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Dispersion forces on Proca electrodynamics

Gabriel Camacho de Pinho

Tese de Doutorado apresentada ao Programa de Pós-Graduação em Física do Instituto de Física da Universidade Federal do Rio de Janeiro - UFRJ, como parte dos requisitos necessários à obtenção do título de Doutor em Ciências (Física).

Orientador: Carlos Augusto Domingues Zarro
Coorientador: Reinaldo Faria de Melo e Souza

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Resumo

Forças dispersivas na eletrodinâmica de Proca

Gabriel Camacho de Pinho

Orientador: Carlos Augusto Domingues Zarro

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Resumo da Tese de Doutorado apresentada ao Programa de Pós-Graduação em Física do Instituto de Física da Universidade Federal do Rio de Janeiro - UFRJ, como parte dos requisitos necessários à obtenção do título de Doutor em Ciências (Física).

Os efeitos de uma massa finita do fóton foram analisados em múltiplos cenários envolvendo forças dispersivas. Para descrever as interações relevantes para tais forças, foi empregada a eletrodinâmica de Proca. Inicialmente, consideramos a interação dispersiva entre dois átomos em seus estados fundamentais, para a qual determinamos o potencial de interação em todos os regimes de distância. Assim como no caso mediado por fótons sem massa, também determinamos o potencial dispersivo em dois regimes assintóticos: o regime não retardado e o regime retardado (aqui referido como regime assintótico). Concluímos ainda que a massa do fóton, ao introduzir uma nova escala de comprimento, modifica as condições necessárias para a validade de cada regime. Por exemplo, para uma dada separação interatômica, quanto maior a massa do fóton, mais precisa se torna a aproximação não retardada. Os efeitos de não aditividade nas forças dispersivas também foram considerados no âmbito da eletrodinâmica de Proca. Esses efeitos foram divididos em dois grupos principais: átomos livres interagindo entre si e átomos interagindo com superfícies condutoras neutras e aterradas. Para o primeiro grupo, verificamos que a massa

do fóton sempre enfraquece a interação não aditiva e reduz sua relevância em comparação com a interação dispersiva total. Para a interação entre átomos e superfícies, entretanto, encontramos cenários nos quais a força não aditiva se torna dominante para valores não nulos da massa do fóton.

Palavras-chave: eletrodinâmica de Proca, forças dispersivas, teoria quântica de campos.

Abstract

Dispersion forces on Proca electrodynamics

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Abstract da Tese de Doutorado apresentada ao Programa de Pós-Graduação em Física do Instituto de Física da Universidade Federal do Rio de Janeiro - UFRJ, como parte dos requisitos necessários à obtenção do título de Doutor em Ciências (Física).

The effects of a finite photon mass have been analyzed in multiple scenarios involving dispersive forces. To describe the interactions relevant to such forces, Proca electrodynamics was employed. Initially, we considered the dispersion interaction between two atoms in their ground states, for which we determined the interaction potential over all distance regimes. As in the case mediated by massless photons, we also determined the dispersive potential in two asymptotic regimes: the non-retarded regime and the retarded (here referred to as asymptotic) regime. We also concluded that the photon mass, by introducing a new length scale, modifies the conditions required for the validity of each regime. For instance, for a given interatomic separation, the larger the photon mass, the more accurate the non-retarded approximation becomes. The effects of non-additivity in dispersive forces were also considered within Proca electrodynamics. These effects were divided into two main groups: free atoms interacting with each other, and atoms interacting with grounded, neutral, conducting surfaces. For the first group, we found that the photon mass always weakens the non-additive interaction and reduces its relevance compared to the total

dispersive interaction. For the interaction between atoms and surfaces, however, we found that there are scenarios in which the non-additive force becomes dominant for nonzero photon mass values.

Keywords: Proca electrodynamics, dispersion forces, quantum field theory

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Chapter 1

Introduction

The theme of this thesis deals with two main areas of physics: dispersion forces and Proca electrodynamics. For this reason, it is instructive to introduce them separately before we proceed with the technicalities of the subject of the present thesis. The author has chosen to provide a concise overview in this chapter, including the historical context of the development of each area, along with specific features of them and their status on the present research scenario. A more complete introduction to the necessary subjects can be found in chapters 2 and 3 and in the appendices.

One of the most remarkable triumphs of the emerging quantum mechanics from the first half of the XX century was the explanation of how neutral objects, with no permanent multipole moment, could attract each other electrically. This was essential to understand intermolecular interaction which are responsible for material cohesion and chemical diversity around us. This feature also lies at the heart of quantum mechanics' foundation, since it deals with the fact that not only the average value corresponding to observables (or eigenvalues of the Hamiltonian) is significant, but the quantum fluctuations around those averages play a fundamental role in physical effects. Unlike thermal fluctuations, which arise from interactions with particles constituting the thermal bath, quantum fluctuations are present even at zero temperature, and they owe their origin to the Heisenberg uncertainty principle. An example is the interaction between two atoms. The expectation value of the

electric dipole (and higher order multipoles) vanishes, which is the quantum counterpart of the fact that atoms do not possess permanent dipole moment. But due to the quantum fluctuations $\langle \mathbf{d}^2 \rangle$ does not vanish, generating a fluctuating electric field that can induce an electric dipole in the other atoms, producing an interaction. Those kinds of interactions are known as *dispersion forces*, one of the three kinds of the well-known Van der Waals forces [1–5]. Such forces are, due to their quantum fluctuating character, genuinely quantum, in the sense that they vanish in the classical limit. The first description of these interactions in terms of quantum operators came still in the early days of quantum mechanics, by the hands of F. London [6], who considered the electric field responsible for the interaction to be electrostatic, which means this interaction works out for a distance regime on which retardation of the electromagnetic field is not relevant. In this scenario, there is no need for the electric field to be quantized, and this regime is called non-retarded regime, and the interaction energy scales as $U_{\text{London}} \sim R^{-6}$. Nevertheless, on the late 40's, Verwey and Overbeek [7], on studying lyophobic colloid suspensions, noticed that for such systems, in which Van der Waals forces plays a significant role, the London potential should be adjusted to fall faster than R^{-6} and they suggested that, in this physical problem, due to large separations between the colloidal particles, retardation of the electromagnetic field should be relevant. Such a suggestion was followed closely by H. Casimir and D. Polder [8], who found an interaction energy which scales $U_{\text{Casimir}} \sim R^{-7}$ for sufficiently large inter-particle separations. Let us better define the scales involved. The dipole fluctuation characteristic time is given by $1/\omega_0$ where ω_0 is the dominant atomic transition frequency (typically corresponding to electronic transitions in the ultraviolet part of the spectrum). If the interatomic distance is denoted by R, then the time the photon takes to travel is given by R/c, and thus retardation can be neglected when $\omega_0 R/c \ll 1$. Casimir-Polder expression is valid when the opposite condition holds: $\omega_0 R/c \gg 1$. This is usually referred to in the literature by the name "retarded" limit, but this is a misleading name since

retardation is also relevant when $\omega_0 R/c \sim 1$, a situation where the interaction energy is not a simple power law. Henceforth, we will adopt "Casimir-Polder regime" or "Asymptotic regime" when referring to the large distancing limit ($\omega_0 R/c \gg 1$) throughout this thesis. In this limit, we must account for relativistic effects, and therefore, the formalism of quantum electrodynamics is required. This is the most complete description known today and contains the London result in the appropriate limit. In this fashion, we can say that the attraction between the atoms happens due to fluctuations on the zero-point electric field operator, corresponding to $\langle \mathbf{E}_0^2 \rangle$ [1]. In fact, this central role of electromagnetic zero-point fluctuations, which is applied not only to the two-atom attraction, but also to atom-surface and surface-surface interaction [8, 9] has attracted considerable interest and is still a subject of modern research [10].

Let us discuss in more detail the physical mechanism involved in dispersion forces. Take, for example, a hydrogen atom, and we shall at first use a classical description for the electron. For it, the dipole moment vector is just the electron charge times the electron position from the nucleus $\mathbf{d} = e\mathbf{x}$. Notably, for every instant, the atom presents a non-zero dipole moment. Nevertheless, since the electron orbits the nucleus very fast, the average dipole moment (for time much longer than the orbit time) is zero. This is not true for $|\mathbf{d}|^2$, which is a positive definite quantity. Nonetheless, the electron dynamics is quantum, and a classical description of the electronic orbit would lead to an unstable atom. How can we understand the dipole fluctuation picture in a quantum mechanical framework? Let us now take a look at the quantum hydrogen atom on its ground state, denoted by $|\phi_0\rangle$. The expectation value $\langle \mathbf{d} \rangle := \langle \phi_0 | \mathbf{d} | \phi_0 \rangle = \int d^3x |\varphi_0(r)|^2 \mathbf{d}$ is zero since $\varphi_0(r)$ depends only on the radial coordinate. On the other hand, the average $\langle \mathbf{d}^2 \rangle \neq 0$ in general. Using the closure relation $\mathbb{I} = \sum_n |n\rangle\langle n|$ where $|n\rangle$ denotes the atomic eigenstates, we can write $\langle \mathbf{d}^2 \rangle = \sum_n |\mathbf{d}^{0n}|^2$. The term $\mathbf{d}^{0n} = \langle \phi_0 | \mathbf{d} | n \rangle$ is a dipole transition element and is a measure of how the dipole operator can connect two different eigenstates of the atomic Hamiltonian.

We see that from a quantum perspective, the fluctuation $\langle \mathbf{d}^2 \rangle$ is given by the sum of transitions $0 \to n \to 0$ mediated by the dipole. So if we have an interaction which couples with the dipole operator (such as dipole-dipole interaction between two Hydrogen atoms), then we can have virtual excitations which lead to a shift in the non-perturbed energy. Another way to understand the dispersion interaction is to realize that in the presence of another Hydrogen atom, the ground-state wavefunction is no longer spherically symmetric (more specifically, in second-order perturbation theory). The electronic cloud around the atom gets deformed, giving rise to an average dipole moment which constitutes the quantum-mechanical description of the induced dipole.

Since the main purpose of this thesis is to study dispersion interactions in the context of Proca quantum electrodynamics, let us now introduce the latter, starting from its historical context. Back in the early XX century, numerous eminent physicists engaged with the challenge of formulating a relativistic framework for the emerging quantum mechanics. The first great success came with Paul Dirac, with his famous equation and, with a (almost) correct fundamental prediction for the magnetic moment of the electron and the prediction of a new particle, the positron, which was readily detected experimentally. However, Dirac's theory presented controversial interpretations, such as the Dirac sea [11]. In this scenario, the Romanian physicist Alexandru Proca suggested an alternative equation for the electron-positron pair [12]. Even though he realized that this equation should represent the dynamics of a spin-1 particle [13], it had grown in popularity, inspiring H. Yukawa on his remarkable meson theory [14–16], as pointed by W. Pauli [17]. Yukawa, on his theory, proposed that a massive boson should mediate the strong interactions, and by doing so, noted that the mass of such mediator creates a shielding pattern on the potential between the nucleons, which should fall like $e^{-\mu r}/r$, where r is the nucleon distance and $\mu = mc/\hbar$, m being the mass of the boson. During the process of formulating his equation, Proca was inspired by the electromagnetic Maxwell's equations, and proposed the Following

Lagrangian (on the neutral case)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu}{2}A^{\mu}A_{\mu}\,, \tag{1.1}$$

where $A^{\mu} = (c\varphi, \mathbf{A})$ is the electromagnetic four-vector and $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$. It can be noticed that (1.1) is very similar to Maxwell's Lagrangian, and it reduces to it as $\mu \to 0$. The electric and magnetic fields are determined in the same manner as in Maxwell's equations by $\mathbf{E} = -\nabla\varphi - \partial_t\mathbf{A}$ and $\mathbf{B} = -\nabla \times \mathbf{A}$, which allows us to write the field equations as

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} - \mu^2\varphi \tag{1.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.4}$$

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} - \mu^2\mathbf{A}\,. \tag{1.5}$$

Either from (1.1) or (1.2), it can be noticed that the usual gauge invariance, experienced by Maxwell's equations, is not present in Proca theory. From the Lagrangian, it is expressed by the fact that, on a Gauge transformation $A^{\mu} \to A^{\mu} = A'^{\mu} + \partial^{\mu}\chi$, the Lagrangian goes to $\mathcal{L} \to \mathcal{L}' + (\mu/2)[2A'^{\mu}\partial_{\mu}\chi + (\partial_{\mu}\chi)^2]$, and it is noticeable on the Proca equations that the vector and scalar potentials appear on a explicit way. Indeed, there is no way one can write those equations without the potentials; thus, on Proca's theory, the potentials are not merely convenient mathematical tools, but undeniable physical quantities.The four-potential must still satisfy a constraint for the field equations to have a unique solution. The condition $\partial_{\mu}A^{\mu} = 0$is then chosen in order to guarantee charge conservation $\partial_{\mu}J^{\mu} = 0$. It is clear that those equations turn into Maxwell's equations for $\mu = 0$. On Maxwells equations in vacuum ($\rho = 0$ and $\mathbf{J} = 0$), the two divergence equations, $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$ directly imply that a wave solution must be transverse, which does not happen for the Proca equations (specifically for $\nabla \cdot \mathbf{E} = -\mu^2\varphi$), where we can anticipate the fact that a vacuum wave solution on Proca's equations possess an extra polarization. If we look at

classical electrostatics, the solution for the potential for a point particle of charge q changes from $\varphi_{\text{Maxwell}} = q/(4\pi\epsilon_0 r)$ to $\varphi = qe^{-\mu r}/(4\pi\epsilon_0 r)$, where r is the distance from the particle to the analyzing point. This is precisely the Yukawa potential commented in the previous paragraph [14]. Even though his theory has been properly replaced by the Anderson-Higgs mechanism in the description of the spin-1 massive bosons of the standard model, the Proca Lagrangian served as an important starting point. The main problem with Proca's theory is that it is not gauge invariant, which excludes it as a fundamental theory of the standard model. Nevertheless, effective theories such as the nucleon-nucleon interaction, with the massive vector mesons ρ and ω [18, 19] still use the Proca model or variations of it. In addition, research on physics beyond the standard model still represents a fertile field for Proca theories applications, running from electromagnetic extensions [20] to the prominent field of Proca applications on cosmology and astrophysics [21–26], which has experienced a substantial increase of publications on the very few years. Furthermore, as discussed below, Proca electrodynamics is also of great interest as an effective description for many phenomena involving light-matter interaction.

In the quantum field theory scenario, the interactions between atoms are mediated by the exchange of photons. In Maxwell's electrodynamics, the photon is known to be transverse, which is a consequence of its zero mass. This is not necessarily the case for other theories, like the Proca electrodynamics, where the photons possess a longitudinal polarization as a possible degree of freedom, which depends on a parameter often identified by the mass of the photon [27]. There are still theories where a mass-like parameter can emerge as a result of environment influence. For example, colloidal suspensions on electrolyte solutions for weak electric potential satisfies the linearized Poisson-Boltzmann equation outside the electric double layer, which can be identified as the electrostatic Proca equation with mass equal to the inverse of the Debye length [28, 29]. Another compelling example are the so-called Proca metamaterials [30], where a suitable choice

for the constitutive relation for the induced current on free Proca theory may exactly reproduce the properties of a theorized material possessing spatial dispersion on the longitudinal sector of the generalized dielectric function dyad. Another example includes cavities, where the inverse mass is associated with the bounded dimension (for example, the distance L of the plates of a plain capacitor) [31–33]. As a final example of a model effectively represented by Proca electrodynamics, we point out superconductors, where the field equations can be effectively written as the Proca-Maxwell's equations [34], by setting $\mu = \lambda_L^{-1}$, where λ_L is the London penetration depth of the superconductor. It is important to mention the fact that this final example is not a good candidate to apply the framework developed in this Thesis, since the superconducting field equations mentioned before only apply to superconducting electrons, which are known to be non-polarizable, so no dispersion forces can arise from them. We can now raise the question of what should be a typical order of magnitude for mu. Such values are strongly model-dependent [35–38], as we summarized in Table 1.1, and they span 23 orders of magnitude from the smallest to the highest. The magnitude of the mass of a free photon, unlike in the effective models, is not directly measured. Hence, we use as an estimate an upper bound a high precision experiment asserts to the photon mass, like fast radio bursts [35].

Model	μ (m^{-1})
Photon in Vacuum*	10^{-12}
Colloid in electrolyte media	$10^{8} - 10^{9}$
Plasmas	$10^{-5} - 10^{11}$
Superconductors	$10^{6} - 10^{8}$

Table 1.1: Typical Values of the μ parameter for various models . *An upper limit for the photon mass in vacuum was used.

The first part of this thesis is devoted to the study of the dispersion forces between two atoms in Proca's electrodynamics. In doing so, we determine not only the effect of the new polarization but also how the introduction of the mass parameter affects the well-established London regime (short interatomic distances or non-retarded regime) and

Casimir-Polder regime (long interatomic distances or asymptotic regime). For dispersion forces on Maxwell's electrodynamics, it is a known fact that the non-retarded regime, defined as the electrostatic one, is defined as the limit where the time for the photon two go from one atom to the other is much less then a characteristic time concerning the atom, in other words, $T_{\text{photon}} \ll \omega_0^{-1}$, for ω_0 a typical transition energy of the atom. The asymptotic regime is mathematically defined in the opposite way as $\omega_0^{-1} \gg T_{\text{photon}}$. Throughout this thesis, we show how those defining relations are modified due to the presence of a massive photon in the theory. In fact, even the notion of a single photon traveling between the atoms turns out to be senseless; in such a way, we will define the new relations in terms of the phase velocity of the massive photon for some wave number. The potential's behavior has been analyzed for those two physical regimes, employing non-relativistic quantum mechanics, which allows us only to reach the potential for the non-retarded regime, and the quantum field theory approach, which makes it possible to obtain both regimes as limiting cases. The latter also clarifies the question of which limiting relations of the physical constants leave us in one of those physical regimes. Additionally, on the quantum field theory approach, we generalize an effective interacting Hamiltonian proposed by P. Milonni [1], which has the advantage of presenting a first-order non-vanishing perturbation, rather than fourth order [39, 40].

Investigating effects between more than just two atoms, dispersion forces exhibit a non-trivial aspect: they are, in general, non-additive. This is the issue of the second part of this thesis. To understand qualitatively what it means, let us take the three atoms A, B, and C as an example. If the interaction between them was to be additive, the force on atom A should be the sum of the forces between A and B plus the force between A and C. Since the force is not additive, we have in general $\mathbf{F}_{\text{Total}} \neq \mathbf{F}_{AB} + \mathbf{F}_{AC}$, where $\mathbf{F}_{\text{Total}}$ is the total force on A and $\mathbf{F}_{KL}$ stands for the force between atoms K and L. This states there must be a term depending simultaneously on the three atoms: a $\mathbf{F}_{ABC}$ must be added, which we

will call the non-additive contribution. Such a problem was first studied by Axilrod and Teller [41] in the 30's. This property of dispersion forces is especially important when one aims to set the validity limits to the usual pair-wise (additive) summation for a many-body description. The non-additivity is usually tiny when we talk about the interaction of a few atoms. This is often the case because the non-additive contribution comes in higher-order perturbation theory (compared to the two-atom dispersion force). Nonetheless, the interaction between many atoms has shown to be stronger and more relevant over the dispersion two-atoms force in some situations, where we emphasize the interaction between two atoms and a conducting macroscopic surface [31], which inspired us to look not only for the non-additivity on three atoms problem but still on the interaction between atoms and conducting surfaces on Proca's electrodynamics as well. The latter, indeed, has proven to admit configurations where the non-additive contribution is even dominant. Additionally, studies from the past two decades concerning non-additivity on Van der Waals forces include improvement on the understanding of intermolecular interactions [42], the role of macroscopic conducting surfaces over intermolecular interactions [43], charged colloids interaction [44], nano-materials [45], besides many others. The present Thesis is inspired by two manuscripts with the participation of the Thesis author. On the former, "Dispersive interaction between two atoms in Proca quantum electrodynamics", published on *Physical Review D* [46], we determined the dispersion interaction for two non-polarized but polarizable particles on the Proca electrodynamics scenario. We split the calculation into a non-relativistic one, where the electric field is considered static and retardation effects are not considered, and a relativistic one, where we consider retardation effects. With the relativistic treatment, we calculated the interacting energy in two regimes: the non-retarded regime (which we proved to be the electrostatic one) and the asymptotic regime (also known as retarded regime in the literature). We also discovered that, even though those regimes still exist in Proca electrodynamics, the limiting conditions to

be satisfied are modified by the mass of the photon. On the last manuscript, named "Non-Additive effects on Proca electrodynamics", which is in the last revision phase to be sent to the publisher, we determined the non-additive parcel of the dispersion forces on Proca electrodynamics. Even though the mass weakens the role of non-additivity for individual atoms, we showed that this role can be enhanced for interactions between atoms and conducting surfaces.

The organization of this thesis is as follows. On Chapter 2, the dispersion interaction for two atoms is developed, starting from the non-relativistic quantum mechanics approach on Sec. 2.1, reaching the interacting energy on the non-retarded limit, and completing the analyses by presenting the quantum field theory approach on Sec. 2.2, reaching the complete potential and obtaining both retarded and asymptotic limits from it. On Chapter. 3, it we analyzed the non-additive contribution for three systems: three atoms interaction (Sec. 3.1), two atoms and a conducting plane (Sec. 3.2) and two atoms inside conducting parallel plates (Sec. 3.3).

Chapter 2

Two-atoms dispersion force

The electric dipole approximation is valid as long as the distance separating the quantum particles is great compared to their size, a condition we assume valid in all our work. We will refer to atoms throughout our work; nonetheless, with minimal modifications, our treatment is also applicable to molecules or nanospheres, as long as the electric dipole approximation applies. To get an intuition of the usual scales involved, for an Hydrogen atom, we can take its size to be the Bohr radius, which is of the order of $a_0 = 5 \times 10^{-11}$ m, and distances of around 5-10 a_0 are typically satisfactorily described by the electric dipole approximation.

2.1 The non relativistic approach and the non-retarded regime

2.1.1 The non-retarded regime

In this section, we calculate the interaction potential between two point-like electric dipoles in the so-called non-retarded regime. This regime is valid whenever the retardation of the electromagnetic field may be neglected, usually when the atoms are placed close enough so the interaction may be regarded as instantaneous. In order to better understand how much is "close enough", we consider that the time a photon takes from one atom to another is much less than some characteristic time scale involving the atoms, which reads:

$T_{\text{photon}} \ll T_{\text{atom}}$. While photons are introduced by means of a relativistic description, for consistency, we will present this relation in Sec. 2.2, where we develop the necessary tools to quantitatively evaluate the criterion for non-retarded regime to be applicable. The electrostatic regime of electromagnetism is defined as the regime where matter interacts instantaneously, which means that the dynamics of the electric field is completely defined by the dynamics of the charge distributions, in this case, the dipolar distributions of atoms A and B. In other words, in this regime the electric field is not directly quantized, whereas the dipole moments, representing the charge distributions here, will be quantized. The interaction Hamiltonian is given by, [47]

$$H_{int} = -\mathbf{d}_B \cdot \mathbf{E}_A(\mathbf{x}_B)\,, \tag{2.1}$$

where $\mathbf{E}_A(\mathbf{R}_B)$ is the electrostatic field generated by the dipole $\mathbf{d}_A$ at the position occupied by atom B, which we denote by $\mathbf{x}_B$. Although the electrostatic field is well-known in the case of Maxwell electrodynamics, we must determine it for the Proca electrodynamics before studying the dispersion forces. In the next section, we begin with the classical case.

2.1.2 Classical electrostatic dipole field on Proca electrodynamics

In order to find the electrostatic potential generated by a point-wise electric dipole governed by Proca electrodynamics, we start by calculating the potential generated by a point-like charge. The equation satisfied by the electrostatic potential is given by the modified Poisson equation given by

$$\left(\nabla^2 - \mu^2\right)\phi(\mathbf{x},\mathbf{x}') = -\frac{q}{\epsilon_0}\delta(\mathbf{x}-\mathbf{x}')\,, \tag{2.2}$$

where q is the particle's electric charge, the potential is evaluated at position $\mathbf{x}$, and the source is located at $\mathbf{x}'$. Note that $(\epsilon_0/q)\phi(\mathbf{x},\mathbf{x}')$ is the Green function $G_I(\mathbf{x},\mathbf{x}')$ of the

Helmholtz equation discussed on Appendix A for $\lambda = i\mu$, so

$$\phi(\mathbf{x}, \mathbf{x}') = \frac{q\, e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}'|}\,. \tag{2.3}$$

We are now in a position to determine the electric potential of a pointwise dipole. It may be computed as a limit of two point charges of modulus q, when the distance $\mathbf{a}$ from each other is set to zero

$$\phi_{\rm dip}(\mathbf{x}, \mathbf{x}') = q\phi(\mathbf{x}, \mathbf{x}' + \mathbf{a}) - q\phi(\mathbf{x}, \mathbf{x}') = \mathbf{d}\cdot\nabla_{\mathbf{a}}\phi(\mathbf{x}, \mathbf{x}' + \mathbf{a})|_{\mathbf{a}=0}\,, \tag{2.4}$$

where $\mathbf{d}$ is $q\mathbf{a}$, so it is recognized as the dipole moment and $\nabla_{\mathbf{a}}$ is the usual gradient with respect to the $\mathbf{a}$ variable. Using Eqs. (2.3) and (2.4), one finds for the dipole Proca field

$$\phi_{\rm dip}(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi\epsilon_0}\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|^3}(\mu|\mathbf{x}-\mathbf{x}'| + 1)[\mathbf{d}\cdot(\mathbf{x}-\mathbf{x}')]\,. \tag{2.5}$$

The dipole field on Maxwells' electrodynamics is obtained by setting $\mu = 0$, as expected. Now we can determine the electrostatic field generated by a pointwise dipole, labeled by A, at a position $\mathbf{x}_A$, evaluated on the position $\mathbf{x}_B$. We do this by computing[1]

$$\mathbf{E}_A(\mathbf{x}_B) = -\,\nabla_{\mathbf{x}}\phi_{\rm dip}(\mathbf{x}, \mathbf{x}_A)|_{\mathbf{x}=\mathbf{x}_B}\,. \tag{2.6}$$

We use the fact that $\phi_{\rm dip}(\mathbf{x}, \mathbf{x}_A) = \phi_{\rm dip}(\mathbf{x}-\mathbf{x}_A)$ so one can define $\mathbf{s} \equiv \mathbf{x}-\mathbf{x}_A$, which allows us to write:

$$\mathbf{E}_A(\mathbf{x}_B) = -\,\nabla_{\mathbf{s}}\phi_{\rm dip}(\mathbf{s})|_{\mathbf{s}=\mathbf{R}}\,, \tag{2.7}$$

where $\mathbf{R} = \mathbf{x}_B - \mathbf{x}_A$. By inserting Eq. (2.5) on Eq. (2.7), and identifying $\mathbf{d} = \mathbf{d}_A$ as the electric dipole moment of atom A, we find:

$$\mathbf{E}_A(\mathbf{x}_B) = \frac{e^{-\mu R}}{4\pi\epsilon_0 R^5}(\mu^2R^2 + 3\mu R + 3)(\mathbf{d}_A\cdot\mathbf{R})\mathbf{R} - \frac{e^{-\mu R}}{4\pi\epsilon_0 R^3}(\mu R + 1)\mathbf{d}_A\,. \tag{2.8}$$

The Maxwell electric field of a static dipole is obtained simply by setting $\mu = 0$.

[1] In Proca electrodynamics, likewise in Maxwell's, the potentials are time-independent in the electrostatic regime.

2.1.3 Non-retarded Dispersion Interaction between two atoms

Substituting Eq.(2.8) into Eq. (2.1) we obtain the interaction energy for two dipoles in Proca electrodynamics, which is given by

$$H_{\rm int} = -\frac{e^{-\mu R}}{4\pi\epsilon_0 R^5}(\mu^2 R^2 + 3\mu R + 3)(\mathbf{d}_A \cdot \mathbf{R})(\mathbf{d}_B \cdot \mathbf{R}) + \frac{e^{-\mu R}}{4\pi\epsilon_0 R^3}(\mu R + 1)(\mathbf{d}_A \cdot \mathbf{d}_B)\,. \quad (2.9)$$

In the non-retarded regime the electric field does not need to be quantized, and thus the only quantum operators are the atomic dipole operators. We define a system's state $|rs\rangle = |r\rangle \otimes |s\rangle$ where $|r\rangle$ and $|s\rangle$ are eigenstates of the non-perturbed Hamiltonians of atoms A and B, respectively. The non-degenerate ground state is represented simply by $|0_A 0_B\rangle = |0_A\rangle \otimes |0_B\rangle$, where $|0_A\rangle$ and $|0_B\rangle$ are the ground states of atoms A and B, respectively. It is intuitive that this Hamiltonian can be treated perturbatively since it stands for the electrostatic interaction between charges in one atom with the charges in the other one, which is far apart since we assumed the distance separating the atoms to be greater than the atomic size. This can be verified explicitly after performing the calculations by substituting the expressions for the matrix elements of the dipole operator and comparing the perturbative energy with the atomic eigenvalues, as is done, *e.g.*, in atomic physics books [48].

Since the atoms do not have permanent electric dipole moment, we have that

$$\langle 0_A 0_B|\mathbf{d}_K|0_A 0_B\rangle = 0\,, \quad (2.10)$$

where $K = \{A, B\}$. Using this last condition, in addition to Eq. (2.9), it is possible to conclude that first order contribution to the interaction energy is zero. Going forward to second order, we write

$$U \;=\; -\sum_{r\neq 0, s\neq 0} \frac{\langle 0_A 0_B|H_{int}|rs\rangle\langle rs|H_{int}|0_A 0_B\rangle}{\hbar(\omega_{r0} + \omega_{s0})}\,, \quad (2.11)$$

where $\hbar\omega_{0k}$ is the energy shift between the k-th energy level of one of the atoms and its ground state. In order to simplify the calculations throughout this thesis, we note that

the interacting Hamiltonian can be written as

$$H_{int} = f_{ij}(\mathbf{R}, \mu) d_A^i d_B^j \,, \tag{2.12}$$

where

$$f_{ij}(\mathbf{R}) = \frac{e^{-\mu R}}{4\pi\epsilon_0 R^3}\Big[-\Big((\mu R)^2 + 3(\mu R) + 3\Big)\hat{R}^i\hat{R}^j + (\mu R + 1)\delta^{ij}\Big]. \tag{2.13}$$

The atoms' isotropy allows us to substitute:

$$\langle 0_K|d_K^i|I\rangle\langle I|d_K^j|0_K\rangle = \frac{\delta_{ij}}{3}|\mathbf{d}_K^{0I}|^2 \,, \tag{2.14}$$

where $K = \{A, B\}$ and $\mathbf{d}_K^{0I}$ is the matrix element between the K atom's ground state and a state I. By inserting Eqs. (2.12), (2.13), (2.14) on (2.11), we end up with a compact form for the interacting energy

$$U = -\frac{f_{ij}(\mathbf{R})^2}{9} \sum_{r\neq 0, s\neq 0} \frac{|\mathbf{d}_A^{0r}|^2 |\mathbf{d}_B^{0s}|^2}{\hbar(\omega_{r0} + \omega_{s0})} \,. \tag{2.15}$$

The summation term on the right-hand side depends only on the atomic internal degrees of freedom, and it is defined as:

$$\Lambda = \sum_{r\neq 0, s\neq 0} \frac{|\mathbf{d}_A^{0r}|^2 |\mathbf{d}_B^{0s}|^2}{\hbar(\omega_{r0} + \omega_{s0})} \,. \tag{2.16}$$

Such a constant is positive and is related to the dynamical atomic polarizabilities $\alpha_A(\omega)$ and $\alpha_B(\omega)$ (which will be better defined in Subsec. 2.2.1) [40]

$$\Lambda = \frac{9}{4\pi} \int_{-\infty}^{\infty} \alpha_A(i\chi)\alpha_B(i\chi) d\chi \,. \tag{2.17}$$

By using Eqs. (2.13) and (2.17), we evaluate the interaction energy (2.15)

$$U_{\mathrm{NR}} = -\Big(\frac{1}{4\pi\epsilon_0}\Big)^2 \Lambda \frac{e^{-2\mu R}}{9R^6}\Big[6 + 12(\mu R) + 10(\mu R)^2 + 4(\mu R)^3 + (\mu R)^4\Big] \,. \tag{2.18}$$

In order to compare this result with the one obtained by F. London [49], we write

$$U_{\mathrm{NR}} = U_{\mathrm{London}}\, e^{-2\mu R}\Big[1 + 2(\mu R) + (5/3)(\mu R)^2 + (2/3)(\mu R)^3 + (1/3)(\mu R)^4\Big] \,, \tag{2.19}$$

where $U_{\text{London}} = \frac{\Lambda}{24\pi^2\epsilon_0^2 R^6}$. As a first test we can verify that if $\mu \to 0$ we re-obtain the known value from Maxwell electrodynamics:

$$\lim_{\mu\to 0} U_{\text{NR}} = U_{\text{London}}\,. \tag{2.20}$$

It is possible do show that $0 < \exp[-2x](1 + 2x + (5/3)x^2 + (2/3)x^3 + (1/3)x^4) \leq 1$ for $x \geq 0$. One direct consequence of this fact is that U_{NR} remains attractive as a non-zero positive μ is introduced. One second conclusion is that the strength of interaction is weakened when compared to the $\mu = 0$ case, which is expected, since massive terms, as in the Yukawa potential, usually shield the interaction. Both these conclusions agree with Fig. 2.1 , where we plotted the ratio between the interactions on Proca and Maxwell scenarios.

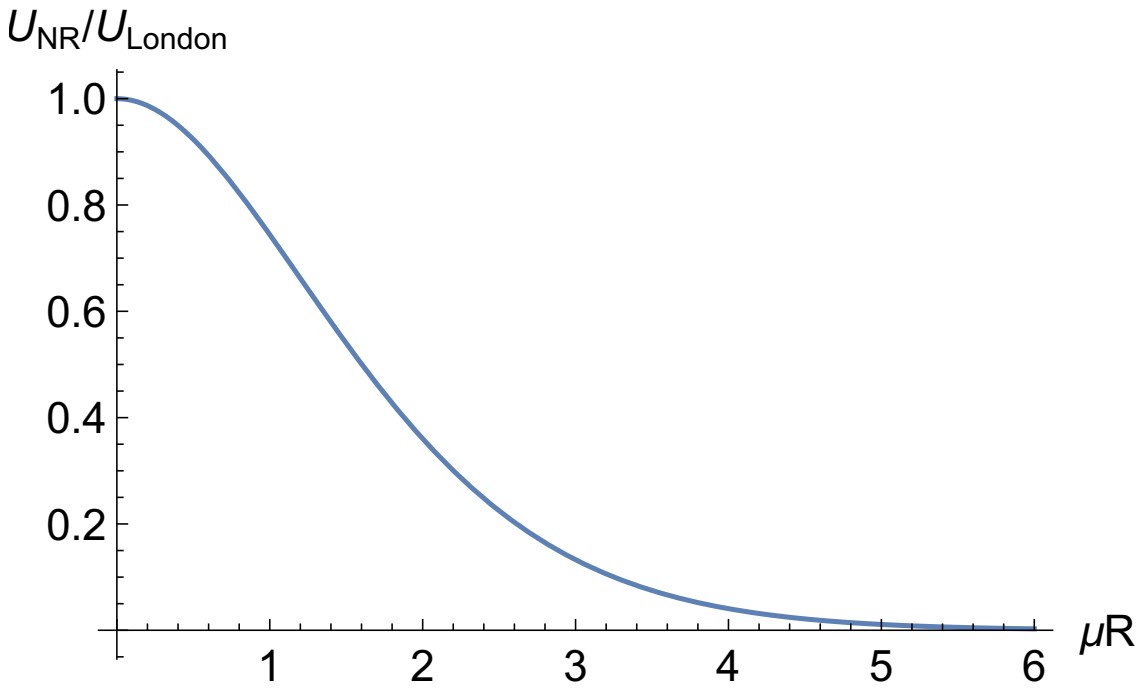


Figure 2.1: Interaction energy in the non-retarded regime divided by the London-van der Waals potential as a function of the dimensionless radius parameter μR.

2.2 Quantum field theory approach and the asymptotical regime

2.2.1 The interacting Hamiltonian: toward a dynamical description

In this section, we present the necessary formalism to include scales where the retardation of the electromagnetic Proca field is relevant. Therefore, we must introduce the massive photon Fock space. This is done by means of the Proca quantum electrodynamics (PQED), presented in quantum field theory textbooks, like Ref. [27]. In what concerns we could use Hamiltonian (2.1), substituting the static electric field by the quantized electric field and the static electric dipole $\mathbf{d}_B$ by $\mathbf{d}_B(t)$. This would lead to a laborious fourth-order perturbation theory calculation, as done on [39, 40] for the Maxwell theory. Here we follow an alternative approach based on a very convenient, effective Hamiltonian. Let us begin with a classical intuition for this effective Hamiltonian, originally proposed by Peter Milonni [1]. In classical electromagnetism, the interaction energy of an induced dipole in the presence of an electric field is given by $-\frac{1}{2}\mathbf{d}_{\text{ind}} \cdot \mathbf{E}$, where the factor $1/2$ reflects the fact that the dipole is induced [50]. In the static case, $\mathbf{d}_{\text{ind}} = \alpha_0 \mathbf{E}$, due to the spherical symmetry of the atom, perhaps, in general, the polarizability is a rank 2 tensor. For a monochromatic time-dependent field $\mathbf{E}_\omega(t) = \tilde{\mathbf{E}}(\omega)e^{i\omega t}$, the polarizability is, due to time dispersion, a function of the frequency ω and its precise form depends on the model. This means that in Fourier space we can write $\tilde{\boldsymbol{d}}_{\text{ind}}(\omega) = \alpha_K(\omega)\tilde{\boldsymbol{E}}(\omega)$, and the interacting energy between a polarizable body and a monochromatic electric field for a frequency mode ω has the form

$$U_\omega = -\frac{1}{2}\alpha_K(\omega)\mathbf{E}^2(\omega)\,. \tag{2.21}$$

For an atom in quantum mechanics, the frequency-dependent (dynamical) polarizability has the following form [51]:

$$\alpha_K(\omega) = \frac{2}{3}\sum_{I\neq 0}\frac{\omega_{I0}|\mathbf{d}_K^{0I}|^2}{\hbar(\omega_{I0}^2-\omega^2)}\,, \tag{2.22}$$

where K labels the atom, $\hbar\omega_{0I}$ is the energy shift between the ground state and the I level, and $\mathbf{d}_K^{0I}$ is the matrix element of the dipole operator on the ground state and the level I.

The above classical discussion suggests the Hamiltonian should present a form close to (2.21), where we must substitute K by A or B and $\tilde{\mathbf{E}}(\omega)$ by the frequency domain electric quantum field on point $\mathbf{x}_A$ or $\mathbf{x}_B$. In fact, the quantum field theory Hamiltonian that we use is a generalization of an effective Hamiltonian proposed by P. Milonni ([1] Eqs. (2.72)-(2.78)) reads:

$$H_{int} = -\frac{1}{2}\sum_{k,\lambda}\alpha_A(\omega_k)\left[\mathbf{E}_{0,\mathbf{k}\lambda}(\mathbf{x}_A,t)+\mathbf{E}_{B,\mathbf{k}\lambda}(\mathbf{x}_A,t)\right]^2\,, \tag{2.23}$$

where $\omega_k = c\sqrt{k^2+\mu^2}$. As it is readily observable, the electric field on the $\mathbf{x}_A$ has been split into two contributions, which possess the following interpretation: $\mathbf{E}_{0,k\lambda}(\mathbf{x}_A,t)$ is the quantum vacuum electric field operator for the Proca electrodynamics (see Ref. [27]); the term $\mathbf{E}_{B,k\lambda}(\mathbf{x}_A,t)$ is a little bit more subtle. It is the electric field operator induced by the vacuum electric field on atom B. More precisely, the vacuum field induces a dipole

$$\mathbf{d}_B(t) = \sum_{k,\lambda}\alpha(\omega_k)\mathbf{E}_{0,\mathbf{k}\lambda}(\mathbf{x}_B,t) \tag{2.24}$$

on atom B, and the field produced by this dipole on atom's A position is $\mathbf{E}_{B,k\lambda}(\mathbf{x}_A,t)$, as presented in Ref. [1]. An essential feature of this prescription is that it does the opposite of the non relativistic treatment of Sec. 2.1: there, the electric field is transported to the dipole moment Hilbert space; here, the dipole moment $\mathbf{d}_B(t)$ is transported to the Hilbert space of the electric field (Fock space). The details of this approach, as well as the electric field PQED properties, will be given afterwards. In the next section, we obtain the quantum electric field. Then, we employ it to obtain the interaction energy between two

atoms in PQED. Using the effective Hamiltonian (2.23), we will show that the interaction energy is given directly in first order of perturbation theory.

2.2.2 The quantized Proca electric field

To account for a finite speed of the interaction between the atoms, one may follow the quantum field theory (QFT) approach, where atoms interact through the exchange of virtual photons. On the Proca theory presented here, the atoms interact via a massive photon exchange: an Abelian vector field with mass proportional to the parameter μ. As commented in the previous subsection, the free vacuum field and its Fock space are crucial to determine the interaction energy between the atoms when the retardation must be taken into account. In this subsection, we present the necessary concepts of this subject needed to evaluate the interacting energy between the atoms in any regime of distances. The approach followed by us is the usual canonical quantization of the Proca field, where most part can be found in more details in Ref. [27].

The first step is to write down the Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu - J_\mu A^\mu \,, \tag{2.25}$$

where A^μ is the usual four vector potential, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and J^μ is the four-current[2]. The potential is related to the electric field by $\mathbf{E}(x) = -\partial_t \mathbf{A}(x) - c\nabla A^0(x)$. A key feature of the above Lagrangian is that it is not gauge invariant, due to the presence of the second term. By using the Euler-Lagrange equation for the above Lagrangian, one finds

$$(\Box + \mu^2)A^\mu - \partial^\mu(\partial_\nu A^\nu) = J^\mu \,, \tag{2.26}$$

with $x = (t, \mathbf{x})$, and $\Box = \frac{1}{c^2}\partial_t^2 - \nabla^2$. Deriving the above equation with respect to x^μ, gives

$$\partial_\mu A^\mu = \frac{1}{\mu^2}\partial_\mu J^\mu \,. \tag{2.27}$$

[2]For brevity, we have absorbed all the S.I. constants on the J^μ definition.

Unlike in Maxwell's theory, charge conservation does not emerge as a direct consequence of the Proca equations (which is expected since charge conservation comes from gauge invariance of Maxwell's equations). It has to be imposed *a priori*. As a consequence of this choice, the following condition holds

$$\partial_\mu A^\mu = 0\,. \tag{2.28}$$

The reader may be aware of the above equation, even though it looks like the Lorentz gauge condition for Maxwell electrodynamics, it is no gauge condition here (since there is no gauge freedom at all), it is mandatory as a result from charge conservation. Inserting condition (2.28) on Eq. (2.26), we have

$$(\Box + \mu^2)A^\mu = J^\mu\,, \tag{2.29}$$

In order to quantize the Proca field, the next natural step is to find the conjugate momentum of $A^\mu(x)$ and impose equal time commutation relations (ETCR). Note that $\partial_0 A_0$ does not appear in the Lagrangian, and thus the momentum conjugate to A_0 vanishes identically. We say that A_0 is a non-dynamical variable. This kind of variable must be treated carefully when quantizing a field theory. Due to the condition (2.28), we see that A_0 is determined from the spatial components of A_i. Thus, we treat only them as independent variables [52]. These latter can be quantized following the standard procedure due to Dirac following the ETCR.

$$[A^i(t,\mathbf{x}), \pi^i(t,\mathbf{x}')] = i\hbar\delta(\mathbf{x}-\mathbf{x}')\,, \tag{2.30}$$

where $\pi^i(x)$ is the conjugate momentum related to $A^i(x)$ as

$$\pi^i(x) = \frac{\partial\mathcal{L}}{\partial(\partial_t A_i)} = -E^i(x)\,. \tag{2.31}$$

It is worth noting that one may define a zero component π^0 such that $\pi^0(x) = \frac{\partial\mathcal{L}}{\partial(\partial_t A_0)} = 0$. On the next step, we search for a set of mode solutions for Eq. (2.29) for $J^\mu = 0$, which

can be cast in the following way:

$$u^{\mu}_{k\lambda}(x) = \epsilon^{\mu}_{\mathbf{k}\lambda} N_k e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} = \epsilon^{\mu}_{k\lambda} N_k e^{ik_\nu x^\nu} \,, \tag{2.32}$$

where

$$\omega_k = c\sqrt{k^2 + \mu^2} \,, \tag{2.33}$$

$x^\mu = (t, \mathbf{x})$ and $k^\mu = (\omega_k/c, \mathbf{k})$. The constant N_k accounts for the normalization and the four-vector $\epsilon^{\mu}_{\mathbf{k}\lambda}$ are unit vectors. We take as basis the unit vectors

$$\begin{aligned}
\epsilon^{\mu}_{\mathbf{k}1} &= (0, \boldsymbol{\epsilon}_{\mathbf{k}1}) \\
\epsilon^{\mu}_{\mathbf{k}2} &= (0, \boldsymbol{\epsilon}_{\mathbf{k}2}) \\
\epsilon^{\mu}_{\mathbf{k}3} &= \left(\frac{k}{\mu}, \frac{\omega_k}{c\mu}\hat{k}\right) \\
\epsilon^{\mu}_{\mathbf{k}0} &= \frac{1}{\mu} k^\mu \,,
\end{aligned} \tag{2.34}$$

where $\hat{k} = \mathbf{k}/k$. The three-vectors $\boldsymbol{\epsilon}_{\mathbf{k}1/2}$ satisfy $\mathbf{k}\cdot\boldsymbol{\epsilon}_{\mathbf{k}1/2} = 0$ (they are transverse). The above vectors are linearly independent and satisfy the normalization condition $g_{\mu\nu}\epsilon^{\mu}_{\mathbf{k}\lambda}\epsilon^{\nu}_{\mathbf{k}\lambda} = g_{\lambda\lambda}$, so they still satisfy a completeness relation

$$\sum_{\lambda=0}^{3} g^{\lambda\lambda}\epsilon^{\mu}_{\mathbf{k}\lambda}\epsilon^{\nu}_{\mathbf{k}\lambda} = g^{\mu\nu} \,. \tag{2.35}$$

It happens that only three of the four above vectors ($\lambda \neq 0$), inserted on (2.32), actually satisfy condition (2.27). Though the basis we take contains only $\lambda = \{1, 2, 3\}$, which we must take into account, it is not a complete basis in four-space. Each of $\lambda = \{1, 2, 3\}$ is identified with a degree of freedom, which is independent of momentum and in some cases can be interpreted as the polarization of the particle that comes out on the quantization procedure, so $\epsilon^{\mu}_{\mathbf{k}\lambda}$ will be called polarization vectors. One important point that deserves to be mentioned is that, for the Maxwell theory, it is impossible to find three space-like linearly independent (LI) polarization vectors, which leads to the conclusion that the photon has only two polarizations (in fact, the polarizations orthogonal to $\mathbf{k}$) in contrast

with the massive photon, or the spin 1 massive vector field, which has three polarization LI vectors. This feature inspires effective theories where the presence of a polarization on $\mathbf{k}$ direction can be interpreted as theories mediated by massive photons with some effective mass.

In this Thesis, we choose to normalize the fields using the box normalization procedure, on which we impose periodic boundary conditions

$$\begin{aligned} A^\mu(t,x,y,z) &= A^\mu(t,x+L,y,z)\,, \\ A^\mu(t,x,y,z) &= A^\mu(t,x,y+L,z)\,, \\ A^\mu(t,x,y,z) &= A^\mu(t,x,y,z+L)\,, \end{aligned} \tag{2.36}$$

so that $\mathbf{k} = \left(\frac{2\pi l}{L}, \frac{2\pi m}{L}, \frac{2\pi n}{L}\right)$, with $\{l, m, n\}$ positive integers. We still choose the following inner product for the mode functions:

$$(f^\mu(x), g^\mu(x)) = i\int d^3x \Big[f^*_\mu(x)\partial_t g^\mu(x) - g^\mu(x)\partial_t f^*_\mu(x)\Big]\,. \tag{2.37}$$

By demanding that

$$\left(u^\mu_{\mathbf{k},\lambda}(x), u^\mu_{\mathbf{k}',\lambda'}(x)\right) = \hbar^{-1}\delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda\lambda'}\,, \tag{2.38}$$

we find the normalization constant to be

$$N_k = \sqrt{\frac{\hbar}{2V\omega_k}}\,. \tag{2.39}$$

In this way, we obtain field commutators as Dirac deltas (to be verified soon). The field operator is given by a linear superposition of the modes

$$A^\mu(x) = \sum_{\mathbf{k}\lambda}\Big[a_{\mathbf{k}\lambda}u^\mu_{\mathbf{k}\lambda}(x) + a^\dagger_{\mathbf{k},\lambda}u^{\mu*}_{\mathbf{k}\lambda}(x)\Big]\,, \tag{2.40}$$

where $a_{\mathbf{k}\lambda}$ and $a^\dagger_{\mathbf{k}\lambda}$ are operators which are called ladder operators or creation ($a^\dagger_{\mathbf{k}\lambda}$) operator and annihilation ($a_{\mathbf{k}\lambda}$) operator, as for the harmonic oscillator in quantum mechanics. The reason will become clear soon. Using Eqs. (2.32) and (2.39), we have

$$A^\mu(x) = \sqrt{\frac{\hbar}{V}}\sum_{\mathbf{k}\lambda}\frac{\epsilon^\mu_{\mathbf{k}\lambda}}{\sqrt{2\omega_k}}\Big[a_{\mathbf{k}\lambda}e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} + a^\dagger_{\mathbf{k}\lambda}e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{x}}\Big]\,. \tag{2.41}$$

The ETCRs for the field and its conjugate (2.30), along with the expansion (2.41) directly imply

$$\left[a_{\mathbf{k},\lambda}, a^{\dagger}_{\mathbf{k}'\lambda'}\right] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda\lambda'}\,. \tag{2.42}$$

The last step to quantization of the free Proca field consists in writing the Hamiltonian in terms of $a_{\mathbf{k}\lambda}$ and $a^{\dagger}_{\mathbf{k}\lambda}$. The Hamiltonian is defined by

$$H = \int d^3x[\boldsymbol{\pi}\cdot\partial_t\mathbf{A} - \mathcal{L}]\,, \tag{2.43}$$

and using Eqs. (2.30), (2.41), it is rewritten as (for a detailed calculation, see for example Ref. [27])

$$H = \sum_{\mathbf{k}\lambda} \hbar\omega_k a^{\dagger}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}\,. \tag{2.44}$$

The Hamiltonian (2.44), with the commutations relations (2.42), is the Hamiltonian for a sum of non-interacting harmonic oscillators $H_{\mathbf{k}\lambda} = \omega_k a^{\dagger}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}$, one for each pair $\mathbf{k}\lambda$. The total Hamiltonian is

$$H = \sum_{\mathbf{k}\lambda} H_{\mathbf{k}\lambda}\,.$$

This form indicates the path to the state space, which must be the same as for multiple oscillators with different frequencies. The state space is formally written as

$$|\psi\rangle = \bigotimes_{\mathbf{k}\lambda} |n_{\mathbf{k}\lambda}\rangle\ ,$$

where $|n_{\mathbf{k}\lambda}\rangle$ satisfies $H_{\mathbf{k}\lambda}\,|n_{\mathbf{k}\lambda}\rangle = n_{\mathbf{k}\lambda}\hbar\omega_k$. We will refer to the ground state, the one which $n_{\mathbf{k}\lambda} = 0$ for every $\mathbf{k}\lambda$, just by $|0\rangle$. On the quantum field theory interpretation, $|n_{\mathbf{k}\lambda}\rangle$ is a state with $n_{\mathbf{k}\lambda}$ particles (or excitations) with momentum $\mathbf{k}$ and polarization λ. For this reason, the operator $a^{\dagger}_{\mathbf{k}\lambda}$ *creates quanta* of momentum $\mathbf{k}$ and polarization $\epsilon^{\mu}_{\mathbf{k}\lambda}$

$$a^{\dagger}_{\mathbf{k}\lambda}\,|0\rangle = |1_{\mathbf{k}\lambda}\rangle\ , \tag{2.45}$$

and the operator $a_{\mathbf{k}\lambda}$ *annihilate quanta* of momentum $\mathbf{k}$ and polarization $\epsilon^{\mu}_{\mathbf{k}\lambda}$

$$a_{\mathbf{k}\lambda}\,|1_{\mathbf{k}\lambda}\rangle = |0\rangle\ ,$$

and it annihilates the vacuum (ground state) as well

$$a_{\mathbf{k}\lambda}\,|0\rangle = 0\,.$$

For a detailed analysis of the non-relativistic state space, see the chapters dedicated to the harmonic oscillator in quantum mechanics textbooks, like [53].

The electric field operator is determined explicitly by the fundamental relation and rescaling a factor $\sqrt{\epsilon_0}$ (so that the energy $U \sim \epsilon_0/2 \int d^3x[\mathbf{E}^2(x) + c^2\mathbf{B}^2(x)$ for $\mu \to 0$ implies Eq. (2.44)):

$$\mathbf{E}(x) = i\sum_{\mathbf{k}\lambda}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda}\Big[a_{\mathbf{k}\lambda}e^{-i\omega_k t+i\mathbf{k}\cdot\mathbf{x}} - a^{\dagger}_{\mathbf{k}\lambda}e^{i\omega_k t-i\mathbf{k}\cdot\mathbf{x}}\Big]\,, \tag{2.46}$$

where the "modified" polarization vectors satisfy

$$\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda} = f_\lambda \boldsymbol{\epsilon}_{\mathbf{k}\lambda}\,, \tag{2.47}$$

with f_λ defined by

$$f_\lambda = \begin{cases} 1 & \text{for} \quad \lambda = 1,2 \\ c^2\mu^2/\omega_k^2 & \text{if} \quad \lambda = 3 \end{cases}\,. \tag{2.48}$$

Finally, the electric vacuum field presented on the interacting Hamiltonian (2.23) is defined as:

$$\mathbf{E}_{0,\mathbf{k}\lambda}(\mathbf{x},t) \equiv i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda}\Big[a_{\mathbf{k}\lambda}e^{-i\omega_k t+i\mathbf{k}\cdot\mathbf{x}} - a^{\dagger}_{\mathbf{k}\lambda}e^{i\omega_k t-i\mathbf{k}\cdot\mathbf{x}}\Big]\,. \tag{2.49}$$

We still have one more useful definition:

$$\begin{aligned}\mathbf{E}^{+}_{0,\mathbf{k}\lambda}(\mathbf{x},t) &\equiv i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}e^{-i\omega_k t+i\mathbf{k}\cdot\mathbf{x}} \\ \mathbf{E}^{-}_{0,\mathbf{k}\lambda}(\mathbf{x},t) &\equiv -i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda}a^{\dagger}_{\mathbf{k}\lambda}e^{i\omega_k t-i\mathbf{k}\cdot\mathbf{x}}\,.\end{aligned} \tag{2.50}$$

2.2.3 Proca dipole's vacuum electric field

For the interaction Hamiltonian given in Eq. (2.23), we need the vacuum electric field operator, obtained in the previous section, and also the electric field produced by a

time-varying point dipole. The electrostatic calculation was performed in Subsec. 2.1.2, here we extend it to the dynamical case. To do so, we must solve the differential equation

$$(\Box + \mu^2)A^\mu(x) = \frac{j^\mu(x)}{c^2\epsilon_0}\,, \tag{2.51}$$

which is Eq. (2.29) with the rescaled source $J^\mu(x) \to \frac{j^\mu(x)}{c^2\epsilon_0}$, where $j^\mu(x) = (c\rho(x), \mathbf{j}(x))$ is the electromagnetic four current for a (fixed in space) oscillating dipole. Due to the linearity of Proca electrodynamics, the solution can be presented in terms of the scalar Green function $G(x, x')$ as:

$$A^\mu(x) = \frac{1}{c^2\epsilon_0}\int d^4x' G(x, x')j^\mu(x')\,, \tag{2.52}$$

where $G(x, x')$ satisfies

$$(\Box + \mu^2)G(x, x') = \delta^4(x - x')\,. \tag{2.53}$$

The Green function in question can be made simpler if we analyze it in the frequency domain, where:

$$G(x, x') = G(t - t', \mathbf{x}, \mathbf{x}') = \int \frac{d\omega}{2\pi} e^{i\omega(t-t')}\tilde{G}(\omega, \mathbf{x}, \mathbf{x}')\,, \tag{2.54}$$

where we used time translation symmetry. The Green's equation follows:

$$(-\nabla^2 + (\mu^2 - \omega^2/c^2))\tilde{G}(\omega, \mathbf{x}, \mathbf{x}') = \delta^3(\mathbf{x} - \mathbf{x}')\,. \tag{2.55}$$

The above equation is the usual Helmholtz equation for the Green function discussed in Appendix A for $\lambda^2 = \omega^2/c^2 - \mu^2$. For $|\omega|/c < \mu$, λ is imaginary and the solution is clearly the same as in eq. (2.3), replacing μ by $\sqrt{\mu^2 - \omega^2/c^2}$. For $|\omega|/c > \mu$, on the other hand, Eq. (2.55) admits two independent solutions, which in this case are oscillatory, that may be interpreted as positive and negative frequency wave solutions (or ingoing and outgoing spherical wave solutions). For a detailed derivation, see Appendix A. The whole solution is cast as:

$$\tilde{G}(\omega, \mathbf{x}, \mathbf{x}') = \begin{cases} \tilde{G}_I(\omega, \mathbf{x} - \mathbf{x}') = \frac{1}{4\pi}\frac{e^{-\sqrt{\mu^2 - \omega^2/c^2}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} & \text{for} \quad |\omega| < \mu c \\ \tilde{G}^{\pm}(\omega, \mathbf{x} - \mathbf{x}') = \frac{1}{4\pi}\frac{e^{\mp i\sqrt{\omega^2/c^2 - \mu^2}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} & \text{for} \quad |\omega| > \mu c \end{cases}\,. \tag{2.56}$$

This means that high-frequency modes can propagate, in distinction to the low-frequency ones that are shielded. A physical interpretation can be obtained by considering a quantum of excitation of the Proca field. It has mass $m = \hbar\mu/c$, and therefore its minimum energy must be $\hbar\omega = E = \mu c$. This means that $\omega < \mu c$ violates energy conservation, implying it must decay exponentially, existing only for an amount of time compatible with the Heisenberg uncertainty energy-time relation. When $\omega > \mu c$, it can propagate without decaying. Another physical interpretation arises when we consider Proca electrodynamics as an effective theory describing a colloid, as discussed in the introduction. Here, the photon's mass is due to the shielding provided by electrolytes, which makes the field produced by a localized charge decay exponentially. However, if a dipole oscillates with a frequency high enough, then the shielding charges cannot follow the motion, and the medium becomes transparent.

The remaining key piece to calculate the vector potential is now the four-current. To model it we begin with the charge density $j^0(\mathbf{x}, t) \equiv \rho(\mathbf{x}, t)$. The dipole charge density is represented by the superposition of two point charges of opposite charges q placed a small oscillating distance $\mathbf{a}(t)$ from each other near $\mathbf{x}_B$

$$\rho(\mathbf{x}, t) = q[\delta(\mathbf{x} - \mathbf{x}_B) + \delta(\mathbf{x} - \mathbf{x}_B - \mathbf{a}(t))] = -\mathbf{d}(t) \cdot \nabla\delta(\mathbf{x} - \mathbf{x}_B)\,, \tag{2.57}$$

where $\mathbf{d}(t) \equiv q\mathbf{a}(t)$ is identified as the time dependent dipole moment of atom B. The $\nabla\delta(\mathbf{x} - \mathbf{x}_B)$ is functional operator as much as $\delta(\mathbf{x} - \mathbf{x}_B)$ itself and it acts on a function as

$$\int d^3x \nabla\delta(\mathbf{x} - \mathbf{x}_B) f(\mathbf{x}) = -\nabla f(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_B}\,. \tag{2.58}$$

The spatial $\mathbf{j}(\mathbf{x}, t)$ may be calculated using the continuity equation:

$$\partial_t \rho = -\nabla \cdot \mathbf{j}\,, \tag{2.59}$$

which leads to:

$$\mathbf{j}(\mathbf{x}, t) = \dot{\mathbf{d}}(t)\delta(\mathbf{x} - \mathbf{x}')\,. \tag{2.60}$$

The next step consists of the identification of the electric vacuum field as responsible for inducing a dipole moment on the B atom. From Eq. (2.24) it is useful to define:

$$\mathbf{d}_B^{\pm}(t) = \sum_{\mathbf{k}\lambda} \alpha_B(\omega_k)\mathbf{E}^{\pm}_{0,\mathbf{k}\lambda}(\mathbf{x}_B, t)\,, \tag{2.61}$$

and similarly for the four-current:

$$j^{\mu\pm}(x) = \Big(-c\mathbf{d}^{\pm}(t)\cdot\nabla\delta(\mathbf{x}-\mathbf{x}_B), \dot{\mathbf{d}}^{\pm}(t)\delta(\mathbf{x}-\mathbf{x}_B)\Big)\,. \tag{2.62}$$

This is particularly useful as we use the positive frequency Green's function for the positive frequency field contribution and the same for the negative frequency, so

$$A^{\mu}(x) = \frac{1}{c^2\epsilon_0}\int d^4x G^{+}(x,x')j^{\mu+}(x') + \frac{1}{c^2\epsilon_0}\int d^4x G^{-}(x,x')j^{\mu-}(x')\,. \tag{2.63}$$

A simplification on the complete calculation of $A^{\mu}(x)$ can be done by noting that $\mathbf{d}(t)$ depends on time simply by $\mathbf{e^{\pm i\omega_k t}}$, so we propose the following preliminary calculation:

$$-\int d^4x' G^{\pm}(x,x')\nabla\delta(\mathbf{x}'-\mathbf{x_B})e^{\pm i\omega_k t} = e^{\pm i\omega_k t}\nabla\tilde{G}^{\pm}(\omega_k,\mathbf{x},\mathbf{x}_B)\,, \tag{2.64}$$

where we have used Eq. (2.58) and the fact that $\tilde{G}^{\pm}(\omega,\mathbf{x},\mathbf{x}')$ is a function of the difference $\mathbf{x}-\mathbf{x}'$. The above pattern, using Eqs. (2.57), (2.60), (2.62) and (2.61), allows us to write:

$$A^0(x) = -\frac{1}{c\epsilon_0}\sum_{k\lambda}\alpha_B(\omega_k)\Big[\mathbf{E}^{+}_{0,\mathbf{k}\lambda}(\mathbf{x}_B,t)\cdot\nabla\tilde{G}^{+}(\omega_k,\mathbf{x},\mathbf{x}_B) + \mathbf{E}^{-}_{0,\mathbf{k}\lambda}(\mathbf{x}_B,t)\cdot\nabla\tilde{G}^{-}(\omega_k,\mathbf{x},\mathbf{x}_B)\Big]\,. \tag{2.65}$$

A similar simplification can be done to the spatial components of the four-vector

$$\int d^4x' G^{\pm}(x,x')\partial_t e^{\pm i\omega_k t} = \mp i\omega_k e^{\pm i\omega_k t}\tilde{G}^{\pm}(\omega_k,\mathbf{x},\mathbf{x}')\,, \tag{2.66}$$

where we can readily write (using Eqs. (2.57), (2.61), (2.62) and (2.63))

$$\mathbf{A}(x) = -\frac{i}{c^2\epsilon_0}\sum_{\mathbf{k}\lambda}\omega_k\Big[\mathbf{E}^{+}_{0,\mathbf{k}\lambda}(\mathbf{x}_B,t)\tilde{G}^{+}(\omega_k,\mathbf{x},\mathbf{x}_B) - \mathbf{E}^{-}_{0,\mathbf{k}\lambda}(\mathbf{x}_B,t)\tilde{G}^{-}(\omega_k,\mathbf{x},\mathbf{x}_B)\Big]\,. \tag{2.67}$$

The electric field vacuum dipole operator can then be written as

$$\mathbf{E}_B(\mathbf{x},t) = -c\nabla A^0(\mathbf{x},t) - \partial_t\mathbf{A}(\mathbf{x},t)\,, \tag{2.68}$$

which can be written in a compact form as:

$$E_B^i(\mathbf{x},t) = \frac{1}{\epsilon_0}\sum_{\mathbf{k}\lambda}\left[e^{-i\omega_k t}\,\tilde{d}_B^j(\omega_k)\Big(\omega_k^2/c^2\delta_{ij} + \partial_i\partial_j\Big)\tilde{G}^+(\omega_k,\mathbf{x},\mathbf{x}_B) + \text{h.c.}\right]. \tag{2.69}$$

h.c. means Hermitian conjugate, and

$$\tilde{\boldsymbol{d}}_B(\omega_k) = \alpha_B(\omega_k)\mathbf{E}^+_{0,\mathbf{k}\lambda}(\mathbf{x}_B,t)e^{i\omega_k t}\,. \tag{2.70}$$

On the right-hand side of Eq. (2.69), it is possible to recognize the Fourier transformed classical electric field generated by a pointwise time-dependent dipole, as can be seen in the appendix C. Indeed, Eq. (C.7) holds for quite a general time dependence of the dipole moment, not only Eq. (2.61). Then, the vacuum dipole field can be written in the form

$$\mathbf{E}_B(\mathbf{x},t) = \sum_{\mathbf{k}\lambda}\left[e^{-i\omega_k t}\tilde{\mathbf{E}}(\mathbf{x},\omega_k) + \text{h.c.}\right]. \tag{2.71}$$

The vacuum dipole electric field operator on $\mathbf{x}_A$, using Eqs. (2.71), (C.10), (2.70) and noting the fact that $\gamma = ik$ for positive frequency $|\omega|/c > \mu$ scenario, is given by

$$E_B^i(\mathbf{x}_A,t) = \sum_{\mathbf{k},\lambda}\alpha_B(\omega_k)\Big[D_{ij}(k,\mathbf{R})E^{j+}_{0,\mathbf{k}\lambda}(\mathbf{x}_B,t) + \text{h.c.}\Big], \tag{2.72}$$

where we defined

$$\begin{cases} D_{ij}(k,\mathbf{R}) = f\delta_{ij} + gR_iR_j \\ f = \frac{e^{-ikR}}{4\pi\epsilon_0 R^3}((k^2+\mu^2)R^2 - ikR - 1) \\ g = \frac{e^{-ikR}}{4\pi\epsilon_0 R^5}(-k^2R^2 + 3ikR + 3) \end{cases}. \tag{2.73}$$

A similar procedure also holds for the component $\boldsymbol{k}\lambda$ of the field, thus leading us to

$$\begin{aligned} E^i_{B,\mathbf{k}\lambda}(\mathbf{x}_A,t) &= \alpha_B(\omega_k)\Big[D_{ij}(k,\mathbf{R})E^{+j}_{0,\mathbf{k}\lambda}(\mathbf{x}_B,t) + D^*_{ij}(k,\mathbf{R})E^{-j}_{0,\mathbf{k}\lambda}(\mathbf{x}_B,t)\Big] \\ &\equiv E^{+i}_{B,\mathbf{k}\lambda}(\mathbf{x}_A,t) + E^{-i}_{B,\mathbf{k}\lambda}(\mathbf{x}_A,t)\,. \end{aligned} \tag{2.74}$$

We stress an important subtlety here. $E^i_{B,\mathbf{k}\lambda}$ does not correspond to the $\boldsymbol{k}\lambda$ component of a Fourier expansion of $E^i_B(\mathbf{x}_A,t)$. Indeed, note that expression (2.74) depends on $\mathbf{R}$! With this notation, we express the dipole electric field produced by a dipole induced only by the Fourier component $\mathbf{E}_{0,\mathbf{k}\lambda}$ of the vacuum electric field. We are now in a position to obtain the desired interaction energy by taking the expectation value of equation (2.23). We perform this calculation in the next section.

2.2.4 The full interaction energy

On the first part of this section we presented both the interacting Hamiltonian in which the atoms interact and the mathematical structure of the quantum fields $\mathbf{E}_{0,\mathbf{k}\lambda}(\mathbf{x}_A,t)$ and $\mathbf{E}_{B,\mathbf{k}\lambda}(\mathbf{x}_A,t)$. In the same spirit of sec.2.1, the interaction energy between the atoms is identified as the leading order of the perturbation performed by H_I, which in this section reads:

$$U=-\frac{1}{2}\sum_{\mathbf{k}\lambda}\alpha_A(\omega_k)\Big\langle 0\Big|\Big[\mathbf{E}_{0,\mathbf{k}\lambda}(\mathbf{x}_A,t)+\mathbf{E}_{B,\mathbf{k}\lambda}(\mathbf{x}_A,t)\Big]^2\Big|0\Big\rangle\,. \tag{2.75}$$

From the four possible terms arising from the square on the above equation, two of them do not contribute to the interacting energy between the atoms in leading order. The $\mathbf{E}^2_{0,\mathbf{k}\lambda}(\mathbf{x}_A,t)$ term does not depend on atom's B degrees of freedom and the term $\mathbf{E}^2_{B,\mathbf{k}\lambda}(\mathbf{x}_A,t)$ is of higher order in the polarizability. We still use the fact that $\langle 0|\mathbf{E}_{0,\mathbf{k}\lambda}(\mathbf{x}_A,t)\cdot \mathbf{E}_{B,\mathbf{k}\lambda}(\mathbf{x}_A,t)|0\rangle = (\langle 0|\mathbf{E}_{B,\mathbf{k}\lambda}(\mathbf{x}_A,t)\mathbf{E}_{0,\mathbf{k}\lambda}(\mathbf{x}_A,t)|0\rangle)^*$ since the fields are Hermitian, so the interaction energy in leading order reads

$$U=-\mathrm{Re}\sum_{\mathbf{k}\lambda}\alpha_A(\omega_k)\Big\langle 0\Big|\mathbf{E}_{B,\mathbf{k}\lambda}(\mathbf{x}_A,t)\cdot\mathbf{E}_{0,\mathbf{k}\lambda}(\mathbf{x}_A,t)\Big|0\Big\rangle\,, \tag{2.76}$$

and we can write it only in terms of vacuum fields by means of Eq. (2.74), so:

$$U=-\mathrm{Re}\sum_{\mathbf{k}\lambda}\alpha_A(\omega_k)\alpha_B(\omega_k)D_{ij}(k,\mathbf{R})\Big\langle 0\Big|E^i_{0,\mathbf{k}\lambda}(\mathbf{x}_B,t)E^j_{0,\mathbf{k}\lambda}(\mathbf{x}_A,t)\Big|0\Big\rangle\,, \tag{2.77}$$

and by using Eq. (2.49) and properties of $a_{\mathbf{k}\lambda}$ and $a^\dagger_{\mathbf{k}\lambda}$, and exploiting the tensorial form of $D_{ij}(k,\mathbf{R})$ we may write

$$U=-\frac{1}{\epsilon_0}\mathrm{Re}\sum_{\mathbf{k},\lambda}\alpha_A(\omega_k)\alpha_B(\omega_k)N_k^2\left[f(\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda}\cdot\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda})^2+gR^2\Big(\hat{R}\cdot\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda}\Big)^2\right]e^{i\mathbf{k}\cdot\mathbf{R}}\,. \tag{2.78}$$

The sum over modes can now be safely taken to the continuous limit by merely making $\sum_{\mathbf{k}}\to((V/2\pi)^3)\int d^3k$, so

$$U=-\frac{V}{(2\pi)^3\epsilon_0}\mathrm{Re}\int d^3k\alpha_A(\omega_k)\alpha_B(\omega_k)N_k^2\sum_{\lambda=1}^{3}\left[f(\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda}\cdot\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda})^2+gR^2\Big(\hat{R}\cdot\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda}\Big)^2\right]e^{i\mathbf{k}\cdot\mathbf{R}}\,. \tag{2.79}$$

In order to perform the sum upon polarizations, we resort to the explicit form of the polarization vectors given by Eqs. (2.34) and (2.47), then

$$\sum_{\lambda=1}^{3}\left(\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda}\cdot\tilde{\boldsymbol{\epsilon}}_{\mathbf{k}\lambda}\right)^2 = 2+\frac{c^2\mu^2}{\omega_k^2}$$
$$\sum\left(\tilde{\boldsymbol{\epsilon}}_{\mathbf{k},\lambda}\cdot\hat{R}\right)^2 = 1+\left(\frac{c^2\mu^2}{\omega_k^2}-1\right)\cos^2\theta_k\,, \tag{2.80}$$

where $\cos\theta_k=\hat{R}\cdot\hat{k}$. It is possible to choose the angular origin of the $\mathbf{k}$ configuration space to be on $\hat{R}$, implying $\theta_k=\theta$ where θ is the polar angle of $\mathbf{k}$ in spherical coordinates. We now make use of Eqs. (2.39), (2.73) to rewrite Eq. (2.79) as

$$U=-\left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{\hbar}{4\pi^2}\mathrm{Re}\int d^3k\,\omega_k\alpha_A(\omega_k)\alpha_B(\omega_k)e^{-ikR}e^{ikR\cos\theta}$$
$$\times\left\{\left(2+\frac{c^2\mu^2}{\omega_k^2}\right)\left(\omega_k^2/c^2R^2-ikR-1\right)\right.$$
$$\left.+\left(\sin^2\theta+\frac{c^2\mu^2}{\omega_k^2}\cos^2\theta\right)\left(-k^2R^2+3ikR+3\right)\right\}\,. \tag{2.81}$$

It is quite direct to solve the angular part of the integral. Additionally, we perform the change of variables $x=kR$ and use the explicit form of ω_k so it is possible to write:

$$U_{\text{total}}=\left(\frac{1}{4\pi\epsilon_0}\right)^2\frac{-\hbar c}{\pi R^7}\int_0^\infty\frac{dx\,x\,\alpha_A(\omega_k(x)))\alpha_B(\omega_k(x))}{2\sqrt{x^2+(\mu R)^2}}\times$$
$$\left\{\left(2x^4+2x^2\left(2(\mu R)^2-5\right)+3\left((\mu R)^4+2\right)\right)\sin(2x)\right.$$
$$\left.+4x\left(x^2-3\right)\cos(2x)\right\}\,, \tag{2.82}$$

where $\omega_k(x)=\omega_k|_{k=k(x)}$. By using simple power counting, the above integral clearly diverges on the $x\to\infty$ limit. This problem can be addressed by integrating over a closed contour on the complex plane, including the point at infinity. Nevertheless, it requires some extra care since the square root inside its integrand has two branch cuts, which are discontinuous regions that must be avoided on the contour for the Cauchy theorem to be valid. A detailed discussion is given in Appendix D. We then identify $P_S(x)=\left(2x^4+2x^2\left(2(\mu R)^2-5\right)+3\left((\mu R)^4+2\right)\right)$ and $P_C(x)=4x\left(x^2-3\right)$ from Eq.

(2.82) and insert Eq. (D.5) on Eq. (D.9) so that we may write:

$$U_{\text{total}} = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{-\hbar c}{\pi R^7} \int_{\mu R}^{\infty} \frac{du\, u\, \alpha_A(\omega_k(u))\alpha_B(\omega_k(u))}{2\sqrt{x^2+(\mu R)^2}} e^{-2u} f(u)\,, \tag{2.83}$$

where

$$f(x) = 6 + 3(\mu R)^4 + 12x + (10 - 4(\mu R)^2)x^2 + 4x^3 + 2x^4\,. \tag{2.84}$$

The above expression (2.83) is convergent, perhaps not analytically solvable in general. However, it is possible to simplify the analysis by approximating it on the two limiting cases, named asymptotic and non-retarded regimes, which restrict the form of the integrand of Eq. (2.83). In the next subsections, we will discuss both in detail. One other form of (2.83) that will prove to be useful is obtained by making the change of variables: $u = \sqrt{z^2+(\mu R)^2}$, that reads

$$U_{\text{total}} = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{-\hbar c}{2\pi R^7} \int_0^{\infty} dz e^{-2\sqrt{z^2+(\mu R)^2}} f\left(\sqrt{z^2+(\mu R)^2}\right) \alpha_A\left(iz\frac{c}{R}\right) \alpha_B\left(iz\frac{c}{R}\right)\,. \tag{2.85}$$

2.2.5 Scales

One of the first provoking questions concerning the topic of this first chapter is what changes in terms of the well-known London and Casimir-Polder regimes when one changes from Maxwell to Proca electrodynamics. Let us consider a two-level atom with a single transition frequency denoted by ω_0. On Maxwell electrodynamics the limiting cases satisfy $[\omega_0 R/c \ll 1]_{\text{Maxwell}}$ (London) and $[\omega_0 R/c \gg 1]_{\text{Maxwell}}$ (Casimir-Polder), primarily defined as the limit where retardation of the electromagnetic field is not relevant (London) and the limit where the atom's response to the EM field is static (Casimir-Polder). We aim to answer whether such regimes exist or not in Proca with the same qualitative definition, and if so, what are the defining limits relating ω_0, R, and μ. The key to finding such a relation is to separate the potential (2.85) into "atomic degrees os freedom" and "field degrees of freedom". In order to simplify the following discussion, let us assume for the moment that the atoms are equal and have only one transition frequency, denoted by ω_0,

so that Eq. (2.22) resumes to:

$$\alpha_A\left(iz\frac{c}{R}\right) = \alpha_B\left(iz\frac{c}{R}\right) \approx \alpha(0)\frac{(\omega_0 R/c)^2}{(\omega_0 R/c)^2 + z^2}. \tag{2.86}$$

The energy can then be written as

$$U_{\text{total}} = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{-\hbar c}{2\pi R^7}\int_0^\infty dz\, A(z)F(z) \tag{2.87}$$

where

$$A(z) = \left[\alpha(0)\frac{(\omega_0 R/c)^2}{(\omega_0 R/c)^2 + z^2}\right]^2, \tag{2.88}$$

represents the atomic contribution, while the field contribution comes from

$$F(z) = e^{-2\sqrt{z^2+(\mu R)^2}} f\left(\sqrt{z^2 + (\mu R)^2}\right). \tag{2.89}$$

Let us first analyze the behavior of each one separately. The atomic part of the integrand, $A(z)$ exhibits a natural scale factor $\omega_0 R/c$, on which

- For $z \ll \omega_0 R/c$, $A(z) \approx [\alpha(0)]^2$, implying that in this regime the atom presents a static response to these (normalized) frequencies;
- For $z \gg \omega_0 R/c$, $A(z) \to 0$, the atom is said to be transparent to these (normalized) frequencies.

The field part of the integrand $F(z)$ has a way less obvious scale factor to be defined. Nevertheless, such a scale factor is supposed to cut more precisely the region of relevance of $F(z)$, since the exponential function decays much faster than any power law, and for this very same reason, we consider only the exponential factor decay as the defining upper limit z_c. At first, it is instructive to inspect what is the (normalized) cut-off z_c in the Maxwell electrodynamics, where we have $F(z) \to F_{\mu=0}(z) = e^{-2z} f(z)$. In this scenario, it is clear that $z_c \sim 1$ is a natural order of magnitude for the exponential to fade out. Heading back to the Proca case, the simple absolute value of the exponential does not seem to be a good measure of the relevance of $F(z)$. Imagine, for instance, that $\mu R \gg 1$.

In this scenario, $\exp\left\{-2\sqrt{z^2+(\mu R)^2}\right\} \ll 1$ for every value of z, and this choice would even restrict the possible values of μR, which is not a restriction of the complete theory discussed in the last section. A more natural definition of the dominant region of $F(z)$ (that recovers Maxwell for $\mu = 0$) is to compare $F(z)$ with its value on $z = 0$. Inspired by this and the half-life of unstable atoms, we define the critical (normalized) frequency z_c as the one that satisfies

$$\frac{F(z_c)}{F(0)} \sim e^{-1}, \tag{2.90}$$

where, in this region, we use only the exponential part of $F(z)$ so that the last equation reads

$$\frac{e^{-2\sqrt{z^2+(\mu R)^2}}}{e^{-2\mu R}} \sim e^{-1}, \tag{2.91}$$

which is satisfied by $z_c \sim \sqrt{1+2\mu R}$. Since we are interested only in the order of magnitude, we take

$$z_c \sim \sqrt{1+\mu R}. \tag{2.92}$$

The conditions over $F(z)$ is completely analogous to the ones posed over $A(z)$, being

- For $z \ll \sqrt{1+\mu R}$, $F(z) \approx F(0)$, so the field's dominant contribution comes from its zero mode; In this regime, the field coincides with the electrostatic field.
- For $z \gg \sqrt{1+\mu R}$, $F(z) \to 0$, the field modes are completely cut by the exponential factor.

As pointed out before, this last condition is even stronger than the analogous one coming from $A(z)$, considering that the exponential factor decays much faster than $1/z^4$. This allows us to write that the upper limit of integration on Eq. (2.87) is relevant at most to z_c, so $\int_0^\infty dz \approx \int_0^{z_c} dz$. We are now in a position to compare the limiting relations involving the two aforementioned scales $\omega_0 R/c$ and z_c. In fact, the dominant contribution to the whole energy comes from the region where both $A(z)$ and $F(z)$ exhibit their dominant individual contributions. This interplay is explored on the diagrams of Fig. 2.2. The slashed regions

denote the region of high relevance of each $A(z)$, for red (downward) slashes and $F(z)$, for blue (upward) slashes. The dominant region comes from the regions of complete blue-red dashes, or in other words, the intersection between the two regions. For the first one, with $\omega_0 R/c \gg z_c$,the relevant region satisfies $z \ll \omega_0 R/c$, so that

$$\int_0^\infty dz\, A(z)F(z) \approx A(0) \int_0^\infty dz\, F(z) \tag{2.93}$$

which we will define as the asymptotic region. The bottom diagram from Fig.2.2, denoting $\omega_0 R \ll z_c$, presents its relevant region on the complete dashes, satisfying $z \ll z_c$, so that:

$$\int_0^\infty dz\, A(z)F(z) \approx F(0) \int_0^\infty dz\, A(z), \tag{2.94}$$

which we define as the electrostatic region, since only the zero mode of the field contributes to the interaction energy. It is expected that this region will be identified with the non-retarded regime studied in Sec. 2.1, where the fundamental assumption of the non-relativistic treatment to be valid remains on the electrostatic nature of the regime. At

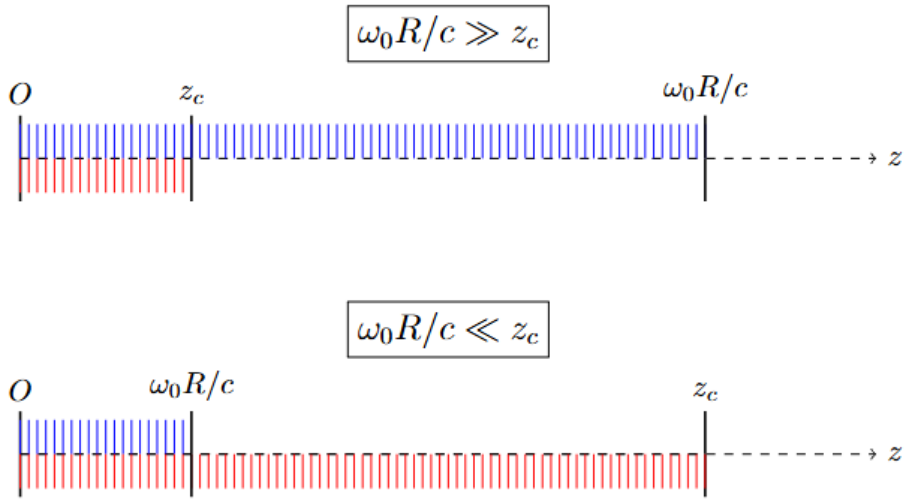


Figure 2.2: Dominant region of $A(z)$ (blue/upward slashes) and $F(z)$ (red/downward slashes) for the two regimes where $\omega_0 R/c \gg z_c$ (on the top) and $\omega_0 R/c \ll z_c$ (on the bottom).

this point, we analyze the physical interpretation of those two mathematical limits on

the integrand of (2.87). We use the Maxwell case as the starting point once again. On Maxwell's wave propagation, a peculiar feature is identified: the phase velocity, given by $v_{\text{phase}} = \omega_k/k$ coincides with the group velocity, given by $v_{\text{group}} = d\omega_k/dk$, valuing c. This allows a simplification in interpreting both limits, since this equality simply unifies the idea of a photon velocity, and $c/R = 1/T_\gamma$, where T_γ is the time elapsed by the photon to travel from one atom to the other. The regimes can be rewritten in three different ways, where we take for example the non-retarded one, that is $\omega_0 R/c \gg 1$, $[\omega_0 R \ll v_{\text{phase}}]_{\text{Maxwell}}$, $[\omega_0 R \ll v_{\text{group}}]_{\text{Maxwell}}$ and $\left[T_\gamma \ll \omega_0^{-1}\right]_{\text{Maxwell}}$. The last one allows for a further interpretation: effects involving oscillations with frequency ω_0, the photon emitted from one atom reaches the other almost immediately. For that reason, this regime, in Maxwell, is called non-retarded. Heading back to the Proca scenario, we analyze whether such interpretations from Maxwell hold. Unlike the massless case, the massive electromagnetic waves' phase and group velocities are different, and using Eq. (2.33), we can determine them as

$$v_{\text{phase}}(k) = c\sqrt{1 + \mu^2/k^2}, \tag{2.95}$$

$$v_{\text{group}}(k) = \frac{c}{\sqrt{1 + \mu^2/k^2}}. \tag{2.96}$$

From Eqs. (2.92) and (2.95) it is straightforward that

$$cz_c = v_{\text{phase}}\left(\sqrt{\frac{\mu}{R}}\right) \equiv \omega_F R, \tag{2.97}$$

so we conclude that the phase velocity is what governs the non-retarded and the asymptotic regimes, limited by the critical wave number $k_c = \sqrt{\mu/R}$, where it is finally possible to write

- Non-retarded regime: $\omega_0 \ll \omega_F$,
- Asymptotic regime: $\omega_0 \gg \omega_F$.

It is important to point out that, despite $v_{\text{phase}}(k_c)$ being supraluminal, there is no causality violation since the exchanged photons leading the interaction between the atoms are virtual.

In the two following subsections, we will explore the approximations (2.93) and (2.94) on the total energy (2.85) in order to explicitly obtain the interaction energy in those regimes.

2.2.6 Asymptotic Regime ($\omega_F \ll \omega_0$)

As we have anticipated in the previous subsection, the asymptotic regime is defined as the one in which the field frequency ω_F is low so that the dominant atomic contribution to the energy comes from its zero argument on $A(z)$. To be more general then on Eq. (2.88), it is enough to choose ω_0 being any ω_{I0} on (2.22), and replacing

$$\tilde{A}(z) = \alpha_A\Big(i\frac{zc}{R}\Big)\alpha_B\Big(i\frac{zc}{R}\Big) \tag{2.98}$$

so that

$$\sum_{IJ}\frac{4|\mathbf{d}_A^{I0}|^2|\mathbf{d}_B^{J0}|^2}{9\hbar^2}\int dz\frac{\omega_{I0}\omega_{J0}c^2}{\Big((\omega_{I0}R/c)^2+z^2\Big)\Big((\omega_{J0}R/c)^2+z^2\Big)}F(z) \approx \alpha_A(0)\alpha_B(0)\int dzF(z), \tag{2.99}$$

which allows us to write

$$U_{\text{total}} = \Big(\frac{1}{4\pi\epsilon_0}\Big)^2\frac{-\hbar c\,\alpha_A(0)\alpha_B(0)}{2\pi R^7}\int_0^\infty dz\, e^{-2\sqrt{z^2+(\mu R)^2}}f\left(\sqrt{z^2+(\mu R)^2}\right), \tag{2.100}$$

which can be exactly solved, using Eq. (2.84), as:

$$\begin{aligned}U_{\text{R}} = -\Big(\frac{1}{4\pi\epsilon_0}\Big)^2\frac{\hbar c\alpha_A(0)\alpha_B(0)}{4\pi R^7}&\Big[8(\mu R)^2\Big((\mu R)^2+3\Big)K_0(2\mu R)+\\ &+\Big(2(\mu R)^5+31(\mu R)^3+24(\mu R)\Big)K_1(2\mu R)+22(\mu R)^2K_2(2\mu R)\Big],\end{aligned} \tag{2.101}$$

where $K_\alpha(x)$ are the modified Bessel functions of the second kind and order α. In order to analyze the small mass limit, we use that, for $|z| \ll 1$:

$$K_\alpha(z) = \begin{cases} -\log\left(\frac{z}{2}\right)-\gamma & \text{if} \quad \alpha = 0 \\ \frac{1}{2}\Gamma(\alpha)\Big(\frac{2}{z}\Big)^\alpha & \text{if} \quad \alpha \neq 0 \end{cases} \tag{2.102}$$

where it is possible to note that the zero mass limit is well behaved and the leading term comes from $K_0(2\mu R)$, giving rise to well know Casimir-Polder potential [8]

$$\lim_{\mu\to 0}U_R = -\Big(\frac{1}{4\pi\epsilon_0}\Big)^2\frac{23\hbar c\alpha_A(0)\alpha_B(0)}{4\pi R^7}. \tag{2.103}$$

The interaction on the next order is:

$$U_R = U_{\rm CP}\left(1 - \frac{15(\mu R)^2}{46}\right) + \mathcal{O}[(\mu R)^4]\,, \tag{2.104}$$

where $U_{\rm CP}$ holds for Casimir-Polder potential. The interpretation of the first mass correction follows the same clue as on the non-retarded regime evaluated in section 2.1.1: the mass tends to shield the interaction. In fact, Figs. 2.3 and 2.4, along with Fig. 2.1 show a very similar binding pattern between the non-retarded and retarded regime potentials, suggesting the shielding mechanism is pretty much independent of the atomic transition frequencies. Even though we have not computed the total energy exactly, it is still possible to estimate it numerically and compare it with the retarded regime approximation, which is plotted in Fig. 2.5, where we use the same simplification as in subsection 2.2.5 for the polarizabilities. The curves show how good the agreement is between the approximation and the total energy for different masses, and it is evident that $\omega_0 R/c \gtrsim 1$ no longer guarantees the validity of the asymptotic approximation (as it happens with Maxwell), and, for a given distance, the greater the mass, the worse the approximation is. These features are in the very spirit of the analysis provided in Subsec. 2.2.5, indicating the asymptotic regime is reached for $\omega_0 R/c \gg \sqrt{1+(\mu R)}$, which is greater than 1 and implying the necessity of greater distances for increasing values of μ.

2.2.7 The non-retarded regime revisited ($\omega_F \gg \omega_0$)

On Subsec. 2.2.5 we have argued that what was there called non-retarded regime, that is $\omega_0 \gg \omega_F$ should coincide with the non-retarded regime obtained by means of non-relativistic quantum mechanics on Sec. 2.1, since it is obtained by the zero mode of $F(z)$, which we identified as the field part of the interacting energy. Nevertheless, it is important to check it explicitly. To do so, we use the same interpretation of ω_0 as on the last subsection, using the form (2.98) and apply approximation (2.93) so that

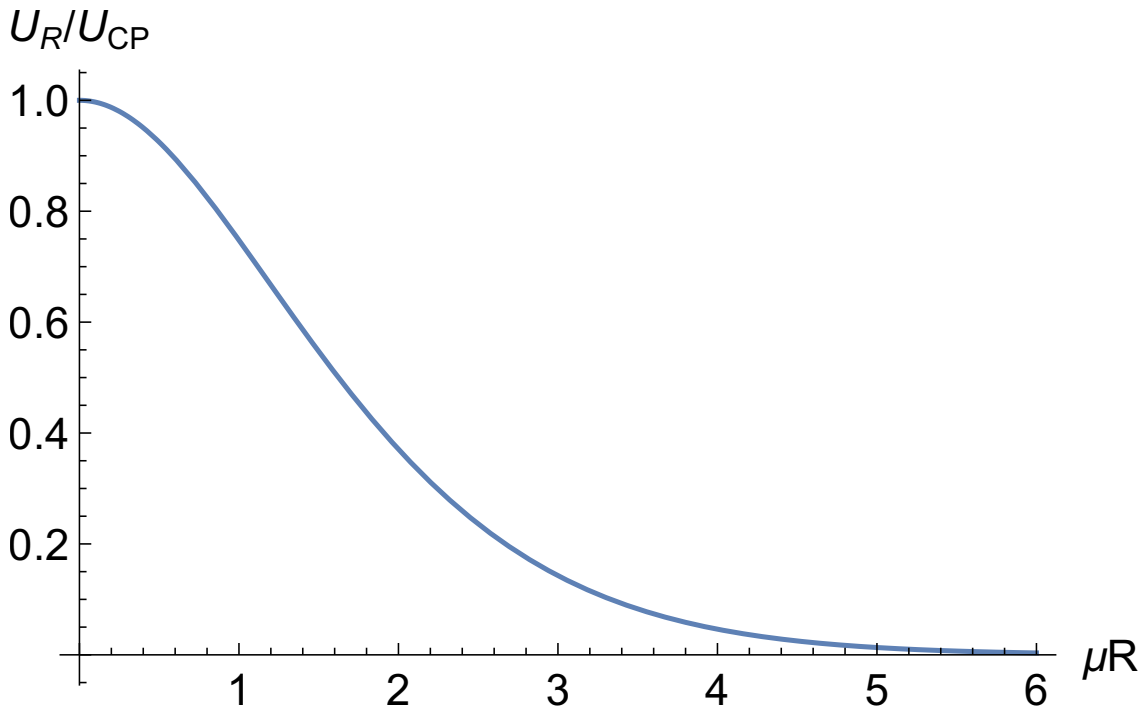


Figure 2.3: Interaction potential normalized by the Casimir-Polder potential $U_{\rm CP}$ as a function of the dimensionless radius parameter μR.

$F(0) = \exp[-2\mu R] f(\mu R)$ and

$$U_{\rm NR} = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{-\hbar}{4\pi R^6} e^{-2\mu R} f(\mu R) \int_{-\infty}^{\infty} d\chi \alpha_A(i\chi)\alpha_B(i\chi)\,, \tag{2.105}$$

where we used $\chi^2 = (c/R)z^2$. By noting that that the integral on Eq. (2.105) is simply $\frac{4\pi}{9}\Lambda$, with Λ defined on Eq. (2.17) and using Eq. (2.84) we can write

$$U_{\rm NR} = -\left(\frac{1}{4\pi\epsilon_0}\right)^2 \Lambda \frac{e^{-2\mu R}}{9R^6}\left[6 + 12(\mu R) + 10(\mu R)^2 + 4(\mu R)^3 + (\mu R)^4\right] \tag{2.106}$$

which exactly coincides with the result (2.18), obtained on Sec. 2.1, as physically expected. As in the previous subsection, we can analyze how good the non-retarded approximation fits on the total interacting energy for a given distance range by comparing the non-retarded energy obtained above with the same numerical estimation discussed at the end of the last subsection. On Fig. 2.6, the curves show how precise is the non-retarded approximation for different masses, and the conclusion is that it keeps being a good approximation even for $\omega_0 R/c \gtrsim 1$ (in contrast with Maxwell). Particularly, for large enough masses, the

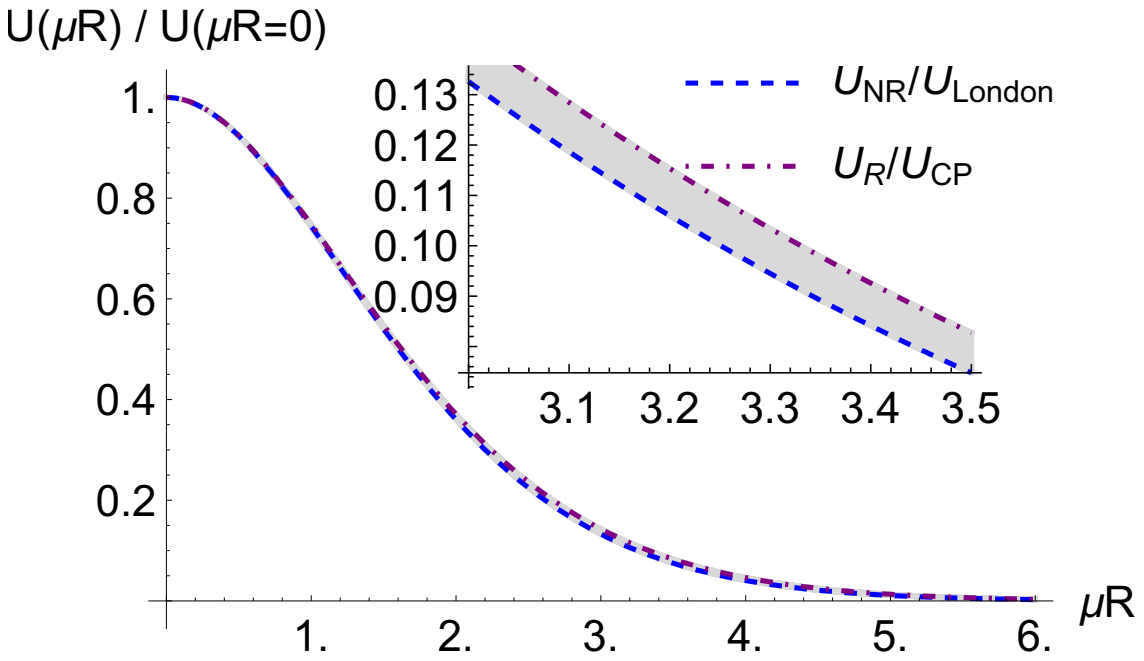


Figure 2.4: Ratio between the interaction energies for massive and massless QED, as a function of the dimensionless radius parameter μR for two regimes: the non-retarded (blue dashed line) and the retarded (purple dot-dashed line) approximations. The gray band visible on the inset displays the curves $U_{\text{total}}(\mu R)/U_{\text{total}}(\mu R = 0)$ for ω_0 in the range $10^{-4}c/R - 10^{4}c/R$.

regime keeps being valid for $\omega_0 R$ far away from 1, which is in agreement with the order we have estimated for the validity of the non-retarded regime, that is, $\omega_0 R/c \ll \sqrt{1+(\mu R)}$.

This completes our analysis of the influence of photon mass on the interatomic interaction. A remarkable feature of the dispersion forces is their non-additivity. In the next Chapter we analyze the effects of a photon mass on the non-additive character of the dispersion forces.

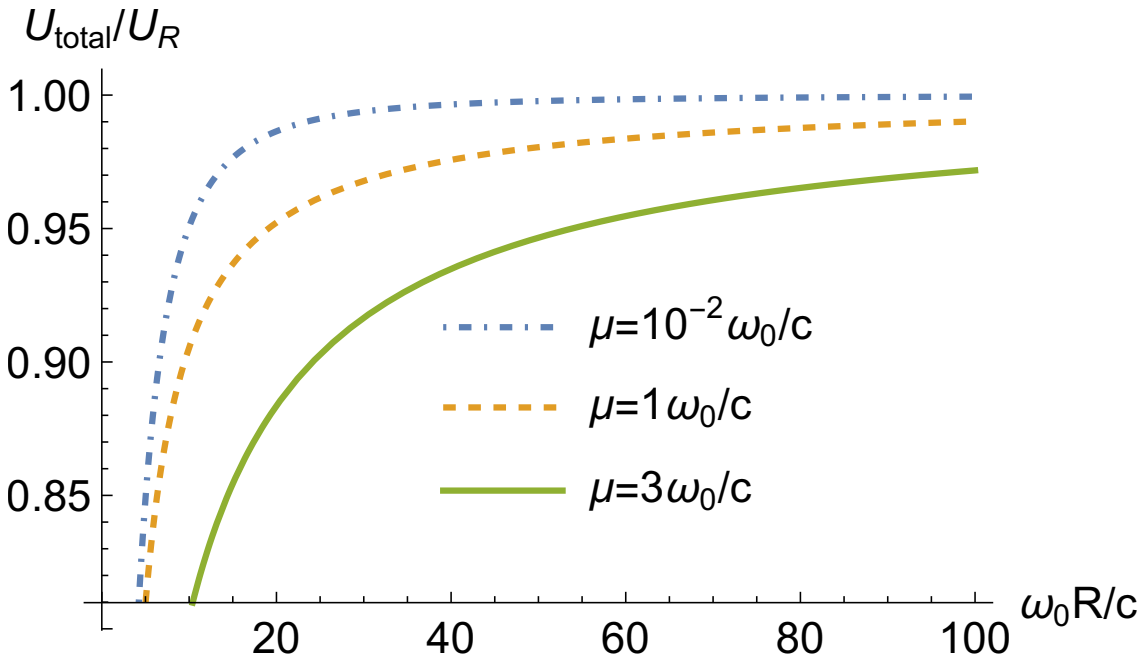


Figure 2.5: Complete interaction energy U_{total}, normalized by the retarded interaction U_R, as a function of $\omega_0 R/c$, for identical two-level atoms with transition frequency $\omega_0 R$ for different values of the photon mass.

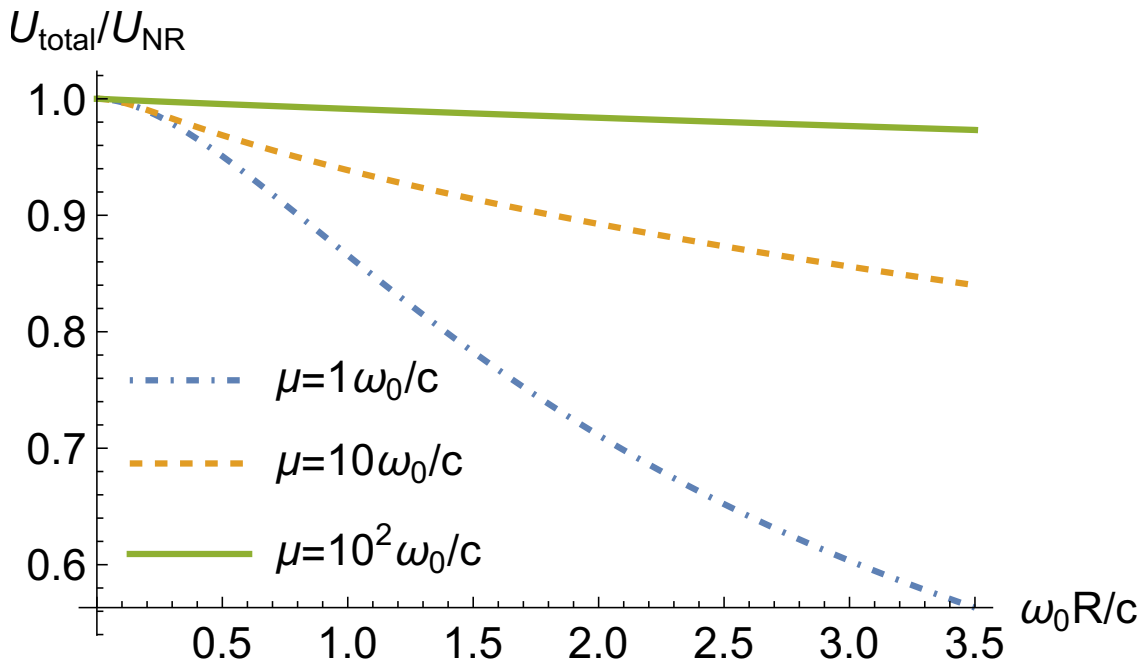


Figure 2.6: Complete interaction energy U_{total}, normalized by the non-retarded regime interaction U_{NR}, as a function of $\omega_0 R/c$ for different values of the photon mass. In this analysis, atoms A and B are considered to be identical two-level systems of energy gap ω_0.

Chapter 3

Non-additive interaction between three atoms

3.1 Non-additivity on electric 3 body problem

3.1.1 General concepts

In this chapter, we discuss the non-additive feature of the dispersion forces and analyze the role of mass on quantitative and qualitative aspects of some specific scenarios, such as dispersion interaction between free atoms and between atoms and conducting surfaces. First, we generalize to Proca electrodynamics the result obtained in the early 40's by Axilrod and Teller [41] for the non-additive contribution to the three-atom interaction, originally done for Maxwell electrodynamics. Throughout this chapter, the atoms and surfaces are supposed to be close enough for the non-retarded regime to be valid. As we discussed in the previous chapter, the range of distances where the non-retarded is valid is increased in Proca electrodynamics in comparison to the Maxwell case. Furthermore, it is for the small distance limit that we expect the non-additivity to be more relevant, and therefore it is reasonable to restrict our attention to this case. We employ a formalism closely related to the one presented in Sec. 2.1. The interacting Hamiltonian is simply the sum of Hamiltonians like (2.1) for each pair formed by atoms A, B and C, which is

$$H_{int} = -\mathbf{d}_A \cdot \mathbf{E}_B(\mathbf{r}_A) - \mathbf{d}_A \cdot \mathbf{E}_C(\mathbf{r}_A) - \mathbf{d}_B \cdot \mathbf{E}_C(\mathbf{r}_B). \tag{3.1}$$

The above expression seems not to be symmetric under atom permutations, but it is, as we can check by substituting the expressions for the dipole electric field derived in Eq. (2.8)

$$\mathbf{E}_I(\mathbf{x}_J) = \frac{e^{-\mu R_{IJ}}}{4\pi\epsilon_0 R_{IJ}^5}(\mu^2 R_{IJ}^2 + 3\mu R_{IJ} + 3)(\mathbf{d}_I \cdot \mathbf{R}_{IJ})\mathbf{R}_{IJ} - \frac{e^{-\mu R_{IJ}}}{4\pi\epsilon_0 R_{IJ}^3}(\mu R_{IJ} + 1)\mathbf{d}_I\,, \quad (3.2)$$

where I and J stands for $\{A, B, C\}$ and $\mathbf{R}_{IJ} = \mathbf{x}_J - \mathbf{x}_I$. This enables us to recast the interaction Hamiltonian in the form

$$H_{int} = \sum_{ij=1}^{3} \left[f_{ij}(\mathbf{R}_{AB}) d_A^i d_B^j + f_{ij}(\mathbf{R}_{BC}) d_B^i d_C^j + f_{ij}(\mathbf{R}_{AC}) d_A^i d_C^j \right] \quad (3.3)$$

where $f_{ij}(\mathbf{R})$ was defined on Eq. (2.13). For instance, we keep f_{ij} a bit more general, stating only that it is symmetric on the $\{ij\}$ variables. This structure for the interacting energy holds for either Maxwell or Proca electrodynamics, and the difference between them is represented only in the form of the $f_{ij}(\mathbf{R})$ function. The non-polarized assumption is similar to Eq. (2.10)

$$\langle 0_A 0_B 0_C |\, d_K^i \,| 0_A 0_B 0_C \rangle = 0\,, \quad (3.4)$$

for $K = \{A, B, C\}$ and $|0_A 0_B 0_C\rangle = |0_A\rangle \otimes |0_B\rangle \otimes |0_C\rangle$ is the tensor product of the ground states of the three free Hamiltonians H_0^A, H_0^B, and H_0^C. As in Sec. 2.1, the interaction energy is defined as the leading order of the perturbed energy generated by Eq. (3.1) on the ground state. Nevertheless, the non-additive contribution is contained only in terms that involve the three atoms simultaneously. In other words, a non-additive term is one in which if someone takes one of the atoms away, the whole term is affected. In order to prospect for such terms, we start by defining

$$H_{int} = H^{(AB)} + H^{(BC)} + H^{(AC)}. \quad (3.5)$$

The first-order correction simply vanishes as it contains only averages on the ground state, which are assumed to be zero. On second order, we have:

$$U^{(2)} = -\sum_{m\neq 0} \frac{\langle 0| \left(H^{(AB)} + H^{(BC)} + H^{(AC)} \right) |m\rangle \langle m| \left(H^{(AB)} + H^{(BC)} + H^{(AC)} \right) |0\rangle}{\hbar\omega_{m0}} \quad (3.6)$$

where $|m\rangle = |m_A\rangle \otimes |m_B\rangle \otimes |m_C\rangle$ are the energy eigenstates of H_0^A, H_0^B and H_0^C respectively, with energy (shift from ground state $\hbar\omega_{m0} = \hbar(\omega_m^A - \omega_0^A) + \hbar(\omega_m^B - \omega_0^B) + \hbar(\omega_m^C - \omega_0^C)$. It is important to point out that the summation over $m \neq 0$ means all values but $m = \{m_A, m_B, m_C\} = \{0, 0, 0\}$. There will be, in principle, nine different terms, where three of them involve only two atoms. Let us take one of the remaining 6 terms as an example

$$\begin{aligned}
&\sum_{m\neq 0} \frac{\langle 0|\, H^{(AB)}\, |m\rangle \, \langle m|\, H^{(BC)}\, |0\rangle}{\omega_{m0}} = \sum_{m\neq 0} \frac{f^{ij}(\mathbf{R}_{AB}) f^{kl}(\mathbf{R}_{BC}) \, \langle 0|\, d_A^i d_B^j\, |m\rangle \, \langle m|\, d_B^k d_C^l\, |0\rangle}{\omega_{m0}} \\
&= \sum_{m\neq 0} \frac{f^{ij}(\mathbf{R}_{AB}) f^{kl}(\mathbf{R}_{BC})}{\omega_{m0}} \times \\
&\times \underbrace{\left(\langle 0_A|\, d_A^i\, |m_A\rangle \overbrace{\langle m_A|0_A\rangle}^{\delta_{m_A}^{0_A}} \right)}_{0} \left(\langle 0_B|\, d_B^j\, |m_B\rangle \, \langle m_B|\, d_B^k\, |0_B\rangle \right) \underbrace{\left(\langle m_C|\, d_C^l\, |0_C\rangle \overbrace{\langle 0_C|m_C\rangle}^{\delta_{m_C}^{0_C}} \right)}_{0} \\
&= 0\,,
\end{aligned} \tag{3.7}$$

where we used the fact that the atoms present no permanent dipole moment (cf. Eq. (3.4)). Every other term of this form will vanish as well, regarding the fact that the above term vanishes has nothing to do with a specific property of d_A^i or the other two operators from atoms B and C. Another important comment is that those terms vanish no matter what the specific form of $f_{ij}(\mathbf{R})$, so until now our results hold equally for both Maxwell and Proca electrodynamics. We then go forward to the third-order perturbation theory, where the energy shift on the ground state is calculated as [47]:

$$U^{(3)} = \sum_{m,k\neq 0} \frac{\langle 0|H_{int}|m\rangle \, \langle m|H_{int}|k\rangle \, \langle k|H_{int}|0\rangle}{\hbar^2 \omega_{m0}\omega_{k0}}. \tag{3.8}$$

where it can be expanded into,

$$U^{(3)} = \sum_{m,k\neq 0} \frac{\left(H_{0m}^{(AB)} + H_{0m}^{(BC)} + H_{0m}^{(AC)}\right)\left(H_{mk}^{(AB)} + H_{mk}^{(BC)} + H_{mk}^{(AC)}\right)\left(H_{k0}^{(AB)} + H_{k0}^{(BC)} + H_{k0}^{(AC)}\right)}{\hbar^2 \omega_{m0}\omega_{k0}}, \tag{3.9}$$

for $H_{kl}^{(IJ)} = \left\langle k \middle| H_{kl}^{(IJ)} \middle| l \right\rangle$. This time we have 27 different (in principle) terms to analyze. Once again, the terms involving only two atoms do not contribute to the non-additive parcel of the energy. The other 24 terms can be split into two categories: products with one repeated index and products with three different indexes. Let us take one term from the first group:

$$\begin{aligned}
\sum_{m,k\neq 0} \frac{H_{0m}^{(AB)} H_{mk}^{(AB)} H_{k0}^{(AC)}}{\hbar^2 \omega_{m0}\omega_{k0}} &= \sum_{m,k} \frac{f_{ij}^{(AB)} f_{ln}^{(BC)} f_{pq}^{(AC)} \langle 0|d_A^i d_B^j|m\rangle \langle 0|d_A^l d_B^n|m\rangle \langle 0|d_A^p d_C^q|m\rangle}{\hbar^2 \omega_{m0}\omega_{k0}} \\
= \sum_{m,k} \frac{f_{ij}^{(AB)} f_{ln}^{(BC)} f_{pq}^{(AC)}}{\hbar^2 \omega_{m0}\omega_{k0}} &\times \langle 0_A|d_A^i|m_A\rangle \langle 0_B|d_B^j|m_B\rangle \overbrace{\langle 0_C|m_C\rangle}^{\delta_{m_C}^{0_C}} \\
&\times \langle m_A|d_A^l|k_A\rangle \langle m_B|d_B^n|k_B\rangle \overbrace{\langle m_C|k_C\rangle}^{\delta_{k_C}^{m_C}} \\
&\times \langle k_A|d_A^p|0_A\rangle \langle k_B|0_B\rangle \overbrace{\langle 0_C|d_C^q|m_C\rangle}^{\langle 0_C|d_C^q|0_C\rangle = 0} . \\
&= 0
\end{aligned} \tag{3.10}$$

Clearly, the same holds for all other terms of this type. If one atom appears only one time on the three multiplications (like atom C on the above equation), it will collapse to a ground state average, which is assumed to be zero. We then end up with only 6 terms: the ones that multiply different indices.

$$\begin{aligned}
\sum_{m,k} \frac{H_{0m}^{(AB)} H_{mk}^{(BC)} H_{k0}^{(AC)}}{\hbar^2 \omega_{m0}\omega_{k0}} &= \sum_{m,k} \frac{f_{ij}^{(AB)} f_{ln}^{(BC)} f_{pq}^{(AC)} \langle 0|d_A^i d_B^j|m\rangle \langle m|d_B^l d_C^n|k\rangle \langle k|d_A^p d_C^q|0\rangle}{\hbar^2 \omega_{m0}\omega_{k0}} \\
= \sum_{m,k\neq 0} \frac{f_{ij}^{(AB)} f_{ln}^{BC} f_{pq}^{(AC)}}{\hbar^2 \omega_{m0}\omega_{k0}} &\times \langle 0_A|d_A^i|m_A\rangle \langle 0_B|d_B^j|m_B\rangle \overbrace{\langle 0_C|m_C\rangle}^{\delta_{m_C}^{0_C}} \times \langle m_B|d_B^l|k_B\rangle \langle m_C|d_C^n|k_C\rangle \\
\times \overbrace{\langle m_A|k_A\rangle}^{\delta_{k_A}^{m_A}} &\times \langle k_A|d_A^p|0_A\rangle \langle k_C|d_C^q|0_C\rangle \overbrace{\langle k_B|0_B\rangle}^{\delta_{k_B}^{0_B}} \\
= \sum_{m\neq 0} \frac{f_{ij}^{(AB)} f_{ln}^{(BC)} f_{pq}^{(AC)}}{\hbar^2 \omega_{m0}\omega_{k0}} &\underbrace{\langle 0_A|d_A^i|m_A\rangle \langle m_A|d_A^p|0_A\rangle}_{|\mathbf{d}_A^{0m_A}|^2\delta_{ip}/3} \underbrace{\langle 0_B|d_B^j|m_B\rangle \langle m_B|d_B^l|0_B\rangle}_{|\mathbf{d}_B^{0m_B}|^2\delta_{jl}/3} \underbrace{\langle 0_C|d_C^q|k_C\rangle \langle k_C|d_C^n|0_C\rangle}_{|\mathbf{d}_C^{0k_C}|^2\delta_{nq}/3} \\
= f_{ij}^{(AB)} f_{il}^{(BC)} f_{jl}^{(AC)} &\sum_{m\neq 0} \frac{|\mathbf{d}_A^{0m_A}|^2 |\mathbf{d}_B^{0m_B}|^2 |\mathbf{d}_C^{0m_C}|^2}{\hbar^2(\omega_{m_A0}+\omega_{m_B0})(\omega_{m_A0}+\omega_{m_C0})}.
\end{aligned} \tag{3.11}$$

On the last line, we have changed the index k_C for m_C, since it is summed over the same states, and on the fourth line, we use the index I to indicate a sum over all variables (in order to shorten the notation). The above contribution is almost symmetric over the exchange of atoms. The only part that is not symmetric is the energy denominator on the last line. It is not difficult to see that the atom A appears two times because it is on the corners of the argument $H_{0m}^{(AB)}H_{0m}^{(BC)}H_{0m}^{(AC)}$. In fact, there are two possibilities for each term of the type since the exchange between $(AB) \leftrightarrow (BC)$ still brings the same result (3.11). So in order to compute the whole non-additive interaction, we must multiply the previously calculated term by 2 and add it to the other two terms, interchanging A by B and C on the energy denominator, so:

$$U_{\rm NA} = \frac{2}{27} f_{ij}(\mathbf{R}_{AB}) f_{il}(\mathbf{R}_{BC}) f_{jl}(\mathbf{R}_{AC}) \sum_m \frac{|\mathbf{d}_A^{0m_A}|^2 |\mathbf{d}_B^{0m_B}|^2 |\mathbf{d}_C^{0m_C}|^2}{\hbar^2 \bar{\omega}_m^2}\,, \tag{3.12}$$

where

$$\bar{\omega}^{-2} = 2 \frac{\omega_{m_A0} + \omega_{m_B0} + \omega_{m_C0}}{(\omega_{m_A0} + \omega_{m_B0})(\omega_{m_A0} + \omega_{m_C0})(\omega_{m_B0} + \omega_{m_C0})}\,. \tag{3.13}$$

We call all the terms after the f's a constant C, since it does not depend on the atoms' positions. So

$$U_{\rm NA} = \frac{C}{3} f_{ij}(\mathbf{R}_{AB}) f_{il}(\mathbf{R}_{BC}) f_{jl}(\mathbf{R}_{AC}) \tag{3.14}$$

where:

$$C = \frac{2}{9} \sum_{m \neq 0} \frac{|\mathbf{d}_A^{0m_A}|^2 |\mathbf{d}_B^{0m_B}|^2 |\mathbf{d}_C^{0m_C}|^2}{\hbar^2 \bar{\omega}_m^2} \tag{3.15}$$

is called the Axilrod-Teller constant, and it depends only on the atomic structure of the three atoms. In order to associate C with the polarizabilities by the same way we did with Λ on Eq. (2.17), let us take the integral [54]:

$$\begin{aligned}
&\int_0^\infty du\, \alpha_A(iu)\alpha_B(iu)\alpha_C(iu) = \\
&= \left(\frac{2}{3\hbar}\right)^3 \sum_{m\neq 0} |\mathbf{d}_A^{0m_A}|^2 |\mathbf{d}_B^{0m_B}|^2 |\mathbf{d}_C^{0m_C}|^2 \int_0^\infty \frac{\omega_{m_A0}\omega_{m_B0}\omega_{m_C0}}{\left(\omega_{m_A0}^2 + u^2\right)\left(\omega_{m_B0}^2 + u^2\right)\left(\omega_{m_C0}^2 + u^2\right)} \\
&= \frac{\pi}{3\hbar} C.
\end{aligned} \tag{3.16}$$

In the following subsections, we will present, separately, the non-additive contribution for the interacting energy for massless photons, first obtained by Axilrod and Teller [41]) and the non-additive contribution for the interacting energy mediated by massive photons.

3.1.2 The non-additive potential for $\mu = 0$: The Axilrod-Teller case

In this section, we will explicitly present the form of $f^{ij}(\mathbf{R}_{AB})$ and fully determine the non-additive dipole-dipole interaction for the problem of three polarizable atoms. To begin with, the interaction energy between two atoms A and B shall be written as:

$$\begin{aligned} H_{int}^{(AB)} &= \frac{-3}{4\pi\epsilon_0 R_{AB}^5}(\mathbf{d}_A \cdot \mathbf{R}_{AB})(\mathbf{d}_B \cdot \mathbf{R}_{AB}) + \frac{1}{4\pi\epsilon_0 R_{AB}^3}(\mathbf{d}_A \cdot \mathbf{d}_B) \\ &= \frac{1}{4\pi\epsilon_0 R_{AB}^3}\Big[-3\Big(\mathbf{d}_A \cdot \hat{\mathbf{R}}_{AB}\Big)\Big(\mathbf{d}_B \cdot \hat{\mathbf{R}}_{AB}\Big) + (\mathbf{d}_A \cdot \mathbf{d}_B)\Big] \\ &= \frac{1}{4\pi\epsilon_0 R_{AB}^3}\Big[-3\hat{R}_{AB}^i \hat{R}_{AB}^j + \delta^{ij}\Big] d_A^i d_B^j \end{aligned} \tag{3.17}$$

where $\mathbf{R}_{AB}$ is the vector from atom A to atom B. From the first two equations, we have:

$$f_{ij}(\mathbf{R}_{AB}) = \frac{1}{4\pi\epsilon_0 R_{AB}^3}\Big(-3\hat{R}_{AB}^i \hat{R}_{AB}^j + \delta^{ij}\Big)\,, \tag{3.18}$$

$$f_{ij}^{(KL)} = \frac{1}{4\pi\epsilon_0 R_{KL}^3}\Big(-3\hat{R}_{KL}^i \hat{R}_{KL}^j + \delta^{ij}\Big)\,, \tag{3.19}$$

for K and L being $\{A, B, C\}$. The sum presented on Eq. (3.14) can then be computed as:

$$\begin{aligned} & f_{ij}^{(AB)} f_{lj}^{(BC)} f_{li}^{(AC)}\Big(R_{AB}^3 R_{BC}^3 R_{AC}^3\Big) = \\ & \Big(-3\hat{R}_{AB}^i \hat{R}_{AB}^j + \delta^{ij}\Big)\Big(-3\hat{R}_{BC}^i \hat{R}_{BC}^l + \delta^{il}\Big)\Big(-3\hat{R}_{AC}^j \hat{R}_{AC}^l + \delta^{jl}\Big) \\ &= \Big[9\Big(\hat{R}_{AB} \cdot \hat{R}_{BC}\Big) R_{AB}^i R_{BC}^l - 3R_{AB}^i R_{AB}^l - 3R_{BC}^i R_{BC}^l \delta^{il}\Big]\Big(-3\hat{R}_{AC}^j \hat{R}_{AC}^l + \delta^{jl}\Big) \\ &= -27\Big(\hat{R}_{AB} \cdot \hat{R}_{BC}\Big)(\hat{R}_{BC} \cdot \hat{R}_{AC})(\hat{R}_{AB} \cdot \hat{R}_{AC}) + 9(\hat{R}_{AB} \cdot \hat{R}_{BC})^2 + \\ &\quad + 9(\hat{R}_{BC} \cdot \hat{R}_{AC})^2 + 9(\hat{R}_{AB} \cdot \hat{R}_{AC})^2 - 6\,. \end{aligned} \tag{3.20}$$

We can then write the above expression in terms of the angles on Fig. 3.1 as:

$$f_{ij}^{(AB)} f_{lj}^{(BC)} f_{li}^{(AC)} = \frac{3}{(4\pi\epsilon_0)^3 (R_{AB}^3 R_{BC}^3 R_{AC}^3)}[3\cos\alpha\cos\beta\cos\gamma + 1] \tag{3.21}$$

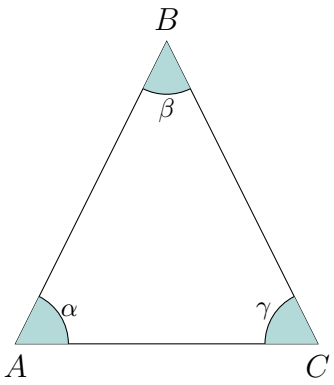


Figure 3.1: Triangle formed by atoms A, B and C, with internal angles α, β e γ.

where used $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = -2\cos\alpha\cos\beta\cos\gamma + 1$ for a triangle [55] and then the final result, based on eq. (3.14) takes the form:

$$U_{\rm NA} = \frac{C}{(4\pi\epsilon_0)^3(R_{AB}^3 R_{BC}^3 R_{AC}^3)}[3\cos\alpha\cos\beta\cos\gamma + 1]\,, \tag{3.22}$$

which is the same result that was originally presented in Ref. [41].

3.1.3 The non-additive potential for $\mu \neq 0$: The Proca scenario

For the Proca's electromagnetism case, the $f_{ij}(\mathbf{R})$ function was determined on Eq. (2.13). The following notation is particularly useful:

$$f_{ij}^{(KL)} = \frac{e^{-\mu R_{KL}}}{4\pi\epsilon_0 R_{KL}^3}\left[-u_{KL}\hat{R}_{KL}^i\hat{R}_{KL}^j + v_{KL}\delta^{ij}\right] \tag{3.23}$$

and

$$u_{KL} = u(\mu R_{KL}) \quad \text{and} \quad v_{KL} = v(\mu R_{KL}) \tag{3.24}$$

in which,

$$u(x) = x^2 + 3x + 3 \quad \text{and} \quad v(x) = x + 1 \tag{3.25}$$

Following the steps of the last section we can calculate the f's contractions as:

$$\begin{aligned} & f_{ij}^{(AB)} f_{lj}^{(BC)} f_{li}^{(AC)} = \\ & \frac{e^{-\mu P_{(ABC)}}}{(4\pi\epsilon_0 R_{AB} R_{BC} R_{AC})^3}\Big(-u_{AB}\hat{R}^i_{AB}\hat{R}^j_{AB} + v_{AB}\delta^{ij}\Big) \\ \times & \Big(-u_{BC}\hat{R}^i_{BC}\hat{R}^l_{BC} + v_{BC}\delta^{il}\Big)\Big(-u_{AC}\hat{R}^j_{AC}\hat{R}^l_{AC} + v_{AC}\delta^{jl}\Big) \\ & = \frac{e^{-\mu P_{ABC}}}{(4\pi\epsilon_0 R_{AB} R_{BC} R_{AC})^3} F(\mathbf{R}_{AB}, \mathbf{R}_{BC}, \mathbf{R}_{AC}) \end{aligned} \tag{3.26}$$

Where P_{ABC} is the perimeter of the triangle with vertices at the atoms' positions A, B, and C, and :

$$\begin{aligned} F(\mathbf{R}_{AB}, \mathbf{R}_{BC}, \mathbf{R}_{AC}) = {} & u_{AB}u_{BC}u_{AC}\cos\alpha\cos\beta\cos\gamma + u_{AB}u_{BC}v_{AC}\cos^2\alpha + \\ & + u_{AB}u_{AC}v_{BC}\cos^2\beta + u_{BC}u_{AC}v_{AB}\cos^2\gamma - u_{AB}v_{BC}v_{AC} \\ & - u_{BC}v_{AB}v_{AC} - u_{AC}v_{AB}v_{BC} + 3v_{AB}v_{BC}v_{AC}\,, \end{aligned} \tag{3.27}$$

and the energy is given by:

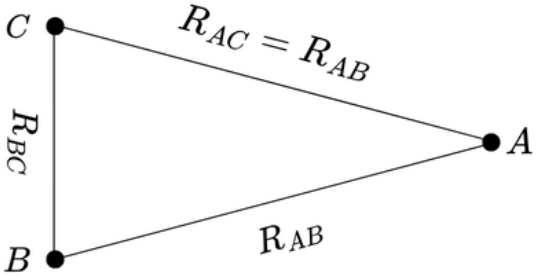
$$U_{NA} = \frac{C}{(4\pi\epsilon_0)^3}\frac{e^{-\mu(R_{AB}+R_{BC}+R_{AC})}}{3R^3_{AB}R^3_{BC}R^3_{AC}} F(\mathbf{R}_{AB}, \mathbf{R}_{BC}, \mathbf{R}_{AC})\,. \tag{3.28}$$

Two things are worth noting. The first one is that the dependence on the interatomic distances is much more complex than the Axilrod-Teller case, since the F functions are basically a polynomial of the three distances, combined with the angles of the triangle determined by the atoms. Even the angles themselves appear in a much more complicated way than the zero mass case. The other important fact to point out is that the damping factor scales with the perimeter of the triangle determined by the atoms.

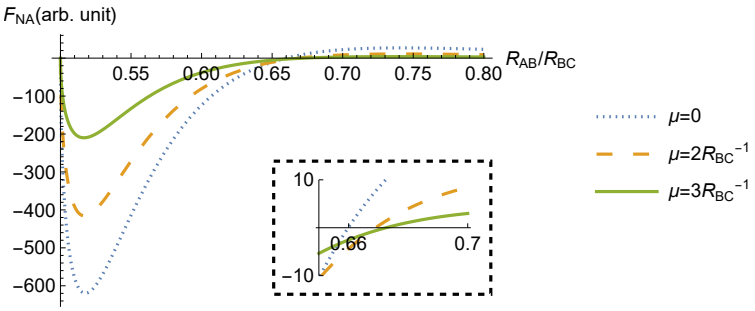
In Fig. 3.2, we observe that the non-additive parcel of the dispersion force on an atom turns weaker as we test greater masses (for a given distance from the other two atoms). It happens for almost all analyzed distances (it breaks down only in a small region where the force turns from attractive to repulsive, where it is relatively small for all mass values).

One other important question concerning the non-additive contribution to the dispersion potential is how much it contributes to the complete dispersion interaction. This comparison

between the additive and non-additive parcels is in general not possible in a universal way, since the additive contribution goes with $U_{NR} \sim \alpha^2$ while the non-additive contribution goes with $U_{NA} \sim \alpha^3$ (for α being the static polarizability for a model consisting in two-level equal atoms). Hence, this analysis depends, in general, on the internal degrees of the atoms. Nevertheless, it is possible to verify how the ratio between additive and non-additive parcels is changed when we introduce a finite photon mass. In other words, even though we can not evaluate this ratio on an absolute way, we can normalize it by the zero mass $U_{NA}(\mu = 0)/U_{NR}(\mu = 0)$, as we did for the simple case where the atoms are equidistant, as shown in Fig. 3.3. The curve shows that the non-additive contribution (in the face of the additive one) turns weaker as the mass grows stronger. This state of affairs prompts a final question: should we expect the same behaviors if, instead of single atoms, we deal with an atom interacting with a macroscopic object? We discuss this issue in the next section.



(a)



(b)

Figure 3.2: Fig. (b) shows the horizontal component of the force on atom A (from configuration on Fig. (a)) normalized by the Axilrod-Teller constant C for a configuration where ABC forms an isosceles triangle with $R_{AB} = R_{AC}$. The dashed frame plot represents a zoom on the specified region, which enables one to conclude that the turning point of the force, in general, depends on the mass parameter.

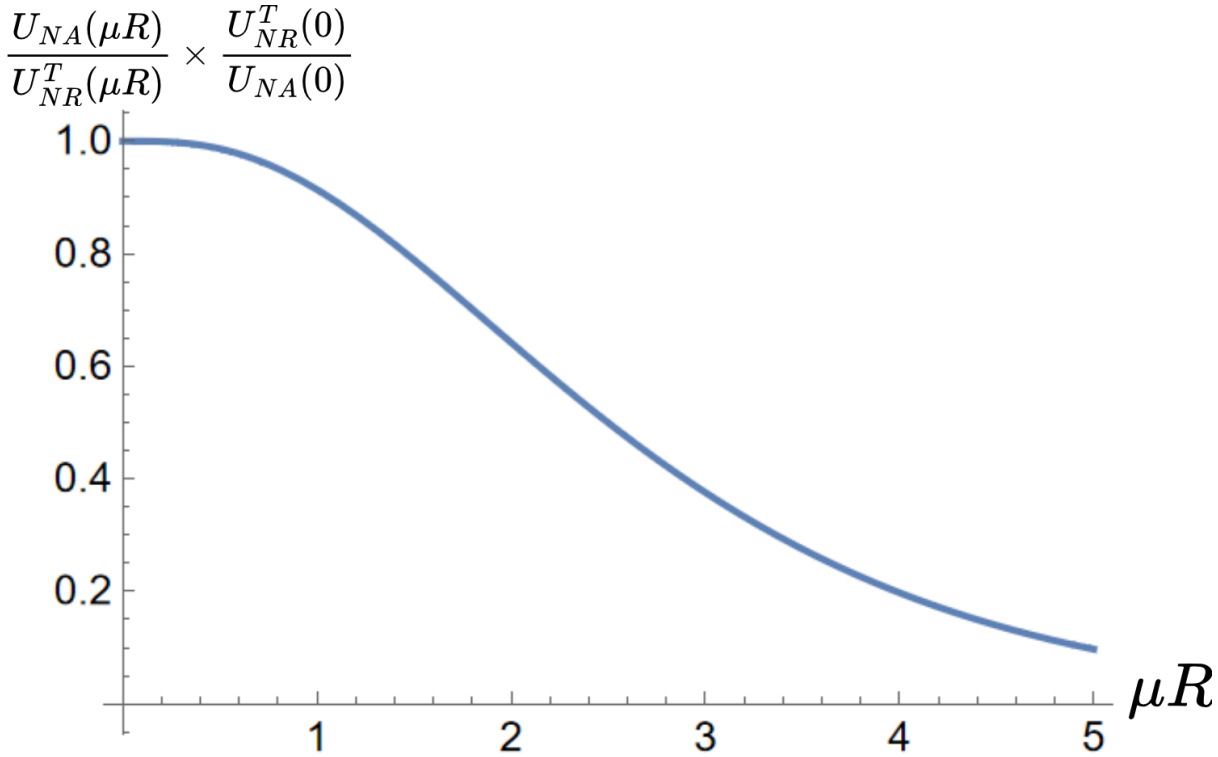


Figure 3.3: Ratio between the non-additive potential, NA and the total two-atoms dispersion, UN_R^T energy normalized by its zero mass function. The three atoms are equidistant with a distance R to each other.

3.1.4 Particular cases

In this subsection, we follow the analysis done in Ref. [41] and determine the shape of Eq. (3.28) for particular and/or asymptotic values of the distances between the atoms and the angles between them.

1. **Equilateral triangle**

   In the most symmetric case, the angles are equal ($\alpha = \beta = \gamma = 60°$), so are the side lengths ($R_{AB} = R_{BC} = R_{AC} \equiv R$). The energy (3.28) reads:

$$U_{\rm NA} = \frac{C}{(4\pi\epsilon_0)^3}\frac{e^{-3\mu R}}{24R^9}\left[\mu^6R^6 + 15\mu^5R^5 + 54\mu^4R^4 + 111\mu^3R^3 + 138\mu^2R^2 + 99\mu R + 33\right] \tag{3.29}$$

   This example reveals that the mass not only damps the interaction by the exponential factor, but also turns the $1/R^9$ original factor into a softer $1/R^3$ falling for long ranges. One other aspect is that the mass does not change the repulsive character of the equilateral triangle as first presented, for a massless photon, in Ref. [41].

2. **Three atoms on the same line**

   In this example, the atoms' positions do not form a triangle anymore. Nevertheless, the constraint on the angles between them remains valid, that is, the sum of the angles is still 180°. In that sense, we choose $\alpha = \beta = 0$ and $\gamma = 180°$ and the interaction, in terms of the u_{KL} and v_{KL} functions defined in Eqs. (3.24) can be found:

$$\begin{aligned} U_{\rm NA} =& C\frac{e^{-\mu(R_{AB}+R_{BC}+R_{AC})}}{(4\pi\epsilon_0)^3 3R_{AB}^3R_{BC}^3R_{AC}^3}\times \\ &\times\Big\{u_{AB}u_{BC}v_{AC} - u_{AB}u_{BC}u_{AC} + u_{AB}u_{AC}v_{BC} - u_{AB}v_{BC}v_{AC} \\ &\quad + u_{AB}u_{AC}v_{BC} - u_{AB}v_{BC}v_{AC} + u_{BC}u_{AC}v_{AB} - u_{BC}v_{AB}v_{AC} \\ &\qquad - u_{AC}v_{AB}v_{BC} + 3v_{AB}v_{BC}v_{AC}\Big\}. \end{aligned} \tag{3.30}$$

It is not obvious, but the above expression is negative for all possible values of the distances and the mass, preserving the same feature from the massless case.

3. **Right angle**

   One other qualitative difference between the massless and massive cases arouses on this example. By a simple comparison between Eqs. (3.22) and (3.28) is that the former is a function of $\cos\alpha\cos\beta\cos\gamma$, while the latter is a more complicated function of the cosines. On the massless case, if one of the angles is a right angle, the angular dependency is all suppressed, which is not true on the massive case, as can be seen when we substitute $\alpha = \pi/2$ in Eqs. (3.27) and (3.28).

4. **One atom far removed from the other two**

   On this example, we consider atom C to be very distant from A and B, so that $R_{AB} \ll R_{BC}$, $R_{AB} \ll R_{AC}$ and $R_{AC} \sim R_{BC}$. With this approximations, we have $\gamma \approx 0$, which leaves $\cos\alpha\cos\beta \approx -\cos^2\alpha$ (as $\alpha+\beta \approx \pi$).Using this on Eq. (3.28), the non-additive energy on this approximation reads:

$$U_{NA} = C\frac{e^{-2\mu R_{BC}}}{(4\pi\epsilon_0)^3 3R_{AB}^3 R_{BC}^6}\left[\cos^2\alpha\left(2u_1u_2v_2 - u_1u_2^2\right) - u_1v_2^2 + u_2^2v_1 - 2u_2v_1v_2 + 3v_1v_2^2\right] \tag{3.31}$$

   where we have the distance from either A or B from to C totally suppressing the interaction. Moreover, the damping factor is the same as if A and B were one single particle.

3.2 Non-additivity interaction energy for two atoms close to a grounded conductive infinite plate

3.2.1 Eberlein-Zietal method

In this section, we generalize Eberlein and Zietal [56] method (EZM) to determine the non-additive contribution of the interacting energy between two atoms and a conducting

grounded surface, but for the Proca electrodynamics. This method (for massless photons) has experienced relevant applications so far [31]. The EZM consists on the idea that a non-retarded interaction between atoms and surfaces can be mapped into a classical electrostatic one, where the link between them lies on the quantum nature of the electric dipole moment. We start the same way as [56]

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} - \mu^2 \varphi \,, \tag{3.32}$$

where φ is the usual electric potential and μ is related to the photon mass m_γ by $\mu = \frac{m_\gamma c}{\hbar}$. In order to reach the electrostatic regime, we use $\partial_t \mathbf{A} = 0$ (where $\mathbf{A}$ is the vector potential), so that

$$\mathbf{E} = -\nabla \varphi \,. \tag{3.33}$$

Inserting Eq. (3.33) into Eq. (3.32), we have

$$\left(\mu^2 - \nabla^2\right) = \frac{\rho}{\epsilon_0}, \tag{3.34}$$

which is the Eq. (2.2) presented on Sec. 2.1 for a general electric density $\rho(\mathbf{x})$. The electrostatic energy follows the usual relation [50]

$$\frac{1}{2} \int d^3x \rho(\mathbf{x}) \varphi(\mathbf{x}), \tag{3.35}$$

and the solution of Eq. (3.34), is

$$\varphi(\mathbf{x}) = \frac{1}{\epsilon_0} \int d^3x' G(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') \,, \tag{3.36}$$

where $G(\mathbf{x}, \mathbf{x}')$ is the Green function associated with the Helmholtz equation, the potential satisfies[1], so

$$\left(\mu^2 - \nabla^2\right) G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'), \tag{3.37}$$

[1]There exist, in general, many Green functions associated with the same differential operator, depending on the physical scenario under consideration. Notwithstanding, the Green function is unique for a given choice of initial or contour conditions.

and then the interaction energy reads:

$$U = \frac{1}{2\epsilon_0} \int d^3x d^3x' \rho(\mathbf{x}) G(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}'). \tag{3.38}$$

One notes that the source and the Green function totally describe the electric energy itself. Following the EZM, we note that any function that satisfies the homogeneous Helmholtz equation can be added to a particular solution

$$G(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}, \mathbf{x}') + G_H(\mathbf{x}, \mathbf{x}')\,, \tag{3.39}$$

where

$$\begin{aligned} \left(\mu^2 - \nabla^2\right) G_0(\mathbf{x}, \mathbf{x}') &= \delta(\mathbf{x} - \mathbf{x}')\,, \\ \left(\mu^2 - \nabla^2\right) G_H(\mathbf{x}, \mathbf{x}') &= 0\,, \end{aligned} \tag{3.40}$$

In this section, we are interested in atoms near a conducting grounded surface, which implies $\varphi(\mathbf{x})_S = 0$ and can then be satisfied by imposing $G(\mathbf{x}, \mathbf{x}')|_S = 0$, where the subscript S means on the surface. If one chooses the particular solution $G_0(\mathbf{x}, \mathbf{x}')$ to be the one particle unity charge potential (which is the Coulomb $\sim 1/r$ for Maxwell and the Yukawa $\sim e^{-\mu r}/r$ potential for Proca), the boundary condition reads

$$\left[\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x} - \mathbf{x}'|} + G_H(\mathbf{x}, \mathbf{x}')\right]_{x \in S} = 0. \tag{3.41}$$

If we put $\mu = 0$ on the above equation, it is clearly a problem solvable by the image method (depending on the surface). For $\mu \neq 0$ we will still call it image method. Nevertheless, one may be aware that, for some source symmetry, the Helmholtz equation does not present the same symmetries as the Poisson equation. For example, for a source with spherical symmetry, the Poisson equation exhibits inversion symmetry, which the Helmholtz equation does not. Then, we focus on the specific structure of the source. As long as we are interested in neutral atoms, we consider them to be pointwise dipoles, so the total charge density is given by (similar to Eq. (2.57))

$$\rho(\mathbf{x}) = -\mathbf{d}_A \cdot \nabla\delta(\mathbf{x} - \mathbf{x}_A) - \mathbf{d}_B \cdot \nabla\delta(\mathbf{x} - \mathbf{x}_B), \tag{3.42}$$

where $\mathbf{d}_A$ and $\mathbf{d}_B$ are the dipole moments of atoms A and B and $\mathbf{x}_A$ and $\mathbf{x}_B$ are the positions of atoms A and B, respectively. The following relation is useful:

$$\int d^3x d^3x' \partial_i \delta(\mathbf{x}-\mathbf{x}_1)\partial'_j \delta(\mathbf{x}'-\mathbf{x}_2) G(\mathbf{x},\mathbf{x}') = \partial_i \partial'_j G(\mathbf{x},\mathbf{x}')|^{\mathbf{x}'=\mathbf{x}_2}_{\mathbf{x}=\mathbf{x}_1}\,, \tag{3.43}$$

where $\partial'_j \equiv \frac{\partial}{\partial x'^j}$. We can still define:

$$\begin{aligned}
\tilde{f}_{ij} &= \partial_i \partial'_j \left.\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\right|^{\mathbf{x}'=\mathbf{x}_B}_{\mathbf{x}=\mathbf{x}_A}\,, \\
g_{ij} &= \partial_i \partial'_j \, G_H(\mathbf{x},\mathbf{x}')|^{\mathbf{x}'=\mathbf{x}_B}_{\mathbf{x}=\mathbf{x}_A}\,, \\
K_{ij} &= \tilde{f}_{ij} + g_{ij}.
\end{aligned} \tag{3.44}$$

Note that $\tilde{f}_{ij} = \epsilon_0 f_{ij}(\mathbf{R}_{AB})$ is defined on Eq. (2.13), and indeed does not depend on any geometrical property of the conducting surface, which is fully contained in g_{ij}. when calculations containing $\mathbf{x}=\mathbf{x}'$ and index 0 is added, like $g^0_{ij} = \partial_i \partial'_j \, G_H(\mathbf{x},\mathbf{x}')|^{\mathbf{x}'=\mathbf{x}_A}_{\mathbf{x}=\mathbf{x}_A}$. The interaction Hamiltonian can then be written as:

$$H_{int} = H_A + H_B + H_X \tag{3.45}$$

where

$$H_A = (1/2\epsilon_0) K^0_{ij} d^i_A d^j_A, \quad H_B = (1/2\epsilon_0) K^0_{ij} d^i_B d^j_B \quad and \quad H_X = (1/\epsilon_0) K_{ij} d^i_A d^j_B. \tag{3.46}$$

We are in a position to investigate the ground state energy change due to the interaction between the atoms and the surface. We will proceed in a similar way as Secs. 2.1 and 3.1.1 since the Hamiltonian (3.45) is of the the form $\sim \mathbf{d}^2$. As long as the atoms have no permanent dipole moment, all terms will vanish in first-order perturbation theory, as it already happens in the aforementioned sections. The least non-vanishing order is the second one, and the energy reads

$$U = -\sum_{m\neq 0} \frac{\langle 0|H_{int}|m\rangle \langle m|H_{int}|0\rangle}{\hbar\omega_{m0}}, \tag{3.47}$$

where the index m stands for $\{m_A, m_B\}$ for the two atoms A and B, so $\hbar\omega_{m0} = \hbar\omega_m - \hbar\omega_0$ is the energy shift of the unperturbed Hamiltonian and $|m\rangle = |m_A\rangle\,|m_B\rangle$.By Substituting Eq. (3.45) on Eq.(3.47) we have:

$$U = -\sum_{m\neq 0} \frac{\langle 0|H_A + H_B + H_X|m\rangle\,\langle m|H_A + H_B + H_X|0\rangle}{\hbar\omega_{m0}}. \tag{3.48}$$

At this point, we make the first selection in order to maintain only non-additive terms, *i.e.* terms which possess information of the two atoms and the surface simultaneously. The terms $\sim |\,\langle 0|H_A|m\rangle\,|^2$ and $\sim |\,\langle 0|H_B|m\rangle\,|^2$ clearly do not satisfy the previously mentioned requirement, since they possess information of only one atom. Let us look at terms of the type $H_A H_B$

$$\sum_{m\neq 0} \frac{H_A^{0m} H_B^{m0}}{\hbar\omega_{m0}} = -\frac{1}{4\epsilon_0^2} \sum_{m\neq 0} K_{ij}^0 K_{kl}^0 \frac{\langle 0_A|d_A^i d_A^j|m_A\rangle\,\langle 0_B|d_B^k d_B^l|m_B\rangle \overbrace{\langle 0_A|m_A\rangle}^{\delta_{m_A}^{0_A}} \overbrace{\langle 0_B|m_B\rangle}^{\delta_{m_B}^{0_B}}}{\hbar\omega_{m0}} = 0\,, \tag{3.49}$$

where $H_K^{ml} = \langle m|H_K|l\rangle$. The above term vanishes because of the two deltas forcing m_A and m_B to be zero simultaneously. By symmetry, it happens the same for the switched term $A \leftrightarrow B$. We then move to the term of the type $H_A H_X$

$$\sum_{m\neq 0} \frac{H_A^{0m} H_X^{m0}}{\hbar\omega_{m0}} = -\frac{1}{4\epsilon_0^2} \sum_{m\neq 0} K_{ij}^0 K_{kl} \frac{\langle 0_A|d_A^i d_A^j|m_A\rangle\,\langle m_A|d_A^k|0_A\rangle \overbrace{\langle 0_B|m_B\rangle}^{\delta_{m_B}^{0_B}} \overbrace{\langle 0_B|d_B^l|m_B\rangle}^{\langle 0_B|d_B^l|0_B\rangle=0}}{\hbar\omega_{m0}} = 0 \tag{3.50}$$

where we used the information that the atom B has no permanent dipole moment, *i.e.* has zero average dipole moment (on ground state). The only term left is the H_X^2 one, so:

$$\begin{aligned} U_{NA} = \sum_{m\neq 0} \frac{H_X^{0m} H_X^{m0}}{\hbar\omega_{m0}} &= -\frac{1}{\epsilon_0^2} \sum_{m\neq 0} K_{ij} K_{kl} \frac{\overbrace{\langle 0_A|d_A^i|m_A\rangle\,\langle m_A|d_A^k|0_A\rangle}^{|\mathbf{d}_A^{0m}|^2\delta_{ik}/3} \overbrace{\langle 0_B|d_B^j|m_B\rangle\,\langle m_B|d_B^l|0_B\rangle}^{|\mathbf{d}_B^{0m}|^2\delta_{jl}/3}}{\hbar\omega_{m0} + \hbar\omega_{m0}} \\ &= -\frac{\Lambda}{9\epsilon_0^2} K_{ij}^2 \end{aligned} \tag{3.51}$$

where Λ depends on the atomic structure of atoms A and B and it is define defined on Eq. (2.16) as:

$$\Lambda = \sum_{r\neq 0, s\neq 0} \frac{|\mathbf{d}_A^{0r}|^2 |\mathbf{d}_B^{0s}|^2}{\hbar(\omega_{r0} + \omega_{s0})} \,. \tag{3.52}$$

Heading back to the non-additive contribution of the interaction energy, we use the definition of D_{ij} on (3.44) and write:

$$K_{ij}^2 = \tilde{f}_{ij}^2 + 2\tilde{f}_{ij} g_{ij} + g_{ij}^2 \,, \tag{3.53}$$

where we have assumed g_{ij} to be symmetric on i, j. The first term contains information only on atoms A and B, since the $\tilde{f}_{ij}$ function has no information about any boundary condition (other than zero at infinity). Therefore, this term can be disregarded for the non-additive contribution, and we finally have

$$U_{NA} = -\frac{\Lambda}{9\epsilon_0^2} 2\tilde{f}_{ij} g_{ij} - \frac{\Lambda}{9\epsilon_0^2} g_{ij}^2 \equiv U_{NA}^{fg} + U_{NA}^{gg}. \tag{3.54}$$

Thus, finding the homogeneous solution $G_H(\mathbf{x}, \mathbf{x}')$ fully determines the non-additive contribution to the potential. This feature inspires the statement that the problem of finding a quantum interacting energy is mapped into the classical electrostatic problem of obtaining $G_H(\mathbf{x}, \mathbf{x}')$, which is, in electrostatic, proportional (by a factor ϵ_0^{-1}) to the potential necessary to cancel a unit pointwise charge particles potential on the referred surface (from Eq.(3.41)). If such a problem is solvable by the image method, $G_H(\mathbf{x}, \mathbf{x}')$ is (proportional to) the image potential associated with the aforementioned distribution, and indeed, a classical electrostatic problem.

3.2.2 Non-additive energy between two atoms and a conducting plate

In this subsection and the subsequent one, we will apply the EZM to determine the non-additive contribution of the interacting energy between two atoms and specific surfaces, such as one infinite conducting plane and two infinite conducting parallel planes. The only

previously unknown quantity, as commented in the last section, is the function $g_{ij}(\mathbf{x}, \mathbf{x}')$, which depends on $G_H(\mathbf{x}, \mathbf{x}')$. In this subsection, we consider this surface to be the infinite plane located at $z = 0$. The configuration is set up in Fig. 3.4 The homogeneous Green

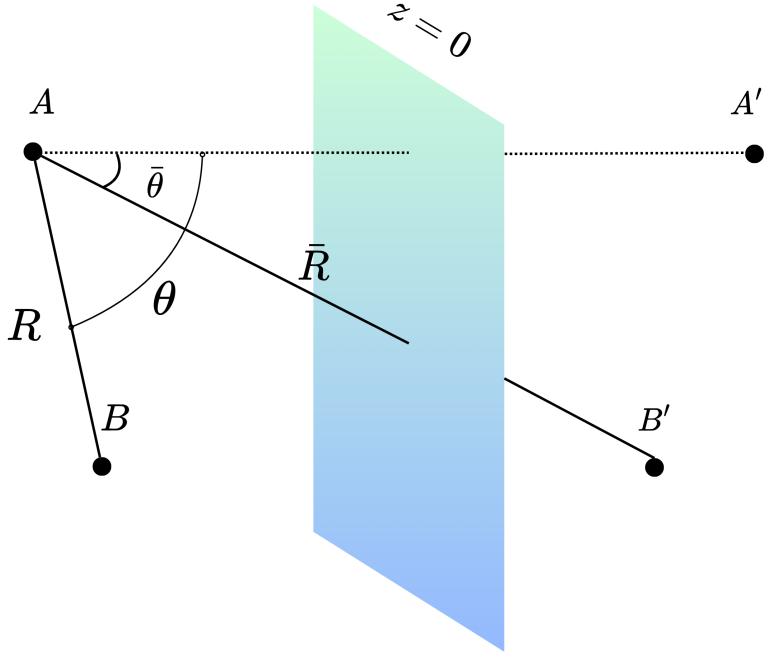


Figure 3.4: Disposition of the two particles A and B near a mirror and their images A' and B'.

function is supposed to cancel the potential of a unit charge, mass μ, and placed on $\mathbf{x}'$. The answer is to reverse the unit charge, bringing a minus sign on the potential, and the z component of $\mathbf{x}'$ to $-z'$ on $\exp(-\mu|\mathbf{x} - \mathbf{x}'|)/(4\pi|\mathbf{x} - \mathbf{x}'|)$, so:

$$G_H(\mathbf{x}, \mathbf{x}') = -\frac{e^{-\mu\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}. \tag{3.55}$$

We must still guarantee that it satisfies the condition $(\mu^2 - \nabla^2)G_H(\mathbf{x}, \mathbf{x}') = 0$ on the region of interest (regarded here to be $z > 0$). Even though the expression for G_H is quite similar to the one particle at $\mathbf{x}'$, the calculations are way more laborious, so additional definitions are going to be presented in order to simplify them. We start with

$$\partial_i G_H(\mathbf{x}, \mathbf{x}')|_{\mathbf{x}=\mathbf{x}_A} = \frac{e^{-\mu\tilde{R}}}{\tilde{R}^3}\left(1 + \mu\tilde{R}\right)\tilde{R}^i\,, \tag{3.56}$$

where $\tilde{\mathbf{R}} = ((x_A - x'), (y_A - y'), (z_A + z'))$. The calculation follows

$$\partial'_j \partial_i G_H(\mathbf{x}, \mathbf{x}')|_{\mathbf{x}=\mathbf{x}_A} = \partial'_j \left(\frac{e^{-\mu\tilde{R}}}{\tilde{R}^3}\Big(1 + \mu\tilde{R}\Big) \right) \tilde{R}^i + \frac{e^{-\mu\tilde{R}}}{\tilde{R}^3}\Big(1 + \mu\tilde{R}\Big)\partial'_j \tilde{R}^i \,. \tag{3.57}$$

Note that the second term involves $\partial'_j \tilde{R}^i$, which is -1 if $i = j = 1$ or $i = j = 2$ and $+1$ if $i = j = 3$ and zero otherwise. It can be written as:

$$\partial'_j \tilde{R}^i = -\delta_{ij} + 2\delta_{i3}\delta_{j3} \,. \tag{3.58}$$

The first term is more subtle. Using the chain rule, we have to evaluate $\partial'_j \tilde{R} = (1/\tilde{R})[-(x_A - x'), -(y_A - y'), +(z_A - z')]$, which inspires the next definitions.

$$\begin{aligned} \bar{\mathbf{R}} &= [(x_A - x_B), (y_A - y_B), (z_A + z_B)] \,, \\ \bar{\mathbf{R}}_L &= [-(x_A - x_B), -(y_A - y_B), +(z_A + z_B)] \,. \end{aligned} \tag{3.59}$$

Hence, the matrix g_{ij} (we will omit (AB) superscript for brevity) can the be written as:

$$g_{ij} = \frac{e^{-\mu\bar{R}}}{4\pi\bar{R}^3}\Big\{ -\bar{u}\hat{\bar{R}}^i \hat{\bar{R}}^j_L + \bar{v}(-\delta_{ij} + 2\delta_{i3}\delta_{j3}) \Big\} \tag{3.60}$$

where hat means unit vector and $\bar{u} \equiv u(\mu\bar{R})$ and $\bar{v} \equiv v(\mu\bar{R})$ for the u, v functions defined on Eq. (3.25). We evaluate the g^2 term as

$$\begin{aligned} g_{ij}g_{ij} &= \frac{e^{-2\mu\bar{R}}}{(4\pi)^2\bar{R}^6}\Big[-\bar{u}\hat{\bar{R}}^i \hat{\bar{R}}^j_L + \bar{v}(-\delta_{ij} + 2\delta_{i3}\delta_{j3}) \Big]\Big[-\bar{u}\hat{\bar{R}}^i \hat{\bar{R}}^j_L + \bar{v}(-\delta_{ij} + 2\delta_{i3}\delta_{j3}) \Big] \\ &= \frac{e^{-2\mu\bar{R}}}{(4\pi)^2\bar{R}^6}\left\{ \bar{u}^2 \overbrace{|\hat{\bar{R}}_L|^2 |\hat{\bar{R}}|^2}^{1} - 2\bar{u}\bar{v} \overbrace{\Big(-\hat{\bar{R}}_L \cdot \hat{\bar{R}} + 2\hat{\bar{R}}_z^2 \Big)}^{1} + 3\bar{v}^2 \right\} . \end{aligned} \tag{3.61}$$

The energy due to this term can be computed using Eq.(3.98)

$$U^{gg}_{NA} = \frac{\Lambda e^{-2\mu\bar{R}}}{144\epsilon_0^2\pi^2\bar{R}^6}\Big\{ 6 + 12\mu\bar{R} + 10\mu^2\bar{R}^2 + 4\mu^3\bar{R}^3 + \mu^4\bar{R}^4 \Big\} \,, \tag{3.62}$$

which is the two-particle non-retarded interaction energy presented in Sec. 2.1, but for atom A and the image of atom B (which has the same atomic properties as atom B but lies on position $\mathbf{x}'_B$). Note that it reduces to the usual massless London interaction for A

and image of B in the limit $\mu \to 0$) in accordance with Ref. [31]. To solve the remaining term, we assume that atoms A, B, and B' (the image of B) are on the same XZ plane (since three points always form a plane and the problem exhibits invariance over the XY plane, we can safely set $y = 0$). Since the previous result does not depend on any specific direction, it will not be affected by this choice. In this part, we use the shorten notation $\mathbf{R}_{AB} = \mathbf{R}$ and $u, v = u(\mu R), v(\mu R)$. We can then calculate:

$$\begin{aligned}\tilde{f}_{ij} g_{ij} &= \frac{e^{-\mu(R+\bar{R})}}{16\pi^2 R^3 \bar{R}^3}\Big\{\Big(-u\hat{R}^i\hat{R}^j + v\delta_{ij}\Big)\Big[-\bar{u}\hat{\bar{R}}^i\hat{R}^j_L + \bar{v}(-\delta_{ij} + 2\delta_{i3}\delta_{j3}\Big]\Big\} \\ &= \frac{e^{-\mu(R+\bar{R})}}{16\pi^2 R^3 \bar{R}^3}\Big\{u\bar{u}\Big(\hat{R}_z^2\hat{\bar{R}}_z^2 - \hat{R}_x^2\hat{\bar{R}}_x^2\Big) - u\bar{v}\Big(-1 + 2\hat{R}_z^2\Big) - \bar{u}v\Big(-1 + 2\hat{\bar{R}}_z^2\Big)\Big\}\end{aligned} \tag{3.63}$$

If we now define θ as the angle between $\hat{R}$ (line AB) and the z axis, normal to the surface, and define $\bar{\theta}$ between $\hat{\bar{R}}$ (line AB') and the z axis, the last term on the energy reads:

$$\begin{aligned}U_{NA}^{fg} =& \frac{\Lambda e^{-\mu(R+\bar{R})}}{72\epsilon_0^2\pi^2 R^3\bar{R}^3}\Big\{u\bar{u}\Big(-\sin^2\theta\sin^2\bar{\theta} + \cos^2\theta\cos^2\bar{\theta}\Big) \\ &- u\bar{v}\Big(-1 + 2\cos^2\theta\Big) - \bar{u}v(-1 + 2\cos^2\bar{\theta}) - v\bar{v}\Big\}.\end{aligned} \tag{3.64}$$

The limit to $\mu \to 0$ is performed as $\{u, \bar{u}\} \to 3$ and $\{v, \bar{v} \to 1\}$. In this limit:

$$\lim_{\mu\to 0} U_{NA}^{fg} = \frac{\Lambda}{72\epsilon_0^2\pi^2 R^3\bar{R}^3}\Big(2 - 3\sin^2\theta - 3\sin^2\bar{\theta}\Big), \tag{3.65}$$

in accordance with Ref. [31]. The last step to obtain the complete potential is to sum the two contributions (3.62) and (3.64). With the complete potential in hand, we bring back the discussion of interpretations of massive non-additive potentials from the last paragraph on Sec. 3.1. The key difference between the present problem and the one from that section is that, this time, the non-additive potential (summing (3.62) and (3.64)) depends on the atoms in the same way as the two-particle potential (2.18), that is, both are linear on Λ. This feature allows us to investigate how the mass affects the relevance of the non-additive term in the face of the pairwise interactions. In other words, we aim to estimate the error

upon assuming the additivity of the interactions. This analysis is done on Fig. 3.5, where we place the two atoms at the same distance d from the plane and a distance R from each other. In this scenario, the resultant in the x-direction (orthogonal to the plane) is simply the x-component of the two-particles' potential (2.18) plus the x-component of the non-additive contribution. An important feature depicted on Fig. 3.5 is that for suitable values of mass, and distance R, the non-additive contribution may cancel the additive one (when $F^x_{NA}/F^x_{NR} = -1$). Moreover, the non-additive contribution can be even dominant (for $F^x_{NA}/F^x_{NR} < -1$) and the atoms can repel each other rather than attracting. This behavior is observed strictly for Proca scenario, which can be seen on Fig. 3.6, where the modulus of non-additive contribution of the x component of the force comes at most to $\sim 10^{-3}$ time the additive one for $\mu = 0$, while this very same parcel may completely cancel the x-component of the total force for interatomic distances approximately 3 times the atoms-plane distance.

3.2.3 Digression on other image problems on Proca electrostatics

The most famous problems solvable by the method of images are the potential generated by a point charge near a conducting uncharged plane and the potential generated by a point charge near an uncharged sphere. Those two problems can be solved by replacing the surface by an image charge, which fulfills both the Laplace equation in the region of interest and the contour condition on the surface. This method relies upon symmetries of the Poison equation with the above mentioned contour conditions. The problem involving the plane uses the symmetry of parity. If the contour condition is that the total potential is zero on a plane on $z = 0$, then a particle P with charge q placed on $z = z_0$ has the very same value of potential of a particle P' of charge q placed on $z = -z_0$, in such a way that the potential generated by P minus the potential generated by P' equals zero. A similar, slightly different situation happens on the conducting sphere problem. For a sphere of radius R and centered on the origin, a particle placed at a distance r from the

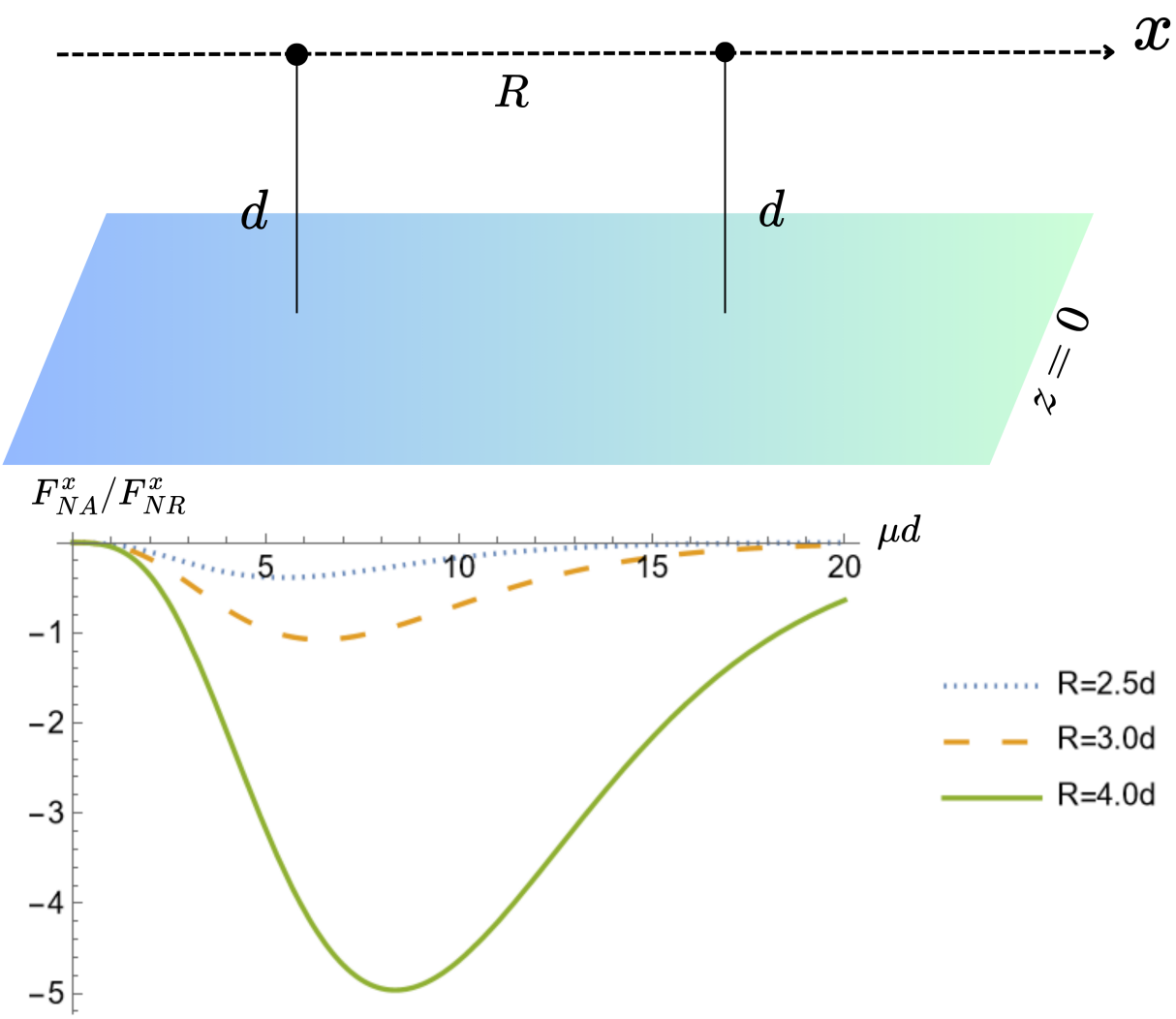


Figure 3.5: The x component of the non-additive force (vertical-axis) normalized by the x component of the additive force between the atoms (horizontal-axis). The d parameter is the distance between one atom and the plane (which is the same for both) and R is the distance between the atoms.

origin presents the same potential value on the surface of the sphere as a particle placed at position R^2/r, but with charge qR/r. This change os position is called an inversion of radius R, and it is a conformal symmetry of the Poisson equation, since it preserves the angles (which is crucial to find another solution with the same value on a specific sphere), but rescales the charge density. This last property is determinant for the difficulty of a generalization for the Helmholtz equation, since the mass term breaks the conformal symmetry mentioned above. Otherwise, it does not prove that there is no analytical

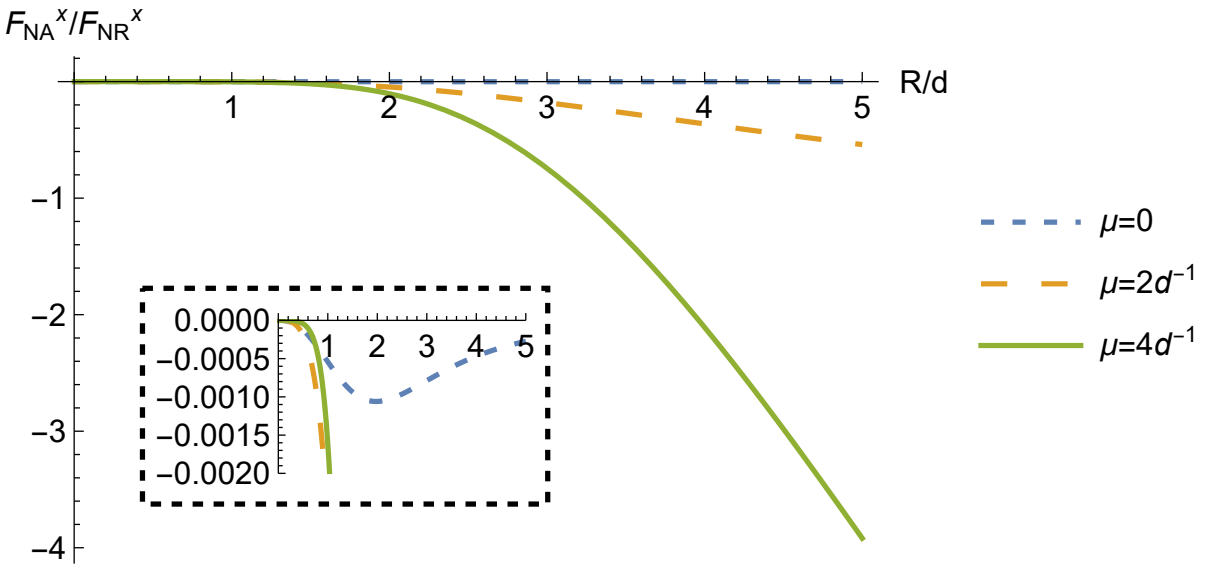


Figure 3.6: The x component of the non-additive force (vertical-axis) normalized by the x component of the NR force between the atoms (horizontal-axis). The d parameter is the distance between one atom and the plane (which is the same for both), and R is the distance between the atoms. The dashed rectangle at the bottom shows a tiny minimum of the forces quotient when $\mu = 0$, (when compared to other mass values).

solution for this problem, nor does it even state that the method of images does not work at all. The only conclusion is that the image may not be determined by an inversion transformation, nor by any conformal transformation for Proca electrodynamics.

3.3 Atoms between infinite parallel plates

In this section, we apply EZM for a system consisting of atoms A and B placed between two parallel infinite plates distancing d one from the other as shown in Fig. 3.7. In order to solve this problem using Eberlein's method, we choose a different pathway to reach the $G_H(x, x')$ function defined on Eq. (3.40). Even though it is possible to solve the massless version of this problem with the method of images Ref. [57] (but for the complete $G(x, x')$ green function), it is not clear that it is possible to do the same for the massive case. This fact drives us to the eigenfunction method presented in Ref. [58], which presents a possible and somewhat simple generalization between massless (first studied

on Ref. [59]) and massive cases. It is important to point out that this method brings us the complete green function $G(\mathbf{x}, \mathbf{x}')$, which is related to $G_H(\mathbf{x}, \mathbf{x}')$ by subtracting $\exp[-\mu|\mathbf{x} - \mathbf{x}'|]/(4\pi|\mathbf{x} - \mathbf{x}'|)$ from the former.

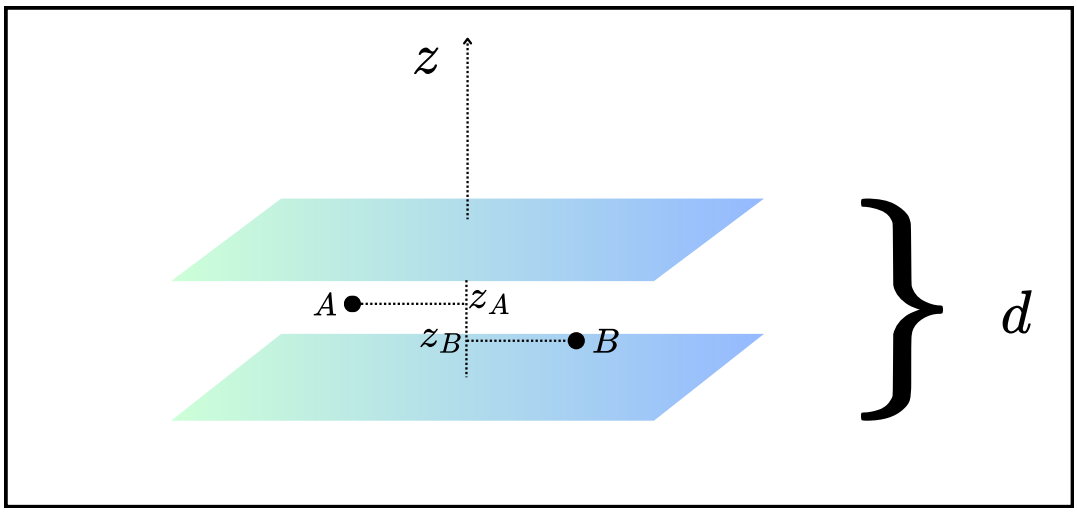


Figure 3.7: Disposition of the two particles A and B between infinite conducting parallel plates.

3.3.1 The $\mu = 0$ case

To obtain a broader perspective, let us start by computing the complete green function following the eigenfunction method exposed on Appendix B for the massless case, i.e. the Poisson equation, where $L = -\nabla^2$ and $f(\mathbf{x}) = \rho(\mathbf{x})/\epsilon_0$. The first step is to determine $u_i(\mathbf{x})$, where we use the conventional cylindrical coordinates such that $\mathbf{x} = (r, \phi, z)$. The contour conditions states that $u_i(z = 0) = 0 = u(z = d)$ and $u_i(r \to \infty) = 0$ and $u_i(r = 0) =$ finite. Thus we have

$$\frac{1}{r}\partial_r(r\partial_r u_i) + \frac{1}{r}\partial_\phi^2 u_i + \partial_z^2 u_i = -\lambda_i u_i(\mathbf{x}) \tag{3.66}$$

The following ansatz solves the above equation and can be adjusted to fulfill the contour conditions :

$$u_i(\mathbf{x}) = N_i \sin(kz) e^{il\phi} f(r). \tag{3.67}$$

By plugging it on Eq. (3.66), we find:

$$\sin(kz) e^{il\phi} \left\{ \left(-k^2 + \lambda - \frac{l^2}{r^2} \right) f(r) + \frac{f'(r)}{r} + f''(r) \right\} = 0\,, \tag{3.68}$$

and by using the conditions on r, we find

$$f(r) = J_l(\beta r)\,, \tag{3.69}$$

where $\beta^2 = \lambda - k^2$ and $J_l(x)$ are Bessel function of the first kind and order l [60]. This relation enables us to write

$$\lambda_{k\beta} = k^2 + \beta^2\,, \tag{3.70}$$

in such a sense that

$$-\nabla^2 u_{kl\beta}(\mathbf{x}) = (k^2 + \beta^2) u_{kl\beta}(\mathbf{x}) = \lambda_{kl\beta} u_{kl\beta}(\mathbf{x})\,, \tag{3.71}$$

where

$$u_{kl\beta}(\mathbf{x}) = N_{kl\beta} \sin(kz) e^{il\phi} J_l(\beta r)\,. \tag{3.72}$$

The boundary condition on z implies that

$$k = n\pi/d\,, \tag{3.73}$$

where $n \in \mathbb{N}$ and the angular usual periodic condition over ϕ constrains $l \in \mathbb{Z}$. The mode function labeled by $\{n, l, \beta\}$ reads:

$$u_{nl\beta}(\mathbf{x}) = N_{nl\beta} \cos\left(\frac{n\pi}{d} z \right) e^{il\phi} J_l(\beta r). \tag{3.74}$$

The normalization constant $N_{nl\beta}$ is found by imposing the orthonormality condition

$$\langle u_{nl\beta}, u_{n'l'\beta'} \rangle = \delta_{n,n'} \delta_{l,l'} \delta(\beta - \beta')\,, \tag{3.75}$$

where we define the inner product between two functions $f(x)$ and $g(x)$ defined over a region Ω as

$$\langle f, g\rangle = \int_\Omega d^3x f^*(x)g(x)\,. \tag{3.76}$$

We then have:

$$\begin{aligned}
&\langle u_{nl\beta}, u_{n'l'\beta'}\rangle = N_{nl\beta}N_{n'l'\beta'} \\
&\times \int_0^{2\pi} d\phi e^{i\phi(l-l')}\int_{-0}^{d} dz \sin\left(\frac{n\pi}{d}z\right)\sin\left(\frac{n'\pi}{d}z\right)\int_0^\infty dr r J_l(\beta r)J_{l'}(\beta' r) \\
&= \frac{\pi d}{\beta}\delta_{nn'}\delta_{ll'}\delta(\beta-\beta')N_{nl\beta}^2\,,
\end{aligned} \tag{3.77}$$

on which follows

$$N_{nl\beta} = \sqrt{\frac{\beta}{\pi d}}\,. \tag{3.78}$$

On the last two lines of Eq. (3.77) we have used that $\delta_{ij}A_{ij} = A_{ii}$ and $\int_0^\infty dr\, r J_\alpha(ax)J_\alpha(bx) = (1/a)\delta(a-b)$ (see Ref. [60]). The modes are explicitly defined as

$$u_{nl\beta}(\mathbf{x}) = \sqrt{\frac{\beta}{\pi d}}\sin(k_n z)e^{il\phi}J_l(\beta r)\,. \tag{3.79}$$

We can then write down the complete Green function using Eqs. (B.13) and (3.79) as

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{\pi d}\sum_{l=-\infty}^{\infty}\sum_{n=1}^{\infty}\int_0^\infty d\beta\beta\frac{J_l(\beta r)J_l(\beta r')}{\beta^2+k^2}e^{il(\phi-\phi')}\sin(kz)\sin(kz')\,. \tag{3.80}$$

At this stage, we use the fact that this problem possesses symmetry of translations and rotations about the (r,ϕ) plane. This means that we can set (r',ϕ') to be the origin, and translate $\mathbf{r} = (r,\phi) \to \mathbf{r}-\mathbf{r}'$. As the unique Bessel function that does not vanish at $r=0$ is $\alpha = 0$, the Green function on (3.80) reduces to:

$$\begin{aligned}
G(\mathbf{x},\mathbf{x}') &= \sum_{n=1}^{\infty}\sin(k_n z)\sin(k_n z')\int_0^\infty d\beta\beta\frac{J_0(\beta|\mathbf{r}-\mathbf{r}'|)}{\beta^2+k_n^2} \\
&= \sum_{n=1}^{\infty}\sin(k_n z)\sin(k_n z')K_0(k_n|\mathbf{r}-\mathbf{r}'|)\,.
\end{aligned} \tag{3.81}$$

where $K_0(z)$ is the modified Bessel function of the second kind and order 0. We have used the identity $\int_0^\infty dx J_0(ax)/(x^2+b^2) = K_0(ab)$ for $|z| \ll 1$ [60]. An asymptotic expansion can

also be done for $|\mathbf{r}-\mathbf{r}'| \gg d$, where we use the asymptotic form $K_n(z) \approx \sqrt{\pi/(2z)}\exp[-z]$ [60] to obtain

$$G(\mathbf{x},\mathbf{x}') \approx \frac{1}{4\pi}\sqrt{\frac{8}{rd}}\sin\left(\frac{\pi z}{d}\right)\sin\left(\frac{\pi z'}{d}\right)e^{-\frac{\pi}{d}|\mathbf{r}-\mathbf{r}'|}\,. \tag{3.82}$$

The expression obtained on Eq. (3.81) is clearly not defined for $\mathbf{r}=\mathbf{r}'$, which represents a problem, since we are not able to represent two particles on any axis perpendicular to the plates. This is a common feature of cylindrical coordinates. To solve it, we must head back to the ansatz (3.67) and impose homogeneity and isotropy on the $\{r,\phi\}$ plane, so that the solution depends only on z, hence:

$$u^0(x) = N^0\sin(kz)\,. \tag{3.83}$$

which is the same mode (3.79) but with $r=0$ and $\phi=0$, which satisfies the contour conditions for $k=k_n=n\pi/d$ and the normalization condition with $N_0=\sqrt{\frac{2}{d}}$. The eigenvalue to this mode is $\lambda=k_n^2$. The Green function then reads:

$$G^0(z,z') = \frac{2}{d}\sum_{n=1}^{\infty}\frac{\sin(k_n z)\sin(k_n z')}{k_n^2}\,, \tag{3.84}$$

and we can cast both solutions in one form as:

$$G(\mathbf{x},\mathbf{x}') = \begin{cases} \sum_{n=1}^{\infty}\sin\left(k_n z\right)\sin\left(k_n z'\right)K_0(k_n|\mathbf{r}-\mathbf{r}'|) & \text{if} \quad \mathbf{r}\neq\mathbf{r}' \\ \frac{2}{d}\sum_{n=1}^{\infty}\frac{\sin(k_n z)\sin(k_n z')}{k_n^2} & \text{if} \quad \mathbf{r}=\mathbf{r}' \end{cases}\,. \tag{3.85}$$

3.3.2 The $\mu \neq 0$ case

Now we consider the modification in the operator $-\nabla^2$ by $-\nabla^2+\mu^2$. The procedure to evaluate $G(\mathbf{x},\mathbf{x}')$ does change from the previous subsection. In fact, we use the same ansatz as on Eq. (3.67), relabeling $f(r)\to f_{(\mu)}(r)$, so the new modes $u_{(\mu)}(\mathbf{x})$ satisfy

$$\sin(kz)e^{il\phi}\left\{\left(-\mu^2-k^2+\lambda_{(\mu)}-\frac{l^2}{r^2}\right)f_{(\mu)}(r)+\frac{f'_{(\mu)}(r)}{r}+f''_{(\mu)}(r)\right\}=0\,, \tag{3.86}$$

and again,

$$f_{(\mu)}(r) = J_l(\beta r)\,, \tag{3.87}$$

but now $\beta^2 = \lambda_{(\mu)} - k^2 - \mu^2$, so

$$\lambda_{(\mu)n\beta} = \beta^2 + k_n^2 + \mu^2 \,. \tag{3.88}$$

The normalization factor remains the same, and so the modes,

$$u_{(\mu)kl\beta}(\mathbf{x}) = u_{kl\beta}(\mathbf{x}') = N_{kl\beta} e^{il\phi} \sin(k_n z) J_l(\beta r) \tag{3.89}$$

which satisfies the following equation

$$\Big(-\nabla + \mu^2\Big) u_{(\mu)kl\beta}(\mathbf{x}) = (\beta^2 + k^2 + \mu^2) u_{(\mu)kl\beta}(\mathbf{x}) \,, \tag{3.90}$$

with shifted eigenvalues This is somewhat expected, since the mass term corresponds to a term proportional to the identity added to the Laplace operator. Even though the modes are the same, the Green function is not. This is because it depends on the eigenvalue, not only on the modes. Using Eqs. (B.13) and (3.89), the Green function reads

$$G_{(\mu)}(\mathbf{x}, \mathbf{x}') = \frac{1}{\pi d} \sum_{l=-\infty}^{\infty} \sum_{n=1}^{\infty} \int_0^{\infty} d\beta \beta \frac{J_l(\beta r) J_l(\beta r')}{\beta^2 + k^2 + \mu^2} e^{il(\phi - \phi')} \sin(k_n z) \sin(k_n z') \,, \tag{3.91}$$

and using exactly the same symmetry arguments as the one used in the last subsection, we can write $G_{(\mu)}(\mathbf{x}, \mathbf{x}')$ as a function of the difference on the (r, ϕ) plane as

$$G_{(\mu)}(\mathbf{x}, \mathbf{x}') = \sum_{n=1}^{\infty} \sin(k_n z) \sin(k_n z') K_0\Big(\sqrt{k_n^2 + \mu^2} |\mathbf{r} - \mathbf{r}'|\Big) \,, \tag{3.92}$$

and it is still possible to perform the same asymptotic approximation for $|\mathbf{r} - \mathbf{r}'| \gg d$ as we did on the last subsection, which is $K_0(z) \approx \sqrt{\pi/(2z)} \exp[-z]$ for $|z| \ll 1$, then:

$$G(\mathbf{x}, \mathbf{x}') \approx \frac{1}{4\pi d} \sqrt{\frac{8\pi}{\mu^* r}} e^{-\mu^* r} \sin\left(\frac{\pi}{d} z\right) \sin\left(\frac{\pi}{d} z\right), \tag{3.93}$$

where $\mu^* = \sqrt{(\pi/d)^2 + \mu^2}$.

Two important remarks may be made concerning this result. The former is that the mass term affects only the (r, ϕ) plane components, without altering the z part of the green function. The second one, coming from the asymptotic expansion, is that the μ

parameter changes the screening from π/d to $\sqrt{(\pi/d)^2+\mu^2}$. Once again, the above results do not hold for $\mathbf{r}=\mathbf{r}'$ (distances on the (r,ϕ) plane). We must adjust the ansatz (3.67) once again to:

$$u^0_{(\mu)}(z) = N^0 \sin(kz), \tag{3.94}$$

which is the same as (3.83), with a shifted eigenvalue $\lambda^0_{(\mu),n} = k_n^2+\mu^2$, so the Green's function reads,

$$G^0_{(\mu)}(z,z') = \frac{2}{d}\sum_{n=1}^{\infty}\frac{\sin(k_n z)\sin(k_n z')}{k_n^2+\mu^2} \tag{3.95}$$

Finally, all the cases are written as

$$G^{(\mu)}(\mathbf{x},\mathbf{x}') = \begin{cases} \sum_{n=1}^{\infty}\sin(k_n z)\sin(k_n z')K_0\Big(\sqrt{k_n^2+\mu^2}|\mathbf{r}-\mathbf{r}'|\Big) & \text{if} \quad \mathbf{r}\neq\mathbf{r}' \\ \frac{2}{d}\sum_{n=1}^{\infty}\frac{\sin(k_n z)\sin(k_n z')}{k_n^2+\mu^2} & \text{if} \quad \mathbf{r}=\mathbf{r}' \end{cases}. \tag{3.96}$$

At this point, we must remember that $G(\mathbf{x},\mathbf{x}')$ is the "complete Green function" (the sum of the point charge Green function plus the $G_H(\mathbf{x},\mathbf{x}')$ function). As discussed in Subsec. 3.2.1, the non-additive contribution to the total potential is characterized by Eq. (3.98). We may, then, define:

$$\tilde{g}_{ij} \equiv \partial_i\partial'_j G(\mathbf{x},\mathbf{x}')\Big|_{\mathbf{x}_A}^{\mathbf{x}_B} = g_{ij} + \tilde{f}_{ij}\,, \tag{3.97}$$

and the non-additive contribution of Eq. (3.98) reads:

$$U_{NA} = -\frac{\Lambda}{9\epsilon_0^2}\Big[[\tilde{g}_{ij}]^2 - \Big[\tilde{f}_{ij}\Big]^2\Big]\,. \tag{3.98}$$

We note, using Eq. (3.23), that

$$\frac{\Lambda}{9\epsilon_0^2}\Big[\tilde{f}_{ij}\Big]^2 = -U_{NR}\,, \tag{3.99}$$

With U_{NR} defined on Eq. (2.18). Thus,

$$U_{NA} = -\frac{\Lambda}{9\epsilon_0^2}[\tilde{g}_{ij}]^2 - U_{NR}\,. \tag{3.100}$$

Once again, we are able to compare both the additive force and the non-additive one, since they both arise from second-order perturbation theory: are of order α^2 (for α being

the static polarizability of the atoms on a two-level model of equal atoms). Inside the plates, we have plotted the interaction behavior on the middle plane of the capacitor ($z = d/2$), so that the additive resultant force coming from the plates on atom A is zero. The resultant force between the additive force and the non-additive one will be of leading order. Fig. 3.8 shows the behavior of three different masses. For distances between $|\vec{r}_A - \vec{r}_B| \ll d$ and $|\vec{r}_A - \vec{r}_B| \sim 10^2 d$, for μ between 0 and $10d^{-1}$, the additive dispersion interaction is almost completely suppressed by the non-additive interaction. The mass plays a tiny contribution to slow down this suppression for $|\vec{r}_A - \vec{r}_B| \sim 1$.

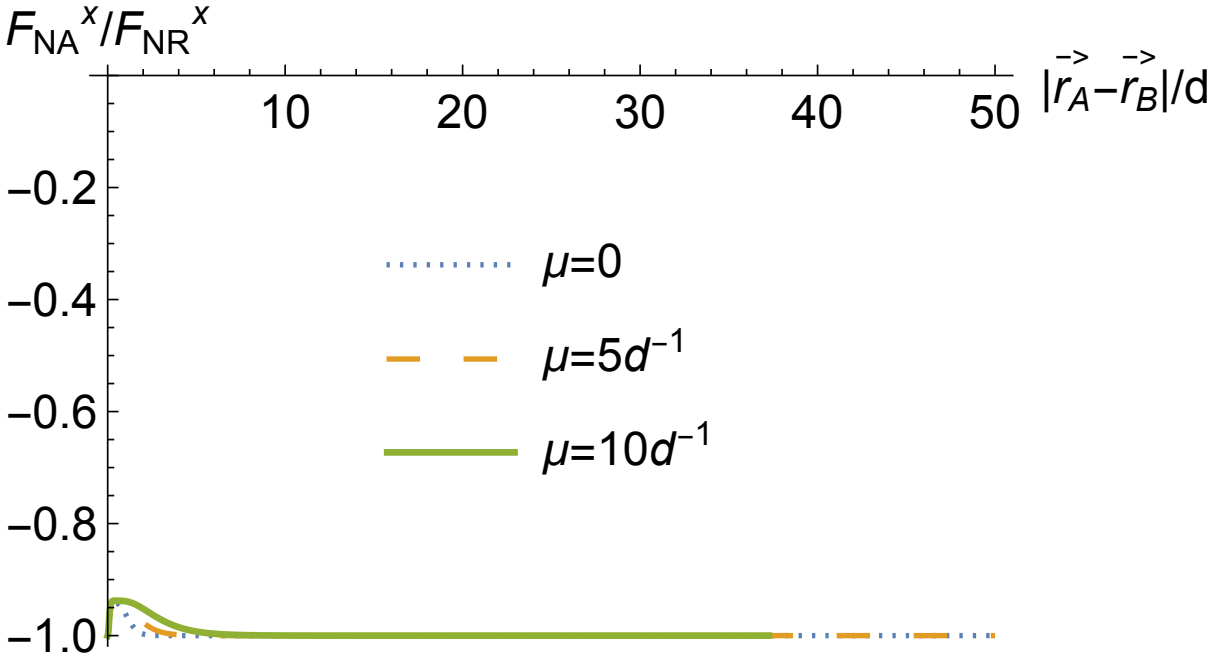


Figure 3.8: The x component of the non-additive force (vertical-axis) normalized by the x component of the NR force between the atoms (horizontal-axis). The d parameter is the distance between the plates, and $\mathbf{r}$ stands for the $\{x, y\}$ position on the plane equidistant to the plates.

Chapter 4

Conclusion and final remarks

Throughout this thesis, we have analyzed the dispersion interatomic energy within Proca electrodynamics framework. In Proca electrodynamics, the photon is a massive particle, which introduces a new physical scale, with fundamental implications for the dispersion interaction. As a first example, we treat the case where only two atoms are involved. Afterwards, we deal with non-additive corrections when we have more than two atoms present.

Concerning the two atoms' dispersion force, we have first treated the field in the electrostatic limit, and we have shown that the presence of a mass for the photon endows the interaction with a finite range (proportional to the inverse of the mass), leading to an overall weakening of the force. A natural question is what condition allows for the use of the electrostatic approximation. To answer this question, we performed a full quantum electrodynamics calculation. We have generalized the Hamiltonian originally proposed by P. Milonni [1] to be able to account for a finite photonic mass. This effective Hamiltonian corresponds physically to considering each atom as possessing a fluctuating electric dipole induced by the total electric field, given by the superposition of the vacuum electric field and the electric field produced by the dipole of the other atom (with this dipole, in turn, induced by the vacuum electric field). In addition to allowing for a physical intuition for the mechanism responsible for the dispersion force, this effective Hamiltonian yields the

dominant contribution in first order of perturbation theory.

We obtain an analytic expression for the Proca dispersion force in terms of an integral form for the complete dispersion energy, *i.e.* the form valid on the whole Proca quantum electrodynamics bounds (and for which the dipole approximation is valid). This integration can be simplified in two different distance regimes. To obtain them, we first split the integral form into two contributions: one involving only atomic variables, with characteristic time scale ω_0^{-1}, and the other containing field degrees of freedom, with characteristic time scale $\omega_F^{-1} = \left(c\sqrt{1/R^2 + \mu/R}\right)^{-1}$.

The regime $\omega_0 \ll \omega_F$ is denominated the non-retarded regime since in this case we have shown that the full QED calculation reproduces the result we have obtained previously, neglecting the electrodynamical retardation. On the other hand, the opposite case, where $\omega_0 \gg \omega_F$ corresponds to the asymptotic limit and the atom's response is so fast that we can substitute the atomic polarizability by its static value.

This identification allows us to understand where the electrostatic regime is valid, which could not be possible only with the non-relativistic treatment. A notable feature concerning the mentioned limits lies on the structure of ω_F. In Maxwell's case, we have $[\omega_F R]_{\text{Maxwell}} = c$. In the Maxwell case, there is an ambiguity since c is both the phase and the group velocity. In the case studied here, this ambiguity is broken, since $\omega_F R = v_{\text{phase}}\left(k = \sqrt{\mu/R}\right)$, revealing that the phase velocity evaluated at the wavenumber $k = \sqrt{\mu/R}$ is the one that matters to establish the limiting regimes. As a final remark on this topic, it is possible from the mass dependency of ω_F to see that in fact, μ has the effect of enlarging the region where the non-retarded regime is valid, and of curtailing the region where the asymptotic regime is applicable. For great enough masses, the non-retarded regime practically works for all relevant interatomic distances. The higher the mass, the more stretched the non-retarded regime becomes, and the more narrowed the asymptotic regime becomes. The physical interpretation is that the photon mass increases

the frequency for each wavelength, thus making the field oscillations more difficult to follow by the atomic dipole, thus enhancing the contribution from the low frequencies from the electric field.

As in the nonretarded regime, the main contribution arising from the photon's mass is to weaken the dispersion force. This shows that the shielding mechanism is pretty much independent of the atomic transition frequency.

In the second part of this Thesis, we have investigated the role of non-additivity on the dispersion energy between atoms and between atoms and surfaces in the non-retarded regime. Two main aspects have been considered in order to understand this contribution to the interaction. The former is the role of the mass on the non-additive parcel for each problem under consideration. The last one is the role of the mass on the comparative importance of the non-additive contribution to the overall force, or over the two atoms' dispersion force. For the three-atom problem, the non-additive term is, in general, pretty small, even for Maxwell's electrodynamics, which can be readily understood just by noting it is one order higher in the atomic polarizabilities than the London-Proca interaction between two atoms. The scenario is the same in Proca electrodynamics, where the non-additive contribution is still shielded.

This fact has led us to consider atoms interacting with macroscopic bodies, which are collections of enormous numbers of, in general, bound atoms. In the Maxwell case, it is well known that for some geometries, it is possible that we have a dominant non-additive contribution [31]. In the Proca context, we have demonstrated that in the case of two atoms near a conducting plane, the insertion of a mass parameter changes not only the overall force, but also leads to an enhancement of the importance of the non-additive term. In this configuration, the component of the force in the atom parallel to the plate can display a dominant contribution arising from non-additivity effects, a feature not possible in Maxwell electrodynamics. In addition, this contribution is repulsive, enabling, in some

scenarios, the obtaining of an overall repulsive dispersion force. Such a behavior suggests that a system of non-polarized atoms, interacting via Proca electrodynamics, disposed relatively close to a conducting plane (a sharp layer of atom suspension on an electrolyte medium placed over a conducting plane) would make the pairwise summation completely fail to approximate the overall interaction. The detailed investigation of this physical situation could be a prominent step for future work, testing both Proca electrodynamics and non-additivity on the Van der Waals forces frameworks. Additionally, for the case where two atoms are placed midway between two conducting parallel plates, the non-additive force almost completely cancels the dispersion force between the two atoms for most distances that have been considered. The plot obtained on Fig. 3.8 suggests that the mentioned pattern is independent of the mass, and it even does not depend on whether the theory mediating the interaction is Maxwell's os Proca's electrodynamics.

In principle, the results developed here constitute a valid test for the search for photon mass, and we hope our findings will inform future investigations toward the subject [61]. However, dispersion forces are known to be weak; in a way, it turns out to be unlikely to improve the actual bounds for the photon mass. Nonetheless, we suggest that macroscopic systems, due to the finite-mass modeling of their microscopic counterpart, could be a more favorable environment to examine those forces than the Van der Waals interaction itself. It would be interesting to explore this possibility in future work.

The concepts discussed throughout this Thesis are not restricted to the canonical realization of Proca's electrodynamics "fundamental interactions" scenario. Any phenomena where the electromagnetic field satisfies a Klein-Gordon equation can be effectively described by Proca electrodynamics (for example, low-concentration colloids).

Due to the ubiquity of dispersion forces, it is expected to find applications to the concepts developed here on most effective theories presented in Chap. 1, depending on whether the realization of the corresponding physical problem is possible there, as, for

example, the modeling of the source function corresponding to each effective theory.

Appendix A

Greens' Functions to Helmholtz equation

In this Appendix, we explicitly evaluate the following problem:

$$-\left(\nabla^2 + \lambda^2\right) G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'), \tag{A.1}$$

and analyze its solutions in terms of λ. Here, We will consider the specific scenarios where λ is a pure real or pure imaginary number, which is enough to compute the integrals of Secs. 2.1.2 and 2.2.3.Let us just start with two assumptions: the Green function's dependency on $\mathbf{x}$ and $\mathbf{x}'$ happens only through $|\mathbf{x} - \mathbf{x}'|$ and the Green's function vanishes for $|\mathbf{x} - \mathbf{x}'| \to \infty$. In the next step we take the Fourier transform the $\mathbf{x}$ variable:

$$G(\mathbf{x}, \mathbf{x}') = \int d^3k \, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \tilde{G}(\mathbf{k}). \tag{A.2}$$

Inserting this on Eq. (A.1) we find

$$G(\mathbf{x}, \mathbf{x}') = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2 - \lambda^2}, \tag{A.3}$$

and the final form before we analyze the comment cases for λ is:

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi^2 |\mathbf{x} - \mathbf{x}'|} \int dk \frac{k \, e^{ik|\mathbf{x}-\mathbf{x}'|}}{(k + \lambda)(k - \lambda)}, \tag{A.4}$$

where the poles are simply $k_0 \pm \lambda$. We start with the case where λ is purely imaginary. Under this assumption, both poles lie on the imaginary axis. The chosen contour is drawn

on Fig. A.1. The integration over the upper semi-circle vanishes on the limit where $R \to \infty$, due to the factor $\exp\{-R\sin\theta\}$, where $\sin\theta \geq 0$ on the upper half-plane (for θ the angle with $\mathrm{Re}(k)$). Then, using the residue theorem, one obtains

$$G_I(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi} \frac{e^{-|\lambda||\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{A.5}$$

The case where λ is real is a little more subtle, since the poles lie on the real axis. There

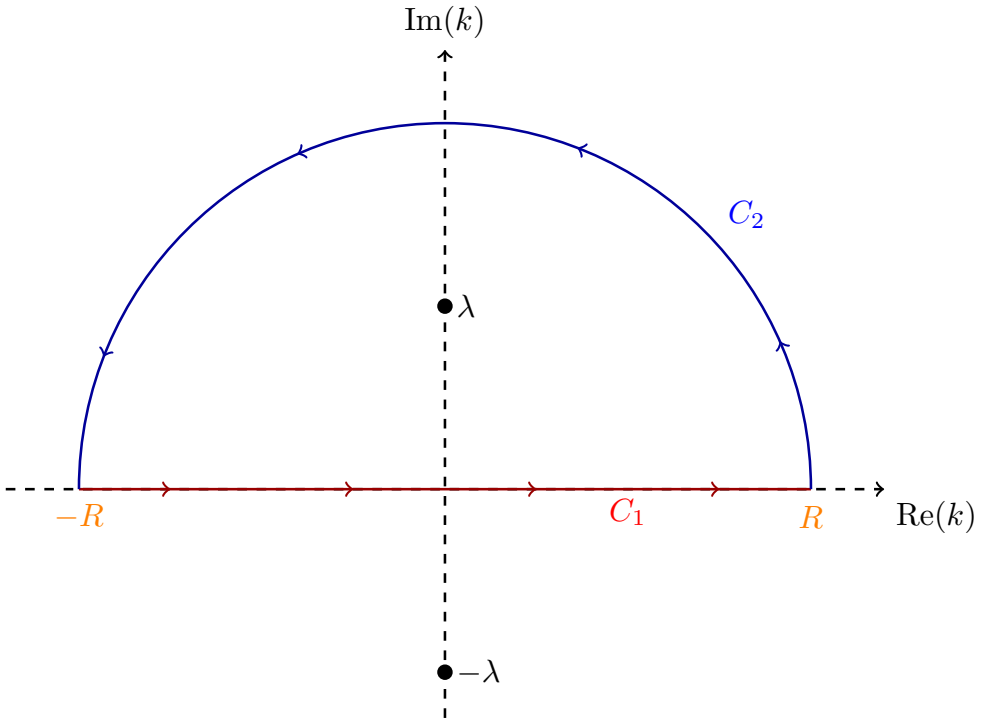


Figure A.1: Contour for the Green's function for imaginary λ.

are two possibilities to avoid these divergences, which can be reached by adding $\pm i\epsilon$ to λ, so the poles move to $k_0^+ = \pm(\lambda - i\epsilon)$ or $k_0^- = \pm(\lambda + i\epsilon)$ as described in Figs A.2 and A.3. When $\epsilon \to 0$, we end up with two linearly independent Green functions, which we will call $G^{\pm}(\mathbf{x}, \mathbf{x}')$ where the sign designates the poles $k_0^{\pm}$. As shown in Figs A.2 and A.3, the poles are no longer on the real axis and only one of them lies inside the contour, so we may use the residue theorem again to find (substituting the poles of (A.4) by $k_0^{\pm}$)

$$G^{\pm}(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi} \frac{e^{\mp i|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,, \tag{A.6}$$

where the limit $\epsilon \to 0$ has already been taken. The ambiguity in the sign on the Green

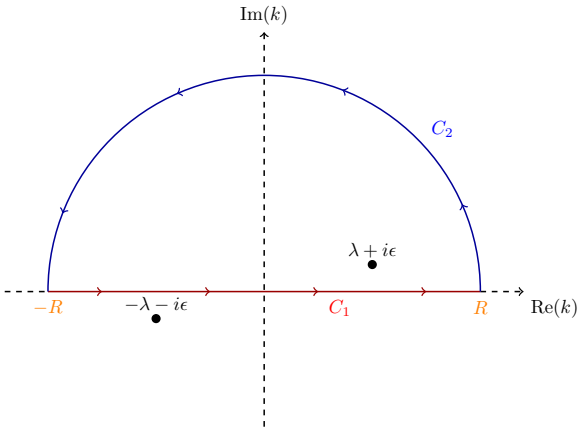


Figure A.2: k_0^- poles disposition on the contour integration of the Green function for real λ.

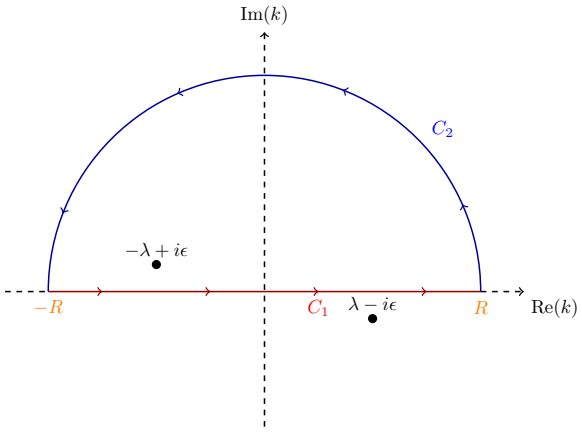


Figure A.3: k_0^+ poles disposition on the contour integration of the Green function for real λ.

function, of course, is only settled by specifying the physical situation under consideration.

Appendix B

Eigenfunction method for Green's functions

In this Appendix, we proceed as in Ref. [58]. Let us first consider the following differential equation

$$L\varphi(\mathbf{x}) = f(\mathbf{x})\,, \tag{B.1}$$

where L is a differential operator. A particular solution for $\phi(\mathbf{x})$ is obtained as:

$$\varphi(\mathbf{x}) = \int d^3x' G(\mathbf{x},\mathbf{x}') f(\mathbf{x}')\,, \tag{B.2}$$

where $G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')$ is the Green function associated with L and it satisfies

$$LG(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')\,, \tag{B.3}$$

.

The eigenfunctions $u_i(\mathbf{x})$ of the differential operator L are a complete set of functions satisfying the eigenvalue equation

$$Lu_i(\mathbf{x}) = \lambda_i u_i(\mathbf{x})\,. \tag{B.4}$$

The functions $u_i(\mathbf{x})$ form a complete orthonormal set on the space function with the following inner product on a region Ω

$$\langle f, g\rangle = \int_\Omega d^3x f^*(\mathbf{x}) g(\mathbf{x}'), \tag{B.5}$$

then, any function can be written as a linear combination of $u_i(\mathbf{x})$. and both $\varphi(\mathbf{x})$ and $f(\mathbf{x})$ can be written in terms of the eigenfunctions as

$$\varphi(\mathbf{x}) = \sum_i a_i u_i(\mathbf{x}) \tag{B.6}$$

$$f(\mathbf{x}) = \sum_i b_i u_i(\mathbf{x})\,, \tag{B.7}$$

so Eq. (B.2) reads

$$\sum_i a_i L u_i(\mathbf{x}) = \sum_i b_i u_i(\mathbf{x}) \tag{B.8}$$

$$= \sum_i \lambda_i a_i u_i(\mathbf{x}) = \sum_i b_i u_i(\mathbf{x}) \tag{B.9}$$

$$\rightarrow b_i = \frac{a_i}{\lambda_i}\,, \tag{B.10}$$

and now we use

$$a_i = \frac{b_i}{\lambda_i} = \int d^3x f(\mathbf{x}) u_i^*(\mathbf{x}) d^3x\,, \tag{B.11}$$

and then use Eq. (B.6) to write

$$\varphi(\mathbf{x}) = \sum_i \frac{b_i}{\lambda} u_i(\mathbf{x}) = \int d^3x' \sum_i \frac{u_i(\mathbf{x}) u_i^*(\mathbf{x}')}{\lambda_i} f(\mathbf{x})\,. \tag{B.12}$$

Comparing the above equation with Eq. (B.2), we obtain

$$G(\mathbf{x}, \mathbf{x}') = \sum_i \frac{u_i(\mathbf{x}) u_i^*(\mathbf{x}')}{\lambda_i} \tag{B.13}$$

Appendix C

Classical electrodynamical dipole field

In Subsec. 2.2.2, we claimed that, in order to determine the vacuum dipole electric field operator, one needs the time Fourier transform of the classical electric field generated by a pointwise oscillating dipole distribution, as summarized in Eq. (2.71). We start this Appendix by proving this claim. Performing a temporal Fourier transform on Eq. (2.51), we find that the four potential satisfies:

$$\left(-\nabla^2 - \omega^2/c^2 + \mu^2\right)\tilde{A}^{\mu}(\omega, \mathbf{x}) = \frac{\tilde{j}^{\mu}(\omega, \mathbf{x})}{c^2\epsilon_0}\,, \tag{C.1}$$

where $\tilde{f}(\omega) = \int_{-\infty}^{+\infty} dt \exp\{-i\omega t\} f(t)$. The source in the frequency space is obtained from Eqs. (2.57) and (2.60) by performing a Fourier transform as

$$\tilde{j}^{\mu}(\omega, \mathbf{x}) = \Big(-c\tilde{\boldsymbol{d}}(\omega)\cdot\nabla\delta(\mathbf{x}-\mathbf{x}_B), i\omega\tilde{\boldsymbol{d}}(\omega)\delta(\mathbf{x}-\mathbf{x}_B)\Big), \tag{C.2}$$

and the relation between the electric field and the potential is simply

$$\tilde{\boldsymbol{E}}(\omega, \mathbf{x}) = -c\nabla\tilde{A}^0(\omega, \mathbf{x}) - i\omega\tilde{\boldsymbol{A}}(\omega, \mathbf{x}). \tag{C.3}$$

The solution to Eq. (C.1) is:

$$\tilde{A}^{\mu}(\omega, \mathbf{x}) = \frac{1}{c^2\epsilon_0}\int d^3x' \tilde{G}(\omega, \mathbf{x}, \mathbf{x}')\tilde{j}^{\mu}(\omega, \mathbf{x}') \tag{C.4}$$

where $\tilde{G}(\omega, \mathbf{x}, \mathbf{x}')$ is given by Eq. (2.56). We may now insert the potential (C.4) with the explicit form of the source (C.2), on Eq. (C.3), so we can find the electric field. Using the

Dirac delta derivative property (2.58), we find

$$-c\nabla\tilde{A}^0(\omega,\mathbf{x}) = \frac{1}{\epsilon_0}\nabla\Big(\tilde{\mathbf{d}}(\omega)\cdot\nabla\tilde{G}(\omega,\mathbf{x},\mathbf{x}_B)\Big), \tag{C.5}$$

and

$$-i\omega\tilde{\mathbf{A}}(\omega,\mathbf{x}) = \frac{\omega^2}{c^2\epsilon_0}\tilde{G}(\omega,\mathbf{x},\mathbf{x}_B). \tag{C.6}$$

Rearranging Eqs. (C.5) and (C.6) on the vectorial index form and adding them, we find the electric field (C.3) as

$$\tilde{E}_i(\omega,\mathbf{x}) = \frac{1}{\epsilon_0}\tilde{d}_j(\omega)\left(\frac{\omega^2}{c^2}\delta_{ij} + \partial_i\partial_j\right)\tilde{G}(\omega,\mathbf{x},\mathbf{x}_B) \tag{C.7}$$

For brevity, we write the Green function (2.56) as:

$$\tilde{G}(\omega,\mathbf{x},\mathbf{x}_B) = \frac{1}{4\pi r}e^{-\gamma r} \equiv f(r), \tag{C.8}$$

where $\gamma \equiv \mp i\sqrt{\omega^2/c^2-\mu^2}$ for $|\omega| > \mu$ and $-\sqrt{\mu^2-\omega^2/c^2}$ otherwise and $r \equiv |\mathbf{x}-\mathbf{x}_B|$. Using the identity

$$\begin{aligned}\partial_j f(r) &= \partial_j r f'(r) = \partial_j (x_k x_k)^{1/2} f'(r)\\ &= 2\delta_{jk}x_k\frac{1}{2}(x_l x_l)^{-1/2}f'(r) = \frac{f'(r)}{r}x_j,\end{aligned} \tag{C.9}$$

where x_i is the $i^{\mathbf{th}}$ component of $\mathbf{x}-\mathbf{x}_B$, the dipole electric field on Fourier frequency domain is

$$\tilde{\mathbf{E}}_B(\omega,\mathbf{x}) = \frac{e^{-\gamma r}}{4\pi\epsilon_0 r^3}\left[(\tilde{\boldsymbol{d}}(\omega)\cdot[\mathbf{x}-\mathbf{x}_B])(\gamma^2r^2+3\gamma r+3)\frac{[\mathbf{x}-\mathbf{x}_B]}{r^2} + \left(\frac{\omega^2}{c^2}r^2-\gamma r-1\right)\tilde{\boldsymbol{d}}(\omega)\right] \tag{C.10}$$

Appendix D

Performing the retarded regime contour integral

In this Appendix, we will explicitly calculate integrals of the form

$$I = A\int_0^\infty \frac{dx}{\sqrt{x^2+y^2}}\left[P_s(x)\sin(2x) + P_c(x)\cos(2x)\right], \tag{D.1}$$

where A is some real constant, $P_s(x)$ is a real odd function and $P_c(x)$ is a real even function with no branch cuts. Those integrals are clearly divergent in the UV regime, so we will need to perform them on the extended complex plane.

The first step is simply to use the parity of the integrand to write the above integral as

$$I = \frac{A}{2}\mathrm{Re}\int_{-\infty}^\infty \frac{dx}{\sqrt{x^2+y^2}}\left[P_c(x) - iP_s(x)\right]e^{2ix}. \tag{D.2}$$

We then move to the complex plane, defining a complex variable z such that $\mathrm{Re}\, z = x$, so that integral (D.2) lies on the real z line. The above integrand has two branch cuts on the complex plane (one running from $x = iy$ to $x = i\infty$ and the other running from $x = -iy$ to $x = -\infty$), as shown in Fig. D.1. To apply the residue theorem, we avoid these branch cuts by employing the contour shown in Fig. D.1. We choose to close the contour on the upper half plane, so that the segments $C_R^\pm$ on this figure evaluate to zero at complex infinity. The only parts of the contour that are non-zero are I, u^+ and u^-, and the residue theorem implies that

$$\int_I g(z)\,dz + \int_{u_+} g(z)\,dz + \int_{u_-} g(z)\,dz = 0, \tag{D.3}$$

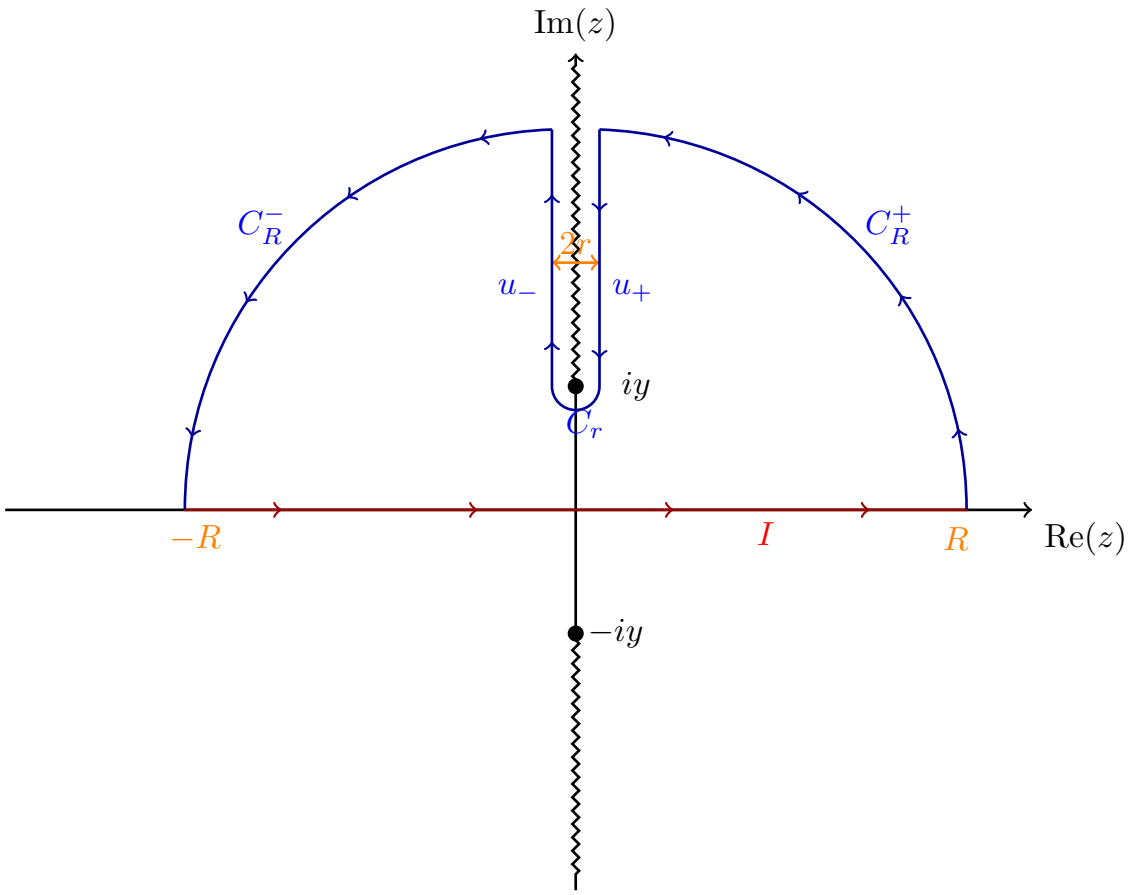


Figure D.1: Contour in the complex plane used to compute Eq. (D.2).

where

$$g(z) = \frac{\tilde{P}(z)\, e^{2iz}}{\sqrt{z^2 + y^2}} \tag{D.4}$$

and

$$\tilde{P}(z) = P_c(z) - iP_s(z). \tag{D.5}$$

The branch cut is chosen such that the square root has a positive imaginary part on the right side (the u^- line) and a negative imaginary part on the left side (the u^- line). Hence,

$$\begin{aligned} \int_{u_+} g(z)dz &= \frac{A}{2}\int_{\infty}^{y} \frac{(idu)}{i\sqrt{u^2 - y^2}} \tilde{P}(iu) e^{-2u} \\ &= -\frac{A}{2}\int_{y}^{\infty} \frac{du}{\sqrt{u^2 - y^2}} \tilde{P}(iu)\, e^{-2u} \end{aligned} \tag{D.6}$$

and

$$\int_{u_-} g(z)dz = \frac{A}{2}\int_y^\infty \frac{(idu)}{-i\sqrt{u^2-y^2}}\tilde{P}(iu)e^{-2u}$$

$$= -\frac{A}{2}\int_y^\infty \frac{du}{\sqrt{u^2-y^2}}\tilde{P}(iu)e^{-2u} \tag{D.7}$$

$$= \int_{u_+} g(z)dz \tag{D.8}$$

So that integral I becomes:

$$I = -A\,\mathrm{Re}\int_y^\infty \frac{du}{\sqrt{u^2-y^2}}\tilde{P}(iu)\,e^{-2u}. \tag{D.9}$$

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