(x)v\| \quad \text{for all } v \in \mathcal{E}^c(x).$$

As a consequence, the Lyapunov exponents of ψ_t and DX_t coincide on $\mathcal{E}^c$, that is,

$$\chi(x, v, \psi_t) = \chi(x, v, DX_t) \quad \text{for all } v \in \mathcal{E}^c.$$

In particular, both systems have the same positive Lyapunov exponents, which implies that the subspaces associated with these exponents are the same.

$$\mathcal{E}^c(x) \subset \bigoplus_{\chi_i(x,\Psi)>0} H_i^{\Psi}(x). \tag{3.6}$$

And by dimension we obtain the equality in (3.6) and (3.5). □

Remark 3.2.1. In some papers, a measure is defined to be hyperbolic if the Lyapunov exponents for the Ψ (Poincaré linear flow) cocycle are nonzero. It follows from the lemma 3.2.1 that if μ is hyperbolic for the derivative cocycle, then it is also hyperbolic for the Ψ cocycle.

4 Entropy of ASH Attractors and Their Applications

In this chapter, we begin to prove our main theorems. Here, we will be specifically interested in the entropy properties of ASH sets on manifolds of any dimension. In addition, we shall apply these to provide a proof for Theorem A, which relates the theories of ASH, SH and star flows.

4.1 Intermediate Entropy Property.

Let us begin by studying the intermediate entropy property of ASH attractors. Next, we shall be concerned with the proof of Theorem B, whose one of the main ingredients is the Theorem 4.1.3.

Let us first recall the concept of *horseshoe* for flows. Let M be a compact metric space and $f : M \to M$ be a homeomorphism. Let $\rho : M \to (0, \infty)$ be a continuous function. The function ρ is called the roof function. Define $M_\rho := M_{f,\rho}$ as the quotient space of $\{(x, l) \mid x \in M, 0 \le l \le \rho(x)\}$, through the equivalence relation

$$(x, \rho(x)) \sim (f(x), 0).$$

It is well known that M_ρ can be endowed with a metric, making it a compact metric space; we refer the reader to [16] for the precise details.

Definition 4.1.1. The suspension flow of f with roof function ρ is the flow $f^\rho : \mathbb{R} \times M_\rho \to M_\rho$ defined as $f_t^\rho(x, s) := f^\rho(t, (x, s)) = (f^n(x), s')$, where n and s' satisfy

$$\sum_{j=1}^{n-1} \rho(f^j(x)) + s' = t + s, \quad 0 \le s' \le \rho(f^n(x)).$$

Definition 4.1.2. A compact invariant set $K \subset M$ for the flow of X is a *horseshoe* if there is a roof function ρ such that $\Phi|_K$ is conjugated to the suspension of the l-symbol shift map $\sigma : \Sigma_l \to \Sigma_l$ with roof ρ. That is, there exists a homeomorphism $h : K \to \Sigma_l^\rho$ such that $h(X_t(x)) = \sigma_t^\rho(h(x))$ for every $x \in K$, $t \in \mathbb{R}$.

$$\begin{array}{ccc} K & \xrightarrow{X_t} & K \\ \downarrow h & & \downarrow h \\ \Sigma_l^\rho & \xrightarrow[\sigma_t^\rho]{} & \Sigma_l^\rho \end{array}$$

A *hyperbolic horseshoe* is a horseshoe admitting a hyperbolic splitting for the tangent flow.

Theorem 4.1.3. Let Λ be an asymptotically sectional-hyperbolic attractor for a C^1-vector field X. Suppose μ is an ergodic measure for the flow of X, supported on Λ with positive entropy. Then, for every $\varepsilon > 0$, there is a hyperbolic horseshoe $K_\varepsilon \subset \Lambda$ such that

$$|h_{top}(X|_{K_\varepsilon}) - h_\mu(X)| \leq \varepsilon.$$

The proof of this theorem is based on three facts: That every ergodic measure is hyperbolic (Theorem 3.1.1), that the dominated decomposition in the normal bundle coincides with the splitting of subspaces associated with the negative and positive exponents given by the Oseledets theorem (Lemma 3.2.1), and given a hyperbolic invariant measure with positive metric entropy, there exists a horseshoe lemma to follow.

Lemma 4.1.4 (Proposition 4.1 in [23])**.** Let $X \in \mathcal{X}^1(M^d)$ and μ be an ergodic hyperbolic measure with $h_\mu(X) > 0$. If the hyperbolic Oseledec splitting

$$\mathcal{N} = \mathcal{E} \oplus \mathcal{E}^c$$

with respect to the ergodic hyperbolic measure μ on the normal bundle is a dominated splitting, then for any $\varepsilon > 0$, there is a horseshoe Λ_ε such that

$$|h_\mu(X) - h_{top}(X\,|_{\Lambda_\varepsilon})| < \varepsilon.$$

For better clarity, I will provide an outline of the proof of the lemma.

Sketch of proof of Lemma 4.1.4. Since the measure is hyperbolic, we can define the Pesin Blocks, though we must be careful with singularities. We define these sets to see that, given a recurrent point, if we take a rectangle parallel to the local stable and unstable manifolds, it is contracted in one direction and expanded in the other.

- Pesin Block for Vector Field

Lemma 4.1.5 (Lemma 5.1 in [50])**.** If the hyperbolic Oseledec splitting of a regular hyperbolic ergodic measure μ is a dominated splitting, then there exist $T_1 > 0$ such that for μ-a.e. $x \in M$ and every $T \geq T_1$, the following limits exist

$$\lim_{k\to\infty} \frac{1}{kT} \sum_{i=0}^{k-1} \log ||\psi_T^*|\mathcal{E}^s(X_{iT}(x))|| < -\eta \quad \text{and}$$
$$\lim_{k\to\infty} \frac{1}{kT} \sum_{i=0}^{k-1} \log ||\psi_{-T}^*|\mathcal{F}^u(X_{-iT}(x))|| < -\eta.$$

where $\eta = \min\{|\lambda_i| : 1 \leq i \leq k\}$ and λ_i are the Lyapunov exponent associated to μ.

Except for countable many $T \in \mathbb{R}$, the measure μ is ergodic with respect to the time T map of the flow. Without loss of generality, we may assume that μ is an ergodic measure for the time 1 map X_1 and the general case is identical. Just as in Pesin theory, the **Pesin Block** Λ_C of vector fields is defined

$$\Lambda_C := \Big\{ x \in Supp(\mu) \Big| \prod_{i=0}^{n-1} \|\psi_1^*|\mathcal{E}(X_i(x))\| \leq Ce^{-n\eta},$$
$$\prod_{i=0}^{n-1} \|\psi_{-1}^*|\mathcal{F}(X_{-i}(x)))\| \leq Ce^{-n\eta}, \forall n \geq 1,\ d(x, Sing(X) \geq \frac{1}{C} \Big\}.$$

Since $\mu(Sing(X)) = 0$, $\mu(\Lambda_C) \to 1$ as $C \to \infty$. Therefore, we can choose C large enough such that $\mu(\Lambda_C)$ is close to 1.

- Locating the horseshoe

  We fix a point $z \in \Lambda_C$. Consider the local cross setion

$$N_z(\delta) = exp(E_z(\delta) \times F_z(\delta)) \quad \text{and} \quad V_z(\delta) = \bigcup_{t \in (-\delta, \delta)} X_t(N_z(\delta)),$$

  (see figure 2). We have $\mu(V_z(\delta) \cap \Lambda_C) > 0$

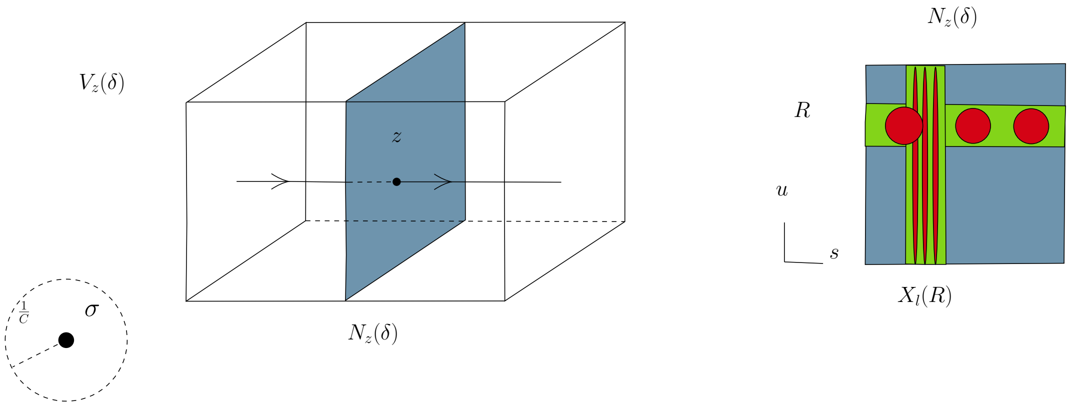

Figure 2 – Constructing a Horseshoe.

- Katok's Argument

  Following the Katok's argument, for $\alpha > 0, l > 0$ and $n \in \mathbb{N}^+$, we shall construct a finite set $K_n(\alpha, l)$ satisfying the following four properties

  1. $K_n(\alpha, l) \subset V_z(\delta/2) \cap \Lambda_C$;
  2. If $x, y \in K_n(\alpha, l)$ and $x \neq y$ then $d_n(x, y) = \max_{0 \leq i \leq n-1} d(X_i(x), X_i(y)) > \frac{1}{l}$;
  3. For every $x \in K_n(\alpha, l)$, there is an integer m_x with $n \leq m_x \leq (1 + \alpha)n$ such that
  $$X_{m_x}(x) \in V_z(\delta/2) \cap \Lambda_C \quad \text{and} \quad d(X_{m_x}(x), x) < \frac{1}{1000l};$$
  4. For every $\alpha > 0$, $\liminf_{l \to \infty} \liminf_{n \to \infty} \frac{1}{n} \log \# K_n(\alpha, l) \geq h_\mu(X) - \alpha$.

The construction of the set $K_n(\alpha, l)$ begins in accordance with Katok's definition of metric entropy $h_\mu(f)$ of f-invariant ergodic measures μ as

$$h_\mu(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_f(n, \varepsilon, \delta)$$

where $N_f(n, \varepsilon, \delta)$ is the minimal number of ε-balls in the d_n^f metric covering the set of measure larger than or equal to $1 - \delta$. Observing that $K_n(\alpha, l)$ can be taken within $N_z(\delta)$.

□

The following lemma states that the suspension of a full shift satisfies the intermediate entropy property.

Lemma 4.1.6 (Proposition 1.2 in [24])**.** Let $(\Sigma_l^\rho, \sigma_t^\rho)$ be a suspension flow of a full shift with $l \geq 2$. Then for any constant $h \in (0, h_{top}(\sigma_t^\rho, \Sigma_l^\rho))$, there exist a σ_t^ρ-ergodic measure $\tilde{\mu}$ satisfying

$$h_{\tilde{\mu}}(\sigma_t^\rho, \Sigma_l^\rho) = h.$$

Before giving the proof of Theorem B, we recall its statement.

Theorem B. Let X be a C^1-vector field on M. Suppose M contains an asymptotically sectional-hyperbolic attractor Λ for X. Then Λ has intermediate entropy property.

Proof of Theorem B. Let $h \in [0, h_{top}(X\mid_\Lambda))$. By the variational principle, there exists an ergodic measure μ such that $h < h_\mu(X\mid_\Lambda)$. By Theorem 3.1.1, μ is a hyperbolic measure.

Now, by Lemma 3.2.1, the splitting given by Oseledets Theorem (Theorem 2.4.1) for cocycle Ψ, Poincaré linear flow, is

$$\mathcal{E}(x) = \bigoplus_{\chi_i(x,\Psi)<0} H_i^\Psi(x) \quad \text{and} \quad \mathcal{E}^c(x) = \bigoplus_{\chi_i(x,\Psi)>0} H_i^\Psi(x),$$

and this splitting is dominated (Lemma 2.5.2). Thus, we can apply Lemma 4.1.4 to μ and obtain a horseshoe $K_h \subset \Lambda$ such that $h < h_{top}(X\mid_{K_h})$. Applying Lemma 4.1.6 to the horseshoe K_h, we obtain an ergodic measure $\tilde{\mu}$ such that $h_{\tilde{\mu}}(X\mid_\Lambda) = h$. □

The previous result was obtained for star flows in [24], but its proof can be completely reproduced in the context of ASH attractors except for one ingredient, which was unknown until the present work. Namely, the hyperbolicity of ergodic regular measures. Fortunately, this was achieved in Theorem 3.1.1.

4.2 Existence of Periodic Orbits

Before presenting the proof of Theorem E, we recall its statement.

Theorem E. If a C^1-vector field X on M contains a asymptotically sectional-hyperbolic attractor with positive topological entropy, then M contains a non-trivial homoclinic class.

Next, we shall present the proof of Theorem E, which states that any ASH attractor with positive topological entropy contains a nontrivial homoclinic class.

Proof of Theorem E. Assume that M contains an asymptotically sectional hyperbolic attractor Λ. By hypothesis we have that topological entropy is positive, $h_{top}(X) > 0$. So, Lemma 4.1.3 implies that Λ contains a hyperbolic horseshoe. Since hyperbolic horseshoes are homoclinic classes, the proof is complete. □

Before presenting the proof of Corollary F, we recall its statement.

Corollary F. Let X be a C^1 vector field on M, and let Λ be an asymptotically sectional-hyperbolic attractor. Then, there is a neighborhood U of Λ and a C^1 neighborhood $\mathcal{U}$ of X such that $X|_\Lambda$ is a point of lower semicontinuity for the entropy function on

$$\mathcal{X}^1(M,U) = \left\{ Y|_{\Lambda_Y} : \Lambda_Y = \bigcap_{t\geq 0} Y_t(U) \right\}.$$

In addition, if M is three-dimensional, then $X|_\Lambda$ is a point of continuity for the entropy function on $\mathcal{X}^1(M,U)$.

Next, we present the proof of Corollary F.

Proof of Corollary F. Let Λ be an ASH attractor for a C^1-vector field X on M. First, notice that if $h_{top}(X|_\Lambda) = 0$, then the result holds trivially. Indeed, since zero is the lowest possible value for the entropy of a flow, then for any other C^1-vector field Y on M and for any compact and Y-invariant subset $\Lambda_Y \subset M$, one has $h_{top}(Y|_{\Lambda_Y}) \geq h_{top}(X|_\Lambda) = 0$.

Next, suppose $h_{top}(X|_\Lambda) > 0$ and let $U \subset M$ be a trapping region of Λ. Fix $\varepsilon > 0$. By Lemma 4.1.3 Λ contains a horseshoe K_ε such that

$$h_{top}(X|_{K_\varepsilon}) \geq h_{top}(X|_\Lambda) - \frac{\varepsilon}{2}. \tag{4.1}$$

On the other hand, since hyperbolicity is a robust property, there is an open $\mathcal{U} \subset \mathcal{X}^1(M)$ such that any vector field $Y \in \mathcal{U}$ admits a hyperbolic set $\Lambda_{Y,\varepsilon} \subset \Lambda_Y \subset U$, which is a continuation of K_ε, such that

$$h_{top}(Y_{\Lambda_Y}) \geq h_{top}(Y|_{\Lambda_{Y,\varepsilon}}) > h_{top}(X|_{K_\varepsilon}) - \frac{\varepsilon}{2}. \tag{4.2}$$

Now, we conclude from (4.1) and (4.2) that $h_{top}(Y_{\Lambda_Y}) \geq h_{top}(X|_\Lambda) - \varepsilon$, and the proof is complete. □

4.2.1 Growth of Periodic Orbits

Before presenting the proof of Theorem G, we recall its statement

Theorem G. Let X be a C^1 vector field on M. Suppose M contains an asymptotically sectional-hyperbolic attractor Λ for X. Then,

$$\limsup_{t\to\infty} \frac{1}{t} \log \# P_t(X|_\Lambda) \geq h_{top}(X|_\Lambda)$$

where $\pi(x)$ is the period of x and $P_t(X) = \{\mathrm{Orb}(x, X) \mid x \in Per(X) \text{ e } 0 < \pi(x) < t\}$.

Now, we are going to prove Theorem G.

Lemma 4.2.1 (*Theorem 3.4* in [50])**.** Let μ be a regular hyperbolic ergodic measure of $X \in \mathcal{X}^1(M)$. If the splitting $\mathcal{N} = \mathcal{E}^s \oplus \mathcal{E}^u$ is dominated for the linear Poincaré flow Ψ, then

$$\limsup_{t\to\infty} \frac{1}{t} \log \# P_t(X) \geq h_\mu(X) := h_\mu(X_1).$$

In order to apply Lemma 4.2.1 for proving Theorem G, we must to lift a dominated splitting for X to a dominated splitting for the linear Poincaré flow.

Proof of Theorem G. Let X be a C^1 vector field. If $h_{top}(X) = 0$, then the result trivially holds. Therefore, let us suppose $h_{top}(X) > 0$. Let μ be an ergodic measure supported on Λ with positive topological entropy. This implies, in particular, that μ is regular. So, by Theorem 3.1.1, μ is a hyperbolic measure. Now, by the Lemma 3.2.1 we have

$$\mathcal{E}(x) = \bigoplus_{\chi_i(x,\Psi)<0} H_i^\Psi(x) \text{ and } \mathcal{E}^c(x) = \bigoplus_{\chi_i(x,\Psi)>0} H_i^\Psi(x),$$

and its splitting is dominated (Lemma 2.5.2). Thus, we can apply Lemma 4.2.1 to μ and obtain:

$$\limsup_{t\to\infty} \frac{1}{t} \log \# P_t(X) \geq h_\mu(X) := h_\mu(X_1).$$

Finally, by the Variational Principle, we conclude that

$$\limsup_{t\to\infty} \frac{1}{t} \log \# P_t(X) \geq h_{top}(X) := h_{top}(X_1),$$

and the proof is complete. □

4.3 Proof of Theorem A

Next, we provide a proof for Theorem A. We will achieve this by applying some of the results previously obtained. Let $X \in \mathcal{X}^1(M)$ and $\sigma \in Sing(X)$ be a hyperbolic singularity, and assume that the Lyapunov exponents of $DX_t(\sigma)$ are

$$\lambda_1 \leq \cdots \leq \lambda_s < 0 < \lambda_{s+1} \leq \cdots \leq \lambda_d.$$

Recall from [44] that the *saddle value* $sv(\sigma)$ *of* σ is defined by $sv(\sigma) = \lambda_s + \lambda_{s+1}$. In this case, we say that a hyperbolic singularity σ is

- *Lorenz-like*: if $\lambda_s + \lambda_{s+1} > 0$.
- *Rovella-like*: if $\lambda_s + \lambda_{s+1} < 0$.
- *Resonant*: if $\lambda_s + \lambda_{s+1} = 0$.

We say that an attractor Λ for a C^1 vector field X is *singular* if $Sing_\Lambda(X) = Sing(X) \cap \Lambda \neq \emptyset$. It is easy to check that if a singular attractor Λ is ASH, then its singularities are of one of the three types mentioned above.

Now, let V be a finite-dimensional vector space. We denote by $\wedge^2 V$ the second exterior power of V, defined as follows: If $\{v_1, \dots, v_n\}$ is a basis of V, then $\wedge^2 V$ is the vector space spanned by the set $\{v_i \wedge v_j\}_{i \neq j}$. In this way, any linear transformation $A : V \to W$ induces a transformation $\wedge^2 A : \wedge^2 V \to \wedge^2 W$. Moreover, the element $v_i \wedge v_j$ can be viewed as the 2-plane generated by v_i and v_j if $i \neq j$. See for instance [7] for more details.

Definition 4.3.1. The *sectional Lyapunov exponents of* x along E^c are the limits

$$\lim_{t\to\infty} \frac{1}{t} \log \| \wedge^2 DX_t(x) \cdot \tilde{v} \|,$$

whenever they exist, where $\tilde{v} \in \wedge^2 E^c_x \setminus \{0\}$

It turns out that if μ is an invariant probability measure, Y is the subset given by Oseledets Theorem (Theorem 2.4.1) and $\{\chi_i(x)\}_{i=1}^{k(x)}$ are the Lyapunov exponents of a point $x \in Y$, then its sectional Lyapunov exponents are given by $\{\chi_i(x) + \chi_j(x)\}_{1 \le i < j \le k(x)}$. Moreover, if L_x is a 2-plane, then it can be seen as $\widetilde{v} \in \Lambda^2 E^c_x \setminus \{0\}$ of norm one. In this way, the asymptotic expansion given in definition of ASH set can be rewritten as follows: There exists $C > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \left\| \wedge^2 DX_t(x) \cdot \tilde{v} \right\| \ge C > 0, \quad \forall x \in \Lambda \setminus W^s(Sing(X)).$$

Next, we present a useful result for proving Theorem A. But before, let us set some notation. Recall that if Λ is ASH, then the ASH splitting of Λ is denoted by $T\Lambda = E \oplus E^c$. On the other hand, any singularity contained on Λ is hyperbolic and it has a hyperbolic splitting. We shall denote the hyperbolic splitting given by $T_\sigma M = \overline{E}^s_\sigma \oplus \overline{E}^u_\sigma$.

Lemma 4.3.2. Let Λ an asymptotically sectional-hyperbolic attractor for a C^1 vector field X. Suppose:

1. All the singularities contained in it are Lorenz-like.

2. For every $\sigma \in Sing(X)$, one has $\dim(E(\sigma)) + 1 = \dim(\overline{E}^s_\sigma)$.

Then, Λ is sectional-hyperbolic.

Before to present the proof of above lemma, we state the following theorem, proved in [6].

Theorem 4.3.3. Let $\{X_t\}_{t\in\mathbb{R}}$ be a flow with a dominated splitting $T_\Lambda M = E \oplus E^c$ over an attracting set Λ whose singularities contained in it are hyperbolic. The flow $\{X_t\}_{t\in\mathbb{R}}$ is a sectional-hyperbolic flow if and only if the Lyapunov exponents in the E direction are negative and the sectional Lyapunov exponents in the E^c direction are positive on a set of total probability. If $\Lambda = M$ and the manifold has no boundary, the flow has no singularities and it is an hyperbolic flow.

Proof of Lemma 4.3.2. Suppose we are under the hypothesis of the Lemma. Let μ be a regular ergodic measure for the flow of X, and let $B \subset M$ be the set of regular points given by Oseledets theorem. Then, it is hyperbolic by Lemma 3.1.1 and, in particular, by Lemma 3.1.3 we have

$$E(x) = \bigoplus_{\chi(x)<0} H_i(x) \text{ and } E^c(x) = \bigoplus_{\chi(x)\geq 0} H_i(x).$$

Then, the Lyapunov exponents in $E(x)$ are negative for $x \in B$. Now, for $x \in B \setminus W^s(Sing(X))$, let $u, v \in E^c(x)$ and let consider $L = \langle u, v\rangle$ the plane spanned by u and v. Denote by $A(u, v)$ the area of the parallelogram defined by the vectors u and v. We have that

$$\begin{aligned} A(DX_t(x)u, DX_t(x)v) &= |\det DX_t(x)\,|_L\,|A(u, v) \\ &= \sin\theta_t \cdot \|DX_t(x)u\| \cdot \|DX_t(x)v\|, \end{aligned} \tag{4.3}$$

where $\theta_t(x)$ represents the angle between $DX_t(x)u$ and $DX_t(x)v$. Then, the asymptotic area expansion on L implies

$$\limsup_{t\to\infty} \frac{1}{t} \log|\det DX_t(x)|_L| \geq C > 0.$$

On the other hand, by definition of the Lyapunov exponents and (4.3),

$$\limsup_{t\to\infty} \frac{1}{t} \log|\det DX_t(x)|_L| = \chi(x, u) + \chi(x, v),$$

so that $\chi(x, u) + \chi(x, v) \geq C > 0$. Thus, the Lyapunov exponent sectional are positive. Since $\mu(W^s(Sing(X)) \setminus Sing(X)) = 0$ for any μ measure invariant, by the recurrence Poincare theorem we assume that $B \cap (W^s(Sing(X)) \setminus Sing(X)) = \emptyset$. So, we need to verify that the property of having positive sectional lyapunov exponents is satisfied in the singularities. For this, recall that by the Lemma 2.2.2 we have $E(\sigma) \subset \overline{E}^s_\sigma$. On the other hand, by hypotheses we have

$$\dim(E(\sigma)) + 1 = \dim(\overline{E}^s_\sigma),$$

which implies that $E^c(\sigma)$ contains one contracting direction. In this way, the Lyapunov exponents in $E(\sigma)$ are $\lambda_1 \leq \cdots \leq \lambda_{s-1} < 0$, and the Lyapunov exponent in $E^c(\sigma)$ are $\lambda_s < 0 < \lambda_{s+1} \leq \cdots \leq \lambda_d$. Therefore, the sectional Lyapunov exponent are $\{\lambda_i + \lambda_j\}_{s-l \leq i < j \leq d}$ which are positive because σ is Lorenz-like. Then, by Theorem 4.3.3, we see that Λ is sectional-hyperbolic. □

The last ingredient in the proof of Theorem A is the connecting lemma, given in [49] and Lemma 4.4 in [44].

Lemma 4.3.4 (Connecting lemma)**.** For any vector field $X \in \mathcal{X}^1(M^d)$ and any neighborhood $\mathcal{U}$ of X,for any point $z \notin Per(X) \cup Sing(X)$, there exist $L > 0$, $\rho > 1$, $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, for any p and q in $M \setminus \Delta$ ($\Delta = \cup_{0 \leq t \leq L} X_t(B_\delta(z))$), if both positive orbit of p and the negative orbit of q enter into $B_{\delta/\rho}(z)$, then there is $Y \in \mathcal{U}$ such that

- q is on the positive orbit of p with respect to the flow Y_t generated by Y.
- $Y(x) = X(x)$ for any $x \in M \setminus \Delta$.

Remark 4.3.1*.* Symmetrically, we can reformulate the above Theorem for a tube along the negative orbit of z.

In [44] the authors prove that for star fields there are no resonant singularities [44, Corollary 4.3] and they define the periodic index, Ind_p, for singularities of a star field as

$$\mathrm{Ind}_p(\sigma) = \begin{cases} \mathrm{Ind}(\sigma) - 1 & if \quad \sigma \text{ is Lorenz-like,} \\ \mathrm{Ind}(\sigma) & if \quad \sigma \text{ is Rovella-like.} \end{cases} \tag{4.4}$$

The following lemma [44, lemma 4.4] asserts that when a periodic orbit is close enough to a homoclinic loop associated to some singularity, then its index has to be equal to the periodic index of the singularity.

Lemma 4.3.5 (Lemma 4.4 in [44])**.** Let $X \in \mathcal{X}^*(M)$ and $\sigma \in Sing(X)$. Let $\Gamma = Orb(x)$ be a homoclinic orbit associated to σ. Assume that there exists a sequence of vector fields $\{X^n\}$ converging to X in the C^1 topology and periodic orbits P_n of X^n such that converges $\{P_n\}$ to $\Gamma \cup \{\sigma\}$ in the Hausdorff topology. Then we have

$$\lim_{n\to\infty} \mathrm{Ind}(P_n) = \mathrm{Ind}_p(\sigma),$$

i.e., for n large enough, $\mathrm{Ind}(P_n)$=$\mathrm{Ind}_p(\sigma)$.

Remark 4.3.2. When we consider another kind of critical element, periodic orbits, this also holds. Specifically, if the periodic orbit Q_n tends to a homoclinic orbit $\Gamma = \{Orb(x)\}$ associated with some periodic orbits P, then we must have $\mathrm{Ind}(Q_n) = \mathrm{Ind}(P)$ for n large enough.

Before presenting the proof of Theorem A, we recall its statement

Theorem A. Every asymptotically sectional-hyperbolic attractor associated with C^1 vector field X on M, having positive topological entropy and satisfying the star property is sectional-hyperbolic.

Next, we are ready to prove Theorem A.

Proof of Theorem A. Let X be a star C^1-vector field, and let Λ be an ASH attractor for X. If Λ is non-singular, the Hyperbolic lemma shows that it is, in fact, hyperbolic of saddle type. So, Λ is sectional-hyperbolic.

Now, suppose that the ASH attractor Λ is singular. Then, since Λ is star, no singularity of Λ can be resonant according to [44, Corollary 4.3]. Therefore, in light of Lemma 4.3.2, to conclude the proof we need to show the following two assertions:

a) Every singularity in Λ is Lorenz-like,

b) $\dim(E(\sigma)) + 1 = \dim(\overline{E}^s_\sigma)$, for every $\sigma \in \Lambda$.

We recall that since X is star, there is a neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X such that every singularity or periodic orbit of $Y \in \mathcal{U}$ is hyperbolic. Assume that Λ contains a singularity σ. We will show that it is Lorenz-like.

Now, by hypothesis the entropy topological is positive, $h_{top}(X|_\Lambda) > 0$, which implies, by Theorem G, that there exists a periodic point $p \in \Lambda$. In addition, the Hyperbolic lemma shows that $O(p)$ is hyperbolic of saddle type. By shrinking $\mathcal{U}$, if needed, we can assume that the continuation σ_Y of σ, for any $Y \in \mathcal{U}$, is also Rovella-like or Lorenz-like, and the continuation p_Y of p is also hyperbolic of saddle type. In this way, by Lemma 2.2.4 and transitivity, for every $\varepsilon > 0$ we can find points

$$x_s^\sigma \in W^s(\sigma), x_u^\sigma \in W^u(\sigma), x_s^p \in W^s(p), x_u^p \in W^u(p)$$

and regular orbits γ_1 and γ_2 such that

$$d\left(\gamma_1, \{x_u^\sigma, x_s^p\}\right), d\left(\gamma_2, \{x_u^p, x_s^\sigma\}\right) < \varepsilon.$$

So, using the Connecting Lemma (Lemma 4.3.4) four times connecting x_u^σ to γ_1, connecting γ_1 to x_s^p, connecting x_u^p to γ_2 and finally connecting γ_2 to x_s^σ (see figure 3), there is a vector field $Y \in \mathcal{U}$ admitting a cycle Γ_Y associated to the continuation σ_Y and p_Y. More precisely, there are regular orbits $\gamma_1^Y \in W^u(\sigma_Y) \cap W^s(p_Y)$ and $\gamma_2^Y \in W^u(p_Y) \cap W^s(\sigma_Y)$ such that

$$\Gamma_Y = \{\sigma_Y\} \cup \{p_Y\} \cup \gamma_1^Y \cup \gamma_2^Y.$$

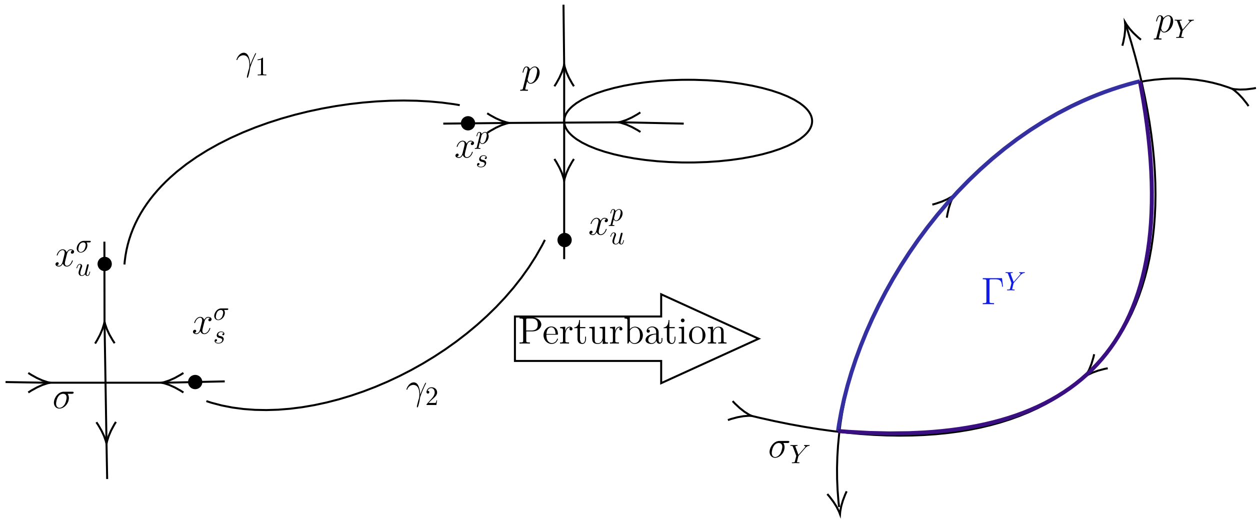

Figure 3 – Before and after of perturbation.

Now, following the arguments of [44, lemma 4.5] we claim

Claim: There is a one-parameter family of vector fields $\{Y_r\}_{r_0 \leq r \leq r_1}$ contained in $\mathcal{U}$, such that Γ_{r_0} is a periodic orbit with $\mathrm{Ind}(\Gamma_{r_0}) = \mathrm{Ind}(p_Y)$ for Y_{r_0} and Γ_{r_1} is a periodic orbit $\mathrm{Ind}(\Gamma_{r_1}) = \mathrm{Ind}_p(\sigma_Y)$ for Y_{r_1}.

Proof of Claim. In two disjoint linearizable neighborhood of σ_Y and p_Y, choose two pairs of points $\{x_s, y_u\}$ and $\{x_u, y_s\}$ such that

- $x_s \in W^s_{loc}(\sigma_Y) \cap Orb(x, Y)$ and $y_u \in W^u_{loc}(\sigma_Y) \cap Orb(y, Y)$,
- $x_u \in W^s_{loc}(p_Y) \cap Orb(x, Y)$ and $y_s \in W^u_{loc}(p_Y) \cap Orb(y, Y)$.

Then we can choose two points $x_{s,r}, y_{u,r}$ such that $x_{s,r} \to x_s$, $y_{u,r} \to y_u$ when $r \to 0$ and two pairs of continuous segments $\{x_{s,r}, y_{u,r}\}$, $0 < r \leq 1$ and $\{x_{u,l}, y_{s,l}\}$, $0 < l \leq 1$ such that

- $Y_{t_r}(x_{s,r}) = y_{u,r}$, $x_{s,0} = x_s$ and $y_{u,0} = y_u$;
- $Y_{t_l}(x_{u,l}) = y_{s,l}$, $x_{u,0} = x_u$ and $y_{s,0} = y_s$.

Using Lemma 4.3.4(Connecting Lemma) we connect x_s to $x_{s,r}$ and y_u to $y_{u,r}$; x_u to $x_{u,l}$ and y_s to $y_{s,l}$ continuously (see figure 4), we get a continuous family of vector fields $\{Y_{r,l} : 0 \leq r, l \leq 1\} \subset \mathcal{U} \subset \mathcal{X}^*(M)$ with two parameters which satisfies

- $\lim_{r,l\to 0} Y_{r,l} = Y$
- $Y_{0,l}$ exhibits a homoclinic orbit associated to σ_Y , denoted by $\Gamma_{0,l}$, for $0 < l \leq 1$.
- $Y_{r,0}$ exhibits a homoclinic orbit associated to p_Y , denoted by $\Gamma_{r,0}$, for $0 < r \leq 1$.
- $Y_{r,l}$ exhibits a periodic orbit $\Gamma_{r,l}$ satisfying

$$\lim_{r\to 0} \Gamma_{r,l} = \Gamma_{0,l} \text{ and } \lim_{l\to 0} \Gamma_{r,l} = \Gamma_{r,0}$$

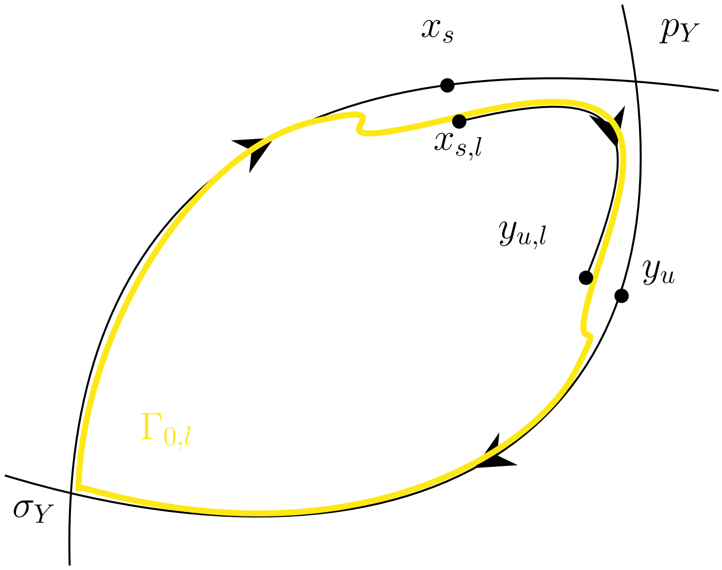

Figure 4 – Perturbation $Y_{0,l}$ and cicle $\Gamma_{0,l}$ in yellow.

We fix some $r_0 > 0$ and let $l \to 0$ be sufficiently small. For $l = l_0$ small enough, Lemma 4.3.5 ensures that

$$\text{Ind}(\Gamma_{r_0,l_0}) = \text{Ind}(p_Y).$$

Then letting $\Gamma_{r,l_0} \to \Gamma_{0,l_0}$ as $r \to 0$, and applying Lemma 4.3.5 again, we know there is some $r_0 < r_1$ such that

$$\text{Ind}(\Gamma_{r_1,l_0}) = \text{Ind}_p(\sigma_Y).$$

Considering $Y_r := Y_{r,l_0}$ for $r \in [r_0, r_1]$ the claim is proven. □

Since the family of vectors fields $\{Y_r : r_0 \leq r \leq r_1\}$ is continuous on r in the C^1 topology, the Lyapunov exponent of Γ_{r,l_0} is also continuous on r. This implies that

$$\text{Ind}(p_Y) = \text{Ind}(\Gamma_{r_0,l_0}) = \text{Ind}(\Gamma_{r_1,l_0}) = \text{Ind}_p(\sigma_Y). \tag{4.5}$$

Since if $\text{Ind}(\Gamma_{r_0,l_0}) \neq \text{Ind}(\Gamma_{r_1,l_0})$ implies that there must be some r_2 with $r_0 < r_2 < r_1$, such that Γ_{r_2,l_0} is a nonhyperbolic periodic orbit, contradicting $Y_{r_2,l_0} \in \mathcal{U} \subseteq \mathcal{X}^*(M)$. As σ_Y and

p_Y are continuations of σ and p by (4.5) we have Ind(p)=Ind$_p(\sigma)$ and remember that the periodic point $p \in \Lambda$, by the property ASH, must satisfy

$$E(p) = \overline{E}^s(p) \text{ and } E^c(p) = \langle X(p) \rangle \oplus \overline{E}^u(p),$$

where $T_pM = \overline{E}^s(p) \oplus \langle X(p) \rangle \oplus \overline{E}^u(p)$ is its hyperbolic splitting. Thus, by the transitivity of Λ we have that $\dim(E)$ is constant and

$$\dim(E) = \dim(E(p)) = \operatorname{Ind}(p) = \operatorname{Ind}_p(\sigma).$$

By Lemma 2.2.7 we have $\dim(E) < \dim(\overline{E}^s_\sigma) = \operatorname{Ind}(\sigma)$. Therefore, by (4.4) we have

$$\dim(E) = \operatorname{Ind}_p(\sigma) = \dim(\overline{E}^s_\sigma) - 1.$$

This proves *a*) and *b*). This concludes the proof. □

5 Three-dimensional ASH attractors

In this chapter, we prove Theorem C and Theorem D.

5.1 Entropy Expansiveness

The proof of Theorem C is essentially based on three results. The first two of them are elementary facts from flow theory, and we present their proofs here for the sake of completeness.

Lemma 5.1.1. Let $\Phi = \{X_t\}_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space M. For every $\alpha > 0$, there exists $\beta > 0$ such that if $y \in B^\infty_\beta(x, X_1)$, then

$$d(X_t(x), X_t(y)) \leq \alpha$$

Proof. Fix $\alpha > 0$. Since M is compact and Φ is a continuous flow, we can find $\beta > 0$ such that if $d(x, y) \leq \beta$, then $d(X_t(x), X_t(y)) \leq \alpha$, for every $t \in [0, 1]$. Take $y \in B^\infty_\beta(x, X_1)$ and fix $t \in \mathbb{R}$. One can write $t = n_t + r_t$, with $n_t \in \mathbb{Z}$ and $0 \leq r_t < 1$. By hypothesis, we have $d(X_{n_t}(x), X_{n_t}(y)) \leq \beta$ and so

$$d(X_t(x), X_t(y)) = d(X_{n_t+r_t}(x), X_{n_t+r_t}(y)) = d(X_{r_t}(X_{n_t}(x)), X_{r_t}(X_{n_t}(y))) \leq \alpha.$$

This concludes the proof. □

Notice that if one denotes

$$B^\infty_\alpha(x, X) = \{y \in M; d(X_t(x), X_t(y)) \leq \alpha, \forall t \in \mathbb{R}\},$$

then the previous lemma says that $B^\infty_\beta(x, X_1) \subset B^\infty_\alpha(x, X)$, if $\beta > 0$ is small enough.

Lemma 5.1.2. Let $\Phi = \{X_t\}_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space M. Then, $h_{top}(X_{[-\varepsilon,\varepsilon]}(x)) = 0$ for every $\varepsilon > 0$ and $x \in M$.

Proof. First, we claim that for every $\eta > 0$ there is $\varepsilon_0 > 0$ such that if $y \in X_{[-\varepsilon_0,\varepsilon_0]}(x)$, then $d(X_t(x), X_t(y)) \leq \eta$, for any $t \in \mathbb{R}$. Indeed, otherwise there is $\eta > 0$ and sequences $x_n \in M$, $t_n \in \mathbb{R}$ and $s_n \in \mathbb{R}$ with $s_n \to 0$ such that

$$d(X_{t_n}(x_n), X_{t_n+s_n}(x_n)) > \eta, \quad \forall n \geq 1. \tag{5.1}$$

By compactness of M and the continuity of the flow, there is $s > 0$ such that $d(X_t(x), x) < \eta$ for any $x \in M$ and $|t| < s$. So, by (5.1) we obtain

$$\eta > d(X_{s_n}(X_{t_n}(x_n)), X_{t_n}(x_n))) = d(X_{s_n+t_n}(x_n), X_{t_n}(x_n)) \geq \eta$$

for n large enough, which leads us to a contradiction.

Now, we fix $\eta > 0$. By the above claim, there is $\varepsilon_0 > 0$ such that

$$d(X_t(x), X_t(y)) \leq \eta, \quad \forall t \in \mathbb{R},$$

for every $x \in M$ and $y \in X_{[-\varepsilon_0,\varepsilon_0]}(x)$. So, if $\varepsilon_0 > \varepsilon$, we have $S(t,\eta) = 1$, where $S(t,\eta)$ denotes the minimum cardinality of a (t,η)-spanning set, whereas if $\varepsilon_0 < \varepsilon$, there is $N \in \mathbb{N}$ such that $N\varepsilon_0 > \varepsilon$, so that $S(t,\eta) \leq N$. Then, we have the desired result by the definition of topological entropy. □

In light of the above lemmas, the proof of Theorem C is reduced to obtain the following result:

Theorem 5.1.3. Every asymptotically sectional-hyperbolic attractor Λ associated to C^1 vector field X on a three-dimensional manifold M is *kinematic expansive*, i.e, for every $\varepsilon > 0$, there is $\delta > 0$ such that if $x, y \in \Lambda$ satisfy

$$d(X_t(x), X_t(y)) \leq \delta \quad \forall t \in \mathbb{R},$$

then $y \in X_{[-\varepsilon,\varepsilon]}(x)$.

Remark 5.1.1. Note that kinematic expansiveness is a weaker form of expansiveness since it does not care about reparametrizations. For a more detailed discussion about the properties of kinematic expansive flows, we refer the reader to the work [9].

We explain why these results imply Theorem C. Before presenting the proof of Theorem C, we recall its statement.

Theorem C. Every asymptotically sectional-hyperbolic attractor Λ associated with a C^1 vector field X on a three-dimensional manifold M is entropy-expansive.

Proof of Theorem C.. Fix $\eta > 0$. By Theorem 5.1.3 there is $\varepsilon > 0$ such that $B^\infty_\varepsilon(x) \subset X_{[-\eta,\eta]}(x)$ for any $x \in M$. Let $\beta > 0$ be as in Lemma 5.1.1 with respect to ε. Then, for every $x \in M$ one has

$$B^\infty_\beta(x, X_1) \subset B^\infty_\varepsilon(x, X) \subset X_{[-\eta,\eta]}(x),$$

and the proof follows from Lemma 5.1.2. □

From now on, we will devote ourselves to obtaining a proof of Theorem 5.1.3. Many parts of the arguments presented here resemble that of the proof of Theorem 2.5 in [38]. First, we state some known results that will be used in the proof.

Let σ be a singularity of Λ. We say that σ is

- *attached* to Λ if it is accumulated by regular orbits in Λ,

- *real index two* if it has three real eigenvalues satisfying $\lambda_{ss} < \lambda_s < 0 < \lambda_u$.

For a real index-two singularity σ, we say that it is *Lorenz-like* if its eigenvalues satisfy the relation $\lambda_u + \lambda_s > 0$, called *central expanding condition.* In [15], it was shown that the singularities contained in a connected singular-hyperbolic set must be Lorenz-like. This is not true for ASH sets in general (see [42] for instance). For ASH sets, we state the following result, whose proof is analogous to that of Theorem A in [32]:

Theorem 5.1.4. Let Λ be a nontrivial asymptotically sectional-hyperbolic set of X and assume that Λ is not hyperbolic. Then, Λ has at least one attached singularity. In addition, if Λ is transitive, the following holds for X: Each singularity σ of Λ is real of index two and satisfies

$$\Lambda \cap W^{ss}(\sigma) = \{\sigma\}, \tag{5.2}$$

where $W^{ss}(\sigma)$ denotes the strong stable manifold associated with λ_{ss}.

Now, let $E^s \oplus E^c$ be the partially hyperbolic splitting associated with the asymptotically sectional-hyperbolic attractor Λ, and consider a continuous extension $\widetilde{E}^s \oplus \widetilde{E}^c$ to the trapping region U_0. According to Proposition 3.2 in [4], the subbundle $\widetilde{E}^s$ can be chosen DX_t-invariant for positive t. Nevertheless, the subbundle $\widetilde{E}^c$ is not invariant in general, but we can consider a cone field C^c_a of size $a > 0$ around $\widetilde{E}^c$ on U_0 defined by

$$C^c_a(x) := \{v = v_s + v_c : v_s \in \widetilde{E}^s, v_c \in \widetilde{E}^c \text{ and } \|v_s\| \le a\|v_c\|\}, \quad \forall x \in U_0,$$

which is invariant for $t > 0$ large enough, i.e., there is $T_0 > 0$ such that

$$DX_t(x)C^c_a(x) \subset C^c_a(X_t(x)), \quad \forall t \ge T_0, \forall x \in U_0.$$

Remark 5.1.2. By possibly shrinking the neighborhood U_0, the number $a > 0$ can be taken arbitrarily small.

To simplify the notation, we write E^s and E^c for $\widetilde{E}^s$ and $\widetilde{E}^c$ respectively in what follows.

Now, we consider a special kind of neighborhood of the singularities contained in an ASH attractor. Let Λ be an ASH attractor on a three-dimensional manifold M. Let $x \in \Lambda$ and Σ' be a cross-section to X containing x in its interior. Define $W^s(x, \Sigma')$ as the connected component of $W^s(x) \cap \Sigma'$. This gives us a foliation $\mathcal{F}_{\Sigma'}$ of Σ'. We can construct a smaller cross section Σ, which is the image of a diffeomorphism $h : [-1, 1] \times [-1, 1] \to \Sigma'$, that sends vertical lines inside $\mathcal{F}_{\Sigma'}$ in a such way that x belongs to the interior of $h([-1, 1] \times [-1, 1])$. In this case, the *s-boundary* $\partial^s\Sigma$ and *cu-boundary* $\partial^{cu}\Sigma$ of Σ are defined by

$$\partial^s\Sigma = h(\{-1, 1\} \times [-1, 1]) \quad \text{and} \quad \partial^{cu}\Sigma = h([-1, 1] \times \{-1, 1\})$$

respectively. We say that Σ is η-*adapted* if

$$d(\Lambda \cap \Sigma, \partial^{cu}\Sigma) > \eta.$$

A consequence of the Hyperbolic lemma (see [41]) is the following:

Proposition 5.1.1. Let Λ be an asymptotically sectional-hyperbolic attractor, and let $x \in \Lambda$ be a regular point. Then, there exists a η_0-adapted cross-section Σ at x for some $\eta_0 > 0$.

Remark 5.1.3. From any $x \in M$ and any cross-section containing x in its interior, one can obtain an η_0-adapted cross-section containing x in its interior.

Next, we recall the construction performed in [45] of partitions for singular cross sections. These partitions give us a detailed picture of the flow dynamics inside small neighborhoods of the singularities of Λ. According to that reference we can find $\beta_1 > 0$ such that:

(a) $B_{\beta_1}(\sigma) \cap B_{\beta_1}(\sigma') = \emptyset$, where $B_r(a)$ denotes the open ball centered in a and radius $r > 0$ and $\sigma, \sigma' \in Sing_\Lambda(X)$.

(b) The map exp_σ is well defined on $\{v \in TM_\sigma : \|v\| \leq \beta_1\}$ for every $\sigma \in Sing_\Lambda(X)$.

(c) There are $L_0, L_1 > 0$ such that

$$L_0 \leq \frac{\|X(x)\|}{d(x,\sigma)} \leq L_1, \quad \forall x \in \overline{B_{\beta_1}(\sigma)}, \quad \forall \sigma \in Sing_\Lambda(X).$$

(d) The flow in $B_{\beta_1}(\sigma)$ is a small C^1 perturbation of the linear flow.

For every $\sigma \in Sing_\Lambda(X)$, define the *singular cross-section*

$$D_\sigma = exp_\sigma(\{v = (v^s, v^u) \in TM_\sigma : \|v\| \leq \beta_1, \|v^s\| = \|v^u\|\}) \subset M,$$

and the following partition of D_σ

$$D_n = D_\sigma \cap (B_{e^{-n}}(\sigma) \setminus B_{e^{-(n+1)}}(\sigma)), \quad \forall n \geq n_0,$$

where n_0 is large enough. As noticed by the authors in [45], the partition $\{D_n\}_{n \geq n_0}$ induces a partition of the cross sections $\Sigma_\sigma^{i,o,\pm}$ given in [5]. More precisely, assume that $\Sigma_\sigma^{i,o,\pm} \subset \partial B_{\beta_1}(\sigma)$. Consider

$$D_n^o = \bigcup_{x \in D_n} X_{t_x^+}(x), \quad D_n^i = \bigcup_{x \in D_n} X_{-t_x^-}(x), \quad \forall n \geq n_0,$$

where

$$t_x^+ = \inf\{\tau > 0 : X_\tau(x) \in \Sigma_\sigma^{o,\pm}\},$$

and

$$t_x^- = \inf\{\tau > 0 : X_{-\tau}(x) \in \Sigma_\sigma^{i,\pm}\}.$$

Note that the sets $\{D_n^i \cap \Sigma_\sigma^{i,\pm}\}_{n \geq n_0}$, illustrated in the Figure 5, form a partition of $\Sigma_\sigma^{i,\pm}$ for which $X_{\tau(x)}(x) \in D_n^o \cap \Sigma_\sigma^{o,\pm}$ for every $x \in D_n^i \cap \Sigma_\sigma^{i,\pm}$ and every $n \geq n_0$, where $\tau(\cdot)$ is the flight time to go from $\Sigma_\sigma^{i,\pm}$ to $\Sigma_\sigma^{o,\pm}$. Moreover, it is shown that this flight time satisfies

$$\tau(x) \approx \left(\frac{\lambda_u + 1}{\lambda_u}\right) \cdot n, \quad \forall x \in D_n^i \cap \Sigma_\sigma^{i,\pm}, \quad \forall n \geq n_0. \tag{5.3}$$

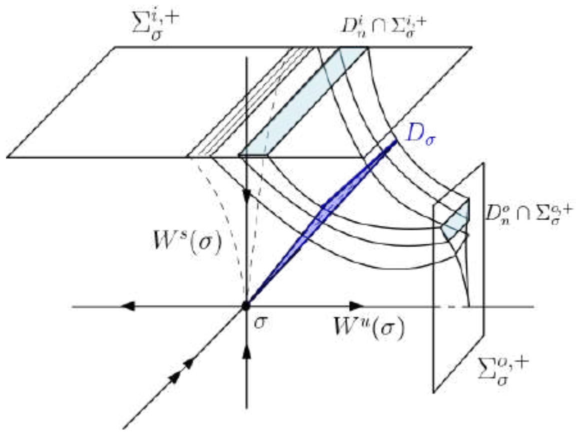

Figure 5 – The sets $D_n^{i,o} \cap \Sigma_\sigma^{i,o,\pm}$.

In this case, let $\widetilde{\Sigma_\sigma^{i,\pm}} = \left(\left(\bigcup_{n \geq n_0} D_n^i \cap \Sigma_\sigma^{i,\pm}\right)\right) \cup \ell_\pm$, where $\ell_\pm$ is contained in $W^s_{loc}(\sigma) \cap \Sigma_\sigma^{i,\pm}$, and let consider

$$V_\sigma = \bigcup_{z \in \widetilde{\Sigma_\sigma^{i,\pm}} \setminus \ell_\pm} X_{(0,\tau(z))}(z) \cup (-e^{-n_0}, e^{-n_0}) \times (-e^{-n_0}, e^{-n_0}) \times (-1,1) \subset U. \tag{5.4}$$

Denote $\widetilde{\Sigma^{i,o,\pm}} = \bigcup_{\sigma \in Sing_\Lambda(X)} \widetilde{\Sigma_\sigma^{i,o,\pm}}$ and $V = \bigcup_{\sigma \in Sing_\Lambda(X)} V_\sigma$.

Remark 5.1.4. We have the following remarks:

- We can assume without loss of generality that every cross section in $\widetilde{\Sigma^{i,o,\pm}}$ is η_0 adapted for some $\eta_0 > 0$.
- The above construction can be made by taking $\widetilde{\beta} < \beta_1$. In this case, we denote

$$\widetilde{\Sigma_{\widetilde{\beta}}^{i,o,\pm}} = \bigcup_{\sigma \in Sing_\Lambda(X)} \widetilde{\Sigma_{\sigma,\widetilde{\beta}}^{i,o,\pm}} \quad \text{and} \quad V_{\widetilde{\beta}} = \bigcup_{\sigma \in Sing_\Lambda(X)} V_{\sigma,\widetilde{\beta}}$$

Before continuing with the proof, we will take a little break to briefly outline the following lemmas to make our argument clearer. Our main goal is to obtain some sort of hyperbolicity for Poincaré maps from the ASH property. In [5], this property is obtained from the uniform area expansion property of the sectional hyperbolicity. Nevertheless, we can only see area expansion on the center bundle for ASH attractors during hyperbolic times.

In what follows, we prove that if two points satisfy the shadowing condition given in the definition of kinematic expansiveness, they share the same hyperbolic times. To do this, we need some previous lemmas to help us control the hyperbolic times of a pair of close points, depending on where they are located in the phase space. More precisely, Lemma 5.1.5 controls the hyperbolic times of points away from singularities, Lemma 5.1.6 controls the hyperbolic times of points before entering a neighborhood of a singularity and Lemma 5.1.9 helps us to control the hyperbolic times of points crossing a neighborhood of a singularity.

First, denote $\Lambda_+ = \bigcap_{t\geq 0} X_{-t}(\Lambda \setminus V)$. Then, we have the following lemma.

Lemma 5.1.5 (*Lemma 3.2 in* [38])**.** Given $\varepsilon_0 > 0$, there are positive numbers $\delta_1(\varepsilon_0), T_0$ and $K_2 = K_2(\varepsilon_0) \approx 1$, and a neighborhood W_0 of Λ_+ such that for any $x, y \in W_0$ with $d(x,y) < \delta_1(\varepsilon_0)$, then

$$\frac{|\det DX_{t_1}(y)|_{E^c_y}|}{|\det DX_{t_1}(x)|_{E^c_x}|} \geq K_2, \quad 0 < t_1 \leq T_0, \tag{5.5}$$

where $t_1 \leq T_0$ is a first hyperbolic time for x and y.

Let us consider the following compact set $\Lambda'' = \Lambda \setminus (V \cup W_0)$, where W_0 is as the above lemma.

Lemma 5.1.6 (*Lemma 3.3 in* [38])**.** Given $\varepsilon_1 > 0$, there are positive numbers $\delta_2(\varepsilon_1), T_1$ and $K_3 = K_3(\varepsilon_1) \approx 1$, and a neighborhood W_1 of Λ'' such that for every $x, y \in W_1$, with $d(x,y) \leq \delta_2(\varepsilon_1)$, there is $0 < s \leq T_1$ such that $X_s(x), X_s(y) \in V$ and

$$\frac{|\det DX_s(y)|_{E^c_y}|}{|\det DX_s(x)|_{E^c_x}|} \geq K_3. \tag{5.6}$$

We need, in addition, the following result

Lemma 5.1.7 (*Lemma 3.4 in* [38])**.** Let $\widetilde{U} = \overline{U \setminus V}$, where U is the basin of attraction of Λ. There are $\beta' > 0$ and $\varepsilon_2 > 0$ such that if $x \in \widetilde{U}$, and $y, z \in B_{\varepsilon_2}(x)$, $z \in \mathcal{O}(y)$, then $z = X_u(y)$, $|u| < 2\beta'$.

Lemma 5.1.8 (*Lemma 3.5 in* [38])**.** There exists $\delta_3 > 0$ with the following property: For $x \in \Sigma^{i,o,\pm}_{\widetilde{\beta}}$, $\sigma \in Sing_\Lambda(X)$ and $z \in M$ with $d(x,z) < \delta_3$ exist $l \in \mathbb{R}$, with $|l| < L$, such that $X_l(z) \in \Sigma^{i,\pm}_{\widetilde{\beta}}$, where $L = \frac{\lambda_u + 1}{\lambda_u}$.

Next lemma is contained in the proof of Theorem 2.5 in [38]. For the reader convenience, we state it here separately and provide a proof.

Lemma 5.1.9. Let $\widetilde{\varepsilon}_0 > 0$. There are a positive number δ_0, independent of $\widetilde{\varepsilon}_0$, and a positive number $\widetilde{\beta}$ such that if $x, y \in \widetilde{\Sigma^{i,\pm}_{\widetilde{\beta}}}$ satisfy

$$d(X_t(x), X_t(y)) \leq \delta_0$$

for every $t \in \mathbb{R}$, then

$$\frac{|\det DX_{\tau(y)}(y)|_{E^c_y}|}{|\det DX_{\tau(x)}(x)|_{E^c_x}|} \geq K_1 C_0^{\tau(x)}, \tag{5.7}$$

where $C_0 = C_0(\widetilde{\varepsilon}_0)$ and K_1 is a fixed positive constant.

Proof. Let σ be an attached singularity of Λ. First, recall that since σ is hyperbolic, we can use the Grobman-Hartman Theorem to conjugate X_t with its linear part in a small neighborhood of σ. More precisely, there are a neighborhood U_σ of σ, a neighborhood V_0 of $0 \in T_\sigma M$ and a homeomorphism $h : U_\sigma \to V_0$ that conjugates X_t with its linear flow L_t. Furthermore, this homeomorphism can be chosen to be Hölder continuous (see [10] for more details). Thus, there are $C = C(\sigma) > 0$ and $\alpha(\sigma) > 0$ such that

$$\|h(x) - h(y)\| \leq C d(x, y)^\alpha, \quad \forall x, y \in U_\sigma. \tag{5.8}$$

Since we have only finitely many singularities in Λ, we can shrink U_σ, $\sigma \in Sing_\Lambda(X)$, if it is necessary to obtain uniform Hölder constants for every singularity, i.e., we can write (5.8) with

$$C = \max_{\sigma \in Sing_\Lambda(X)} \{C(\sigma)\} \text{ and } \alpha = \min_{\sigma \in Sing_\Lambda(X)} \{\alpha(\sigma)\}.$$

Now, let us consider

$$W^{ss}(0) = \{(a, b, c) : b = c = 0\}, \quad W^s(0) = \{(a, b, c) : b = 0\}$$

and

$$W^u(0) = \{(a, b, c) : a = c = 0\}.$$

inside of the neighborhood V_0. We can assume that $V_\sigma \subset U_\sigma$, where V_σ is as in (5.4). In particular, we have that $\Sigma^{i,o,\pm}_\sigma \subset U_\sigma$ for every $\sigma \in \Lambda$. Since h is a homeomorphism, it induces a partition of $h(\Sigma^{i,\pm}_\sigma) \subset U_0$. Indeed, we just need to consider the family

$$\mathcal{F} = \{h(D^i_n \cap \Sigma^{i,\pm}_\sigma)\}_{n \geq n_0}.$$

By the remarks in page 380 of [45], there is $K' > 0$ such that for every $x \in D^i_n \cap \Sigma^{i,\pm}_\sigma$, we have

$$d(x, W^s(\sigma)) \leq K' e^{-(\lambda_u + 1)n}.$$

Then, since $h(W^s(\sigma)) = W^s(0)$, the Hölder estimates of h gives us

$$\begin{aligned} d(h(x), W^s(0)) &\leq C d(x, W^s(\sigma))^\alpha \\ &\leq C K' e^{-\alpha(1+\lambda_u)n} \\ &= e^{\left(\frac{\ln(CK')}{n} - \alpha(1+\lambda_u)\right)n}. \end{aligned}$$

On the other hand, let $0 < \varepsilon_0 < \alpha(1+\lambda_u)$ and $n_0 \geq 1$ such that

$$|\ln(CK')/n| < \varepsilon_0, \forall n \geq n_0.$$

Then,

$$d(h(x), W^s(0)) \leq e^{\rho n}, \quad \forall x \in D^i_n \cap \Sigma^{i,\pm}_\sigma,$$

where $\rho = \varepsilon_0 - \alpha(1+\lambda_u) < 0$. In fact, since $\mathcal{F}$ is a partition of $h(\Sigma^{i,\pm}_\sigma)$ we have that

$$e^{\rho(n+1)} \leq d(h(x), W^s(0)) \leq e^{\rho n}, \quad \forall x \in D^i_n \cap \Sigma^{i,\pm}_\sigma, \forall n \geq n_0. \tag{5.9}$$

Therefore, if

$$h(\Sigma^{o,\pm}_\sigma) = \Sigma^{\pm}_0 = \{p = (\pm 1, b, c) : |b|, |c| < 1\} \subset U_0,$$

we obtain by (5.9) that the flight time $\tau(p)$, $p \in h(\Sigma^{i,\pm}_\sigma)$, to go from $h(\Sigma^{i,\pm}_\sigma)$ to $\Sigma^{\pm}_0$ satisfies

$$-\frac{\rho}{\lambda_u} n \leq \tau(p) \leq -\frac{\rho}{\lambda_u}(n+1). \tag{5.10}$$

Now, since h conjugates X_t and L_σ, by applying Lemma 5.1.7 we obtain that for every $\beta > 0$ there is $\varepsilon > 0$ such that if $v \in V_0$ and $w \in h(\Sigma^{i,\pm}_\sigma)$ satisfy $\|v - w\| < \varepsilon$, then $L_u(v) \in h(\Sigma^{i,\pm}_\sigma)$, with $u \in (-\beta, \beta)$.

Take a compact neighborhood C' of V_0 inside U_0. By uniform continuity of h on $h^{-1}(C')$, there is $\delta > 0$ such that $\|h(z) - h(w)\| < \varepsilon$ if $z, w \in h^{-1}(C')$ satisfy $d(z, w) < \delta$. Besides, by uniform continuity of $DX_1(\cdot)$ and E^c on $\overline{U}$ there exists $0 < \delta_0 < \delta$ such that

$$\text{if} \quad d(x,y) < \tilde{\delta}_0, \quad \text{then} \quad \frac{|\det DX_1(y)|_{E^c_y}|}{|\det DX_1(x)|_{E^c_x}|} \geq C_0 \approx 1. \tag{5.11}$$

Let us consider

$$0 < \beta < -\rho/\lambda_u, \quad 0 < \varepsilon < 1 - e^{\rho + \lambda_u \beta},$$

and $x, y \in \Lambda$ such that

$$d(X_t(x), X_t(y)) \leq \delta_0 \quad \forall t \in \mathbb{R}.$$

Suppose that there is $t \geq 0$ such that

$$X_t(x) \in D^i_n \cap \Sigma^{i,\pm}_\sigma,$$

for some $\sigma \in Sing_\Lambda(X)$ and $n \geq n_0$. Then, there is $s \in \mathbb{R}$ such that $y' = X_{s+t}(y) \in \Sigma^{i,\pm}_\sigma$. Moreover, both points y' and $X_t(x)$ belongs to the same connected component of $\Sigma^{i,\pm}_\sigma \setminus \ell_\pm$.

Claim: $y' \in \left(D^i_{n-1} \cap \Sigma^{i,\pm}_\sigma\right) \cup (D^i_n \cap \Sigma^{i,\pm}_\sigma) \cup \left(D^i_{n+1} \cap \Sigma^{i,\pm}_\sigma\right)$.

Indeed, assume that $y' \in D^i_{n+k} \cap \Sigma^{i,\pm}_\sigma$ for some $k > 1$. By the choice of δ there is $u \in (-\beta, \beta)$ such that

$$h(y') = (a_0, b_0, c_0) = (e^{\lambda_{ss} u} a, e^{\lambda_u u} b, e^{\lambda_s u} c) \in h(D^i_{n+k} \cap \Sigma^{i,\pm}_\sigma).$$

Moreover, assume that $a, c > 0$. Since $h(x) \in D_n^i \cap \Sigma_\sigma^{i,\pm}$, we have by (5.9) and (5.10) that

$$\begin{aligned} \varepsilon > \|L_{\tau(h(x))}(h(x)) - L_{\tau(h(x))}(h(y))\| &\geq 1 - e^{\lambda_u \tau(h(x))} a \\ &= 1 - e^{-\lambda_u u} e^{\lambda_u \tau(h(x))} a_0 \\ &\geq 1 - e^{\rho + \lambda_u \beta}, \end{aligned}$$

which contradicts the choice of ε, and the claim is proved.

So, by the above claim, we have that.

$$|\tau(x) - \tau(y)| \approx \frac{\lambda_u + 1}{\lambda_u} = L. \tag{5.12}$$

Let $n_1 \geq 0$ be the largest integer such that $X_{n_1}(y) \in V_\sigma$. Thus we have $\tau(y) = n_1 + r_y$ with $0 \leq r_y \leq 1$ and $\tau(x) = n_1 + r_x$ with $|r_x| \leq L + 1$. Therefore, by the chain rule, we have

$$\frac{|\det DX_{\tau(y)}(y)|_{E_y^c}|}{|\det DX_{\tau(x)}(x)|_{E_x^c}|} \geq K_1 C_0^{\tau(x)},$$

for some $K_1 > 0$. □

Once we have the previous lemmas, we can quickly adapt the proof of lemma 3.6 in [38] to obtain the desired control of the hyperbolic times.

Lemma 5.1.10. There exist positive numbers δ_4, T, c_* such that if $x \in \Lambda'$, and $y \in U$ satisfy

$$d(X_t(x), X_t(y)) \leq \delta_4, \quad \forall t \in \mathbb{R},$$

and given a C-hyperbolic time $t_x \geq T$, we have

$$|\det DX_{t_x}(y)|_{E_y^c}| \geq e^{c_* t_x}. \tag{5.13}$$

Proof. Let $\tilde{\varepsilon}_0, \varepsilon_0, \varepsilon_1$ be given by the previous lemmas and let $c^*, \alpha > 0$ satisfying

$$C - (\alpha + |\ln(C_0)| + |\ln K_2| + |\ln K_3|) > c^* > 0, \tag{5.14}$$

where $K_2 = K_2(\varepsilon_0)$ and $K_3 = K_3(\varepsilon_1)$. Since $C_0 \approx 1$, $K_2 \approx 1$, $K_3 \approx 1$, it is possible to choose such c^*. Let $\delta_0, \delta_1 = \delta_1(\varepsilon_0), \delta_2 = \delta_2(\varepsilon_1)$ and δ_3 be given by Lemmas 5.1.9, 5.1.5 and 5.1.8, respectively. Let consider

$$0 < \delta_4 < \min\{\delta_0, \delta_1, \delta_2, \delta_3, \delta'\},$$

where δ' is the Lebesgue's number of the open cover V, W_0, W_1 of Λ and fix $T = \max\{T_0, T_1\}$.

Let $x \in \Lambda'$ and $y \in U$ be two points satisfying the condition given the statement of lemma. Next, we will take a sequence of points x_n in the orbit of x which is contained in the union of

$$((W_0 \cup W_1) \setminus V) \cup \widetilde{\Sigma^{i,\pm}_{\widetilde{\beta}}}.$$

Let $s_0 \geq 0$ (if it exists) be the first time satisfying

$$x_0 = X_{s_0}(x) \in \widetilde{\Sigma^{i,\pm}_{\widetilde{\beta}}},$$

where $\widetilde{\beta}$ is given by lemma 5.1.9. Then, define

$$x_1 = X_{s_0+\tau(x_0)}(x)$$

and note that $x_1 \in (W_0 \cup W_1) \setminus V$. Now we split the definition of x_2 into two cases:

1. If $x_1 \in W_1 \setminus V$, then Lemma 5.1.6 implies the existence of $0 < s \leq T$ such that $X_s(x) \in V$, and therefore we can take $0 < s_1 < s$ such that $X_{s_1}(x_1) \in \widetilde{\Sigma^{i,\pm}_{\widetilde{\beta}}}$. In this case, define $x_2 = X_{s_1}(x_1)$.

2. If $x_1 \in W_0 \setminus V$, then we consider $0 < t_1 \leq T$ given by Lemma 5.1.5 and define $x_2 = X_{t_1}(x_1)$, if $X_{t_1}(x_1) \notin V$ or $x_2 = X_{r_1}(x_1)$, if $X_{t_1}(x_1) \in V$ and $0 < r_1 < t_1$ is such that
$$X_{r_1}(x_1) \in \widetilde{\Sigma^{i,\pm}_{\widetilde{\beta}}}.$$

Proceeding inductively for $n \geq 3$, we define

$$x_n = \begin{cases} X_{\tau(x_{n-1})}(x_{n-1}) & if \quad x_{n-1} \in \widetilde{\Sigma^{i,\pm}_{\widetilde{\beta}}}, \\ X_{s_{n-1}}(x_{n-1}) & if \quad x_{n-1} \in W_1 \setminus V, \\ X_{t_{n-1}}(x_{n-1}) & if \quad x_{n-1} \in W_0 \setminus V \text{ and } X_{t_{n-1}}(x_{n-1}) \notin V, \\ X_{r_{n-1}}(x_{n-1}) & if \quad x_{n-1} \in W_0 \setminus V \text{ and } X_{t_{n-1}}(x_{n-1}) \in V. \end{cases}$$

For a fixed $n \geq 1$, let us consider the following sets

- $O_n = \{0 \leq i \leq n : x_i \in \widetilde{\Sigma^{i,\pm}_{\widetilde{\beta}}}\}$, $n_O = \#O_n$,
- $A_n = \{0 \leq i \leq n : x_i \in W_1\}$, $n_A = \#A_n$,
- $B_n = \{0 \leq i \leq n : x_i \in W_0 \text{ and } X_{t_i}(x_i) \notin V\}$, $n_B = \#B_n$,
- $C_n = \{0 \leq i \leq n : x_i \in W_0 \text{ and } X_{t_i}(x_i) \in V\}$ and $n_C = \#C_n$.

Note that for every x_n, there exists $t'(n) > 0$ such that $x_n = X_{t'(n)}(x)$ where

$$t'(n) = s_0 + \sum_{i \in A_n} s_i + \sum_{i \in B_n} t_i + \sum_{i \in C_n} r_i + \sum_{i \in O_n} \tau(x_i). \tag{5.15}$$

If y satisfies

$$d(X_t(x), X_t(y)) \leq \delta_4, \forall t \in \mathbb{R}, \tag{5.16}$$

define $y_n = X_{t'(n)}(y)$.

Let consider $x_n \in \widetilde{\Sigma_{\widetilde{\beta}}^{i,\pm}}$. By hypothesis, we have that

$$d(x_n, y_n) \leq \delta_4 \text{ and } d(X_{\tau(x_n)}(x_n), X_{\tau(x_n)}(y_n)) \leq \delta_4.$$

By using twice Lemma 5.1.8, there exists $l_0, l_1 \in (-L, L)$ such that

$$y' = X_{l_0}(y_n) \in \widetilde{\Sigma_{\widetilde{\beta}}^{i,\pm}} \text{ and } X_{\tau(x_n)+l_1}(y_n) \in \widetilde{\Sigma_{\widetilde{\beta}}^{o,\pm}}.$$

Thus,

$$\tau(x_n) = \tau(y') + l_0 - l_1, \tag{5.17}$$

and by (5.17) and lemma (5.1.9) we have that

$$\begin{aligned} |\det DX_{\tau(x_n)}(y_n)|_{E^c_{y_n}}| \quad &\geq \quad |\det DX_{-l_1}(X_{\tau(y')+l_0}(y_n))|_{E^c_{X_{\tau(y')+l_0}(y_n)}}| \\ &\quad\quad |\det DX_{\tau(y')}(y')|_{E^c_{y'}}||\det DX_{l_0}(y_n)|_{E^c_{y_n}}| \\ &\geq \quad K' C_0^{\tau(x_n)} |\det DX_{\tau(x_n)}(x_n)|_{E^c_{x_n}}|, \end{aligned} \tag{5.18}$$

where

$$K' = K_0^2 K_1 \text{ and } K_0 = \min_{(z,s) \in \widetilde{U} \times [-L, T+L]} |\det DX_s(z)|_{E^c_z}|.$$

Let $\{t_k\}_{k \geq 1}$ be an unbounded and increasing sequence of C-hyperbolic times for x. Up to a slight change of C, we can assume that $X_{t_k}(x) \notin V$. Thus, for every $k \geq 1$ one can write $t_k = t'_k + u_k$, with $u_k \in [0, T)$ and t'_k of the form (5.15). Let us denote $\varphi_t(z) = |\det DX_t(z)|_{E^c_z}|$. By the relations (5.5), (5.6) and (5.18),

$$\begin{aligned} |\det DX_{t_x}(y)|_{E^c_y}| \quad &= \quad |\det DX_u(X_{t'_x}(y))|_{E^c_{X_{t'_x}(y)}}||\det DX_{t'_x}(y)|_{E^c_y}| \\ &\geq \quad K_0 \prod_{i \in O_n} \varphi_{\tau(x_i)}(y_i) \prod_{i \in A_n} \varphi_{s_i}(y_i) \prod_{i \in B_n} \varphi_{t_i}(y_i) \prod_{i \in C_n} \varphi_{r_i}(y_i) \\ &\geq \quad \overline{K}(K')^{n_O} C_0^{\sum_{i \in O_n} \tau(x_i)} K_3^{n_A} K_2^{n_B} \left(\frac{K_0}{K_5}\right)^{n_C+1} \varphi_{t_x}(x), \end{aligned}$$

where

$$\overline{K} = \frac{|\det DX_{s_0}(y)|_{E^c_y}|}{|\det DX_{s_0}(x)|_{E^c_x}|} \text{ and } K_5 = \max_{(z,s) \in \widetilde{U} \times [-L, T+L]} |\det DX_s(z)|_{E^c_z}|.$$

Since t_x is a C-hyperbolic time for x, by the above estimate we have that

$$|\det DX_{t_x}(y)|_{E^c_y}| \geq \overline{K} e^{(C + N(C_0) + N(K') + N(K_2) + N(K_3) + N(K_0, K_5))t_x}, \tag{5.19}$$

where

- $N(C_0) = \frac{\sum_{i\in O_n}\tau(x_i)}{t_x}\ln(C_0)$,
- $N(K') = \frac{n_O}{t_x}\ln(K')$
- $N(K_2) = \frac{n_A}{t_x}\ln(K_2(\epsilon_0))$,
- $N(K_3) = \frac{n_B}{t_x}\ln(K_3(\epsilon_0))$ and
- $N(K_0, K_5) = \frac{n_C+1}{t_x}\ln\left(\frac{K_0}{K_5}\right)$.

Now, since $t_x > \sum_{i\in O_n}\tau(x_i)$ it follows that

$$|N(C_0)| \le |\ln C_0|.$$

Furthermore, since $n_C + 1 \le n_O$ by the construction of the sequence x_n we have

$$\left|\frac{n_O}{t_x}\ln K' + \frac{n_C+1}{t_x}\ln\frac{K_0}{K_5}\right| \le \frac{n_O}{t_x}\left|\ln K' + \ln\frac{K_0}{K_5}\right| \le \frac{|\ln K'| + |\ln\frac{K_0}{K_5}|}{n_0}.$$

Then, by shrinking $V_{\widetilde{\beta}}$ if it is necessary, we have

$$\left|\frac{n_O}{t_x}\ln K' + \frac{n_C+1}{t_x}\ln\frac{K_0}{K_5}\right| \le \frac{|\ln K'| + |\ln\frac{K_0}{K_5}|}{n_0} \le \alpha.$$

On the other hand, since $\frac{(n_A+n_B+n_C+n_O)}{t_x} \le 1$, we obtain

$$|N(K_2)| \le |\ln K_2(\varepsilon_0)| \text{ and } |N(K_3)| \le |\ln K_3(\varepsilon_1)|,$$

and by (5.14) and (5.19) we have

$$|\det DX_{t_x}(y)|_{E^c_y}| \ \ge\ \overline{K}e^{(C-(\alpha+|\ln C_0|+|\ln K_2(\varepsilon_0)|+|\ln K_3(\varepsilon_1)|))t_x} \ge \overline{K}e^{c^*t_x}.$$

Finally, there exist $0 < c_* < c^*$ and $\overline{T} > 0$ such that $\overline{K}e^{c^*t} \ge e^{c_*t}$ for all $t \ge \overline{T}$. Therefore, for every y satisfying (5.16) and every C-hyperbolic time $t_x \ge \overline{T}$ we have

$$|\det DX_{t_x}(y)|_{E^c_y}| \ge \overline{K}e^{c^*t_x} \ge e^{c_*t_x},$$

and this concludes the proof. □

Next, we can finally prove the kinematic expansiveness of Λ. The proof we present here is very similar to the proof of Theorem A of [5], as most of the steps from that work can be directly applied here. The only exception is Theorem 3.1 of [5], which cannot be replicated directly in our context. Hence, we will be mainly focused on providing a detailed proof of a version of Theorem 3.1 from [5] adapted to our setting, and we will briefly explain the other steps of the argument.

Proof of Theorem 5.1.3. We begin assuming that X is not kinematic expansive on an ASH attractor Λ. Then, there are $\varepsilon > 0$, $x_n, y_n \in \Lambda$ and $\delta_n \to 0$ such that $y_n \notin X_{[-\varepsilon,\varepsilon]}(x_n)$ and

$$d(X_t(x_n), X_t(y_n)) \leq \delta_n \text{ for every } t \in \mathbb{R}. \tag{5.20}$$

Following the arguments in [5] we can find a regular point $z \in \Lambda$ and $z_n \in \omega(x_n)$ such that $z_n \to z$. Let us consider Σ_η an η-adapted cross-section trough z. By using flow boxes in a small neighborhood of $\Sigma_\eta \cup \widetilde{\Sigma^{i,o,\pm}}$ we can find positive numbers δ', t_0 such that for any $\Sigma' \subset \Sigma_\eta \cup \widetilde{\Sigma^{i,o,\pm}}$, $z \in \Sigma'$ and $w \in M$ with $d(z, w) < \delta'$, there is $t_w \leq t_0$ such that $w' = X_{t_w}(w) \in \Sigma'$ and $d_{\Sigma'}(z, w') < K'\delta'$, where $d_{\Sigma'}$ is the intrinsic distance in Σ', for some constant positive constant K' which depends on $\Sigma_\eta \cup \widetilde{\Sigma^{i,o,\pm}}$.

Let $N > 0$ be large enough such that

$$0 < \delta_N < \min\{\delta_4, \eta, \eta_0, \delta'\},$$

where η_0 and δ_4 are given by Remark 4.2, Lemma 5.1.10 respectively. Let us denote $x = x_N$ and $y = y_N$. The next claim is our version of Theorem 3.1 in [5].

Claim: There is $s \in \mathbb{R}$ such that $X_s(y) \in W^{ss}_\varepsilon(X_{[s-\varepsilon,s+\varepsilon]}(x))$.

Once the claim holds, by following the proof of Theorem A in [5] step by step, one can conclude the proof of the theorem. Because of this, we now devote ourselves to proving the previous claim. By construction, the orbit of x must intersect Σ_η in infinitely many positive times t_j. Let us denote $x_j = X_{t_j}(x)$. Thus, we can find a sequence of times s_j close to t_j such that $y_j = X_{s_j}(y)$ also intersect Σ_η. Now, we briefly recall the construction of the tube-like domain presented in [5]. For any $j \geq 0$ we can find a smooth immersion

$$\rho^j : [0,1] \times [0,1] \to M$$

such that the following holds:

1. $\rho^j([0,1] \times \{0\})$ is the orbit arc from x_j to x_{j+1} and $\rho^j([0,1] \times \{1\})$ is the orbit arc from y_j to y_{j+1} and
2. $\rho^j(\{0\} \times [0,1])$ is a curve contained in Σ_η, everywhere tranverse to $W^s(\Sigma_\eta)$ and joining x_j and y_j.
3. $\rho^j(\{1\} \times [0,1])$ is a curve contained in Σ_η, everywhere tranverse to $W^s(\Sigma_\eta)$ and joining x_{j+1} and y_{j+1}.
4. Denote $S_j = \rho^j([0,1] \times [0,1])$. Then the intersection of S_j with any $\Sigma^{i,o,\pm}_\sigma$ is transverse to stable foliation of $\Sigma^{i,o,\pm}_\sigma$.

Denote

$$\mathcal{T}_j = \bigcup_{p \in S_j} W^s_{loc}(p).$$

The results in [5] show that $\mathcal{T}_j$ does not contain singularities. Moreover, if $\mathcal{T}_j$ intersects some $\Sigma^{i,o,\pm}_\sigma$, this intersection is contained in the same connected component of $\Sigma^{i,o,\pm}_\sigma \setminus W^s(\sigma)$. Finally, in [5] it was also shown the existence of a Poincaré map R_j between the whole strip between the stable manifolds of x_j and y_j inside Σ_η. Unfortunately, we cannot guarantee that this Poincaré maps are hyperbolic in the same way as in [5] because we are in the ASH setting. Nevertheless, we will obtain some expansion for these maps using the previous lemmas. To prove the claim we first assume that

$$X_s(y) \notin W^{ss}_\varepsilon(X_{[s-\varepsilon,s+\varepsilon]}(x)),$$

for every $s \in \mathbb{R}$. In this case, we have $y \notin \mathcal{O}(x)$; otherwise by Lemma 5.1.7 and Remark 5.1.4 we obtain that y belongs to $X_{[-\varepsilon,\varepsilon]}(x)$, and this is a contradiction.

In addition, unless to take a subsequence of x_j, we can find a sequence of arbitrarily large C-hyperbolic times $\{t_j\}_{j\geq 1}$ of x such that $x_j = X_{t_j+r_j}(x) \in \Sigma_\eta$, where $0 \leq r_j < T$, with

$$T = \max\{T_0, T_1\}.$$

Besides, we have that $y_j = X_{t_j+r_j+v_j}(y) \in \Sigma_\eta$, where $|v_j| \leq t_0$. Moreover, by shrinking U if it is necessary, by Lemma 2.7 in [5] there is $\kappa_0 > 0$ such that

$$\ell(\gamma_n) \leq \kappa_0 d_{\Sigma'}(x_j, y_j) \leq \kappa_0 K'\delta', \tag{5.21}$$

where γ_j is any curve joining x_j and y_j, for every $n \geq 1$. In particular, this holds for the curves

$$\gamma_j = \rho^j(\{1\} \times [0,1]) \subset \Sigma_\eta.$$

On the other hand, take

$$\kappa = \min_{(z,s)\in \overline{B_{\delta'}(\Sigma_\eta)}\times[-t_0,t_0]} |\det DX_s(z)|_{E^c_z}| \cdot \min_{(z,s)\in\overline{U}\times[0,T]} |\det DX_s(z)|_{E^c_z}| > 0.$$

Let $\lambda > \kappa_0 K'\delta'$, and let j_1 large enough such that $\kappa e^{c_* t_{j_1}} > \lambda$. Let R be a Poincaré map whit return time

$$s(x) \approx t_{j_2} + T \quad s(z') \approx t_0 + t_{j_2} + T, \quad \forall z' \in \gamma_0,$$

where $j_2 > j_1$ is large enough. Figure 6 helps visualize the definition of R.

Following the proof of Lemma 5.1.10, we deduce, by shrinking ε' if it is necessary, that the relation (5.13) is satisfied for any $z \in \gamma_{j_1}$. So, by definition of R we have that $R(\gamma_{j_1})$ is a curve in Σ_η that connects $x_{j_1+j_2}$ with $y_{j_1+j_2}$ and satisfies

$$\ell(R(\gamma_{j_1})) \geq ke^{c_* t_{n_1}} \geq \lambda > \kappa_0 K'\delta',$$

which contradicts (5.21). So, we have that $y_{j_1+j_2} \in W^s(x_{j_1+j_2}, \Sigma_\eta)$. Therefore, we obtain the claim by following step by step the argument given in [5], p. 2456. □

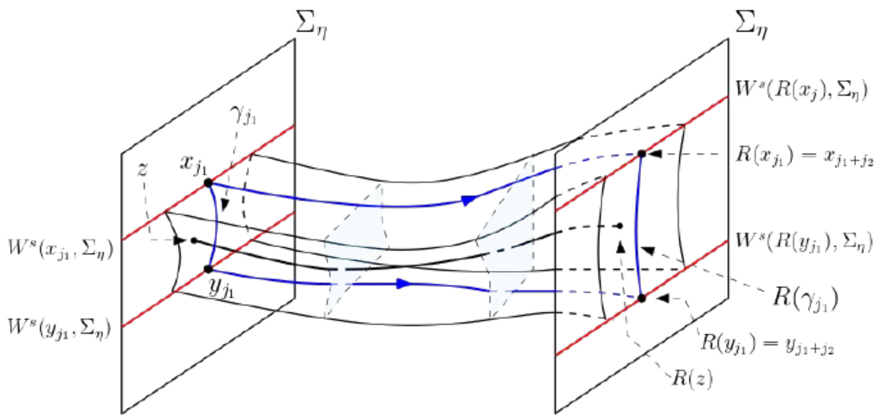

Figure 6 – The Poincaré map R.

5.2 Positive Entropy and Periodic Orbits

Next we shall proceed to prove Theorem D. Our argument here is inspired by the work [43]. To begin with, we recall that we are again with 3 dimensional ambient espace.

let us consider $E^s \oplus E^c$ and U_0. As a first step for the proof of Theorem D, we show that the ASH property can be extended to the trapping region U_0 in the following way:

Lemma 5.2.1. Let Λ be an asymptotically sectional-hyperbolic attractor associated to a C^1 vector field X. Then, we have

$$\limsup_{t\to\infty} \frac{1}{t} \log |\det DX_t(x)|_{E^c_x}| > 0, \quad \forall x \in U' = U_0 \setminus W^s(Sing(X)).$$

Proof. First, we need to get an estimate of $|\det DX_t(\cdot)|_{E^c_{(\cdot)}}|$ in a neighborhood of the singularities of Λ. Let σ be either a Rovella-like or resonant singularity, and let V_σ be a neighborhood of σ. By shrinking V_σ if it is necessary, by continuity of $DX_{(\cdot)}(x)$ and the choice of the cone field there are $\theta'_\sigma \leq 0$ and $T_0 > 0$ such that for every $x \in V_\sigma$, whose trajectory remains in V_σ for $t \in [0, T_0]$, $X_{[0,T_0]}(x) \subset V_\sigma$, the following inequality holds:

$$|\det DX_t(x)|_{L_x}| \geq \frac{1}{2}|\det DX_t(\sigma)|_{E^c_\sigma}| = \frac{1}{2}e^{\theta'_\sigma t},$$

for every plane $L_x \subset C^c_a(x)$ and every $t \in [0, T_0]$ such that $X_t(x) \in V_\sigma$. So, for $x \in V_\sigma$ and every $t > 0$ such that $X_t(x) \in V_\sigma$ we have $t = mT_0 + r$, $m \in \mathbb{N}$ and $0 \leq r < T_0$, so that, by the above estimation, we have for every plane $L_x \subset C^c_a(x)$ that

$$|\det DX_t(x)|_{L_x}| \geq \left(\frac{1}{2}e^{\theta'_\sigma r}\right)\left(\frac{1}{2}e^{\theta'_\sigma T_0}\right)^m \geq K_-(\sigma)e^{\theta_\sigma t},$$

where $\theta_\sigma = \frac{1}{T_0}\log\left(\frac{1}{2}e^{\theta'_\sigma T_0}\right) < 0, K_-(\sigma) > 0$. In a similar way, by taking T_0 large enough, there are $K_+ > 0$ and $\theta_\sigma > 0$ such that

$$|\det DX_t(x)|_{L_x}| \geq K_+(\sigma)e^{\theta_\sigma t},$$

for every Lorenz-like singularity σ, every $x \in V_\sigma$ such that $X_t(x) \in V_\sigma$, and for every plane $L_x \subset C_a^c(x)$. We set $V_{Sing(X)} = V_R \cup V_L$, where R denotes the set of Rovella-like or resonant singularities and L denotes the set of Lorenz-like singularities. In this case, let

$$\theta_+ = \min_{\sigma \in L} \theta_\sigma > 0, \quad \theta_- = \min_{\sigma \in R} \{\theta_\sigma\} \leq 0, \text{ and } K = \min_{\sigma \in V_{Sing(X)}} \{K_-(\sigma), K_+(\sigma)\} > 0.$$

On the other hand, by the ASH property there is a positive constant $c < C$, where C is given by (1.1), and $T_1 > T_0$ large enough with the following property: for every $x \in \Lambda' = \Lambda \setminus W^s(Sing(X))$ there is a neighborhood V_x of x such that

$$|\det DX_t(y)|_{L_y}| \geq e^{ct_1}, \quad \forall y \in V_x, t_1 = t_1(x) \geq T_1,$$

for every plane $L_y \subset C_a^c(y)$, where t_1 is the first hyperbolic time for x. In this case, denote

$$W_x = \bigcup_{0 \leq t \leq t_1(x)} X_t(V_x).$$

Then, the set $W = \bigcup_{x \in \Lambda'} W_x$ defines an open cover of Λ'.

By compactness of $\overline{U_0 \setminus (V_{Sing(X)} \cup W)}$, there are $T_2 > 0$ and $b \in \mathbb{R}$ such that

$$|\det DX_r(z)|_{L_z}| \geq e^{br}, \quad \forall (z, r) \in \left(\overline{U_0 \setminus (V_{Sing(X)} \cup W)}\right) \times [0, T_2], \quad L_z \subset C_a^c(z).$$

Now, for every $x \in U_0$, let consider the following numbers:

- the maximal average time that the orbit of x spends in $\overline{U_0 \setminus (V_{Sing(X)} \cup W)}$ is given by
$$\beta_0(x) = \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \chi_{\overline{U_0 \setminus (V_{Sing(X)} \cup W)}}(X_s(x))ds,$$
- the minimal average time that the orbit of x spends in $V_L \cup W$ is given by
$$\beta_1(x) = \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \chi_{V_L \cup W}(X_s(x))ds \text{ and}$$
- the maximal average time that the orbit of x spends in $V_R \setminus W$ is given by
$$\beta_2(x) = \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \chi_{V_R \setminus W}(X_s(x))ds.$$

By definition of these numbers, for every $\varepsilon > 0$ there is $r > 0$ such that for any $t > r$, one has

$$\frac{1}{t} \int_0^t \chi_{\overline{U_0 \setminus (V_{Sing(X)} \cup W)}}(X_s(x))ds \leq \beta_0(x) + \varepsilon, \quad \frac{1}{t} \int_0^t \chi_{V_L \cup W}(X_s(x))ds \geq \beta_1(x) - \varepsilon$$

and

$$\frac{1}{t} \int_0^t \chi_{V_R \setminus W}(X_s(x))ds \leq \beta_2(x) + \varepsilon.$$

So, for any $x \in U_0$, by splitting the orbit into orbit segments and by joining the above estimations, we have that

$$|\det DX_t(x)|_{E^c_x}| \geq e^{\psi(x,t)t}, \quad \forall t > r, \tag{5.22}$$

with

$$\psi(x,t) = \frac{(k_1(t)+k_2(t))\ln K}{t} + b\beta_0(x) + c'\beta_1(x) + \theta_-\beta_2(x) + (\theta_- - c' + b)\varepsilon.$$

Here $c' = \min\{c, \theta_+\} > 0$ and the numbers $k_1(t)$ and $k_2(t)$ denote how many times the orbit of x crosses V_R and V_L in the interval $[0,t]$, respectively, and these crossing segment of orbit de not belong to W.

Now, let

$$Z = \{x \in U_0 : d(x) = \rho(x) + b\beta_0(x) + c'\beta_1(x) + \theta_-\beta_2(x) \leq 0\} \subset U_0,$$

where $\rho(x) = \limsup_{t\to\infty}\left(\frac{k_1(t)+k_2(t)}{t}\ln K\right) < \infty$. Since $\rho(\cdot)$, $\beta_0(\cdot)$, $\beta_1(\cdot)$ and $\beta_2(\cdot)$ are invariant by the flow we have that Z is an invariant subset of U_0. So, since Λ is attracting, it follows that $Z \subset \Lambda$. Note that for any $x \in \Lambda' \cup L$ it holds that $k_1(t) = 0$, $k_2(t) = 0$ or 1, $\beta_0(x) = 0$, $\beta_2(x) = 0$ and $\beta_1(x) = 1$, since the orbit of x remains entirely in $V_L \cup W$,so that $d(x) = c' > 0$. It follows that $Z \subset W^s(R)$. Then, $d(x) > 0$ for $x \in U' = U_0 \setminus W^s(Sing(X))$. Moreover, note that $U' = \bigcup_{n\in\mathbb{N}} U'_n$, where $U'_n = \{x \in U' : d(x) > 1/n\}$. Therefore, if $x \in U'$ there is $n \in \mathbb{N}$ such that $d(x) > 1/n$. Thus, if we choose $\varepsilon_n > 0$ satisfying

$$1/n + (\theta_- + b - c')\varepsilon_n > d_n > 0,$$

we obtain by (5.22) that

$$\limsup_{t\to\infty} \frac{1}{t}\log|\det DX_t(x)|_{E^c_x}| \geq d_n > 0.$$

This concludes the proof. □

Now, recall that for a continuous map $f : M \to M$, the *empirical probabilities of orbit of a point* $x \in M$ are defined as

$$m_{n,x} = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^j(x)}, \quad n \in \mathbb{N}, \tag{5.23}$$

where δ_y is the Dirac measure supported at $y \in M$. Let $p\omega_f(x)$ be the set of accumulation points of the sequence (5.23) in the weak* topology.

Lemma 5.2.2. For every point $x \in \Lambda \setminus W^s(Sing(X))$, the set R is not in the support of any measure $\nu \in p\omega_f(x)$, where $f = X_1$.

Proof. Assume that $R \cap supp(\nu) \neq \emptyset$, and let consider $\sigma \in R \cap supp(\nu)$. Take a neighborhood V of σ. By definition of R and by shrinking V if necessary, there exists $C_0 > 0$ and $N > 0$ such that

$$|\det Df^n(x)|_{E^c_x}| \leq C_0 e^{\underline{\alpha} n} \leq e^{\overline{\alpha} n}, \tag{5.24}$$

where $\underline{\alpha} \leq \overline{\alpha} \leq 0$, for every $n \geq N$ and every $x \in V$ satisfying $f^i(x) \in V$, for $i = 0, \ldots, n$.

Let $x \in \Lambda'$ and let t_k be a sequence of C-hyperbolic times for x. It is easy to check that one can take $t_k = n_k \in \mathbb{N}$, so that

$$|\det Df^{n_k}(x)|_{E^c_u}| \geq e^{Cn_k}, \quad k \geq 1. \tag{5.25}$$

Since V^c is a compact set, there exists $a > 1$ such that

$$|\det Df(z)|_{E^c_z}| \leq a \ \forall z \in V^c. \tag{5.26}$$

So, if $B_n = \{1 \leq m \leq n : f^m(x) \in V^c\}$, we have by (5.24), (5.25) and (5.26),

$$\begin{aligned} a^{\#B_{n_k}} e^{\#B^c_{n_k}\overline{\alpha}} &\geq \prod_{i \in B_{n_k}} |\det Df(f^i(x))|_{E^c_{f^i(x)}}| \prod_{i \in B^c_{n_k}} |\det Df(f^i(x))|_{E^c_{f^i(x)}}| \\ &= \prod_{i=0}^{n_k-1} |\det Df(f^i(x))|_{E^c_{f^i(x)}}| \\ &= |\det Df^{n_k}(x)|_{E^c_x}| \\ &\geq e^{Cn_k}. \end{aligned} \tag{5.27}$$

So,

$$\frac{\#B_{n_k}}{n_k} \geq \frac{\#B_{n_k}}{n_k} + \frac{\#B^c_{n_k}}{n_k \log a}\overline{\alpha} \geq \frac{C}{\log a} > 0,$$

and hence

$$\limsup_{n\to\infty} \frac{\#B_n}{n} = \alpha(x) > 0.$$

Now, by the above relation,

$$0 \leq \alpha'(x) = \limsup_{n\to\infty} \frac{\#B^c_n}{n} < 1.$$

In particular, if $\varepsilon > 0$ satisfies $\alpha'(x) + \varepsilon < \alpha < 1$, there is $N \in \mathbb{N}$ such that

$$\frac{\#B^c_n}{n} < \alpha, \quad \forall n \geq N. \tag{5.28}$$

Let V' be a compact neighborhood of σ contained in V. By Urysohn's lemma there is a continuous function $\varphi : M \to \mathbb{R}$ such that $\varphi(V') = 1$ and $\varphi(M \setminus V) = 0$. So, it follows that $\int \varphi d\nu \geq 1$. Therefore,

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \int \varphi d\nu \right| \geq 1 - \frac{\#B^c_n}{n} > 1 - \alpha > 0, \quad n \geq N.$$

This shows that $\nu \notin p\omega_f(x)$, which is a contradiction. This proves the result. □

Before proving Theorem D, we recall the following result from [17], which will be used in its proof.

Theorem 5.2.3 (Theorem F in [17])**.** For any C^1 dioffeomorphism f, for any compact invariant set Λ admiting a dominated splitting $E \oplus F$, and for Lebesgue almost every point $x \in M$, if $\omega(x) \subset \Lambda$, then each limit measure $\mu \in p\omega_f(x)$ satisfies

$$h_\mu(f) \geq \int \log |detDf\,|_F\,|\,d\mu.$$

Before presenting the proof of Theorem D, we recall its statement.

Theorem D. Any asymptotically sectional-hyperbolic attractor Λ associated to three-dimensional vector fields X of class C^1 has a periodic orbit. Actually it contains a nontrivial homoclinic class. Thus its topological entropy is positive. If the periodic orbits are dense on Λ, then it is a homoclinic class.

Proof of Theorem D. Let $f = X_1$. By Lemma 5.2.2 we have that $supp(\nu) \subset \Lambda' \cup L$, where $\Lambda' = \Lambda \setminus W^s(Sing(X))$, for every $\nu \in p\omega_f(x)$, $x \in U'$. Besides, since $\omega(x) \subset \Lambda$ for any $x \in U'$, we have by Theorem 5.2.3 that

$$h_\nu(f) \geq \int [\chi_1(x) + \chi_2(x)]\,d\nu,$$

where $\chi_i(x)$, $i = 1, 2$, are the Lyapunov exponents of Df on E^c_x.

Now, since $\chi_1(\sigma) + \chi_2(\sigma) = \lambda_s(\sigma) + \lambda_u(\sigma) > 0$ for every $\sigma \in L$ and

$$\chi_1(x) + \chi_2(x) = \lim_{n\to\infty} \frac{1}{n} \log |\det(Df^n(x)|_{E^c_x})| \geq C, \quad \forall x \in \Lambda', \tag{5.29}$$

we have

$$h_\nu(f) \geq \int_{\Lambda' \cup L} \chi_1(x) + \chi_2(x) d\nu \geq C + \#(L) \min_{\sigma \in L}[\chi_1(\sigma) + \chi_2(\sigma)] > 0.$$

So, by the Variational Principle, we have $h_{top}(X) > 0$. Given that the topological entropy is positive, Theorem G ensures the existence of a periodic orbit, and Theorem E guarantees the existence of a nontrivial homoclinic class.

For the last part, Let $z \in \Lambda$ such that $\omega(z) = \Lambda$. By hypothesis, there is a sequence of periodic orbits p_n, with period $\tau(p_n)$, such that $p_n \to z$. In particular, we have $\tau(p_n) \to \infty$. So, by Hyperbolic lemma, all these orbits are hyperbolic of saddle type, so that there is $N \in \mathbb{N}$ large enough such that $\gamma_{p_n} \sim \gamma_{p_m}$ for $n, m \geq N$, i.e., $W^s(p) \pitchfork W^u(p)$ and $W^s(q) \pitchfork W^u(p)$. This shows that $z \in H(p_N)$, since $H(p_N)$ is closed. Hence, the result is obtained by the denseness of $O(z)$. $\square$

6 Concluding Remarks and Future Directions.

This chapter outlines several future directions inspired by the results of this thesis.

In this doctoral thesis, we have advanced the understanding of asymptotically sectional-hyperbolic (ASH) attractors, a class that generalizes both classical and sectional hyperbolicity. We showed that, under natural conditions such as positive topological entropy and the star property, these attractors must be sectional-hyperbolic.

Moreover, we explored entropy flexibility in this context, establishing that ASH attractors satisfy the intermediate entropy property. In dimension three, we proved that every nontrivial ASH attractor is entropy-expansive and has positive topological entropy, which implies the existence of a periodic orbit and, in fact, a nontrivial homoclinic class. In higher dimensions, we also showed the existence of periodic orbits and nontrivial homoclinic classes under the assumption of positive topological entropy.

Finally, we proved that the exponential growth rate of periodic orbits is greater than or equal to the system's topological entropy.

As future work, it is natural to further explore the dynamical and ergodic properties of ASH attractors in higher dimensions, particularly:

- **Existence of periodic orbits:** An important step in the proof of Theorem A was establishing the existence of a periodic orbit, which was guaranteed by the assumption of positive topological entropy. In dimension three, we observed that this hypothesis is not necessary, as nontrivial ASH attractors already exhibit positive entropy. This suggests that, in higher dimensions, it is plausible that ASH attractors also contain periodic orbits without assuming positive entropy. Proving that every ASH attractor admits a periodic orbit would lead to an improvement of Theorem A by removing the positive entropy assumption.

  Once this question is addressed, a natural follow-up arises: are periodic orbits dense in the ASH attractor? In the case of three-dimensional sectional-hyperbolic attractors, this is known to be true [8].

- **Expansivity:** We showed that in dimension three, ASH attractors are entropy-expansive, which suggests that this property might also hold in higher dimensions. It is known that higher-dimensional sectional-hyperbolic attractors are entropy-expansive [36].

  Another notion of expansivity worth exploring is *Komuro expansivity.* In dimension three, sectional-hyperbolic attractors are known to be Komuro-expansive [5], and in higher dimensions, under an additional condition, [3] showed that there exists an open set (in the C^1 topology) in which sectional-hyperbolic sinks are Komuro-expansive.

- **Flexibility:** Recall that flexibility refers to the following question: Given a X vector field, Λ set compact invariant and a real number $h \in [0, h_{top}(X \mid_\Lambda))$, does there exist Γ an subsystem ($\Gamma \subset \Lambda$ set compact invariant) such that $h_{top}(X \mid_\Gamma) = h$?

  This problem remains open even for general hyperbolic flows. Existing progress concerns the *intermediate entropy property*, which asserts that, for any value $h \in [0, h_{top}(X))$, there exists an invariant ergodic measure μ supported in Λ whose $h_\mu(X) = h$.

  It is known that star flows satisfy the intermediate entropy property [24], as do ASH attractors, as shown in this thesis. Therefore, we find it quite plausible that the answer to the flexibility question is affirmative.

- **Multidimensional Rovella:** It is known that the Rovella attractor [39] is the archetypal example of an ASH attractor and, moreover, that it is purely ASH—that is, ASH but not star. This example exists in dimension 3, which motivates the question of whether there exist ASH attractors in higher dimensions, a question that remains open.

  In the case of sectional-hyperbolic attractors, the multidimensional Lorenz attractor [14] provides an example in higher dimensions.

  However, the example proposed in [2] does not exhibit a dominated splitting on the entire maximal invariant set.

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