WEIGHTED COMPOSITION OPERATORS THAT ARE CHAOTIC IN THE SENSE OF LI AND YORKE

Fernanda Mendonça de Vasconcellos

Rio de Janeiro

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Tese de doutorado apresentada ao Programa de Pós-graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutora em Matemática.

Orientador: D.Sc. Nilson da Costa Bernardes Jr.

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Aprovada por:

____
(Presidente) Prof. D.Sc. Nilson da Costa Bernardes Jr. - IM/UFRJ

____
Prof. D.Sc. Carlos S. Kubrusly - PUC-Rio

____
Prof. D.Sc. Dinamérico Pereira Pombo Junior - UFF

____
Prof. D.Sc. Vinícius Vieira Fávaro - UFU

____
Prof. D.Sc. Xavier Carvajal Paredes - IM/UFRJ

____
Profa. D.Sc. Jaqueline Siqueira Rocha - IM/UFRJ (Suplente)

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Aos meus pais.

To my parents.

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Resumo

Estabelecemos caracterizações completas da noção de caos Li-Yorke para operadores de composição com peso sobre os espaços $C_0(X)$ e $L^p(\mu)$ $(1 \leq p < \infty)$. Como consequência, apresentamos caracterizações simples para o caos Li-Yorke de operadores de deslocamento com pesos sobre c_0 e sobre ℓ^p $(1 \leq p < \infty)$ que complementam resultados previamente conhecidos. Também investigamos a noção de caos Li-Yorke para operadores de deslocamento com pesos sobre espaços de Fréchet de sequências. Como aplicações, obtemos caracterizações do caos Li-Yorke para operadores de deslocamento com pesos sobre espaços de sequências de Köthe que dependem apenas dos elementos da matriz de Köthe e dos pesos do operador.

Palavras-chave: dinâmica linear, caos Li-Yorke, operadores de composição com peso, operadores de deslocamento com pesos.

Abstract

We establish complete characterizations of the notion of Li-Yorke chaos for weighted composition operators on $C_0(X)$ spaces and on $L^p(\mu)$ spaces ($1 \leq p < \infty$). As a consequence, we obtain simple characterizations of the Li-Yorke chaotic weighted shifts on c_0 and on ℓ^p ($1 \leq p < \infty$) that complement previously known results. We also investigate the notion of Li-Yorke chaos for weighted shifts on Fréchet sequence spaces. As applications, we obtain characterizations of the Li-Yorke chaotic weighted shifts on Köthe sequence spaces depending only on the entries of the Köthe matrix and the weights of the shift.

Keywords: linear dynamics, Li-Yorke chaos, weighted composition operators, weighted shifts.

Contents

Chapter 1

Introduction

The goal of this work is to establish complete characterizations of the concept of Li-Yorke chaos for weighted composition operators on certain specific Banach spaces. More precisely, we are interested in studying the chaotic behavior in the sense of Li and Yorke for operators of the form

$$C_{w,f}(\varphi) := (\varphi \circ f) \cdot w$$

on $C_0(X)$, where X is a locally compact Hausdorff space, and on $L^p(\mu)$, with $p \in [1, \infty)$, where $(X, \mathfrak{M}, \mu)$ is a positive measure space. As relevant applications, we will obtain simple characterizations of Li-Yorke chaos for weighted shifts (unilateral and bilateral) on the classical Banach spaces c_0 and ℓ^p, which complement previously known results. Additionally, we will study the notion of Li-Yorke chaos for weighted shifts on Fréchet sequence spaces. The main results in this work appeared in the paper [11].

The interest in the topic addressed in this work arises from the fact that the area of linear dynamics has seen a significant development in recent decades and has relevant connections with other branches of dynamics, such as topological dynamics, for example. Regarding the specific type of chaotic behavior addressed in this work, it is noteworthy because the concept of "chaos" was first mentioned in mathematical texts in the article published by Tien-Yien Li and James A. Yorke in 1975, entitled "Period Three Implies Chaos" (see [26]). In particular, we owe these two mathematicians the definition of "Li-Yorke chaos", which we will explore throughout this work. For readers interested in linear dynamics, we recommend the books [1, 22] and the more recent articles [2, 7, 9, 10, 12, 20], where many additional references can be found.

Regarding composition operators, they naturally arise in many different contexts (analytic, measure-theoretic, topological, etc.) and, therefore, they actually constitute a family of classes of operators. They play a very important role in operator theory and its applications. The dynamics of such operators has been investigated by several authors in different settings (see, for instance, [4, 5, 13, 15, 17, 18, 19, 21, 23, 32]). In particular, a detailed study of Li-Yorke chaos for composition operators on $L^p(\mu)$ spaces was developed in [8].

As for the structure of this work, some prerequisites in Analysis and General Topology will be assumed. Others, considered important for a good understanding of the text, such as the notions of irregular and semi-irregular vectors and results related to Li-Yorke chaos for operators on Fréchet spaces, will be presented in Chapter 2.

In Chapter 3, we will present a complete characterization of Li-Yorke chaos for operators $C_{w,f}$ on $C_0(X)$ spaces (Theorem 3.6) and the applications to weighted shifts on c_0.

In Chapter 4, we will provide a complete characterization of Li-Yorke chaos for operators $C_{w,f}$ on $L^p(\mu)$ spaces (Theorem 4.8). As a consequence, we will obtain simple characterizations of the chaotic behavior of the Li-Yorke type for weighted shifts on ℓ^p. It is also worth mentioning that, in the case of $L^p(\mu)$ spaces, our results generalize previous results from [8] in the unweighted case, and we were strongly motivated by this previous work.

Finally, in Chapter 5, we will turn our attention to the study of Li-Yorke chaos for weighted shifts on Fréchet sequence spaces. A trichotomy (Theorem 5.2) will be obtained as one of the main results of this chapter, complementing the study of Li-Yorke chaos for linear operators developed in [6]. As applications, we will obtain characterizations for the Li-Yorke chaos of weighted shifts on Köthe sequence spaces, which have a computational character, that is, they depend only on the entries of the Köthe matrix and the weights of the shift (Theorems 5.11 and 5.12). Additionally, at the end of Chapter 5, we will present examples of Köthe spaces and use Theorem 5.11 to demonstrate that a specific weighted shift on a particular Köthe space is Li-Yorke chaotic but not hypercyclic.

Chapter 2

Preliminaries

In this chapter, we will present some definitions and basic results that are prerequisites for understanding the main theorems of this work and their resulting corollaries. Throughout this chapter, $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

2.1 Topology and functional analysis

Given a set X, recall that a *topology* on X is a collection τ of subsets of X having the following properties:

- $\varnothing$ and X are in τ;
- the union of the elements of any subcollection of τ is in τ;
- the intersection of the elements of any finite subcollection of τ is in τ.

In this case, the pair (X, τ) is called a *topological space* and the elements of τ are called the *open sets* of X. If V is an open set of X, then $X \setminus V$ is called a *closed set* of X. In our context, throughout this work, given $x \in X$ and U an element of τ such that $x \in U$, we will say that U is a *neighborhood* of x.

Examples of topological spaces and their properties can be found in detail in [29] and will not be presented here.

The next proposition will be useful, in Chapter 3. But first, we will need the following definition and lemma.

Definition 2.1. *Let X be a topological space. An indexed family $\{A_\alpha\}_{\alpha\in J}$ of subsets of X is said to be a* locally finite family *in X if every $a \in X$ has a neighborhood that intersects A_α for only finitely many values of α.*

Lemma 2.2. *Let $\{A_\alpha\}_{\alpha\in J}$ be a locally finite family in a topological space X. Then,*

$$\overline{\bigcup_{\alpha\in J} A_\alpha} = \bigcup_{\alpha\in J} \overline{A_\alpha}.$$

Proof. Let $A := \bigcup_{\alpha\in J} A_\alpha$. If $x \in \bigcup_{\alpha\in J} \overline{A_\alpha}$, then $x \in \overline{A_\beta}$, for some $\beta \in J$. Thus, every neighborhood V of x in X is such that $V \cap A_\beta \neq \varnothing$. Therefore, $V \cap \left(\bigcup_{\alpha\in J} A_\alpha\right) \neq \varnothing$, for all neighborhood V of x, whence it follows that $x \in \overline{\bigcup_{\alpha\in J} A_\alpha} = \overline{A}$.

Conversely, let $y \in \overline{A}$. Let U be a neighborhood of y that intersects only finitely many elements of $\{A_\alpha\}_{\alpha\in J}$, say $A_{\alpha_1}, \ldots, A_{\alpha_k}$. We assert that $y \in \bigcup_{i=1}^{k} \overline{A_{\alpha_i}}$. Indeed, if $y \notin \bigcup_{i=1}^{k} \overline{A_{\alpha_i}}$, then $W := U \setminus \bigcup_{i=1}^{k} \overline{A_{\alpha_i}}$ would be a neighborhood of y that does not intersect any of the A_α's and thus does not intersect A. This would contradict the statement that $y \in \overline{A}$. □

Proposition 2.3. *Let $\{A_\alpha\}_{\alpha\in J}$ be a locally finite family in a topological space X. If A_α is closed for all $\alpha \in J$, then $A = \bigcup_{\alpha\in J} A_\alpha$ is closed.*

Proof. Since A_α is closed, we have that $A_\alpha = \overline{A_\alpha}$ $(\alpha \in J)$. Hence, by Lemma 2.2

$$A = \bigcup_{\alpha\in J} A_\alpha = \bigcup_{\alpha\in J} \overline{A_\alpha} = \overline{\bigcup_{\alpha\in J} A_\alpha} = \overline{A}.$$

□

In the next subsections, we will provide other definitions and results that will be useful later.

2.1.1 Locally compact Hausdorff spaces

Let X be a topological space. Recall that X is a *Hausdorff space* if, for each pair of distinct points a and b in X, there exist neighborhoods U and V of a and b, respectively, such that $U \cap V = \varnothing$.

Recall also that a collection $\mathcal{C}$ of subsets of a topological space X is said to be a *covering* of X if the union of the elements of $\mathcal{C}$ is equal to X. This covering is called an *open covering* of X if its elements are open subsets of X. A topological space X is said to be *compact* if every open covering $\mathcal{C}$ of X admits a finite subcovering.

Definition 2.4. *Let X be a topological space. We say that X is a* locally compact Hausdorff space *if it is a Hausdorff space and given $x \in X$ and given a neighborhood V of x, there is a neighborhood U of x such that $\overline{U}$ is compact and $\overline{U} \subset V$.*

Example 2.5. The real line $\mathbb{R}$ endowed with standard topology is a locally compact Hausdorff space. Indeed, given $a \in \mathbb{R}$ and a neighborhood V of a, there exists $\varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon) \subset V$. Hence, $U := \left(a - \dfrac{\varepsilon}{2}, a + \dfrac{\varepsilon}{2}\right)$ satisfies:

(i) U is a neighborhood of a;

(ii) $\overline{U}$ is compact in $\mathbb{R}$;

(iii) $\overline{U} \subset (a-\varepsilon, a+\varepsilon) \subset V$.

2.1.2 Baire categories

If X is a topological space and A is a subset of X, we define the *interior of* A ($\mathrm{int}(A)$) as the union of all open subsets of X that are contained in A. Thus, saying that A has *empty interior* is the same as saying that A contains no open subset of X beyond the empty set. Equivalently, we can say that if every point of A is a accumulation point[1] of $X \setminus A$, then A has empty interior, which means that the complement of A is dense[2] in X.

[1] Let B be a subset of a topological space X and $x \in X$. We say that x is a *accumulation point* of B if, for every neighborhood V of x, we have $B \cap (V \setminus \{x\}) \neq \varnothing$.

[2] A set A is *dense* in X if $\overline{A} = X$.

Definition 2.6. *Let X be a topological space and A be a subset of X. We say that:*

(a) A is a nowhere dense set *if $\overline{A}$ has empty interior.*

(b) A is a set of the first category *if A is the union of a countable family of nowhere dense sets.*

(c) A is a set of the second category *if A is not a set of the first category.*

(d) A is a residual set *if $X \setminus A$ is a set of the first category.*

Proposition 2.7. *For every topological space X, the following statements are equivalent:*

(i) Every countable intersection of dense open sets in X is a dense set in X;

(ii) Every countable union of closed sets with empty interior in X is a set with empty interior in X;

(iii) Every non-empty open set in X is a set of the second category in X;

(iv) Every residual set in X is a dense set in X.

Remark 2.8. Let X be a topological space. The following properties are true and can be found in [30]:

(a) any closed set $A \subset X$ whose interior is empty is of the first category in X;

(b) if $f : X \to X$ is a homeomorphism and $A \subset X$, then A and $f(A)$ have the same category in X.

Definition 2.9. *A topological space X is a* Baire space *if X satisfies the equivalent conditions of Proposition 2.7.*

Theorem 2.10. [Baire] *Every completely metrizable space (that is, admitting a complete metric compatible with the topology) and every locally compact Hausdorff space are Baire spaces.*

The proof of Baire's Theorem can be found in [30] and we won't do it here. A curious fact is that the term "category" was introduced by René-Louis Baire (1874-1932) through the formulation given by item (*ii*) of Proposition 2.7 (cf. [29]).

Definition 2.11. *A subset A of a topological space X is called a* set G_δ*(or a G_δ-set*[3]*) if A can be written as a countable intersection of open sets in X.*

Proposition 2.12. *If X is a Baire space and $A \subset X$, then A is a residual set in X if and only if A contains a dense G_δ-set in X.*

The proof of Proposition 2.12 falls outside the main topic of this text and can be found in [29].

Example 2.13. Both sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$. However, $\mathbb{Q}$ is not a residual set in $\mathbb{R}$ (in fact, $\mathbb{Q}$ is of the first category in $\mathbb{R}$) and $\mathbb{R} \setminus \mathbb{Q}$ is a residual set in $\mathbb{R}$.

Intuitively, a residual set in a Baire space X is commonly seen as a set that contains the "topological majority of points" of X. In this sense, the Example 2.13 tells us that most real numbers are irrational.

2.1.3 Urysohn's lemma

Let X be a topological space. Given a map $\varphi : X \to \mathbb{K}$, we denote by $\operatorname{supp} \varphi$ its *support*, that is,

$$\operatorname{supp} \varphi := \overline{\{x \in X : \varphi(x) \neq 0\}}.$$

We denote by $C_c(X)$ the collection of all continuous maps $\varphi : X \to \mathbb{K}$ which have compact support.

Next, we will present the version of Urysohn's Lemma that will be used later in this work, whose proof will be omitted here as it falls outside the scope of this text, but can be found in [31].

[3] The notation " G_δ" comes from German, where "G" stands for "gebiet", which in this context means "open set", and "δ" is from "durchschnitt", which means "intersection" (see [29], p.194).

Lemma 2.1.4. [Urysohn's lemma] *Suppose X is a locally compact Hausdorff space. If V is an open subset of X and $K \subset V$ is compact, then there is $f \in C_c(X)$ such that*

$$f(X) \subset [0,1], \quad f = 1 \text{ em } K \quad \text{and} \quad \operatorname{supp} f \subset V.$$

2.1.4 Banach-Steinhaus Theorem

Before stating the version of the Banach-Steinhaus Theorem that we will use, we will recall some useful definitions, which can be found in [30]. Recall that a *topological vector space* X is a vector space endowed with a topology τ in which two conditions are satisfied:

(1) every point of X is a closed set, and

(2) the vector space operations are continuous with respect to τ.

Next, we state an important result regarding the topology of topological vector spaces, which will be used later and can be found in [30].

Theorem 2.1.5. *If $\mathcal{B}$ is a local base for a topological vector space X, then every member of $\mathcal{B}$ contains the closure of some member of $\mathcal{B}$.*

Recall that a topological vector space X is an *F-space* if its topology is induced by a complete invariant metric d. A metric on X is said to be *invariant* if

$$d(x+z, y+z) = d(x,y), \quad \forall\, x,y,z \in X.$$

Throughout this subsection, consider X and Y be topological vector spaces over $\mathbb{K}$ and $\mathcal{L}$ a collection of continuous linear mappings from X to Y.

Definition 2.1.6. *A subset $E \subset X$ is said to be* bounded *if for each neighborhood V of 0 in X, there exists a real number $t > 0$ such that $E \subset sV$, for all $s > t$.*

Definition 2.1.7. *A set $V \subset X$ is said to be* balanced *if $\alpha V \subset V$ for every $\alpha \in \mathbb{K}$, with $|\alpha| \leq 1$.*

Definition 2.18. *We say that a map $T : X \to Y$ is* uniformly continuous *if for every neighborhood W of 0 in Y, there corresponds a neighborhood V of 0 in X such that*

$$y - x \in V \Rightarrow Tx - Ty \in W.$$

Definition 2.19. *We say that $\mathcal{L}$ is* equicontinuous *if for every neighborhood W of 0 in Y, there is a neighborhood V of 0 in X such that $T(V) \subset W$, for all $T \in \mathcal{L}$.*

Remark 2.20. In the case where X and Y are normed spaces, $\mathcal{L}$ being equicontinuous is equivalent to $\sup\limits_{T\in\mathcal{L}} \|T\| < \infty$.

The next theorem will be useful for the proof of the Banach-Steinhaus Theorem that we will see below, and its proof can be found in [30].

Theorem 2.21. *Suppose $\mathcal{L}$ equicontinuous. If E is a bounded subset of X, then Y has a bounded subset F such that $T(E) \subset F$, for all $T \in \mathcal{L}$.*

Proof. Let E a bounded subset of X and $F := \bigcup\limits_{T\in\mathcal{L}} T(E) \subset Y$. Let W be a neighborhood of 0 in Y. Since $\mathcal{L}$ is equicontinuous, there is a neighborhood V of 0 in X such that $T(V) \subset W$, for all $T \in \mathcal{L}$. Since E is bounded, $E \subset tV$, for all sufficiently large t. For these t,

$$T(E) \subset T(tV) = tT(V) \subset tW, \quad \text{for all} \quad T \in \mathcal{L},$$

whence $F \subset tW$. Thus, F is bounded. □

Theorem 2.22 (Banach-Steinhaus)**.** *Suppose $\mathcal{L}$ is the collection mentioned above and B is the set of all $x \in X$ such that*

$$\mathcal{L}(x) := \{T(x) : T \in \mathcal{L}\}$$

is bounded in Y. If B is of the second category in X, then $B = X$ and $\mathcal{L}$ is equicontinuous.

Proof. Let $\mathfrak{B}$ be the collection of all balanced neighborhoods of 0 in Y and pick an element W of $\mathfrak{B}$. Since $\mathfrak{B}$ is a local base[4] for Y and addition is continuous in Y, there exists $V \in \mathfrak{B}$

[4] If M is a topological space and $x \in M$, a *local base at x* is a collection $\mathcal{B}(x)$ of neighborhoods of x with the property that every neighborhood of x contains some element of $\mathcal{B}(x)$. By a *local base for M*, we mean a local base at 0. For instance, in the case where M is a metric space, the collection $\mathcal{B}(x) := \{B(x; r); r > 0\}$ of all open balls centered at x with radius r is a local base at x.

such that $V+V\subset W$. By Theorem 2.15, there exists $U\in\mathfrak{B}$ such that

$$\overline{U}+\overline{U}\subset V+V\subset W.$$

Put

$$E:=\bigcap_{T\in\mathcal{L}} T^{-1}(\overline{U}).$$

If $x\in B$, then there exists $n\in\mathbb{N}$ such that $T(x)\in nU$ for all $T\in\mathcal{L}$. Since $U\subset\overline{U}$, we have that $T(x)\in n\overline{U}$, whence $x\in nT^{-1}(\overline{U})$, for all $T\in\mathcal{L}$. Thus, $x\in nE$. Therefore,

$$B\subset\bigcup_{n=1}^{\infty} nE.$$

Since B is of the second category by hypothesis, it follows that nE is of the second category in X for some $n\in\mathbb{N}$. Since

$$x\in X\mapsto nx\in X$$

is a homeomorphism, E is itself of the second category in X (by Remark 2.8-(b)). But E is closed because each T is continuous. By item *(a)* of Remark 2.8, E does not have empty interior, hence there exists $x\in \mathrm{int}(E)$. Therefore, there exists a neighborhood V of 0 in X such that $V\subset x-E$. Hence,

$$T(V)\subset T(x-E)=T(x)-T(E)\subset\overline{U}-\overline{U}\subset W,$$

for all $T\in\mathcal{L}$. This proves that $\mathcal{L}$ is equicontinuous. Let us prove that $X\subset B$. Let x be an arbitrary element of X. By Theorem 2.21, there is a bounded subset J of Y such that $T(x)\in J$, for all $T\in\mathcal{L}$. In particular, $\mathcal{L}(x)\subset J$. Since J is bounded, we have that $\mathcal{L}(x)$ is bounded in Y. Therefore, $x\in B$, which implies $X\subset B$. □

We know that continuous linear maps are bounded. By Theorem 2.21, we see that equicontinuous collections exhibit the boundedness property in a uniform manner. For this reason, the Banach-Steinhaus Theorem is also widely known as the "uniform boundedness principle".

2.2 Banach and Fréchet Spaces

Recall that a *Banach space* is a vector space X over $\mathbb{K}$ endowed with a norm, usually denoted by $\|\cdot\|$, whose topology is induced by the metric

$$d(x,y) := \|x-y\| \quad (x,y \in X)\,,$$

in which X is complete.

In this section, we will recall the definition of a Fréchet space and present some examples.

Definition 2.23. *We say that X is a* locally convex space (LCS) *if X is a topological vector space that possesses a local base whose elements are convex.*

Recall that a *seminorm* on a vector space X is a functional $p : X \to \mathbb{R}$ such that

$$\begin{array}{lll} \text{(S1)} & p(\alpha x) = |\alpha|\, p(x), & \forall \alpha \in \mathbb{K} \text{ e } \forall x \in X; \\ \text{(S2)} & p(x+y) \leq p(x) + p(y), & \forall x, y \in X. \end{array}$$

Note that the conditions for a functional p to be a seminorm exclude the equivalence "$x = 0 \Leftrightarrow p(0) = 0$". However, setting $\alpha = 0$ in *(S1)*, we have that $p(0) = 0$. In particular, every norm is a seminorm.

Definition 2.24. *A family $\mathcal{P}$ of seminorms on a vector space X is* separating *if, for each $x \neq 0$ in X, there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$.*

An important fact about separating families of seminorms is that if $\mathcal{P}$ is such a family on a vector space X, then the topology induced by $\mathcal{P}$ makes X a locally convex space. The converse of this fact is also true: if X is a LCS, then there exists a separating family of seminorms that induces its topology. The proof of these facts can be found in [30] and it is beyond the main focus of this work. However, such results are important for proving the following characterization of metrizability for locally convex spaces, which can also be found in [30]:

Theorem 2.25. *Let X be a locally convex space. Then X is metrizable if and only if there exists a sequence $(p_n)_{n\in\mathbb{N}}$ of seminorms that generates the topology of X. In this case,*

$$d(x,y) := \sum_{k=1}^{\infty} \frac{1}{2^k} min\{1, p_k(x-y)\} \qquad (x, y \in X) \tag{2.1}$$

is an invariant metric that induces the topology of X.

With the above result, we can finally define the concept of a Fréchet space.

Definition 2.26. *We say that X is a* Fréchet space *if X is a locally convex metrizable space such that the metric* (2.1) *is complete.*

Remark 2.27. In Definition 2.26, it is possible to prove that the completeness of the metric d does not depend on the choice of the sequence $(p_n)_{n\in\mathbb{N}}$ of seminorms that induces the topology of X. In particular, if X is a Fréchet space, we can choose $(p_n)_{n\in\mathbb{N}}$ to be increasing[5]. Moreover, the following facts are true:

1) If $(p_n)_{n\in\mathbb{N}}$ is increasing and induces the topology of X, then every subsequence $(p_{n_k})_{k\in\mathbb{N}}$ also induces the topology of X.

2) If $(p_n)_{n\in\mathbb{N}}$ induces the topology of X, then $(a_n p_n)_{n\in\mathbb{N}}$ also induces it, provided that $(a_n)_{n\in\mathbb{N}}$ is a sequence of non-zero scalars.

We also highlight that every Banach space can be seen as a Fréchet space: it is enough to take $(p_n)_{n\in\mathbb{N}}$ as the constant sequence $(\|\cdot\|)_{n\in\mathbb{N}}$, where $\|\cdot\|$ is the norm of the Banach space.

Example 2.28. Let Ω be a non-empty open subset of $\mathbb{K}^n$ and define

$$C(\Omega) := \{f : \Omega \to \mathbb{K} : f \text{ is continuous}\}.$$

Note that $C(\Omega)$ is a vector space over $\mathbb{K}$. For each $n \in \mathbb{N}$, consider

$$K_n := \left\{x \in \overline{B}(0;n) : d(x, \Omega^c) \geq \frac{1}{n}\right\},$$

where $\overline{B}(0;n)$ is the closed ball centered at 0 with radius n in $\mathbb{K}^n$. The following facts are true:

[5] The sequence of seminorms $(p_n)_{n\in\mathbb{N}}$ is *increasing* if $p_1(x) \leq p_2(x) ... \leq p_n(x) \leq ...$, $\forall x \in X$.

ii) K_n is compact for all $n \in \mathbb{N}$;

iii) $\Omega = \bigcup_{n=1}^{\infty} K_n$;

iiii) $K_n \subset int(K_{n+1})$ for all $n \in \mathbb{N}$.

If we consider the separating sequence of seminorms[6] given by

$$p_n(f) = sup\left\{|f(x)| : x \in K_n\right\},$$

we have that $p_1 \leq p_2 \leq ... \leq p_n \leq ...$ and the sets $V_n = \left\{f \in C(\Omega) : p_n(f) < \frac{1}{n}\right\}$, with $n \in \mathbb{N}$, form a local base for $C(\Omega)$ consisting of balanced convex neighborhoods. Since the metric induced by the sequence $(p_n)_{n\in\mathbb{N}}$ is complete (cf. [30]), it follows that $C(\Omega)$ is a Fréchet space.

Example 2.29. If $\Omega \subset \mathbb{C}$, it is possible to prove that

$$H(\Omega) := \{f : \Omega \to \mathbb{C} : f \text{ is holomorphic}\}$$

is a closed subspace of $C(\Omega)$ and, therefore, it is complete with respect to the metric induced by $C(\Omega)$. Thus, $H(\Omega)$ is also a Fréchet space. In particular, if $\Omega = \mathbb{C}$, then

$$H(\mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C} : f \text{ is entire}\}$$

is a Fréchet space.

Remark 2.30. The spaces $C(\Omega)$ and $H(\Omega)$ are not normable (cf. [30], p.34). Hence, they are not Banach spaces.

[6] The topology induced by the family of seminorms defined in example (2.28) is called the *topology of uniform convergence on the compact subsets of* Ω.

2.3 Li-Yorke Chaos

In this section, we will study the concept of Li-Yorke chaos for bounded linear operators on Fréchet (and Banach) spaces. Throughout this section, X denotes an arbitrary infinite-dimensional Fréchet space, endowed with the metric in (2.1), and $T : X \to X$ will be a bounded linear operator, unless otherwise specified. All definitions and results regarding Li-Yorke chaos for linear operators on Fréchet spaces stated in this section can be found in [6].

Definition 2.31. *A pair $(x, y) \in X \times X$ is called a* Li-Yorke pair *for T if*

$$\liminf_{n\to\infty} d(T^n x, T^n y) = 0 \quad \textit{and} \quad \limsup_{n\to\infty} d(T^n x, T^n y) > 0.$$

Definition 2.32. *A subset $Y \subset X$ is said to be a* scrambled set *for T if $(x, y) \in Y \times Y$ is a Li-Yorke pair for T, whenever $x \neq y$.*

Definition 2.33. *We say that T is* Li-Yorke chaotic *if there exists an uncountable scrambled set for T.*

Definition 2.34. *We say that T is* densely *(respectively* generically*) Li-Yorke chaotic if there exists an uncountable dense (respectively residual) scrambled set for T.*

Definition 2.35. *A vector $x \in X$ is said to be an* irregular vector *for T if the sequence $(T^n x)_{n\in\mathbb{N}}$ is unbounded and it has a subsequence converging to zero.*

The following notion of irregularity is somewhat weaker but will be very useful throughout this work.

Definition 2.36. *A vector $x \in X$ is said to be a* semi-irregular vector *for T if the sequence $(T^n x)_{n\in\mathbb{N}}$ does not converges to zero, but it has a subsequence converging to zero.*

We observe that Definitions 2.31, 2.32, 2.33, 2.34, 2.35 and 2.36 hold for a Banach space $(X, \|\cdot\|)$ instead of a Fréchet space (X, d). In particular, for a Banach space $(X, \|\cdot\|)$, we say that $x \in X$ is an *irregular vector for T* if

$$\liminf_{n\to\infty} \|T^n x\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \|T^n x\| = \infty,$$

and we say that $x \in X$ is a *semi-irregular vector for T* if

$$\liminf_{n\to\infty} \|T^n x\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \|T^n x\| > 0.$$

It is worth noting that the notions of irregular and semi-irregular vectors apply only to infinite-dimensional spaces, since an application of the Jordan form implies that there are no semi-irregular vectors for operators on finite-dimensional spaces.

The next theorem, whose proof can be found in [6], provides an equivalence between the existence of an irregular (semi-irregular) vector and Li-Yorke chaos for linear operators on an arbitrary infinite-dimensional Fréchet spaces. In all subsequent results, $L(X)$ denotes the set of all bounded linear operators $T : X \to X$.

Theorem 2.37. *Let X be an infinite-dimensional Fréchet space. If $T \in L(X)$, then the following assertions are equivalent:*

(i) T is Li-Yorke chaotic.

(ii) T admits a Li-Yorke pair.

(iii) T admits a semi-irregular vector.

(iv) T admits an irregular vector.

Next, we will state some results from [6] that will be equally useful later on.

Proposition 2.38. *Let $T \in L(X)$. The set of all vectors $x \in X$ such that $(T^n x)_{n\in\mathbb{N}}$ has a subsequence converging to zero is a G_δ-set in X.*

Corollary 2.39. *Let $T \in L(X)$. If the set of all points $x \in X$ such that $(T^n x)_{n\in\mathbb{N}}$ has a subsequence converging to zero is dense in X, then it is residual in X.*

Proposition 2.40. *Let $T \in L(X)$. If T has a vector with unbounded orbit, then T has a residual set of vectors with unbounded orbits.*

Recall that the *orbit of a vector $x \in X$* under $T \in L(X)$ is given by the set $\{x, Tx, T^2x, T^3x, ...\}$.

Theorem 2.41. *If $T \in L(X)$, then every neighborhood of a semi-irregular vector for T contains an irregular vector for T.*

Theorem 2.42. *Assume X separable. If $T \in L(X)$, then the following assertions are equivalent:*

(i) T is densely Li-Yorke chaotic;

(ii) T admits a dense set of semi-irregular vectors;

(iii) T admits a dense set of irregular vectors;

(iv) T admits a residual set of irregular vectors.

2.3.1 Li-Yorke chaos criterion

In order to introduce another useful characterization for Li-Yorke chaos in the context of linear operators on Fréchet spaces, we present here the Li-Yorke Chaos Criterion (LYCC), defined in [6], which provides a necessary and sufficient condition for a bounded linear operator to be Li-Yorke chaotic.

Definition 2.43 (LYCC). *Let $T \in L(X)$. We say that T satisfies the* Li-Yorke chaos criterion *(LYCC) if there exists a subset X_0 of X with the following properties:*

(a) $(T^n x)_{n\in\mathbb{N}}$ has a subsequence converging to zero, for every $x \in X_0$;

(b) There is a bounded sequence $(a_n)_{n\in\mathbb{N}}$ in $\overline{span(X_0)}$ such that the sequence $(T^n a_n)_{n\in\mathbb{N}}$ is unbounded.

Theorem 2.44. *An operator $T \in L(X)$ is Li-Yorke chaotic if and only if it satisfies the Li-Yorke chaos criterion.*

Definition 2.45. *Let $T \in L(X)$. We say that T satisfies the* dense Li-Yorke chaos criterion *if there exists a dense subset X_0 of X with properties (a) and (b) of Definition 2.43.*

Theorem 2.46. *Assume X separable. An operator $T \in L(X)$ is densely Li-Yorke chaotic if and only if it satisfies the dense Li-Yorke chaos criterion.*

Theorem 2.47. *If $T \in L(X)$, then the following assertions are equivalent:*

(i) *T is generically Li-Yorke chaotic;*

(ii) *Every non-zero vector is semi-irregular for T;*

(iii) *X is a scrambled set for T.*

Chapter 3

Li-Yorke chaotic weighted composition operators on $C_0(X)$ spaces

3.1 Weighted composition operators on $C_0(X)$ spaces

Throughout this chapter, X denotes an arbitrary locally compact Hausdorff space, unless otherwise specified.

Recall that a map $\varphi : X \to \mathbb{K}$ *vanishes at infinity* if, for every $\varepsilon > 0$, there is a compact subset K of X such that

$$|\varphi(x)| < \varepsilon, \quad \text{for all} \ \ x \in X \setminus K.$$

Let $C_0(X)$ denote the Banach space over $\mathbb{K}$ of all continuous maps $\varphi : X \to \mathbb{K}$ that vanish at infinity, endowed with the supremum norm

$$\|\varphi\| := \sup_{x \in X} |\varphi(x)| \, .$$

More generally, for any subset B of X, we define

$$\|\varphi\|_B := \sup_{x \in B} |\varphi(x)| \, ,$$

where we consider this supremum to be 0 whenever $B = \varnothing$.

We denote by $C_c(X)$ the normed subspace of $C_0(X)$ formed by the continuous maps $\varphi : X \to \mathbb{K}$ which have compact support.

Throughout this chapter we will fix a *weight function* $w : X \to \mathbb{K}$, that is, a continuous map such that

$$\varphi \cdot w \in C_0(X) \quad \text{for all } \varphi \in C_0(X). \tag{3.1}$$

The next result shows that the weight functions are exactly the continuous maps from X into $\mathbb{K}$ which are bounded on X.

Proposition 3.1. *If $u : X \to \mathbb{K}$ is a continuous map, then the following statements are equivalent:*

(i) *u is bounded on X;*

(ii) *$\varphi \cdot u \in C_0(X)$, for all $\varphi \in C_0(X)$.*

In order to prove Proposition 3.1, we will need two lemmas. The first is Lemma 2.14 and the second is a basic fact about General Topology, presented below.

Lemma 3.2. *Let Y and Z be topological spaces. Assume that $(Y_i)_{i\in I}$ is a locally finite closed cover of Y. For each $i \in I$, let $f_i : Y_i \to Z$ be a continuous map. If*

$$f_i = f_j \quad \text{on} \quad Y_i \cap Y_j, \quad \text{for all} \quad i, j \in I, \tag{3.2}$$

then the map $f : Y \to Z$ given by

$$f(x) := f_i(x) \quad (x \in Y_i, i \in I) \tag{3.3}$$

is well-defined and continuous.

Proof. Condition (3.2) ensures that the function f, given by (3.3), is well-defined. In order to show that f is continuous, we take a closed subset F of Z and prove that $f^{-1}(F)$ is a closed subset of Y. For each $i \in I$, the set $f_i^{-1}(F)$ is closed in Y_i (because f_i is continuous), whence it is closed in Y (because Y_i is closed in Y). Since the family $(Y_i)_{i\in I}$ is locally finite and $f_i^{-1}(F) \subset Y_i$, for all $i \in I$, the family $(f_i^{-1}(F))_{i\in I}$ is also locally finite. Therefore, by Proposition 2.3, the set $\bigcup_{i\in I} f_i^{-1}(F)$ is closed in Y. Since

$$f^{-1}(F) = \bigcup_{i\in I}(f^{-1}(F) \cap Y_i) = \bigcup_{i\in I} f_i^{-1}(F),$$

we have that $f^{-1}(F)$ is closed in Y, as we wanted to prove. $\square$

Now, let us see the proof of Proposition 3.1.

Proof. $(i) \Rightarrow (ii)$: Obvious.
$(ii) \Rightarrow (i)$: Suppose that u is not bounded on X. Then, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$u(x_1) \neq 0 \quad \text{and} \quad |u(x_n)| + 4 < |u(x_{n+1})|, \forall n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, let $F_n := \{x \in X : |u(x) - u(x_n)| \leq 1\}$, which is a closed set in X, with $x_n \in \operatorname{Int} F_n$. By Lemma 2.14, there exists $\varphi_n \in C_c(X)$ such that

$$\varphi_n(X) \subset \left[0, |u(x_n)|^{-1}\right], \quad \varphi_n(x_n) = |u(x_n)|^{-1} \quad \text{and} \quad \operatorname{supp} \varphi_n \subset \operatorname{Int} F_n.$$

Let's prove that the family $(F_n)_{n\in\mathbb{N}}$ is discrete. Given $x \in X$, choose a neighborhood V of x such that $|u(y) - u(x)| < 1$, for all $y \in V$. We claim that V intersects at most one of the sets F_n. Indeed, if $n < m$, $y \in V \cap F_n$ and $z \in V \cap F_m$, then

$$|u(x_m)| \leq |u(z)| + 1 < |u(x)| + 2 < |u(y)| + 3 \leq |u(x_n)| + 4,$$

which is a contradiction. Let $F := X \setminus \bigcup_{n=1}^{\infty} \operatorname{Int} F_n$. The family $(F_n)_{n\in\mathbb{N}}$, together with F, forms a locally finite closed cover of X. Define $\varphi : X \to \mathbb{R}$ by

$$\varphi := 0 \text{ in } F \quad \text{and} \quad \varphi := \varphi_n \text{ in } F_n, \ \forall n \in \mathbb{N}.$$

By Lemma 3.2, φ is continuous. If $\varepsilon > 0$ and we choose $k \in \mathbb{N}$, with $|u(x_k)|^{-1} < \varepsilon$, then $K := \operatorname{supp} \varphi_1 \cup \cdots \cup \operatorname{supp} \varphi_k$ is a compact set and $|\varphi(x)| < \varepsilon$, for all $x \in X \setminus K$, proving that $\varphi \in C_0(X)$. However, since the sequence $(x_n)_{n\in\mathbb{N}}$ can not be contained in a compact subset of X and $|\varphi(x_n) \cdot u(x_n)| = 1$, for all $n \in \mathbb{N}$, we conclude that $\varphi \cdot u \notin C_0(X)$. □

Proposition 3.3. *Given a continuous map $f : X \to X$, the weighted composition operator*

$$C_{w,f}(\varphi) := (\varphi \circ f) \cdot w$$

is well-defined and is a bounded linear operator on $C_0(X)$ if and only if

$$f^{-1}(K) \cap X_\varepsilon \text{ is compact, for all } \varepsilon > 0 \text{ and } K \subset X \text{ compact}, \tag{3.4}$$

where $X_\varepsilon := \{x \in X : |w(x)| \geq \varepsilon\}$.

Proof. Suppose that there exist $\varepsilon > 0$ and a compact set $K \subset X$ such that

$$f^{-1}(K) \cap X_\varepsilon \text{ is not compact.}$$

Let U be an open subset of X, with $K \subset U$ and $\overline{U}$ a compact set. By Lemma 2.14, there is a continuous map $\varphi : X \to [0, 1]$ such that

$$\varphi(x) = 1, \quad \forall x \in K \quad \text{and} \quad \varphi(x) = 0, \quad \forall x \in X \setminus U.$$

Thus, for all $\delta > 0$, $\overline{U}$ satisfies

$$|\varphi(x)| < \delta, \quad \forall x \in X \setminus \overline{U}, \quad \text{since} \quad U \subset \overline{U}.$$

Therefore, $\varphi \in C_0(X)$. However, given any compact $H \subset X$, since

$$f^{-1}(K) \cap X_\varepsilon \not\subset H,$$

there is $x \in X \setminus H$ such that $x \in X_\varepsilon$ and $f(x) \in K$, whence $|\varphi(f(x))w(x)| = |w(x)| \geq \varepsilon$. Therefore, $(\varphi \circ f) \cdot w \notin C_0(X)$, demonstrating that $C_{w,f}$ is not well-defined.

Conversely, if $w = 0$ on X, the condition (3.4) is trivially satisfied and $C_{w,f}$ is the null operator.

Thus, we can assume that w is not identically zero. Let $\varphi \in C_0(X) \setminus \{0\}$. Given $\varepsilon > 0$, let $\varepsilon' := \dfrac{\varepsilon}{\|w\|_X}$ and $\varepsilon'' := \dfrac{\varepsilon}{\|\varphi\|}$. Since φ vanishes at infinity, there is $K \subset X$ compact such that

$$|\varphi(x)| < \varepsilon', \ \forall x \in X \setminus K. \tag{3.5}$$

By (3.4), $H := f^{-1}(K) \cap X_{\varepsilon''}$ is compact, where $X_{\varepsilon''} := \{x \in X : |w(x)| \geq \varepsilon''\}$. Furthermore,

$$|\varphi(f(x))w(x)| < \varepsilon, \quad \text{for all} \quad x \in X \setminus H. \tag{3.6}$$

Indeed, if $x \in X \setminus H$, then $x \notin f^{-1}(K) \cap X_{\varepsilon''}$, i.e. $x \notin f^{-1}(K)$ or $x \notin X_{\varepsilon''}$.

- If $x \notin f^{-1}(K)$, then $f(x) \notin K$. By (3.5), we have that $|\varphi(f(x))| < \varepsilon' = \dfrac{\varepsilon}{\|w\|_X}$, which implies $|\varphi(f(x))w(x)| < \varepsilon$.
- If $x \notin X_{\varepsilon''}$, then $|w(x)| < \varepsilon'' = \dfrac{\varepsilon}{\|\varphi\|}$, whence $|\varphi(f(x))|\,|w(x)| < \dfrac{\varepsilon}{\|\varphi\|} \cdot \|\varphi\|$ and so, $|\varphi(f(x))w(x)| < \varepsilon$.

Hence, since (3.6) is satisfied, it follows that $(\varphi \circ f) \cdot w \in C_0(X)$. Therefore, $C_{w,f}$ is a well-defined function from $C_0(X)$ to itself. The linearity and boundedness of $C_{w,f}$ are evident. $\square$

Corollary 3.4. *If $w \in C_0(X)$, then the weighted composition operator $C_{w,f}$ is well-defined and is a bounded linear operator on $C_0(X)$, for all $f : X \to X$ continuous.*

Proof. Indeed, assume $w \in C_0(X)$ and take any continuous map $f : X \to X$. In view of Proposition 3.3, given $\varepsilon > 0$ and $K \subset X$ compact, it is enough to show that

$$f^{-1}(K) \cap X_\varepsilon \text{ is compact,}$$

where $X_\varepsilon := \{x \in X : |w(x)| \geq \varepsilon\}$. But this follows from the facts that $f^{-1}(K)$ is closed (because f is continuous) and X_ε is compact (because $w \in C_0(X)$). In fact, given $\varepsilon > 0$, there is $H \subset X$ compact such that

$$|w(x)| < \varepsilon, \ \forall x \in X \setminus H.$$

Hence, X_ε is a closed subset of H, which is compact. □

Corollary 3.5. *Given a continuous map $f : X \to X$, the (unweighted) composition operator*

$$C_f(\varphi) := \varphi \circ f$$

is well-defined and is a bounded linear operator on $C_0(X)$ if and only if

$$f^{-1}(K) \text{ is compact for every } K \subset X \text{ compact.}$$

Proof. Note that $C_f = C_{w,f}$, where w is the constant function 1 on X. In this case,

$$X_\varepsilon := \{x \in X : |w(x)| \geq \varepsilon\} = X, \ \forall \varepsilon \in (0, 1].$$

Hence, the result follows from Proposition 3.3. □

3.2 Li-Yorke chaos for weighted composition operators on $C_0(X)$ spaces

For the remainder of this chapter, we fix a continuous map $f : X \to X$ satisfying condition (3.4). Hence, the weighted composition operator

$$C_{w,f} : \varphi \in C_0(X) \to (\varphi \circ f) \cdot w \in C_0(X)$$

is a well-defined bounded linear operator. We associate to w and f the following sequence of continuous maps from X into $\mathbb{K}$:

$$w^{(1)} := w \quad \text{and} \quad w^{(n)} := \left(w \circ f^{n-1}\right) \cdot \ldots \cdot (w \circ f) \cdot w, \ \text{ for } n \geq 2.$$

If w is the constant function 1, then so is $w^{(n)}$ for all $n \geq 1$.

We now establish the main result of this chapter, a characterization of the weighted composition operators on $C_0(X)$ that are Li-Yorke chaotic.

Theorem 3.6. *The weighted composition operator $C_{w,f}$ on $C_0(X)$ is Li-Yorke chaotic if and only if there is a sequence $(B_i)_{i\in\mathbb{N}}$ of relatively compact open subsets of X such that the following conditions hold:*

(i) There is an increasing sequence $(n_j)_{j\in\mathbb{N}}$ of positive integers such that

$$\lim_{j\to\infty} \left\|w^{(n_j)}\right\|_{f^{-n_j}(B_i)} = 0, \forall i \in \mathbb{N}.$$

(ii) $\sup\left\{\left\|w^{(n)}\right\|_{f^{-n}(B_i)} : i, n \in \mathbb{N}\right\} = \infty.$

Proof. Suppose that the operator $C_{w,f}$ is Li-Yorke chaotic on $C_0(X)$. Let $\varphi \in C_0(X)$ be an irregular vector for $C_{w,f}$, that is,

$$\liminf_{n\to\infty} \|(C_{w,f})^n(\varphi)\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \|(C_{w,f})^n(\varphi)\| = \infty.$$

For each $i \in \mathbb{N}$, let

$$B_i := \left\{x \in X : |\varphi(x)| > \frac{1}{i}\right\},$$

which is a relatively compact open subset of X. First, let's check condition *(i)*. Since φ is an irregular vector for $C_{w,f}$, there is an increasing sequence $(n_j)_{j\in\mathbb{N}}$ of positive integers such that $\lim_{j\to\infty} \|(C_{w,f})^{n_j}(\varphi)\| = 0$, that is,

$$0 = \lim_{j\to\infty} \|(C_{w,f})^{n_j}(\varphi)\| = \lim_{j\to\infty} \left\|(\varphi \circ f^{n_j})\, w^{(n_j)}\right\| = \lim_{j\to\infty} \sup_{x\in X} \left|\varphi\left(f^{n_j}(x)\right) w^{(n_j)}(x)\right|.$$

Note that, for each $i \in \mathbb{N}$, we have that

$$|\varphi\left(f^{n_j}(x)\right)| > \frac{1}{i}, \quad \forall x \in f^{-n_j}(B_i). \tag{3.7}$$

Therefore,

$$0 = \lim_{j\to\infty}\sup_{x\in X}\left|\varphi\left(f^{n_j}(x)\right)w^{(n_j)}(x)\right| \geq \limsup_{j\to\infty}\sup_{x\in f^{-n_j}(B_i)}\left|\varphi\left(f^{n_j}(x)\right)w^{(n_j)}(x)\right|$$

$$\geq \frac{1}{i}\cdot\limsup_{j\to\infty}\sup_{x\in f^{-n_j}(B_i)}\left|w^{(n_j)}(x)\right| \qquad \text{(because (3.7))}$$

$$= \frac{1}{i}\cdot\limsup_{j\to\infty}\left\|w^{(n_j)}\right\|_{f^{-n_j}(B_i)},$$

which implies condition *(i)*.

Let us now assume that condition *(ii)* is false. So there is a constant $\beta < \infty$ such that

$$\left\|w^{(n)}\right\|_{f^{-n}(B_i)} \leq \beta, \ \forall i, n \in \mathbb{N},$$

that is,

$$\sup_{x\in f^{-n}(B_i)}\left|w^{(n)}(x)\right| \leq \beta, \ \forall i, n \in \mathbb{N}. \tag{3.8}$$

For each $n \in \mathbb{N}$, let $I_n := \bigcup_{i\in\mathbb{N}} f^{-n}(B_i)$. We obtain from (3.8) that

$$\sup_{x\in I_n}\left|w^{(n)}(x)\right| \leq \beta, \ \forall n \in \mathbb{N}.$$

Since φ is an irregular vector for $C_{w,f}$, we have:

$$\begin{aligned}
\infty &= \limsup_{n\to\infty}\|(C_{w,f})^n(\varphi)\| = \limsup_{n\to\infty}\left\|(\varphi\circ f^n)w^{(n)}\right\| \\
(*) &= \limsup_{n\to\infty}\left\|(\varphi\circ f^n)w^{(n)}\right\|_{I_n} \\
&= \limsup_{n\to\infty}\sup_{x\in I_n}\left|(\varphi\left(f^n(x)\right))w^{(n)}(x)\right| \\
&\leq \limsup_{n\to\infty}\left(\sup_{x\in I_n}\left|(\varphi\left(f^n(x)\right))\right|\cdot\sup_{x\in I_n}\left|w^{(n)}(x)\right|\right) \\
&\leq \limsup_{n\to\infty}\|\varphi\circ f^n\|_{I_n}\cdot\beta \\
&\leq \|\varphi\|\cdot\beta,
\end{aligned}$$

which is a contradiction. Therefore, condition *(ii)* holds.

Remark 3.7. In $(*)$, we use the fact that $(\varphi \circ f^n)$ is zero outside of I_n, $\forall n \in \mathbb{N}$.

Conversely, suppose that there is a sequence $(B_i)_{i\in\mathbb{N}}$ with all the properties described in the statement of the theorem. For each $i \in \mathbb{N}$ and each $n \in \mathbb{N}$ such that $f^{-n}(B_i)$ is not empty, choose $a_{i,n} \in f^{-n}(B_i)$ such that

$$\left|w^{(n)}(a_{i,n})\right| \geq \left\|w^{(n)}\right\|_{f^{-n}(B_i)} - 1. \tag{3.9}$$

By Lemma 2.14, there exists $\varphi_{i,n} \in C_c(X)$ such that:

- $0 \leq \varphi_{i,n}(x) \leq 1$, $\forall x \in X$;
- $\varphi_{i,n}(f^n(a_{i,n})) = 1$;
- $\operatorname{supp} \varphi_{i,n} \subset B_i$.

Note that the family of functions $\{\varphi_{i,n}\}$ defined above lies in $C_0(X)$. Indeed, since $\operatorname{supp} \varphi_{i,n}$ is compact, for any $\varepsilon > 0$, we have $|\varphi_{i,n}(x)| < \varepsilon$ for all $x \in X \setminus (\operatorname{supp} \varphi_{i,n})$, which implies $\varphi_{i,n} \in C_0(X)$. Moreover,

$$\|\varphi_{i,n}\| = \sup_{x\in X} |\varphi_{i,n}(x)| = 1.$$

Let Z be the linear span of the $\varphi_{i,n}$'s and $Y := \overline{Z}$. Consider the set

$$R_1 := \{\varphi \in Y; \text{the } C_{w,f}\text{-orbit of } \varphi \text{ has a subsequence converging to zero}\}.$$

We claim that $Z \subset R_1$. In fact,

$$\begin{aligned}
\lim_{j\to\infty} \|(C_{w,f})^{n_j}(\varphi_{i,n})\| &= \lim_{j\to\infty} \sup_{x\in X} \left|\varphi_{i,n}(f^{n_j}(x))w^{(n_j)}(x)\right| \\
&= \lim_{j\to\infty} \sup_{x\in f^{-n_j}(B_i)} \left|\varphi_{i,n}(f^{n_j}(x))w^{(n_j)}(x)\right| \quad (\varphi_{i,n}(x) = 0, \forall x \in X \setminus B_i) \\
&\leq \lim_{j\to\infty} \left(\sup_{x\in f^{-n_j}(B_i)} |\varphi_{i,n}(f^{n_j}(x))| \cdot \sup_{x\in f^{-n_j}(B_i)} \left|w^{(n_j)}(x)\right| \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \underbrace{\|\varphi_{i,n}\|}_{=1} \cdot \lim_{j\to\infty} \sup_{x\in f^{-n_j}(B_i)} \left|w^{(n_j)}(x)\right| \\
&= \lim_{j\to\infty} \left\|w^{(n_j)}\right\|_{f^{-n_j}(B_i)} \\
&= 0 \quad \text{(by hypothesis } (i)\text{)}.
\end{aligned}
$$

Therefore, by linearity, given $\varphi \in Z$, we have $(C_{w,f})^{n_j}(\varphi) \to 0$ as $j \to +\infty$, since each $\varphi \in Z$ is a finite linear combination of the $\varphi_{i,n}$'s. Thus, $Z \subset R_1 \subset Y = \overline{Z}$, whence $\overline{Z} \subset \overline{R_1} \subset Y = \overline{Z}$, which implies that R_1 is dense in Y, and so it is residual in Y, by Proposition 2.38.

Now, observe that

$$
\begin{aligned}
\|(C_{w,f})^n(\varphi_{i,n})\| &= \sup_{x\in X}\left|\varphi_{i,n}(f^n(x))w^{(n)}(x)\right| \geq \left|\underbrace{\varphi_{i,n}(f^n(a_{i,n}))}_{=1} w^{(n)}(a_{i,n})\right| \\
&= \left|w^{(n)}(a_{i,n})\right| \\
&\geq \left\|w^{(n)}\right\|_{f^{-n}(B_i)} - 1 \text{ (by (3.9))}.
\end{aligned}
$$

Then,

$$
\sup_{i,n} \|(C_{w,f})^n(\varphi_{i,n})\| \geq \sup_{i,n} \left\|w^{(n)}\right\|_{f^{-n}(B_i)} - 1 = \infty, \text{ by hypothesis } (ii).
$$

Hence,

$$
\sup_n \|(C_{w,f})^n\,|_Y\| = \infty. \tag{3.10}
$$

Let $R_2 := \{\varphi \in Y : \text{the } C_{w,f}\text{-orbit of } \varphi \text{ is unbouded}\}$. The equation in (3.10) means that the sequence of continuous linear operators $((C_{w,f})^n\,|_Y)_{n\in\mathbb{N}}$ is not equicontinuous, by Remark 2.20. So, by Theorem 2.22, $Y \setminus R_2$ is of the first category in Y, whence the set R_2 is residual in Y. Thus, $R_1 \cap R_2$ is also residual in Y. Since each $\varphi \in R_1 \cap R_2$ is an irregular vector for $C_{w,f}$, it follows that $C_{w,f}$ is Li-Yorke chaotic, by Theorem 2.37. □

Corollary 3.8. *A composition operator C_f on $C_0(X)$ is never Li-Yorke chaotic.*

Proof. Indeed, $C_f = C_{w,f}$, where w is the constant function 1. Hence, it is clear that condition *(ii)* in Theorem 3.6 fails. $\square$

Remark 3.9. Note that the sequence $(B_i)_{i\in\mathbb{N}}$ constructed in the proof of Theorem 3.6 has the following additional property:

(iiii) $\overline{B_i} \subset B_{i+1}, \forall i \in \mathbb{N}$.

If we include this additional condition in the statement of Theorem 3.6, then we can replace condition (i) by the following weaker condition:

(i') $\liminf_{n\to\infty} \left\|w^{(n)}\right\|_{f^{-n}(B_i)} = 0, \forall i \in \mathbb{N}$.

In other words: The weighted composition operator $C_{w,f}$ on $C_0(X)$ is Li-Yorke chaotic if and only if there is an sequence $(B_i)_{i\in\mathbb{N}}$ of relatively compact open subsets of X satisfying conditions (i'), (iii) and (iiii).

Indeed, since the necessity is clear, let us prove the sufficiency. If

$$\limsup_{n\to\infty} \left\|w^{(n)}\right\|_{f^{-n}(B_j)} > 0, \text{ for a certain } j \in \mathbb{N},$$

take a continuous map $\varphi : X \to [0,1]$ with supp $\varphi \subset B_{j+1}$ and $\varphi = 1$ on $\overline{B_j}$. Since

$$\begin{aligned} \left\|w^{(n)}\right\|_{f^{-n}(B_j)} &= \left\|(\varphi\circ f^n) w^{(n)}\right\|_{f^{-n}(B_j)} \qquad \left(\varphi = 1 \text{ in } \overline{B_j}\right) \\ &= \left\|(C_{w,f})^n(\varphi)\right\|_{f^{-n}(B_j)} \\ &\leq \left\|(C_{w,f})^n(\varphi)\right\|, \end{aligned}$$

we obtain

$$0 < \limsup_{n\to\infty} \left\|w^{(n)}\right\|_{f^{-n}(B_j)} \leq \limsup_{n\to\infty} \left\|(C_{w,f})^n(\varphi)\right\|. \tag{3.11}$$

On the other hand,

$$\begin{aligned} \|(C_{w,f})^n(\varphi)\| &= \left\|(\varphi \circ f^n)w^{(n)}\right\| \\ &= \left\|(\varphi \circ f^n)w^{(n)}\right\|_{f^{-n}(B_{j+1})} \qquad (\mathrm{supp}\varphi \subset B_{j+1}) \\ &\le \left\|w^{(n)}\right\|_{f^{-n}(B_{j+1})} \qquad (\varphi(X) \subset [0,1]) \end{aligned}$$

Hence,

$$\liminf_{n\to\infty} \|(C_{w,f})^n(\varphi)\| \le \liminf_{n\to\infty} \left\|w^{(n)}\right\|_{f^{-n}(B_{j+1})} = 0 \quad \text{by } \textit{(i')}. \tag{3.12}$$

Thus, by (3.11) and (3.12), we have that

$$\limsup_{n\to\infty} \|(C_{w,f})^n(\varphi)\| > 0 \quad \text{and} \quad \liminf_{n\to\infty} \|(C_{w,f})^n(\varphi)\| = 0.$$

Hence, φ is a semi-irregular vector for $C_{w,f}$, and so $C_{w,f}$ is Li-Yorke chaotic.

Now, assume

$$\limsup_{n\to\infty} \left\|w^{(n)}\right\|_{f^{-n}(B_i)} = 0, \forall i \in \mathbb{N}.$$

Therefore,

$$\lim_{n\to\infty} \left\|w^{(n)}\right\|_{f^{-n}(B_i)} = 0, \forall i \in \mathbb{N}.$$

In this case, the condition (i) holds with $(n_j)_{j\in\mathbb{N}} := (n)_{n\in\mathbb{N}}$. Since we are assuming that condition (iii) holds, $C_{w,f}$ is Li-Yorke chaotic by Theorem 3.6.

3.3 Li-Yorke chaos for weighted shifts on c_0-spaces

As applications of Theorem 3.6, we will present in this section necessary and sufficient conditions for unilateral (case 1) and bilateral (case 2) weighted backward shifts to be Li-Yorke chaotic on $c_0(\mathbb{N})$ and on $c_0(\mathbb{Z})$, respectively.

3.3.1 Unilateral weighted backward shifts on $c_0(\mathbb{N})$

In this subsection, let $\mathbb{N}$ be endowed with the discrete topology and consider the Banach space of sequences

$$c_0(\mathbb{N}) := \left\{x = (x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}};\ \lim_{n\to\infty} x_n = 0\right\}, \text{ with } \|x\| := \sup_n |x_n|\,.$$

Note that $c_0(\mathbb{N}) = C_0(\mathbb{N})$. Indeed, for all $x \in c_0(\mathbb{N})$, we have:

$$x : n \in \mathbb{N} \mapsto x(n) = x_n \in \mathbb{K},$$

where $(x(n))_{n\in\mathbb{N}} = (x_n)_{n\in\mathbb{N}}$ is a notation for the function $x : \mathbb{N} \to \mathbb{K}$, which is always continuous. Furthermore, the norm in $c_0(\mathbb{N})$ is exactly the norm in $C_0(\mathbb{N})$, since

$$\|x\| = \sup_n |x_n| = \sup_{n\in\mathbb{N}} |x(n)|\,.$$

Also, observe that a map $x : \mathbb{N} \to \mathbb{K}$ vanishes at infinity if and only if $\lim_{n\to\infty} x_n = 0$. Indeed,

$$\begin{aligned}
\lim_{n\to\infty} x_n = 0 \quad &\Leftrightarrow \quad \forall\varepsilon > 0, \exists n_0 \in \mathbb{N}; |x_n| < \varepsilon, \forall n > n_0 \\
&\Leftrightarrow \quad \forall\varepsilon > 0, \exists K = \{1, 2, \cdots n_0\} \subset \mathbb{N}; |x(n)| < \varepsilon, \forall n \in \mathbb{N} \setminus K \\
&\Leftrightarrow \quad x : \mathbb{N} \to \mathbb{K} \text{ vanishes at infinity,}
\end{aligned}$$

because the compact subsets of $\mathbb{N}$ in the discrete topology are exactly the finite subsets.

Now, given $f : i \in \mathbb{N} \mapsto i + 1 \in \mathbb{N}$ and $w = (w_n)_{n\in\mathbb{N}}$ a bounded sequence of scalars in $\mathbb{K}$, consider the following weighted composition operator on $C_0(\mathbb{N})$:

$$C_{w,f} : x \in C_0(\mathbb{N}) \mapsto (x \circ f)w \in C_0(\mathbb{N}). \tag{3.13}$$

Note that $C_{w,f}$ is exactly the unilateral weighted backward shift on $c_0(\mathbb{N})$ given by:

$$B_w : x = (x_1, \cdots, x_i, \cdots) \in c_0(\mathbb{N}) \mapsto (x_2 w_1, x_3 w_2, \cdots, x_{i+1} w_i, \cdots) \in c_0(\mathbb{N}). \tag{3.14}$$

The next result is a characterization of Li-Yorke chaos for unilateral weighted backward shifts on $c_0(\mathbb{N})$. This result was established in ([3], Proposition 27) in the case w is a bounded sequence of nonzero scalars.

Proposition 3.10. *A unilateral weighted backward shift B_w on $c_0(\mathbb{N})$, where $w := (w_n)_{n\in\mathbb{N}}$ is a bounded sequence of scalars, is Li-Yorke chaotic if and only if*

$$\sup\{|w_i \cdots w_j|\,; i,j \in \mathbb{N}, i \leq j\} = \infty. \tag{3.15}$$

Proof. Suppose B_w is a Li-Yorke chaotic operator. If $\sup\{|w_i \cdots w_j|\,; i,j \in \mathbb{N}, i \leq j\} < \infty$, there is a constant $c > 0$ such that $\sup\{|w_i \cdots w_j|\,; i,j \in \mathbb{N}, i \leq j\} = c$.

Since $f : i \in \mathbb{N} \mapsto i+1 \in \mathbb{N}$, for each $n \in \mathbb{N}$, we have that

$$w^{(n)}(i) = \left(\left(w \circ f^{n-1}\right) \cdots (w \circ f) \cdot w\right)(i) = w_i \cdot w_{i+1} \cdots w_{i+n-1}, \forall i \in \mathbb{N},$$

which implies $\left|w^{(n)}(i)\right| \leq c$, $\forall n, i \in \mathbb{N}$ (because $\left|w^{(n)}(i)\right| \in \{|w_i \cdots w_j|\,; i,j \in \mathbb{N}, i \leq j\}$, $\forall n, i \in \mathbb{N}$). Hence,

$$\sup_{i\in\mathbb{N}} \left|w^{(n)}(i)\right| \leq c, \forall n \in \mathbb{N}.$$

In particular, for every subset M of $\mathbb{N}$:

$$\sup_{i\in M} \left|w^{(n)}(i)\right| \leq c,$$

so that

$$\left\|w^{(n)}\right\|_M \leq c, \;\; \forall M \subset \mathbb{N}. \tag{3.16}$$

By Theorem 3.6, there is a sequence $(B_i)_{i\in\mathbb{N}}$ of open and relatively compact subsets of $\mathbb{N}$, such that

$$\sup_{i,n} \left\|w^{(n)}\right\|_{f^{-n}(B_i)} = \infty. \tag{3.17}$$

By (3.16), for each $n \in \mathbb{N}$ and each $i \in \mathbb{N}$, we have

$$\left\|w^{(n)}\right\|_{f^{-n}(B_i)} \leq c,$$

since $f^{-n}(B_i) \subset \mathbb{N}$, for all $n, i \in \mathbb{N}$. As this contradicts (3.17), we conclude that

$$\sup\{|w_i \cdots w_j|\,; i,j \in \mathbb{N}, i \leq j\} = \infty.$$

Conversely, assume that $\sup\{|w_i \cdots w_j|\,; i,j \in \mathbb{N}, i \leq j\} = \infty$ and consider B_w written as in (3.13). For each $k \in \mathbb{N}$, let be $B_k := \{k\}$. Then, $(B_k)_{k\in\mathbb{N}}$ is a sequence of open and relatively compact subsets of $\mathbb{N}$. Since, for each $k \in \mathbb{N}$, $f^{-n}(B_k) = \varnothing$ for all $n > k$, we have that condition *(i)* of Theorem 3.6 is trivially satisfied in the present case.

Furthermore,

$$f^{-n}(B_k) = f^{-n}(\{k\}) = \{k-n\}, \forall k \geq n.$$

Therefore, for each fixed $n \in \mathbb{N}$ and each $k > n$, we obtain:

$$\left\|w^{(n)}\right\|_{f^{-n}(B_k)} = \left\|w^{(n)}\right\|_{\{k-n\}} = \left|w^{(n)}(k-n)\right| = |w_{k-n}w_{k-n+1}\cdots w_{k-1}|. \tag{3.18}$$

Writing $k := j+1$ and $n := j-i+1$, with $i, j \in \mathbb{N}$ and $i \leq j$ in (3.18), we have:

$$\left\|w^{(n)}\right\|_{f^{-n}(B_k)} = |w_i \cdots w_j|, \forall i, j \in \mathbb{N}, i \leq j.$$

By hypothesis, $\sup\{|w_i \cdots w_j|; i, j \in \mathbb{N}, i \leq j\} = \infty$, so the above equality gives us

$$\sup_{k,n} \left\|w^{(n)}\right\|_{f^{-n}(B_k)} = \infty,$$

showing that condition *(ii)* of Theorem 3.6 also holds. So, B_w is Li-Yorke chaotic. $\square$

3.3.2 Bilateral weighted backward shifts on $c_0(\mathbb{Z})$

In this subsection, we will present another application of Theorem 3.6. Next, we will provide necessary and sufficient conditions for a bilateral weighted backward shift on $c_0(\mathbb{Z})$ to be Li-Yorke chaotic.

Throughout this subsection, consider $\mathbb{Z}$ endowed with the discrete topology, and the following space of sequences:

$$c_0(\mathbb{Z}) := \left\{x = (x_n)_{n\in\mathbb{Z}} \in \mathbb{K}^{\mathbb{Z}}; \lim_{n\to\pm\infty} x_n = 0\right\}, \text{ with } \|x\| := \sup_{n\in\mathbb{Z}} |x_n|.$$

Similarly to what was seen in the previous section, we have $c_0(\mathbb{Z}) = C_0(\mathbb{Z})$. Moreover, for each $x \in c_0(\mathbb{Z})$, we have

$$x : n \in \mathbb{Z} \mapsto x(n) = x_n \in \mathbb{K},$$

where $(x(n))_{n\in\mathbb{Z}} = (x_n)_{n\in\mathbb{Z}}$ denotes the map $x : \mathbb{Z} \to \mathbb{K}$, which is always continuous.

Again, we fix $f : i \in \mathbb{Z} \mapsto i+1 \in \mathbb{Z}$ and $w := (w_n)_{n\in\mathbb{Z}}$ a bounded sequence of scalars in $\mathbb{K}$. We define the following weighted composition operator:

$$C_{w,f} : x \in C_0(\mathbb{Z}) \mapsto (x \circ f)w \in C_0(\mathbb{Z}). \tag{3.19}$$

The operator $C_{w,f}$ defined in (3.19) is exactly the bilateral weighted backward shift operator $\mathcal{B}_w : c_0(\mathbb{Z}) \mapsto c_0(\mathbb{Z})$, defined by

$$\mathcal{B}_w : (x_n)_{n\in\mathbb{Z}} \in c_0(\mathbb{Z}) \mapsto (w_n x_{n+1})_{n\in\mathbb{Z}} \in c_0(\mathbb{Z}). \tag{3.20}$$

The next result is a characterization of Li-Yorke chaos for bilateral weighted backward shifts on $c_0(\mathbb{Z})$. To the best of the author knowledge, a characterization of the Li-Yorke chaotic bilateral weighted shifts on $c_0(\mathbb{Z})$ has not been given before.

Proposition 3.11. *A bilateral weighted backward shift $\mathcal{B}_w$ on $c_0(\mathbb{Z})$, where $w := (w_n)_{n\in\mathbb{Z}}$ is a bounded sequence of nonzero scalars, is Li-Yorke chaotic if and only if the following conditions hold:*

(a) $\liminf_{n\to\infty} |w_{-n}\cdots w_{-1}| = 0$*;*

(b) $\sup\{|w_i\cdots w_j|\,; i,j\in\mathbb{Z}, i\le j\} = \infty$*.*

Proof. Consider $\mathcal{B}_w$ as in (3.19), that is, $\mathcal{B}_w = C_{w,f}$, where $w = (w_n)_{n\in\mathbb{Z}}$ satisfies the statements of this Proposition.

Recall that, for each $k\in\mathbb{Z}$ and $m\in\mathbb{N}$,

$$w^{(m)}(k) = \big((w\circ f^{m-1})\cdots(w\circ f)\cdot w\big)(k) = w_k\cdot w_{k+1}\cdots w_{k+m-1}, \tag{3.21}$$

so that

$$\left\|w^{(m)}\right\|_{f^{-m}(\{k\})} = \left|w^{(m)}(-m+k)\right| = |w_{-m+k}w_{-m+k+1}\cdots w_{-1+k}|\,, \forall m\in\mathbb{N}, \forall k\in\mathbb{Z}. \tag{3.22}$$

$(\Rightarrow)$ Suppose that $C_{w,f}$ is a Li-Yorke chaotic operator. Let $(m_j)_{j\in\mathbb{N}}$ and $(B_i)_{i\in\mathbb{N}}$ be as in the statement of Theorem 3.6. By choosing $l\in B_i$, for some $i\in\mathbb{N}$, we have that

$$\begin{aligned} \left|w_{-m_j+l}\cdots w_{-1+l}\right| &= \left\|w^{(m_j)}\right\|_{f^{-m_j}(\{l\})} \qquad \text{(by (3.22))} \\ &\le \left\|w^{(m_j)}\right\|_{f^{-m_j}(B_i)} \to 0 \quad \text{when } j\to\infty, \end{aligned} \tag{3.23}$$

by hypothesis *(i)* of Theorem 3.6. For each $j\in\mathbb{N}$, let $n_j := -(-m_j+l)$. By (3.23),

$$\lim_{j\to\infty}\left|w_{-n_j}\cdots w_{-1}\right| = \lim_{j\to\infty}\left|w_{-m_j+l}\cdots w_{-1+l}\right| = 0,$$

which gives *(a)*.

Now, suppose that *(b)* is false, that is, $\beta := \sup\{|w_i \cdots w_j|\,; i,j \in \mathbb{Z}, i \leq j\} < \infty$. For each $n \in \mathbb{N}$ and $i \in \mathbb{N}$, we have that:

$$\left\|w^{(n)}\right\|_{f^{-n}(B_i)} = \sup_{k \in B_i} |w_{-n+k} \cdots w_{-1+k}|\,. \tag{3.24}$$

For each $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, let $j := -1 + k$ and $i := -n + k$. Then,

$$|w_{-n+k} \cdots w_{-1+k}| \in \{|w_i \cdots w_j|\,; i,j \in \mathbb{Z}, i \leq j\}\,.$$

And so,

$$\sup_{k \in B_i} |w_{-n+k} \cdots w_{-1+k}| \leq \sup\{|w_i \cdots w_j|\,; i,j \in \mathbb{Z}, i \leq j\}\,.$$

By (3.24), we have that

$$\left\|w^{(n)}\right\|_{f^{-n}(B_i)} \leq \beta, \forall i, n \in \mathbb{N},$$

contradicting condition *(ii)* of Theorem 3.6.

($\Leftarrow$) Suppose $C_{w,f} = \mathcal{B}_w$ satisfies conditions *(a)* and *(b)* above. Thus, we have two cases to consider.

Case 1. $\limsup\limits_{n\to\infty} |w_{-n} \cdots w_{-1}| > 0$.

In this case, the characteristic function $\varphi = \chi_{\{0\}}$ is a semi-irregular vector for $C_{w,f}$. Indeed,

$$\begin{aligned}
\left\|(C_{w,f})^n\left(\chi_{\{0\}}\right)\right\| &= \left\|\left(\chi_{\{0\}} \circ f^n\right) \cdot w^{(n)}\right\| = \sup_{i \in \mathbb{Z}} \left|\chi_{\{0\}}\left(f^n(i)\right) \cdot w^{(n)}(i)\right| \\
&= \left|w^{(n)}(-n)\right| \\
&= |w_{-n} \cdots w_{-1}|\,.
\end{aligned}$$

Hence,

$$0 < \limsup_{n\to\infty} |w_{-n} \cdots w_{-1}| = \limsup_{n\to\infty} \left\|(C_{w,f})^n\left(\chi_{\{0\}}\right)\right\| \tag{3.25}$$

and, by condition *(a)*,

$$0 = \liminf_{n\to\infty} |w_{-n} \cdots w_{-1}| = \liminf_{n\to\infty} \left\|(C_{w,f})^n\left(\chi_{\{0\}}\right)\right\|\,. \tag{3.26}$$

Therefore, by (3.25) and (3.26), $\chi_{\{0\}}$ is a semi-irregular vector for $C_{w,f}$, whence $C_{w,f} = \mathcal{B}_w$ is a Li-Yorke chaotic operator by Theorem 2.37.

Case 2. $\limsup_{n\to\infty} |w_{-n}\cdots w_{-1}| = 0$.

In this case, in view of hypothesis *(a)*, we have that

$$\lim_{n\to\infty} |w_{-n}\cdots w_{-1}| = 0. \tag{3.27}$$

Define $(n_j)_{j\in\mathbb{N}} := (n)_{n\in\mathbb{N}}$ and $(B_i)_{i\in\mathbb{N}}$ an enumeration of the sets $\{k\}$, for $k \in \mathbb{Z}$. We claim that condition *(i)* of Theorem 3.6 holds. Indeed, for all $n \in \mathbb{N}$, (3.22) implies that

$$\left\|w^{(n)}\right\|_{f^{-n}(\{k\})} = \begin{cases} |w_{-n}\cdots w_{-1}|, & k = 0 \\ \\ |w_{-n+k}\cdots w_{-1}|\,|w_0\cdots w_{-1+k}|, & k > 0 \\ \\ \dfrac{|w_{-n+k}\cdots w_{-1+k}w_k\cdots w_{-1}|}{|w_k\cdots w_{-1}|}, & k < 0 \end{cases}.$$

Therefore, for every $k \in \mathbb{Z}$, we have that

$$\lim_{n\to\infty} \left\|w^{(n)}\right\|_{f^{-n}(\{k\})} = 0,$$

which implies

$$\liminf_{n\to\infty} \left\|w^{(n)}\right\|_{f^{-n}(B_i)} = 0, \forall i \in \mathbb{N}.$$

Moreover, by hypothesis *(b)*,

$$\sup\left\{|w_i\cdots w_j|\,; i, j \in \mathbb{Z}, i \le j\right\} = \infty. \tag{3.28}$$

For all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let $j := -1 + k$ and $i := -n + k$ in (3.28). It follows that

$$\sup\left\{|w_{-n+k}\cdots w_{-1+k}| : n \in \mathbb{N}, k \in \mathbb{Z}\right\} = \infty.$$

Since $(B_i)_{i\in\mathbb{N}}$ is an enumeration of the singleton sets of $\mathbb{Z}$, it follows that for each $i \in \mathbb{N}$, we have $B_i = \{k\}$ for some $k \in \mathbb{Z}$, and therefore

$$\left\|w^{(n)}\right\|_{f^{-n}(B_i)} = \left\|w^{(n)}\right\|_{\{-n+k\}} = \left|w^{(n)}(-n+k)\right| = |w_{-n+k}\cdots w_{-1+k}|, \forall n \in \mathbb{N}. \tag{3.29}$$

Hence,

$$\sup\left\{\left\|w^{(n)}\right\|_{f^{-n}(B_i)} : i, n \in \mathbb{N}\right\} = \sup\left\{|w_{-n+k} \cdots w_{-1+k}| : n \in \mathbb{N}, k \in \mathbb{Z}\right\} = \infty,$$

proving that condition *(ii)* of Theorem 3.6 holds. Thus, $\mathcal{B}_w = C_{w,f}$ is a Li-Yorke chaotic operator. □

Remark 3.12. In the statement of Proposition 3.11, we assumed that none of the w_n is zero. What happens if some w_n is zero? We will see this in the next proposition.

Proposition 3.13. *Consider a bilateral weighted backward shift*

$$\mathcal{B}_w : (x_n)_{n\in\mathbb{Z}} \in c_0(\mathbb{Z}) \mapsto (w_n x_{n+1})_{n\in\mathbb{Z}} \in c_0(\mathbb{Z}),$$

where $w := (w_n)_{n\in\mathbb{Z}}$ is a bounded sequence of scalars. Let

$$\beta^- := \sup\left\{|w_i \cdots w_j| : i \leq j \leq -1\right\} \quad \textit{and} \quad \beta^+ := \sup\left\{|w_i \cdots w_j| : 0 \leq i \leq j\right\}.$$

Then $\mathcal{B}_w$ is Li-Yorke chaotic if and only if one the following conditions hold:

(I) $\beta^+ = \infty$ and $\liminf_{n\to\infty} |w_{-n} \cdots w_{-k}| = 0$, for some $k \in \mathbb{Z}$.

(II) $\beta^+ < \infty$, $\beta^- = \infty$ and $\liminf_{n\to\infty} |w_{-n} \cdots w_{-k}| = 0$, for all $k \in \mathbb{N}$.

Proof. Indeed, let X and f be as in the proof of Proposition 3.11. Suppose that $C_{w,f}$ is Li-Yorke chaotic and let $(B_i)_{i\in\mathbb{N}}$ and $(n_j)_{j\in\mathbb{N}}$ be as in the statement of Theorem 3.6. Let

$$\beta := \sup\left\{|w_i \cdots w_j| : i, j \in \mathbb{Z}, i \leq j\right\} \quad \text{and} \quad B := \bigcup_{i\in\mathbb{N}} B_i.$$

We claim that

$$\liminf_{n\to\infty} |w_{-n} \cdots w_{-1+l}| = 0, \quad \text{for all } \; l \in B. \tag{3.30}$$

Indeed, take $l \in B_i$ for some $i \in \mathbb{N}$. By Theorem 3.6*(i)*, we have that

$$\left|w_{-n_j+l} \cdots w_{-1+l}\right| = \left\|w^{(n_j)}\right\|_{f^{-n_j}(\{l\})} \leq \left\|w^{(n_j)}\right\|_{f^{-n_j}(B_i)} \to 0 \;\text{ as }\; j \to \infty,$$

whence (3.30) holds. Moreover, we must have $\beta = \infty$. In fact, if $\beta < \infty$, then

$$\left\|w^{(n)}\right\|_{f^{-n}(B_i)} = \sup_{k\in B_i} |w_{-n+k} \cdots w_{-1+k}| \leq \beta, \quad \text{for all } \; i \in \mathbb{N},$$

contradicting Theorem 3.6*(ii)*.

If $\beta^+ = \infty$, then (3.30) implies that condition *(I)* holds. Assume $\beta^+ < \infty$. Since $\beta \leq \max\{\beta^-, \beta^+, \beta^-\beta^+\}$, we must have $\beta^- = \infty$.

If B is unbounded below, then (3.30) implies that *(II)* holds. In fact, if *(II)* does not hold in this case, then there is $k_0 \in \mathbb{N}$ such that

$$\liminf_{n\to\infty} |w_{-n} \cdots w_{-k_0}| > 0.$$

Hence, there exists $l_0 \in \mathbb{Z}^- \cup \{0\}$ such that $-k_0 = -1 + l_0$. Since $B \subset \mathbb{Z}$ is unbounded below, we can assume that $l_0 \in B$, without loss of generality. Therefore,

$$\liminf_{n\to\infty} |w_{-n} \cdots w_{-1+l_0}| > 0,$$

contradicting (3.30).

Assume B bounded below and let $m := \min B$. We claim that

$$w_n \neq 0, \quad \text{for all } n < m. \tag{3.31}$$

Indeed, assume $w_t = 0$ for a certain $t < m$ and let $C := \sup\{|w_i \cdots w_j| : t < i \leq j\}$. Since $\beta^+ < \infty$, we have that $C < \infty$. Since $t < m = \min B$ and $w_t = 0$, then for all $k \in B_i$, we have that $t \leq -1 + k$. In fact,

$$k \in B_i \Rightarrow k \in B \Rightarrow m \leq k.$$

But since $t < m$, it follows that $t < k$, whence $t \leq -1 + k$, for all $k \in B$. Thus, for each $n \in \mathbb{N}$, we have two possibilities:

(A) If $-n + k \leq t$, then w_t appears in the product $|w_{-n+k} \cdots w_{-1+k}|$, whence

$$|w_{-n+k} \cdots w_{-1+k}| = 0.$$

(B) If $-n + k > t$, then $t < -n + k < -1 + k$ and this implies that

$$|w_{-n+k} \cdots w_{-1+k}| \in \{|w_i \cdots w_j| : t < i \leq j\}.$$

Therefore, by (A) and (B), we have that

$$\left\|w^{(n)}\right\|_{f^{-n}(B_i)} = \sup_{k\in B_i} |w_{-n+k} \cdots w_{-1+k}| \leq \sup\{|w_i \cdots w_j| : t < i \leq j\} = C, \quad \text{for all } \ i, n \in \mathbb{N},$$

which contradicts Theorem 3.6*(ii)*. Hence, (3.31) holds.

By (3.30), $\liminf\limits_{n\to\infty} |w_{-n} \cdots w_{-1+m}| = 0$. We will show that (3.31) implies *(II)*.

Again, for each $n \in \mathbb{N}$, we have two possibilities:

(C) If $-k \geq -1+m$, then for sufficiently large $n \in \mathbb{N}$, we have that $-n < -1+m \leq -k$, whence

$$\begin{aligned}
\liminf_{n\to\infty} |w_{-n}\cdots w_{-k}| &= \liminf_{n\to\infty} |w_{-n}\cdots w_{-1+m}w_{-m}\cdots w_{-k}| \\
&= \liminf_{n\to\infty} |w_{-n}\cdots w_{-1+m}|\,|w_{-m}\cdots w_{-k}| \\
&= 0.
\end{aligned}$$

(D) If $-k < -1+m$, then for sufficiently large $n \in \mathbb{N}$, we have that $-n < -k < -k+1 \leq -1+m$, whence

$$\begin{aligned}
\liminf_{n\to\infty} |w_{-n}\cdots w_{-k}| &= \liminf_{n\to\infty} |w_{-n}\cdots w_{-k}| \cdot \frac{|w_{-k+1}\cdots w_{-1+m}|}{|w_{-k+1}\cdots w_{-1+m}|} \\
&= \frac{1}{|w_{-k+1}\cdots w_{-1+m}|} \liminf_{n\to\infty} |w_{-n}\cdots w_{-k}w_{-k+1}\cdots w_{-1+m}| \\
&= 0.
\end{aligned}$$

Therefore, by (C) and (D), it follows that *(II)* holds.

Conversely, suppose that *(I)* or *(II)* hold. We will prove that $C_{w,f}$ is Li-Yorke chaotic. We can replace "lim inf" in *(I)* and *(II)* by "lim", since

$$\liminf_{n\to\infty} |w_{-n}\cdots w_{-k}| = 0 \quad \text{and} \quad \limsup_{n\to\infty} |w_{-n}\cdots w_{-k}| > 0$$

imply that $\chi_{\{-k+1\}}$ is a semi-irregular vector for $C_{w,f}$, and so, $C_{w,f}$ is Li-Yorke chaotic by Theorem 2.37. Indeed, for each $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have that

$$\begin{aligned}
\left\|(C_{w,f})^n\left(\chi_{\{-k+1\}}\right)\right\| = \left\|\left(\chi_{\{-k+1\}}\circ f^n\right)\cdot w^{(n)}\right\| &= \sup_{i\in\mathbb{Z}} \left|\chi_{\{-k+1\}}\left(f^n(i)\right)\cdot w^{(n)}(i)\right| \\
&= \left|w^{(n)}(-n-k+1)\right| \\
&= |w_{-n-k+1}\cdots w_{-k}|\,.
\end{aligned}$$

If *(I)* holds, let $B_i := \{-k+i\}$, for $i \in \mathbb{N}$. Since $\lim_{n\to\infty} |w_{-n}\cdots w_{-k}| = 0$,

$$\lim_{n\to\infty} \left\|w^{(n)}\right\|_{f^{-n}(B_i)} = \lim_{n\to\infty} \left\|w^{(n)}\right\|_{f^{-n}(\{-k+i\})} = \lim_{n\to\infty} \left\|w^{(n)}\right\|_{\{-n-k+i\}}$$

$$= \lim_{n\to\infty} \left|w^{(n)}(-n-k+i)\right|$$

$$= \lim_{n\to\infty} |w_{-n-k+i}\cdots w_{-1-k+i}|$$

$$= \lim_{n\to\infty} |w_{-n-k+i}\cdots w_{-k}\cdots w_{-1-k+i}|$$

$$= 0.$$

Therefore, condition *(i)* of Theorem 3.6 holds with $(n_j)_{j\in\mathbb{N}} := (n)_{n\in\mathbb{N}}$.

If $-k \leq i' \leq j'$ and we define

$$i := j' + k + 1 \in \mathbb{N} \quad \text{and} \quad n := j' - i' + 1 \in \mathbb{N},$$

then

$$-1 - k + i = j' \quad \text{and} \quad -n - k + i = -(j' - i' + 1) - k + (j' + k + 1) = i',$$

and so

$$|w_{i'}\cdots w_{j'}| = |w_{-n-k+i}\cdots w_{-1-k+i}| = \left\|w^{(n)}\right\|_{f^{-n}(B_i)}.$$

This shows that

$$\sup\left\{\left\|w^{(n)}\right\|_{f^{-n}(B_i)} : i, n \in \mathbb{N}\right\} \geq \sup\left\{|w_{i'}\cdots w_{j'}| : -k \leq i' \leq j'\right\} = \infty,$$

because

$$\beta^+ = \sup\left\{|w_{i'}\cdots w_{j'}| : 0 \leq i' \leq j'\right\} = \infty.$$

Therefore, condition *(ii)* of Theorem 3.6 holds, whence $C_{w,f}$ is a Li-Yorke chaotic operator.

If *(II)* holds, for each $i \in \mathbb{N}$, let $B_i := \{-i\}$. Since $\lim_{n\to\infty} |w_{-n}\cdots w_{-k}| = 0$, for all $k \in \mathbb{N}$, we have that

$$\lim_{n\to\infty} \left\|w^{(n)}\right\|_{f^{-n}(B_i)} = \lim_{n\to\infty} \left\|w^{(n)}\right\|_{f^{-n}(\{-i\})} = \lim_{n\to\infty} \left\|w^{(n)}\right\|_{\{-n-i\}}$$

$$= \lim_{n\to\infty} \left|w^{(n)}(-n-i)\right|$$

$$= \lim_{n\to\infty} |w_{-n-i}\cdots w_{-1-i}|$$

$$= 0, \quad \forall i \in \mathbb{N}.$$

Thus, condition *(i)* of Theorem 3.6 holds with $(n_j)_{j\in\mathbb{N}} := (n)_{n\in\mathbb{N}}$.

If $i' \le j' < -1$ and we define

$$i := -1 - j' \in \mathbb{N} \ \text{ and } \ n := j' - i' + 1 \in \mathbb{N},$$

then

$$i' = -n - i \ \text{ and } \ j' = -1 - i,$$

and so

$$|w_{i'} \cdots w_{j'}| = |w_{-n-i} \cdots w_{-1-i}| \, .$$

Hence,

$$\begin{aligned}
\sup\left\{\left\|w^{(n)}\right\|_{f^{-n}(B_i)} : i, n \in \mathbb{N}\right\} &= \sup\left\{\left\|w^{(n)}\right\|_{f^{-n}(\{-i\})} : i, n \in \mathbb{N}\right\} \\
&= \sup\left\{\left\|w^{(n)}\right\|_{\{-n-i\}} : i, n \in \mathbb{N}\right\} \\
&= \sup\left\{|w_{-n-i} \cdots w_{-1-i}| : i, n \in \mathbb{N}\right\}. \\
&= \sup\left\{|w_{i'} \cdots w_{j'}| : i' \le j' \le -1\right\} \\
&= \beta^{-} \\
&= \infty.
\end{aligned}$$

Therefore, condition *(ii)* of Theorem 3.6 holds, whence $C_{w,f}$ is a Li-Yorke chaotic operator. $\square$

Chapter 4

Li-Yorke chaotic weighted composition operators on $L^p(\mu)$ spaces

4.1 Weighted composition operators on $L^p(\mu)$ spaces

In this chapter, we fix $p \in [1, \infty)$ and an arbitrary positive measure space $(X, \mathfrak{M}, \mu)$. As usual, $L^p(\mu)$ denotes the Banach space over $\mathbb{K}$ of (equivalence classes of) $\mathbb{K}$-valued p-integrable functions on $(X, \mathfrak{M}, \mu)$ endowed with the p-norm

$$\|\varphi\|_p := \left(\int_X |\varphi|^p \, d\mu \right)^{\frac{1}{p}}.$$

In the case where $p = \infty$, we define $L^\infty(\mu)$ as the set of (equivalence classes of) measurable functions $f : X \to \mathbb{K}$ that are essentially bounded with respect to μ, that is, there exist a measurable set $N \in \mathfrak{M}$ and a constant $K \geq 0$ such that $\mu(N) = 0$ and $|f(x)| \leq K$, for all $x \notin N$. Given $f \in L^\infty(\mu)$ and a set $N \in \mathfrak{M}$ with $\mu(N) = 0$, we define

$$S_f(N) := \sup \{|f(x)| : x \notin N\}$$

and the norm

$$\|f\|_\infty := \inf \{S_f(N) : N \in \mathfrak{M} \text{ and } \mu(N) = 0\}.$$

Endowed with this norm, $L^\infty(\mu)$ is also a Banach space over $\mathbb{K}$ (see [14]).

We also fix a *weight function* $w : X \to \mathbb{K}$, which now means a measurable map such that

$$\varphi \cdot w \in L^p(\mu) \quad \text{for all} \quad \varphi \in L^p(\mu). \tag{4.1}$$

The following result shows that, in a very broad sense, the weight functions are exactly the functions on $L^\infty(\mu)$.

Proposition 4.1. *If μ has the property that every set with infinite measure contains a set with positive finite measure (in particular, if μ is σ-finite), then the following statements are equivalent:*

a) $w \in L^\infty(\mu)$;

b) $\varphi \cdot w \in L^p(\mu)$, *for all* $\varphi \in L^p(\mu)$.

Proof. Suppose $w \in L^\infty(\mu)$ and let $\alpha := \|w\|_\infty$. Hence, $|w|^p \leq \alpha^p$ μ-a.e. If $\varphi \in L^p(\mu)$, then

$$\int_X |\varphi \cdot w|^p \, d\mu = \int_X |\varphi|^p \cdot |w|^p \, d\mu \leq \alpha^p \int_X |\varphi|^p \, d\mu = \alpha^p \|\varphi\|_p^p < \infty.$$

Thus, $\varphi \cdot w \in L^p(\mu)$.

Conversely, suppose $w \notin L^\infty(\mu)$. For each $n \in \mathbb{N}$, let

$$A_n := \left\{x \in X : n^2 \leq |w(x)| < (n+1)^2\right\} \in \mathfrak{M}.$$

Let $I := \{n \in \mathbb{N} : \mu(A_n) > 0\}$. Since $w \notin L^\infty(\mu)$, I is an infinity set. For each $n \in I$, let $B_n \in \mathfrak{M}$ such that $B_n \subset A_n$ and $0 < \mu(B_n) < \infty$ (such a set B_n exists because the hypothesis guarantees that every set with infinite measure contains a set with positive finite measure).

Define

$$\varphi := \sum_{n \in I} \left(\frac{1}{n^{2p}\mu(B_n)}\right)^{\frac{1}{p}} \chi_{B_n}.$$

Note that $\varphi \in L^p(\mu)$, since the above series is absolutely convergent. Indeed:

$$\begin{aligned}
\sum_{n\in I}\left\|\left(\frac{1}{n^{2p}\mu(B_n)}\right)^{\frac{1}{p}}\chi_{B_n}\right\|_p &= \sum_{n\in I}\left(\int_X \frac{1}{n^{2p}\mu(B_n)}\left|\chi_{B_n}\right|^p d\mu\right)^{\frac{1}{p}} \\
&= \sum_{n\in I}\left(\frac{1}{n^{2p}\mu(B_n)}\int_X \left|\chi_{B_n}\right|^p d\mu\right)^{\frac{1}{p}} \\
&= \sum_{n\in I}\frac{1}{n^2\left(\mu(B_n)\right)^{\frac{1}{p}}}\left(\int_{B_n}\left|\chi_{B_n}\right|^p d\mu\right)^{\frac{1}{p}} \\
&= \sum_{n\in I}\frac{1}{n^2\left(\mu(B_n)\right)^{\frac{1}{p}}}\left(\mu(B_n)\right)^{\frac{1}{p}} \\
&= \sum_{n\in I}\frac{1}{n^2} < \infty.
\end{aligned}$$

However,

$$\begin{aligned}
\int_X |\varphi\cdot w|^p = \int_X \sum_{n\in I}\frac{1}{n^{2p}\mu(B_n)}\left|\chi_{B_n}\right|^p |w|^p\, d\mu &= \sum_{n\in I}\frac{1}{n^{2p}\mu(B_n)}\int_{B_n}|w|^p\, d\mu \\
&\geq \sum_{n\in I}\frac{1}{n^{2p}\mu(B_n)}\int_{B_n} n^{2p}\, d\mu \\
&= \sum_{n\in I}\frac{1}{n^{2p}\mu(B_n)}n^{2p}\mu(B_n) \\
&= \sum_{n\in I}1 = \infty.
\end{aligned}$$

Therefore, $\int_X |\varphi\cdot w|^p > \infty$, whence $\varphi\cdot w \notin L^p(\mu)$. $\square$

Remark 4.2. The hypothesis in Proposition 4.1 are essential. We will see this in the next example.

Example 4.3. Suppose that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points in X such that $\{x_n\} \in \mathfrak{M}$ and $\mu(\{x_n\}) = \infty$, for all $n \in \mathbb{N}$. Let $Y := X \setminus \{x_1, x_2, \cdots\}$ and consider $w : X \to \mathbb{R}$ such that

$$w(x) = \begin{cases} 1, & \text{if } x \in Y; \\ n, & \text{if } x = x_n, \quad \text{for some } n \in \mathbb{N}. \end{cases}$$

We claim that $w \notin L^\infty(\mu)$. Indeed, if $w \in L^\infty(\mu)$, then there is $c > 0$ such that

$$|w(x)| \leq c, \ \ \mu\text{-a.e.}$$

Hence,

$$n = |w(x_n)| \leq c, \forall n \in \mathbb{N},$$

which is a contradiction. Now, let $\varphi \in L^p(\mu)$. Then $\varphi(x_n) = 0$, for all $n \in \mathbb{N}$, which implies that $\varphi \cdot w = \varphi \cdot \chi_Y \in L^p(\mu)$. In fact, if $\varphi(x_n) \neq 0$, for some $n \in \mathbb{N}$, there is $b \in \mathbb{K} \setminus \{0\}$ such that $\varphi(x_n) = b$. Moreover,

$$\int_{\{x_n\}} |\varphi|^p \, d\mu \leq \int_X |\varphi|^p \, d\mu = \|\varphi\|_p^p < \infty.$$

However,

$$\int_{\{x_n\}} |\varphi|^p \, d\mu = \int_{\{x_n\}} |b|^p \, d\mu = |b|^p \, \mu(\{x_n\}) = \infty,$$

whence we conclude that $\varphi(x_n) = 0$, for all $n \in \mathbb{N}$. In order to prove that $\varphi \cdot w = \varphi \cdot \chi_Y$, since $Y \subset X$, we have two cases to consider:

1) If $x \in Y$, then $(\varphi \cdot w)(x) = \varphi(x)w(x) = \varphi(x)$;

2) If $x \notin Y$, then $x = x_n$, for some $n \in \mathbb{N}$. Therefore,

$$(\varphi \cdot w)(x) = \varphi(x)w(x) = \varphi(x_n)w(x_n) = 0,$$

since $\varphi(x_n) = 0$.

On the other hand, we have that

1') If $x \in Y$, then $(\varphi \cdot \chi_Y)(x) = \varphi(x)\chi_Y(x) = \varphi(x)$, since $\chi_Y(x) = 1$;

2') If $x \notin Y$, then $(\varphi \cdot \chi_Y)(x) = \varphi(x)\chi_Y(x) = 0$, since $\chi_Y(x) = 0$.

Hence, $\varphi \cdot w = \varphi \cdot \chi_Y$, as we wanted to demonstrate.

Next, we will state two results from Integration Theory, which will be useful for demonstrating the next proposition. The proofs of these results can be found in [31].

Theorem 4.4. *Let $f : X \to [0, \infty)$ be measurable. There exist simple measurable functions s_n on X such that*

(a) $0 \leq s_1 \leq s_2 \leq ... \leq f$,

(b) $s_n(x) \to f(x)$ *as* $n \to \infty$, *for every* $x \in X$.

Theorem 4.5. [Lebesgue's Monotone Convergence Theorem] *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions on X, and suppose that*

(a) $0 \leq f_1(x) \leq f_2(x) \leq ... \leq \infty$, *for every* $x \in X$,

(b) $f_n(x) \to f(x)$ *as* $n \to \infty$, *for every* $x \in X$.

Then f is measurable, and

$$\int_X f_n d\mu \to \int_X f d\mu \quad as \quad n \to \infty.$$

Proposition 4.6. *Given a bimeasurable map $f : X \to X$ (i.e., $f(A) \in \mathfrak{M}$ and $f^{-1}(A) \in \mathfrak{M}$, for all $A \in \mathfrak{M}$), the weighted composition operator*

$$C_{w,f}(\varphi) := (\varphi \circ f) \cdot w$$

is a well-defined bounded linear operator on $L^p(\mu)$ if and only if there exists a constant $c \in (0, \infty)$ such that

$$\int_A |w|^p \, d\mu \leq c \; \mu(f(A)), \quad \textit{for every } A \in \mathfrak{M}. \tag{4.2}$$

Moreover, in this case, $\|C_{w,f}\| \leq c^{\frac{1}{p}}$.

Proof. Suppose that $C_{w,f}$ is a bounded linear operator on $L^p(\mu)$ and let $\beta := \|C_{w,f}\|$. If $A \in \mathfrak{M}$ and $\mu(f(A)) < \infty$, then $\chi_{f(A)} \in L^p(\mu)$ and

$$\begin{aligned}
\int_A |w|^p\,d\mu \;\le\; \int_{f^{-1}(f(A))} |w|^p\,d\mu &= \int_X \left|\chi_{f(A)}\circ f\right|^p |w|^p\,d\mu \\
&= \left\|\left(\chi_{f(A)}\circ f\right)\cdot w\right\|_p^p \\
&= \left\|C_{w,f}(\chi_{f(A)})\right\|_p^p \\
&\le \beta^p \left\|\chi_{f(A)}\right\|_p^p \\
&= \beta^p\,\mu(f(A)).
\end{aligned}$$

Therefore, the condition (4.2) holds, with $c := \beta^p$, whenever $\beta \neq 0$, and $c := 1$, if $\beta = 0$.

Conversely, suppose that condition (4.2) holds. Let $s \in L^p(\mu)$ be a simple function and write

$$s = a_1\chi_{A_1} + a_2\chi_{A_2} + \cdots + a_k\chi_{A_k},$$

with $k \geq 1$, $A_1, A_2, \cdots, A_k \in \mathfrak{M}$ pairwise disjoint and $a_1, a_2, \cdots, a_k \in \mathbb{K}$ non-zero. By condition (4.2), we have that

$$\begin{aligned}
\int_X |s\circ f|^p\,|w|^p\,d\mu &= \int_X |a_1(\chi_{A_1}\circ f)+a_2(\chi_{A_2}\circ f)+\cdots+a_k(\chi_{A_k}\circ f)|^p\,|w|^p\,d\mu \\
&= \int_X \left|\sum_{j=1}^{k} a_j\chi_{f^{-1}(A_j)}\right|^p |w|^p\,d\mu \\
&= \sum_{j=1}^{k}|a_j|^p\int_X \chi_{f^{-1}(A_j)}\,|w|^p\,d\mu \\
&= \sum_{j=1}^{k}|a_j|^p\int_{f^{-1}(A)} |w|^p\,d\mu \\
&\le \sum_{j=1}^{k}|a_j|^p\ \, c\mu(f(f^{-1}(A_j))) \\
&\le c\sum_{j=1}^{k}|a_j|^p\,\mu(A_j).
\end{aligned} \tag{4.3}$$

But, since $A_1, A_2, \cdots, A_k$ are pairwise disjoint, we have that

$$|s|^p = |a_1\chi_{A_1}+a_2\chi_{A_2}+\cdots+a_k\chi_{A_k}|^p = |a_1|^p\,\chi_{A_1}+|a_2|^p\,\chi_{A_2}+\cdots+|a_k|^p\,\chi_{A_k}.$$

Hence,

$$\int_X |s|^p\,d\mu = \int_X |a_1|^p\,\chi_{A_1}+|a_2|^p\,\chi_{A_2}+\cdots+|a_k|^p\,\chi_{A_k}d\mu = \sum_{j=1}^{k}|a_j|^p\int_X \chi_{A_j}d\mu = \sum_{j=1}^{k}|a_j|^p\,\mu(A_j).$$

Therefore,

$$\int_X |s\circ f|^p\,|w|^p\,d\mu \le c\ \sum_{j=1}^{k}|a_j|^p\,\mu(A_j) = c\ \int_X |s|^p\,d\mu. \tag{4.4}$$

Let $\varphi\in L^p(\mu)$. Since $|\varphi| : X\to[0,\infty]$ is a measurable function, by Theorem 4.4, there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of measurable simple functions on X such that

$$0\le s_1(x)\le s_2(x)\le\cdots\le|\varphi(x)|\ \text{ and }\ s_n(x)\to|\varphi(x)|,\ \forall x\in X.$$

Therefore, the sequences $(|s_n \circ f|^p\,|w|^p)_{n\in\mathbb{N}}$ and $(|s_n|^p)_{n\in\mathbb{N}}$ are increasing and converge pointwise to $|\varphi\circ f|^p\,|w|^p$ and $|\varphi|^p$, respectively. By (4.4),

$$\int_X |s_n \circ f|^p\,|w|^p\,d\mu \le c\int_X |s_n|^p\,d\mu, \quad \forall n\in\mathbb{N}.$$

Hence, by the monotone convergence theorem (Theorem 4.5), we obtain

$$\|C_{w,f}(\varphi)\|_p^p = \int_X |\varphi\circ f|^p\,|w|^p\,d\mu \le c\int_X |\varphi|^p\,d\mu = c\,\|\varphi\|_p^p.$$

Thus, $C_{w,f}(\varphi)\in L^p(\mu)$ and $\|C_{w,f}\|\le c^{\frac{1}{p}}$. □

4.2 Li-Yorke chaos for weighted composition operators on $L^p(\mu)$ spaces

Throughout this section, we fix a bimeasurable map $f : X \to X$ satisfying condition (4.2). Hence, the weighted composition operator

$$C_{w,f} : \varphi \in L^p(\mu) \mapsto (\varphi\circ f)\cdot w \in L^p(\mu)$$

is a well-defined bounded linear operator. If w is the constant function 1, then we obtain the (unweighted) composition operator

$$C_f : \varphi \in L^p(\mu) \mapsto \varphi\circ f \in L^p(\mu).$$

We associate to μ, w and f, the following sequence of positive measures on $(X,\mathfrak{M})$:

$$\mu_1(A) := \int_A |w|^p\,d\mu, \quad \mu_n(A) := \int_A \left|w\circ f^{n-1}\right|^p \cdots |w\circ f|^p\,|w|^p\,d\mu \quad (A\in\mathfrak{M}, n\ge 2).$$

If w is the constant function 1, then $\mu_n = \mu$ for all $n\ge 1$.

Lemma 4.7. *$\mu_n(A) \le c^n\mu(f^n(A))$, for all $A\in\mathfrak{M}$ and $n\in\mathbb{N}$.*

Proof. For $n = 1$, we have that

$$\mu_1(A) := \int_A |w|^p \, d\mu \leq c \; \mu(f(A)),$$

by condition (4.2). If $n \geq 2$ and $A \in \mathfrak{M}$, then

$$\begin{aligned}
\mu_n(A) \;\leq\; \mu_n(f^{-n}(f^n(A))) &= \int_{f^{-n}(f^n(A))} \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p \, |w|^p \, d\mu \\
&= \int_X \left|\chi_{f^n(A)} \circ f^n\right|^p \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p \, |w|^p \, d\mu \\
&= \left\|(C_{w,f})^n \left(\chi_{f^n(A)}\right)\right\|_p^p \\
&\leq \|(C_{w,f})^n\|^p \left\|\chi_{f^n(A)}\right\|_p^p \\
&\leq \left(c^{n/p}\right)^p \int_X \left|\chi_{f^n(A)}\right|^p d\mu \\
&= c^n \mu(f^n(A)),
\end{aligned}$$

as it was to be shown. □

We now establish the main result of this section, a characterization of the weighted composition operators on $L^p(\mu)$ that are Li-Yorke chaotic.

Theorem 4.8. *The weighted composition operator $C_{w,f}$ on $L^p(\mu)$ is Li-Yorke chaotic if and only if there is a nonempty countable family $(B_i)_{i \in I}$ of measurable sets of finite positive μ-measure such that the following conditions hold:*

(A) There is a increasing sequence $(n_j)_{j \in \mathbb{N}}$ of positive integers such that

$$\lim_{j \to \infty} \mu_{n_j}(f^{-n_j}(B_i)) = 0, \quad \text{for all } \; i \in I.$$

(B) $\sup\left\{\dfrac{\mu_n(f^{-n}(B_i))}{\mu(B_i)} : i \in I, n \in \mathbb{N}\right\} = \infty.$

Proof. Suppose that $C_{w,f}$ is a Li-Yorke chaotic operator. Then, it admits an irregular vector $\psi \in L^p(\mu)$. For each $i \in \mathbb{Z}$, let

$$B_i := \left\{x \in X : 2^{i-1} \leq |\psi(x)| < 2^i\right\} \in \mathfrak{M}.$$

Consider the countable set $I := \{i \in \mathbb{Z} : \mu(B_i) > 0\}$, which is nonempty because ψ is not the zero vector in $L^p(\mu)$ (the zero vector cannot be an irregular vector for any operator). In this way, $(B_i)_{i\in I}$ is a nonempty countable family of measurable sets of positive μ-measure. Moreover, the sets B_i have finite measure, because

$$\sum_{i\in\mathbb{Z}} 2^{(i-1)p}\mu(B_i) \leq \int_X |\psi|^p \, d\mu = \|\psi\|_p^p < \infty. \tag{4.5}$$

Since ψ is an irregular vector for $C_{w,f}$, there is an increasing sequence $(n_j)_{j\in\mathbb{N}}$ of positive integers such that

$$\left\|(C_{w,f})^{n_j}(\psi)\right\|_p \to 0 \ \text{ as } \ j \to \infty.$$

Since, for each $i \in I$,

$$\begin{aligned}
\left\|(C_{w,f})^{n_j}(\psi)\right\|_p^p &= \int_X |\psi \circ f^{n_j}|^p \left|w \circ f^{n_j-1}\right|^p \cdots |w \circ f|^p \, |w|^p \, d\mu \\
&\geq \int_{f^{-n_j}(B_i)} |\psi \circ f^{n_j}|^p \left|w \circ f^{n_j-1}\right|^p \cdots |w \circ f|^p \, |w|^p \, d\mu \\
&\geq 2^{(i-1)p} \int_{f^{-n_j}(B_i)} \left|w \circ f^{n_j-1}\right|^p \cdots |w \circ f|^p \, |w|^p \, d\mu \\
&= 2^{(i-1)p} \mu_{n_j}\left(f^{-n_j}(B_i)\right),
\end{aligned}$$

we obtain $\lim\limits_{j\to\infty} \mu_{n_j}\left(f^{-n_j}(B_i)\right) = 0$, for all $i \in I$, which is condition (A). If condition (B) is false, then there exists $\beta \in \mathbb{R}$ such that

$$\mu_n\left(f^{-n}(B_i)\right) \leq \beta\mu(B_i), \text{ for all } i \in I \text{ and } n \in \mathbb{N}. \tag{4.6}$$

If $i \in \mathbb{Z} \setminus I$, we have that (4.6) holds too, because $\mu(B_i) = 0$ implies $\mu_n\left(f^{-n}(B_i)\right) = 0$, by Lemma 4.7. Hence, we have that (4.6) holds for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
\|(C_{w,f})^n(\psi)\|_p^p &= \sum_{i\in\mathbb{Z}}\int_{f^{-n}(B_i)} |\psi\circ f^n|^p \left|w\circ f^{n-1}\right|^p \cdots |w\circ f|^p |w|^p \, d\mu \\
&\leq \sum_{i\in\mathbb{Z}} 2^{ip}\int_{f^{-n}(B_i)} \left|w\circ f^{n-1}\right|^p \cdots |w\circ f|^p |w|^p \, d\mu \\
&= \sum_{i\in\mathbb{Z}} 2^{ip}\mu_n(f^{-n}(B_i)) \leq \beta\sum_{i\in\mathbb{Z}} 2^{ip}\mu(B_i) = \beta\sum_{i\in\mathbb{Z}} 2^{ip+p-p}\mu(B_i) \\
&= \beta\sum_{i\in\mathbb{Z}} 2^p 2^{(i-1)p}\mu(B_i) = 2^p\beta\sum_{i\in\mathbb{Z}} 2^{(i-1)p}\mu(B_i) \leq 2^p\beta\,\|\psi\|_p^p,
\end{aligned}$$

which contradicts the fact that ψ is an irregular vector for $C_{w,f}$.

Conversely, let Z be the linear span of $\{\chi_{B_i} : i \in I\}$ in $L^p(\mu)$ and $Y := \overline{Z}$. Let

$$S_1 := \{\varphi \in Y : \text{ the } C_{w,f}\text{-orbit of } \varphi \text{ has a subsequence converging to zero}\}.$$

By condition (A), for each $i \in I$, we have that

$$\begin{aligned}
\|(C_{w,f})^{n_j}(\chi_{B_i})\|_p^p &= \int_X |\chi_{B_i}\circ f^{n_j}|^p \left|w\circ f^{n_j-1}\right|^p \cdots |w\circ f|^p |w|^p \, d\mu \\
&= \int_{f^{-n_j}(B_i)} \left|w\circ f^{n_j-1}\right|^p \cdots |w\circ f|^p |w|^p \, d\mu \\
&= \mu_{n_j}(f^{-n_j}(B_i)) \to 0 \text{ as } j\to\infty.
\end{aligned}$$

Hence, by linearity, for any $\varphi \in Z$, $(C_{w,f})^{n_j}(\varphi) \to 0$ as $j \to \infty$. Therefore, $Z \subset S_1$, whence we conclude that S_1 is dense in Y and so it is residual in Y, by Proposition 2.38. Now, for each $i \in I$, let

$$\varphi_i := \frac{1}{(\mu(B_i))^{1/p}}\,\chi_{B_i} \in Y.$$

Then,

$$\|\varphi_i\|_p^p := \int_X |\varphi_i|^p \, d\mu = \int_X \left|\frac{1}{(\mu(B_i))^{1/p}}\right|^p |\chi_{B_i}|^p \, d\mu = \frac{1}{\mu(B_i)}\int_X \chi_{B_i} d\mu = \frac{1}{\mu(B_i)}\,\mu(B_i) = 1,$$

and

$$
\begin{aligned}
\|(C_{w,f})^n(\varphi_i)\|_p^p &= \int_X |\varphi_i \circ f^n|^p \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p |w|^p \, d\mu \\
&= \int_X \left|\frac{1}{(\mu(B_i))^{1/p}}\right|^p |\chi_{B_i} \circ f^n|^p \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p |w|^p \, d\mu \\
&= \frac{1}{\mu(B_i)} \int_{f^{-n}(B_i)} \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p |w|^p \, d\mu \\
&= \frac{1}{\mu(B_i)} \, \mu_n(f^{-n}(B_i)), \quad n \in \mathbb{N}.
\end{aligned}
$$

Therefore, $\|(C_{w,f})^n(\varphi_i)\|_p^p = \dfrac{\mu_n(f^{-n}(B_i))}{\mu(B_i)}$, for all $n \in \mathbb{N}$. By condition (B), we have that the sequence $(\|(C_{w,f})^n|_Y\|)_{n\in\mathbb{N}}$ is unbounded and so, it is not equicontinuous. Hence, by the Theorem 2.22, the set

$$S_2 := \{\varphi \in Y : \text{ the } C_{w,f}\text{-orbit of } \varphi \text{ is unbounded}\}$$

is residual in Y. Thus, $S_1 \cap S_2$ is also residual in Y. Since each vector $\psi \in S_1 \cap S_2$ is an irregular vector for $C_{w,f}$, we conclude that $C_{w,f}$ is Li-Yorke chaotic. $\square$

Remark 4.9. Note that condition (A) in Theorem 4.8 can be replaced by the following weaker condition:

$$\text{(A')} \quad \liminf_{n\to\infty} \mu_n\left(f^{-n}(B_i)\right) = 0, \text{ for all } i \in I.$$

In fact, suppose that there is a nonempty countable family $(B_i)_{i\in I}$ of measurable sets of finite positive μ-measure satisfying conditions *(A')* and *(B)*. We have to prove that $C_{w,f}$ is Li-Yorke chaotic. If

$$\limsup_{n\to\infty} \mu_n\left(f^{-n}(B_k)\right) > 0, \text{ for some } k \in I,$$

then χ_{B_k} is a semi-irregular vector for $C_{w,f}$. Indeed,

$$
\begin{aligned}
\|(C_{w,f})^n(\chi_{B_k})\|_p^p &= \int_X |\chi_{B_k} \circ f^n|^p \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p |w|^p \, d\mu \\
&= \int_{f^{-n}(B_k)} \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p |w|^p \, d\mu \\
&= \mu_n(f^{-n}(B_k)).
\end{aligned}
$$

Thus,

$$0 = \liminf_{n\to\infty} \mu_n\left(f^{-n}(B_k)\right) = \liminf_{n\to\infty} \|(C_{w,f})^n(\chi_{B_k})\|_p^p$$

and

$$0 < \limsup_{n\to\infty} \mu_n\left(f^{-n}(B_k)\right) = \limsup_{n\to\infty} \|(C_{w,f})^n(\chi_{B_k})\|_p^p,$$

which shows that χ_{B_k} is a semi-irregular vector for $C_{w,f}$, and so, $C_{w,f}$ is Li-Yorke chaotic.

On the other hand, if

$$\limsup_{n\to\infty} \mu_n\left(f^{-n}(B_k)\right) = 0, \text{ for all } k \in I,$$

then

$$\lim_{n\to\infty} \mu_n\left(f^{-n}(B_k)\right) = 0, \text{ for all } k \in I.$$

In this case, the condition *(A)* holds with $(n_j)_{j\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$. Since we are assuming that condition *(B)* holds, $C_{w,f}$ is Li-Yorke chaotic by Theorem 4.8.

Conversely, if $C_{w,f}$ is Li-Yorke chaotic, *(A)* and *(B)* of Theorem 4.8 hold. Since *(A)* implies *(A')*, we obtain the claim in Remark 4.9.

As an immediate consequence of Theorem 4.8, we obtain the following result, which is exactly Theorem 1.1 in [8].

Corollary 4.10. *The composition operator C_f on $L^p(\mu)$ is Li-Yorke chaotic if and only if there is a nonempty countable family $(B_i)_{i\in I}$ os measurable sets of finite positive μ-measure such that the following conditions hold:*

- *There is a increasing sequence $(n_j)_{j\in\mathbb{N}}$ of positive integers such that*

$$\lim_{j\to\infty} \mu(f^{-n_j}(B_i)) = 0, \quad \textit{for all } \; i \in I.$$

- $\sup\left\{\dfrac{\mu(f^{-n}(B_i))}{\mu(B_i)} : i \in I, n \in \mathbb{N}\right\} = \infty.$

Proof. It is enough to observe that $C_f = C_{w,f}$, with $w(x) = 1$, for all $x \in X$, and apply Theorem 4.8. $\square$

Remark 4.11. In view of Remark 4.9, we have that the first condition in Corollary 4.10 can be replaced by the following weaker condition:

- $\liminf_{n\to\infty} \mu(f^{-n}(B_i)) = 0$, for all $i \in I$.

4.3 Li-Yorke chaos for weighted shifts on ℓ^p-spaces

As an application of Theorem 4.8, we will present in this section necessary and sufficient conditions for unilateral and bilateral weighted backward shifts to be Li-Yorke chaotic on $\ell^p(\mathbb{N})$ and on $\ell^p(\mathbb{Z})$, respectively.

The next proposition was established in ([3], Proposition 27) in the case w is a bounded sequence of nonzero scalars.

Proposition 4.12. *A unilateral weighted backward shift*

$$B_w : (x_1, x_2, x_3, \cdots) \in \ell^p(\mathbb{N}) \mapsto (w_1x_2, w_2x_3, w_3x_4, \cdots) \in \ell^p(\mathbb{N}),$$

where $w := (w_n)_{n\in\mathbb{N}}$ is a bounded sequence of scalars, is Li-Yorke chaotic if and only if

$$\sup\{|w_i \cdots w_j| : i, j \in \mathbb{N}, i \leq j\} = \infty. \tag{4.7}$$

Proof. Consider $X := \mathbb{N}$, $\mathfrak{M} := \mathcal{P}(X)$ (the power set of X), μ the counting measure on $\mathfrak{M}$, and $f : n \in \mathbb{N} \mapsto n+1 \in \mathbb{N}$. Then $L^p(\mu) = \ell^p(\mathbb{N})$ and $C_{w,f} = B_w$. In the present case, condition *(A)* of Theorem 4.8 is superfluous (indeed, for each $B \subset X$ finite, we have $f^{-n}(B) = \varnothing$ for all sufficiently large $n \in \mathbb{N}$). If (4.7) fails, that is, $\alpha := \sup\{|w_i \cdots w_j| : i, j \in \mathbb{N}, i \leq j\} < \infty$, then

$$\mu_n(A) = \int_A \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p\, |w|^p\, d\mu = \sum_{i\in A} |w_i \cdots w_{i+n-1}|^p \leq \alpha^p \mu(A), \text{ for all } A \in \mathfrak{M},$$

and so condition *(B)* of Theorem 4.8 fails no matter how we choose the family $(B_i)_{i\in I}$.

On the other hand, if (4.7) is true, since condition *(A)* of Theorem 4.8 is superfluous, it is sufficient to prove condition *(B)*. We claim that condition *(B)* of Theorem 4.8 holds with $B_i := \{i\}$ for each $i \in \mathbb{N}$. Indeed, in this case, $\mu(B_i) = 1$, for all $i \in \mathbb{N}$. Moreover, $f^{-n}(B_i) = f^{-n}(\{i\}) = \{i-n\}$, for all $i > n$. For each $i, n \in \mathbb{N}$ such that $i > n \geq 2$, we define

$$i' := i - n \in \mathbb{N} \text{ and } j' := i - 1 \in \mathbb{N}.$$

Thus,

$$\begin{aligned}
\mu_n(f^{-n}(B_i)) \quad = \quad \mu_n(\{i-n\}) \quad &= \int_{\{i-n\}} \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p \, |w|^p \, d\mu \quad (n \geq 2) \\
&= \left|w_{i-n} \cdots w_{i-n+n-1}\right|^p \\
&= \left|w_{i-n} \cdots w_{i-1}\right|^p \\
&= \left|w_{i'} \cdots w_{j'}\right|^p .
\end{aligned} \tag{4.8}$$

Hence, (4.7) implies condition *(B)* of Theorem 4.8, whence B_w is a Li-Yorke chaotic operator.

□

Proposition 4.13. *A bilateral weighted backward shift*

$$B_w : (x_n)_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z}) \mapsto (w_n x_{n+1})_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z}),$$

where $w := (w_n)_{n\in\mathbb{Z}}$ is a bounded sequence of nonzero scalars, is Li-Yorke chaotic if and only if the following conditions hold:

(a) $\liminf_{n\to\infty} |w_{-n} \cdots w_{-1}| = 0.$

(b) $\sup\{|w_i \cdots w_j| : i, j \in \mathbb{Z}, i \leq j\} = \infty.$

Proof. Consider $X := \mathbb{Z}$, $\mathfrak{M} := \mathcal{P}(X)$, μ the counting measure on $\mathfrak{M}$, and $f : n \in \mathbb{Z} \mapsto n+1 \in \mathbb{Z}$. Then $L^p(\mu) = \ell^p(\mathbb{Z})$ and $C_{w,f} = B_w$. Suppose *(a)* and *(b)* hold. If

$$\limsup_{n\to\infty} |w_{-n} \cdots w_{-1}| > 0, \tag{4.9}$$

then $\chi_{\{0\}}$ is a semi-irregular vector for $C_{w,f}$, and so, $C_{w,f}$ is Li-Yorke chaotic. In fact,

$$\begin{aligned}
\left\|(C_{w,f})^n \left(\chi_{\{0\}}\right)\right\|_p^p &= \int_X \left|\chi_{\{0\}} \circ f^n\right|^p \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p \, |w|^p \, d\mu \\
&= \int_{f^{-n}(\{0\})} \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p \, |w|^p \, d\mu \\
&= \left|w_{-n} \cdots w_{-n+n-1}\right|^p \\
&= \left|w_{-n} \cdots w_{-1}\right|^p .
\end{aligned}$$

Thus, the condition *(a)* implies

$$\liminf_{n\to\infty} \left\|(C_{w,f})^n\left(\chi_{\{0\}}\right)\right\|_p = 0$$

and (4.9) implies

$$\limsup_{n\to\infty} \left\|(C_{w,f})^n\left(\chi_{\{0\}}\right)\right\|_p > 0,$$

which proves that $\chi_{\{0\}}$ is a semi-irregular vector for $C_{w,f}$.

If $\limsup\limits_{n\to\infty} |w_{-n}\cdots w_{-1}| = 0$, then

$$\lim_{n\to\infty} |w_{-n}\cdots w_{-1}| = 0. \tag{4.10}$$

In this case, conditions *(A)* and *(B)* of Theorem 4.8 hold with $(n_j)_{j\in\mathbb{N}} := (n)_{n\in\mathbb{N}}$ and $B_i := \{i\}$, for all $i \in \mathbb{Z}$, and so $C_{w,f}$ is Li-Yorke chaotic. Indeed, let $I := \mathbb{Z}$. If $i' \leq j'$ and we define

$$i := j' + 1 \in I \quad \text{and} \quad n := j' - i' + 1 \in \mathbb{N},$$

then

$$j' = -1 + i \quad \text{and} \quad i' = -n + j' + 1 = -n + i,$$

and so

$$\mu_n(f^{-n}(B_i)) = \mu_n(f^{-n}(\{i\})) = \mu_n(\{i-n\}) = |w_{-n+i}\cdots w_{-1+i}|^p = |w_{i'}\cdots w_{j'}|^p\,. \tag{4.11}$$

Since $\mu(B_i) = 1$, for all $i \in I$ and *(b)* holds, we conclude that condition *(B)* of Theorem 4.8 holds. To see that condition *(A)* of Theorem 4.8 is also satisfied, let's analyze the three possible cases.

- Case 1: $i = 0$.

  For $i = 0$, $\mu_n(f^{-n}(B_0)) = |w_{-n}\cdots w_{-1}|^p$, for all $n \in \mathbb{N}$, whence (4.10) implies *(A)* of Theorem 4.8.

- Case 2: $i < 0$.

  If $i < 0$, then $-n < -i$, for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}
\lim_{n\to\infty} \mu_n(f^{-n}(B_i)) &= \lim_{n\to\infty} |w_{-n+i}\cdots w_{-1+i}|^p \\
&= \lim_{n\to\infty} \frac{|w_{-n+i}\cdots w_{-1+i}w_i\cdots w_{-1}|^p}{|w_i\cdots w_{-1}|^p} \qquad (4.12)\\
&= \frac{1}{|w_i\cdots w_{-1}|^p}\lim_{n\to\infty} |w_{-n+i}\cdots w_{-1}|^p = 0, \text{ by (4.10).}
\end{aligned}$$

- Case 3: $i > 0$.

  If $i > 0$, then $\mu_n(f^{-n}(B_i)) = |w_{-n+i} \cdots w_{-1} w_0 \cdots w_{-1+i}|^p$, for all $n \in \mathbb{N}$. Thus,

$$\lim_{n\to\infty} \mu_n(f^{-n}(B_i)) = \lim_{n\to\infty} |w_{-n+i} \cdots w_{-1}|^p |w_0 \cdots w_{-1+i}|^p = 0, \text{ by (4.10)}.$$

Therefore, for all $i \in I = \mathbb{Z}$, we have that $\lim_{n\to\infty} \mu_n(f^{-n}(B_i)) = 0$, which implies condition *(A)* in Theorem 4.8, as desired.

On the other hand, suppose that $C_{w,f}$ is Li-Yorke chaotic and let $(B_i)_{i\in I}$ and $(n_j)_{j\in\mathbb{N}}$ be as in the statement of Theorem 4.8. By choosing $i \in I$ and $k \in B_i$, we have that

$$\left|w_{-n_j+k} \cdots w_{-1+k}\right|^p = \mu_{n_j}(f^{-n_j}(\{k\})) \le \mu_{n_j}(f^{-n_j}(B_i)) \to 0 \text{ as } j \to \infty, \tag{4.13}$$

which gives *(a)*. If *(b)* is false, that is, $\beta := \sup\{|w_i \cdots w_j| : i, j \in \mathbb{Z}, i \le j\} < \infty$, then

$$\mu_n(f^{-n}(B_i)) = \sum_{k\in B_i} |w_{-n+k} \cdots w_{-1+k}|^p \le \beta^p \mu(B_i), \text{ for all } i \in I,$$

contradicting condition *(B)*. $\square$

Proposition 4.13 was established in ([8], Corollary 1.6) in the case w is a bounded sequence of positive scalars.

Remark 4.14. We can remove the hypothesis that the weights w_n are nonzero in Proposition 4.13 provided we state the result as follows in the Proposition below.

Proposition 4.15. *Consider a bilateral weighted backward shift*

$$B_w : (x_n)_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z}) \mapsto (w_n x_{n+1})_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z}),$$

where $w := (w_n)_{n\in\mathbb{Z}}$ is a bounded sequence of scalars. Let

$$\alpha^- := \sup\{|w_i \cdots w_j| : i \le j \le -1\} \quad \text{and} \quad \alpha^+ := \sup\{|w_i \cdots w_j| : 0 \le i \le j\}.$$

Then B_w is Li-Yorke chaotic if and only if one the following conditions hold:

(I) $\alpha^+ = \infty$ *and* $\liminf_{n\to\infty} |w_{-n} \cdots w_{-k}| = 0$, *for some* $k \in \mathbb{Z}$.

(II) $\alpha^+ < \infty$, $\alpha^- = \infty$ *and* $\liminf_{n\to\infty} |w_{-n} \cdots w_{-k}| = 0$, *for all* $k \in \mathbb{N}$.

Proof. Let X, $\mathfrak{M}$, μ and f be as in the proof of Proposition 4.13. Suppose that $C_{w,f}$ is Li-Yorke chaotic and let $(B_i)_{i\in I}$ and $(n_j)_{j\in\mathbb{N}}$ be as in the statement of Theorem 4.8. Let

$$\alpha := \sup\{|w_i \cdots w_j| : i, j \in \mathbb{Z}, i \le j\} \text{ and } B' := \bigcup_{i\in I} B_i.$$

We claim that

$$\liminf_{n\to\infty} |w_{-n} \cdots w_{-1+\ell}| = 0, \ \forall \ell \in B'. \tag{4.14}$$

Indeed, let $\ell \in B_i$ for some $i \in I$. Since

$$\left|w_{-n_j+\ell} \cdots w_{-1+\ell}\right|^p = \mu_{n_j}(f^{-n_j}(\{\ell\})) \le \mu_{n_j}(f^{-n_j}(B_i)) \to 0 \text{ as } j \to \infty \text{ (by Theorem 4.8\textit{(A)})},$$

we conclude that (4.14) holds. Since $B_i \subset B' \in \mathcal{P}(\mathbb{Z})$, for all $i \in I$, we have that

$$\liminf_{n\to\infty} |w_{-n} \cdots w_{-k}| = 0, \text{ for some } k \in \mathbb{Z}. \tag{4.15}$$

Moreover, we must have $\alpha = \infty$. In fact, if $\alpha := \sup\{|w_i \cdots w_j| : i, j \in \mathbb{Z}, i \le j\} < \infty$, then

$$\mu_n(f^{-n}(B_i)) = \sum_{\ell\in B_i} |w_{-n+\ell} \cdots w_{-1+\ell}|^p \le \alpha^p \mu(B_i), \ \forall i \in I,$$

whence $\dfrac{\mu_n(f^{-n}(B_i))}{\mu(B_i)} \le \alpha^p$, for all $i \in I$ and for all $n \in \mathbb{N}$. This contradicts *(B)* in Theorem 4.8. Therefore, $\alpha = \infty$. If $\alpha^+ = \infty$, this fact, together with (4.15), leads us to condition *(I)* above. Thus, assume that $\alpha^+ < \infty$. Since $\alpha \le \max\{\alpha^-, \alpha^+, \alpha^-\alpha^+\}$, we must have $\alpha^- = \infty$. Thus, we have two possibilities for B' to consider.

- If B' is unbounded below, then (4.14) implies that condition *(II)* hold. Indeed, if condition *(II)* not hold, there is $k_0 \in \mathbb{N}$ such that

$$\liminf_{n\to\infty} |w_{-n} \cdots w_{-k_0}| > 0.$$

Hence, there exists $\ell_0 \in \mathbb{Z}^- \cup \{0\}$ such that $-k_0 = -1 + \ell_0$. Since $B' \in \mathcal{P}(\mathbb{Z})$ and we are considering B' to be unbounded below, we can assume, without loss of generality, that $\ell_0 \in B'$. Therefore,

$$\liminf_{n\to\infty} |w_{-n} \cdots w_{-1+\ell_0}| > 0, \text{ for } \ell_0 \in B',$$

which contradics (4.14).

- If B' is bounded below, let $m' := \min B'$. We claim that

$$w_n \ne 0, \text{ for all } n < m'. \tag{4.16}$$

In fact, suppose that exists $s < m'$ such that $w_s = 0$. Let

$$D := \sup\{|w_i \cdots w_j| : i, j \in \mathbb{Z}, s < i \leq j\}.$$

Since $\alpha^+ < \infty$, we have that $D < \infty$. Moreover, since $s < m' = \min B$, then $s \leq -1 + k$, for all $k \in B_i$ and for each $i \in I$ (because $s < m' \leq k$, $\forall k \in B'$). Now, fix any $i \in I$. For each $k \in B_i$ and each $n \in \mathbb{N}$, we have two possibilities:

(i') If $-n + k \leq s$, then $w_s = 0$ appears in the product $|w_{-n+k} \cdots w_{-1+k}|$. Thus,

$$|w_{-n+k} \cdots w_{-1+k}| = 0 \leq D.$$

(ii') If $-n + k > s$, then $s < -n + k < -1 + k$, whence

$$|w_{-n+k} \cdots w_{-1+k}| \in \{|w_i \cdots w_j| : i, j \in \mathbb{Z}, s < i \leq j\}.$$

Thus, $|w_{-n+k} \cdots w_{-1+k}| \leq D$. Therefore, by (i') and (ii'), we conclude that

$$\sup_{k \in B_i} |w_{-n+k} \cdots w_{-1+k}| \leq D, \ \forall n \in \mathbb{N},$$

which implies that

$$\mu_n(f^{-n}(B_i)) = \sum_{k \in B_i} |w_{-n+k} \cdots w_{-1+k}|^p \leq D^p \mu(B_i),$$

contradicting *(B)* of Theorem 4.8. By (4.14), we have that

$$\liminf_{n \to \infty} |w_{-n} \cdots w_{-1+m'}| = 0. \tag{4.17}$$

We will prove that (4.16) and (4.17) imply that *(II)* holds. Again, for each $k \in \mathbb{N}$, we have two possibilities to consider:

(iii') If $-k > -1 + m'$, then $-n < -1 + m' < -k$, for all $n \in \mathbb{N}$ sufficiently large. Thus, by (4.17), we have that

$$\begin{aligned}
\liminf_{n \to \infty} |w_{-n} \cdots w_{-k}| &= \liminf_{n \to \infty} |w_{-n} \cdots w_{-1+m'} w_{m'} \cdots w_{-k}| \\
&= |w_{m'} \cdots w_{-k}| \liminf_{n \to \infty} |w_{-n} \cdots w_{-1+m'}| \\
&= 0.
\end{aligned}$$

(iv') If $-k < -1 + m'$, then $-n < -k < -k + 1 \leq m'$, for all $n \in \mathbb{N}$ sufficiently large, whence (4.16) ensures that $|w_{-k+1} \cdots w_{-1+m'}| \neq 0$. Therefore, by (4.17), we have that

$$\begin{aligned}
\liminf_{n\to\infty} |w_{-n} \cdots w_{-k}| &= \liminf_{n\to\infty} |w_{-n} \cdots w_{-k}| \frac{|w_{-k+1} \cdots w_{-1+m'}|}{|w_{-k+1} \cdots w_{-1+m'}|} \\
&= \frac{1}{|w_{-k+1} \cdots w_{-1+m'}|} \liminf_{n\to\infty} |w_{-n} \cdots w_{-1+m'}| \\
&= 0.
\end{aligned}$$

Hence, for each $k \in \mathbb{N}$, we conclude that $\liminf_{n\to\infty} |w_{-n} \cdots w_{-k}| = 0$, which implies *(II)*.

Conversely, suppose that *(I)* or *(II)* hold. We will prove that $C_{w,f}$ is Li-Yorke chaotic. We can replace "lim inf" by "lim" in *(I)* and *(II)*, since

$$\liminf_{n\to\infty} |w_{-n} \cdots w_{-k}| = 0 \quad \text{and} \quad \limsup_{n\to\infty} |w_{-n} \cdots w_{-k}| > 0$$

imply that $\chi_{\{-k+1\}}$ is a semi-irregular vector for $C_{w,f}$, and so $C_{w,f}$ is Li-Yorke chaotic. Indeed, for each $n \in \mathbb{N}$ and each $k \in \mathbb{Z}$, we have that

$$\begin{aligned}
\left\|(C_{w,f})^n \left(\chi_{\{-k+1\}}\right)\right\|_p^p &= \int_X \left|\chi_{\{-k+1\}} \circ f^n\right|^p \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p |w|^p \, d\mu \\
&= \int_{f^{-n}(\{-k+1\})} \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p |w|^p \, d\mu \\
&= \int_{\{-k+1-n\}} \left|w \circ f^{n-1}\right|^p \cdots |w \circ f|^p |w|^p \, d\mu \\
&= \left|w_{-n-k+1} \cdots w_{-k+1-n+n-1}\right|^p \\
&= |w_{-n-k+1} \cdots w_{-k}|^p .
\end{aligned}$$

Hence,

$$\liminf_{n\to\infty} \left\|(C_{w,f})^n \left(\chi_{\{-k+1\}}\right)\right\|_p = \liminf_{n\to\infty} |w_{-n-k+1} \cdots w_{-k}| = \liminf_{n\to\infty} |w_{-n} \cdots w_{-k}| = 0,$$

and

$$\limsup_{n\to\infty} \left\|(C_{w,f})^n \left(\chi_{\{-k+1\}}\right)\right\|_p = \limsup_{n\to\infty} |w_{-n-k+1} \cdots w_{-k}| = \limsup_{n\to\infty} |w_{-n} \cdots w_{-k}| > 0,$$

which confirms that $\chi_{\{-k+1\}}$ is a semi-irregular vector for $C_{w,f}$, for all $k \in \mathbb{Z}$.

If *(I)* holds, let $I := \{i \in \mathbb{Z} : i > -k\}$. For each $i \in I$, let $B_i := \{i\}$. Since $\lim\limits_{n\to\infty} |w_{-n} \cdots w_{-k}| = 0$, then

$$\begin{aligned}
\lim_{n\to\infty} \mu_n(f^{-n}(B_i)) &= \lim_{n\to\infty} \mu_n(f^{-n}(\{i\})) &&= \lim_{n\to\infty} \mu_n(\{-n+i\}) \\
\lim_{n\to\infty} |w_{-n+i} \cdots w_{-n+i+n-1}|^p &= \lim_{n\to\infty} |w_{-n+i} \cdots w_{-1+i}|^p &&= \lim_{n\to\infty} |w_{-n} \cdots w_{-k} \cdots w_{-1+i}|^p \\
& &&= 0.
\end{aligned}$$

Hence, condition *(A)* in Theorem 4.8 holds. Now, for each $n \in \mathbb{N}$, choose $i \in \mathbb{N}$ large enough so that

$$i' := -n + i > 0 \quad \text{and} \quad j' := -1 + i.$$

Hence, $j' \geq i'$ and so

$$\begin{aligned}
\sup\left\{ \frac{\mu_n(f^{-n}(B_i))}{\mu(B_i)} : i \in I, n \in \mathbb{N} \right\} &= \sup\{\mu_n(f^{-n}(B_i)) : i \in I, n \in \mathbb{N}\} \\
&= \sup\{|w_{-n+i} \cdots w_{-1+i}|^p : i \in I, n \in \mathbb{N}\} \\
&= \sup\{|w_{i'} \cdots w_{j'}|^p : 0 \leq i' \leq j'\} \\
&= \alpha^+ = \infty.
\end{aligned}$$

Thus, we have that *(I)* implies condition (B) of Theorem 4.8, whence $C_{w,f}$ is Li-Yorke chaotic.

If *(II)* holds, let $I := \{i \in \mathbb{Z} : i \leq -1\}$. For each $i \in I$, let $B_i := \{i\}$. Since $\lim\limits_{n\to\infty} |w_{-n} \cdots w_{-k}| = 0$ for all $k \in \mathbb{N}$, then

$$\begin{aligned}
\lim_{n\to\infty} \mu_n(f^{-n}(B_i)) = \lim_{n\to\infty} \mu_n(f^{-n}(\{i\})) &= \lim_{n\to\infty} \mu_n(\{-n+i\}) \\
&= \lim_{n\to\infty} |w_{-n+i} \cdots w_{-n+i+n-1}|^p \\
&= \lim_{n\to\infty} |w_{-n+i} \cdots w_{-1+i}|^p \\
&= 0.
\end{aligned}$$

Again, condition *(A)* of Theorem 4.8 holds with $(n_j)_{j\in\mathbb{N}} := (n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$ and $i \in I$, we define

$$i' := -n + i \in \mathbb{Z} \quad \text{and} \quad j' := -1 + i \in \mathbb{Z}.$$

Thus, $i' \leq j' \leq -1$. Since $\mu(B_i) = 1$, for all $i \in I$, we have that

$$\begin{aligned}
\sup\left\{\frac{\mu_n(f^{-n}(B_i))}{\mu(B_i)} : i \in I, n \in \mathbb{N}\right\} &= \sup\{\mu_n(f^{-n}(B_i)) : i \in I, n \in \mathbb{N}\} \\
&= \sup\{|w_{-n+i} \cdots w_{-1+i}|^p : i \in I, n \in \mathbb{N}\} \\
&= \sup\{|w_{i'} \cdots w_{j'}|^p : i' \leq j' \leq -1\} \\
&= \alpha^- = \infty.
\end{aligned}$$

Hence, condition *(B)* of Theorem 4.8 holds. Therefore, *(II)* implies conditions *(A)* and *(B)* of Theorem 4.8, whence $C_{w,f}$ is a Li-Yorke chaotic operator. $\square$

Chapter 5

Weighted shifts on Fréchet sequence spaces

In this chapter we complement the previous results on Li-Yorke chaos for weighted shifts on the classical Banach sequence spaces c_0 and ℓ^p by considering weighted shifts on the more general setting of abstract Fréchet sequence spaces. More precisely, we will present a trichotomy that complements the study of Li-Yorke chaos for linear operators developed in [6]. As applications, we will obtain characterizations of the Li-Yorke chaotic weighted shifts on Köthe sequence spaces that depend only on the entries of the Köthe matrix and the weights of the shift.

5.1 Trichotomy

Before presenting the main result of this section, we will make some relevant considerations. In that follows, we will consider $J := \mathbb{N}$ or $\mathbb{Z}$.

Definition 5.1. *A Fréchet sequence space over J is a Fréchet space X which is a vector subspace of the product space $\mathbb{K}^J$ such that the inclusion map*

$$i : X \to \mathbb{K}^J$$

is continuous, i.e., convergence in X implies coordinatewise convergence.

For $J = \mathbb{N}$, if $w := (w_n)_{n\in\mathbb{N}}$ is a sequence of nonzero scalars, the closed graph theorem implies that the unilateral weighted backward shift

$$B_w(x_1, x_2, x_3, \cdots) := (w_1 x_2, w_2 x_3, w_3 x_4, \cdots)$$

is a continuous linear operator on X as soon as it maps X into itself. Indeed, suppose that B_w maps X into itself. Since B_w is clearly linear, it remains to show that B_w is continuous. Since convergence in X implies coordinatewise convergence, it follows that the graph $\mathrm{Gr}(B_w)$ of B_w is closed in $X \times X$. Hence, B_w is continuous by the closed graph theorem[1].

The canonical vectors $e_n := (\delta_{n,j})_{j\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ $(n \in \mathbb{N})$ form a *basis* of X if they belong to X and

$$x = \sum_{n=1}^{\infty} x_n e_n, \quad \text{for all} \quad x := (x_n)_{n\in\mathbb{N}} \in X.$$

Now, consider $J = \mathbb{Z}$. As before, if $w := (w_n)_{n\in\mathbb{Z}}$ is a sequence of nonzero scalars, then the bilateral weighted backward shift

$$B_w\left((x_n)_{n\in\mathbb{Z}}\right) := (w_n x_{n+1})_{n\in\mathbb{Z}}$$

is a continuous linear operator on X as soon as it maps X into itself. The canonical vectors $e_n := (\delta_{n,j})_{j\in\mathbb{Z}} \in \mathbb{K}^{\mathbb{Z}}$ $(n \in \mathbb{Z})$ form a *basis* of X if they belong to X and

$$x = \sum_{n=-\infty}^{\infty} x_n e_n, \quad \text{for all} \quad x := (x_n)_{n\in\mathbb{Z}} \in X.$$

Theorem 5.2 (Trichotomy). *Let T be a continuous linear operator on a Fréchet space X. If*

$$NS(T) := \left\{x \in X : (T^n x)_{n\in\mathbb{N}} \text{ has a subsequence converging to zero}\right\},$$

then either

(a) T has a residual set of irregular vectors, or

(b) $\lim_{n\to\infty} T^n x = 0$ for every $x \in X$, or

(c) $NS(T)$ is not dense in X.

[1] Let X, Y be Fréchet spaces. The **closed graph theorem** states that if a linear map $T : X \to Y$ has closed graph, that is, if $x_n \to x$ in X and $T(x_n) \to y$ in Y imply that $T(x) = y$, then T is continuous. (c.f [22]).

Moreover:

(A) If $NS(T)$ is not closed in X, then T is Li-Yorke chaotic.

(B) If X is separable and $NS(T)$ is dense and not closed in X, then T is densely Li-Yorke chaotic.

Proof. It is clear that three properties (a), (b) and (c) are mutually exclusive.

If (c) is true, we have nothing to prove. Thus, let us assume that $NS(T)$ is dense in X (hence, by Corollary 2.39, $NS(T)$ is residual in X) and we will prove that either (a) or (b) must be true.

If T has an irregular vector, then the dense Li-Yorke chaos criterion holds (Definition 2.45). Indeed, let $X_0 := NS(T)$ and $a_n = x$, for all $n \in \mathbb{N}$, where x is an irregular vector for T. Then, X_0 satisfies (a) of Definition 2.43. Since x is an irregular vector for T, we have that $(T^n a_n)_{n\in\mathbb{N}}$ is unbounded, showing that (b) of Definition 2.43 holds. Hence, by Theorem 2.46, (a) is true.

If T does not have an irregular vector, then Theorem 2.41 guarantees that T does not have a semi-irregular vector either. Let

$$\widehat{NS}(T) := \left\{x \in X : \lim_{n\to\infty} T^n x = 0\right\}.$$

Then,

$$NS(T) = \widehat{NS}(T). \tag{5.1}$$

In fact, the inclusion $\widehat{NS}(T) \subset NS(T)$ is clear. If $x \in NS(T)$, since no vector $x \in X$ is semi-irregular, it follows from Definition 2.36 that $T^n x \to 0$ as $n \to \infty$. Hence, $x \in \widehat{NS}(T)$. Since $NS(T)$ is a residual subset of X and (5.1) holds, we have that $\widehat{NS}(T) = X$. Indeed, given $x \in X$, since $\widehat{NS}(T)$ and $x + \widehat{NS}(T)$ are both residual in X, the intersection

$$\widehat{NS}(T) \cap (x + \widehat{NS}(T))$$

is nonempty. If $y \in \widehat{NS}(T) \cap (x + \widehat{NS}(T))$, then $y \in x + \widehat{NS}(T)$, i.e., there is $m \in \widehat{NS}(T)$ such that $y = x + m$, which implies $x \in \widehat{NS}(T)$ (because $y \in \widehat{NS}(T)$). Thus, $\widehat{NS}(T) = X$ and (b) is true.

In order to prove (A), we assume that T is not Li-Yorke chaotic. Let $M := \widehat{NS}(T)$ and $Y := \overline{M}$. Consider the operator $S : Y \to Y$ obtained by restricting T to Y. Since T is not Li-Yorke chaotic, T has no semi-irregular vector, and so

$$NS(T) = M. \tag{5.2}$$

Furthermore, since S is not Li-Yorke chaotic, it follows that S has no irregular vectors, hence (a) does not hold for S. In addition, (c) is also false for S, because $NS(S) = NS(T) = M$ is dense in Y. Thus, (b) must hold for the operator S, which implies that

$$NS(T) = M = \Big\{x \in X : \lim_{n\to\infty} T^n x = 0\Big\} = \Big\{x \in Y : \lim_{n\to\infty} S^n x = 0\Big\} = Y = \overline{M}.$$

Hence, $NS(T)$ is closed and this concludes the proof of (A).

Finally, assume the hypotheses of (B). Since $NS(T)$ is dense in X, (c) does not hold. Since $NS(T)$ is not closed in X, (A) ensures that T is Li-Yorke chaotic, and so (b) is also false. By the above trichotomy, (a) must be true. Since we are assuming that X is separable, T is densely Li-Yorke chaotic by Theorem 2.42. $\square$

Remark 5.3. The converses of (A) and (B) are false in general. For instance, recall that every generically Li-Yorke chaotic operator is also a densely Li-Yorke chaotic operator. But for every generically Li-Yorke chaotic operator T, we have that $NS(T) = X$ by item *(ii)* of Theorem 2.47, which states that T being generically Li-Yorke chaotic is equivalent to every non-zero vector being semi-irregular for T.

Corollary 5.4. *Suppose that X is a Fréchet sequence space in which the sequence $(e_n)_{n\in\mathbb{N}}$ of canonical vectors is a basis, $w := (w_n)_{n\in\mathbb{N}}$ is a sequence of nonzero scalars, and the unilateral weighted backward shift B_w is a well-defined operator on X. Then either*

(a') B_w is densely Li-Yorke chaotic, or

(b') $\lim_{n\to\infty}(B_w)^n x = 0$ for every $x \in X$.

Proof. Note that $NS(B_w)$ is dense in X, because

$$SF(X) := \{x \in X : x \text{ has a finite support}\}$$

is dense in X and $SF(X) \subset NS(B_w)$. Hence, it follows from the above trichotomy that *(a)* or *(b)* must be true.

If *(a)* holds, then *(a')* is true, by Theorem 2.42.

If *(b)* holds, then *(b')* is automatically true. $\square$

In other words, the unilateral weighted backward shift B_w on X is Li-Yorke chaotic (provided it is well-defined, of course) if and only if there is a vector $(x_n)_{n\in\mathbb{N}} \in X$ such that

$$(w_1 \cdots w_n x_{n+1}, w_2 \cdots w_{n+1} x_{n+2}, w_3 \cdots w_{n+2} x_{n+3}, \cdots) \nrightarrow 0 \text{ in } X \text{ as } n \to \infty.$$

For example, this is not the case if $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, for some $p \in [1, \infty)$, but it is true if $X = \mathbb{K}^{\mathbb{N}}$. In particular, the unilateral (unweighted) backward shift B on X is Li-Yorke chaotic (provided it is well-defined, of course) if and only if there is a vector $(x_n)_{n\in\mathbb{N}} \in X$ such that

$$(x_{n+1}, x_{n+2}, x_{n+3}, \cdots) \nrightarrow 0 \text{ in } X \text{ as } n \to \infty.$$

Corollary 5.5. *Suppose that X is a Fréchet sequence space over $\mathbb{Z}$ in which the sequence $(e_n)_{n\in\mathbb{Z}}$ of canonical vectors is a basis, $w := (w_n)_{n\in\mathbb{Z}}$ is a sequence of nonzero scalars, and the bilateral weighted backward shift B_w is a well-defined operator on X. If the sequence $(w_{-n} \cdots w_{-1} e_{-n})_{n\in\mathbb{N}}$ has a subsequence converging to zero, then either*

(a) B_w is densely Li-Yorke chaotic, or

(b) $\lim_{n\to\infty} (B_w)^n x = 0$, for every $x \in X$.

Proof. The additional hypothesis on the weights means that there is an increasing sequence $(n_j)_{j\in\mathbb{N}}$ of positive integers such that

$$(B_w)^{n_j} e_0 = w_{-n_j} \cdots w_{-1} e_{-n_j} \to 0 \text{ as } j \to \infty. \tag{5.3}$$

For each $k \geq 1$, there is a constant $a_k \neq 0$ such that

$$(B_w)^k e_k = a_k e_0, \text{ where } a_k = w_0 \cdots w_{k-1}.$$

For each $k \leq -1$, there is a constant $b_k \neq 0$ such that

$$(B_w)^{-k} e_0 = b_k e_k, \text{ where } b_k = w_k \cdots w_{-1}.$$

Hence, for every $k \in \mathbb{Z}$,

$$(B_w)^{n_j+k} e_k \to 0 \text{ as } j \to \infty. \tag{5.4}$$

Indeed, if $k = 0$, we have nothing to prove, by (5.3). If $k \geq 1$, then

$$(B_w)^{n_j+k} e_k = (B_w)^{n_j} \left((B_w)^k e_k \right) = (B_w)^{n_j} (a_k e_0) = a_k (B_w)^{n_j} e_0 \to 0 \text{ as } j \to \infty, \text{ by (5.3)}.$$

If $k \leq -1$, then

$$\begin{aligned}(B_w)^{n_j+k} e_k &= (B_w)^{n_j+k} \left(b_k^{-1} (B_w)^{-k} e_0\right) \\ &= b_k^{-1} (B_w)^{n_j} e_0 \to 0 \text{ as } j \to \infty, \text{ by } (5.3).\end{aligned}$$

Given $m \geq 1$ and $k \in \{-m, \cdots, m\}$, by applying $(B_w)^{m-k}$ in (5.4), we see that

$$(B_w)^{n_j+m} e_k \to 0 \text{ as } j \to \infty.$$

Thus, $(B_w)^{n_j+m} e_k \to 0$ as $j \to \infty$, for every $x \in \operatorname{span}(\{e_{-m}, \cdots, e_m\})$. This proves that the set $NS(B_w)$ contains every bilateral sequence of finite support, and so it is dense in X. Therefore, the result follows from Theorem 5.2. □

Remark 5.6. The additional hypothesis on the weights is essential for the validity of the above Corollary. Indeed, let B be the unweighted backward shift on $\ell^2(\mathbb{Z})$. In this case, the weight sequence w is the constant map 1, which does not satisfies conditions *(a)* and *(b)* of Proposition 4.13. Hence, B is not Li-Yorke chaotic. In particular, Corollary 5.5*(a)* is not true. Moreover, Corollary 5.5*(b)* is also not valid, because

$$\lim_{n\to\infty} B^n x = 0 \Leftrightarrow x = 0.$$

Theorem 5.7. *Suppose that X is a Fréchet sequence space over $\mathbb{Z}$ in which the sequence $(e_n)_{n\in\mathbb{Z}}$ of canonical vectors is a basis, $w := (w_n)_{n\in\mathbb{Z}}$ is a sequence of nonzero scalars, and the bilateral weighted backward shift B_w is a well-defined operator on X. Then B_w is Li-Yorke chaotic if and only if the following conditions hold:*

- *(a) The sequence $(w_{-n} \cdots w_{-1} e_{-n})_{n\in\mathbb{N}}$ has a subsequence converging to zero.*
- *(b) There is a vector $(x_n)_{n\in\mathbb{Z}} \in X$ such that*

$$(w_j \cdots w_{j+n-1} x_{j+n})_{j\in\mathbb{Z}} \nrightarrow 0 \text{ in } X \text{ as } n \to \infty.$$

Proof. Suppose that B_w is Li-Yorke chaotic and let $x := (x_n)_{n\in\mathbb{Z}} \in X$ be a semi-irregular vector for B_w. Since $(B_w)^n(x) \nrightarrow 0$ as $n \to \infty$, *(b)* holds (because $(B_w)^n(x) = (w_j \cdots w_{j+n-1} x_{j+n})_{j\in\mathbb{Z}}$). Choose $k \in \mathbb{Z}$ such that $x_k \neq 0$ and let $(n_j)_{j\in\mathbb{N}}$ be an increasing sequence of positive integers such that $(B_w)^{n_j}(x) \to 0$ as $j \to \infty$ (such a sequence exists because x is a semi-irregular vector for B_w). Given a neighborhood V of 0 in X, the equicontinuity of the family of operators

$$y \in X \mapsto y_n e_n \in X, \; n \in \mathbb{Z}$$

implies the existence of a neighborhood U of 0 in X such that

$$y \in U \Rightarrow y_n e_n \in V, \; \forall n \in \mathbb{Z}.$$

Since $(B_w)^{n_j}(x) \to 0$ as $j \to \infty$, there exists $j_0 \in \mathbb{N}$ such that

$$j \geq j_0 \Rightarrow (B_w)^{n_j}(x) \in U \Rightarrow w_{k-n_j} \cdots w_{k-1} x_k e_{k-n_j} \in V.$$

This proves that $w_{k-n_j} \cdots w_{k-1} x_k e_{k-n_j} \to 0$ as $j \to \infty$, which implies *(a)*.

Conversely, if *(a)* and *(b)* hold, then Corollary 5.5*(b)* is not true. Thus, condition *(a)* of Corollary 5.5 holds, i.e., B_w is (densely) Li-Yorke chaotic. $\square$

In particular, the bilateral (unweighted) backward shift B on X is Li-Yorke chaotic (provided it is well-defined, of course) if and only if the sequence $(e_{-n})_{n\in\mathbb{N}}$ has a subsequence converging to zero in X and there is a vector $(x_n)_{n\in\mathbb{Z}} \in X$ such that

$$(x_{j+n})_{j\in\mathbb{Z}} \nrightarrow 0 \text{ in } X \text{ as } n \to \infty.$$

For example, this is true for $X = \mathbb{K}^{\mathbb{Z}}$, but this is not the case if $X = c_0(\mathbb{Z})$ or $\ell^p(\mathbb{Z})$ for some $p \in [1, \infty)$. It is easy verify that condition *(b)* of Theorem 5.7 holds in the tree cases: for $X = c_0(\mathbb{Z})$ or $\ell^p(\mathbb{Z})$, just take any nonzero vector $x \in X$; for $X = \mathbb{K}^{\mathbb{Z}}$ just take $x = (x_n)_{n\in\mathbb{Z}} \in X$, with $x_n = 1$, for all $n \in \mathbb{Z}$, for example. However, condition *(a)* of Theorem 5.7 fails for $X = c_0(\mathbb{Z})$ or $\ell^p(\mathbb{Z})$: in this case, $\|e_{-n}\| = 1$, for all $n \in \mathbb{N}$. But, $(e_{-n})_{n\in\mathbb{N}}$ has a subsequence converging to zero in product topology over $\mathbb{K}^{\mathbb{Z}}$.

It is known that B is *hypercyclic* (i.e., it has a dense orbit) if and only if there is an increasing sequence $(n_j)_{j\in\mathbb{N}}$ of positive integers such that, for any $k \in \mathbb{Z}$,

$$e_{k-n_j} \to 0 \text{ and } e_{k+n_j} \to 0 \text{ in } X \text{ as } j \to \infty \tag{5.5}$$

(see Theorem 4.12 of [22]). With these characterizations it is easy to exhibit such a shift B which is Li-Yorke chaotic but not hypercyclic (although stronger and more sophisticated examples can be found in [27]). An illustration of this is shown in the following example.

Example 5.8. Consider the weight sequence $\nu := (\nu_n)_{n\in\mathbb{Z}}$, where $\nu_{-n} := \dfrac{1}{n}$ for all $n \geq 1$, and

$$(\nu_n)_{n\geq 0} := (1, 2, 1, 2, 2^2, 2, 1, 2, 2^2, 2^3, 2^2, 2, 1, \cdots, 1, 2, 2^2, \cdots, 2^n, \cdots, 2^2, 2, 1, \cdots),$$

a concatenation of blocks of the form $(1, 2, 2^2, \cdots, 2^n, \cdots, 2^2, 2)$, for $n \geq 1$. Fix $p \in [1, \infty)$ and consider the weighted ℓ^p space

$$\ell^p(\mathbb{Z}, \nu) := \left\{ x := (x_n)_{n\in\mathbb{Z}} \in \mathbb{K}^{\mathbb{Z}} : \|x\|_p^p := \sum_{n=-\infty}^{\infty} |x_n|^p \nu_n < \infty \right\}.$$

Note that the bilateral backward shift B is a well-defined operator on $\ell^p(\mathbb{Z}, \nu)$. In fact, as seen in [22], B is a well-defined operator on $\ell^p(\mathbb{Z}, \nu)$ if and only if $\sup\limits_{n\in\mathbb{Z}} \dfrac{\nu_n}{\nu_{n+1}} < \infty$. Thus, in the case of the sequence of weights considered in this example, we have that

$$\frac{\nu_{-1}}{\nu_0} = 1, \quad \sup_{n\geq 2} \frac{\nu_{-n}}{\nu_{-n+1}} = \sup_{n\geq 2} \frac{1/n}{\frac{1}{n-1}} = \sup_{n\geq 2} \frac{n-1}{n} = \sup_{n\geq 2} \left(1 - \frac{1}{n}\right) = 1 \quad \text{and} \quad \sup_{n\geq 0} \frac{\nu_n}{\nu_{n+1}} = 2,$$

whence B is a well-defined operator.

We claim that B is not hypercyclic. Indeed, for each $n \geq 0$, we have that

$$\|e_n\|_p = \nu_n \geq 1.$$

Then, condition (5.5) is not satisfied, hence B is not hypercyclic.

However, B is Li-Yorke chaotic. In order to see this, let $j_n := n(n+1) + 1$ for each $n \in \mathbb{N}$, and consider the vector

$$x := \sum_{n=1}^{\infty} 2^{-\frac{n}{p}} e_{j_n} \in \ell^p(\mathbb{Z}, \nu).$$

Since $e_{-n} \to 0$ as $n \to \infty$ (because $\|e_{-n}\|_p = 1/n$), then condition *(a)* of Theorem 5.7 is satisfied. Moreover, for each $m \in \mathbb{N}$,

$$\begin{aligned} \|B^m x\|_p^p &= \left\| B^m \left(\sum_{n=1}^{\infty} 2^{-\frac{n}{p}} e_{j_n} \right) \right\|_p^p = \left\| \sum_{n=1}^{\infty} 2^{-\frac{n}{p}} B^m (e_{j_n}) \right\|_p^p \geq \left\| 2^{-\frac{m}{p}} B^m (e_{j_m}) \right\|_p^p \\ &= \left\| 2^{-\frac{m}{p}} e_{j_m - m} \right\|_p^p = \left\| 2^{-\frac{m}{p}}\ e_{m^2+1} \right\|_p^p = 2^{-m} v_{m^2+1} = \frac{1}{2}. \end{aligned}$$

Hence, $B^n x \nrightarrow 0$ as $n \to \infty$, showing that condition *(b)* of Theorem 5.7 also holds. Thus, B is Li-Yorke chaotic, as we wanted to demonstrate.

5.2 Weighted backward shifts on $\lambda_p(A, J)$

In this section, we shall focus on a more concrete class of Fréchet sequence spaces known as Köthe sequence spaces. Let $J := \mathbb{N}$ or $\mathbb{Z}$.

Definition 5.9. A *Köthe matrix (on J)* is a family of scalars of the form $A := (a_{j,k})_{j\in J, k\in\mathbb{N}}$ such that, for each $j \in J$, there exists $k \in \mathbb{N}$ with $a_{j,k} > 0$, and $0 \le a_{j,k} \le a_{j,k+1}$ for all $j \in J$ and $k \in \mathbb{N}$.

Given a real number $p \in \{0\} \cup [1, \infty)$ and such a Köthe matrix A, the associated *Köthe sequence space* $\lambda_p(A, J)$ is the Fréchet sequence space defined as follows:

- If $p \in [1, \infty)$, then

$$\lambda_p(A, J) := \left\{ (x_j)_{j\in J} \in \mathbb{K}^J : \sum_{j\in J} |a_{j,k} x_j|^p < \infty,\ \forall k \in \mathbb{N} \right\}$$

  endowed with the seminorms

$$\|x\|_k := \left(\sum_{j\in J} |a_{j,k} x_j|^p \right)^{\frac{1}{p}} \quad \text{for } x := (x_j)_{j\in J} \in \lambda_p(A, J) \ \text{ and } \ k \in \mathbb{N}.$$

- If $p = 0$, then

$$\lambda_p(A, J) := \left\{ (x_j)_{j\in J} \in \mathbb{K}^J : \lim_{j\in J, |j|\to\infty} a_{j,k} x_j = 0,\ \forall k \in \mathbb{N} \right\}$$

  endowed with the seminorms

$$\|x\|_k := \sup_{j\in J} |a_{j,k} x_j| \quad \text{for } x := (x_j)_{j\in J} \in \lambda_p(A, J) \ \text{ and } \ k \in \mathbb{N}.$$

Note that the sequence $(e_j)_{j\in J}$ of canonical vectors in $\mathbb{K}^J$ is a basis of $\lambda_p(A, J)$ for every $p \in \{0\} \cup [1, \infty)$.

Throughout the remaining of this section, we adopt the convention $\frac{0}{0} = 1$.

Proposition 5.10. *Let $w := (w_j)_{j\in J}$ be a sequence of nonzero scalars. The weighted backward shift*

$$B_w\left((x_j)_{j\in J}\right) := (w_j x_{j+1})_{j\in J}$$

is a well-defined operator on $\lambda_p(A,J)$ if and only if, for each $k\in\mathbb{N}$, there exists $m\in\mathbb{N}$ such that $a_{j,k}=0$ whenever $a_{j+1,m}=0$ ($j\in J$), and

$$\sup_{j\in J}\frac{a_{j,k}|w_j|}{a_{j+1,m}}<\infty. \tag{5.6}$$

Proof. If B_w is a well-defined operator on $\lambda_p(A,J)$, given $k\in\mathbb{N}$, there are $m\in\mathbb{N}$ and $C\in(0,\infty)$ such that

$$\left\|B_w\left((x_j)_{j\in J}\right)\right\|_k \le C\left\|(x_j)_{j\in J}\right\|_m,\ \forall (x_j)_{j\in J}\in\lambda_p(A,J).$$

Then, for each $j\in J$, we have that

$$|w_j|a_{j,k} = \|w_je_j\|_k = \|B_w(e_{j+1})\|_k \le C\|e_{j+1}\|_m = Ca_{j+1,m}. \tag{5.7}$$

Since $w_j\neq 0$, for all $j\in J$, if $a_{j+1,m}=0$, then $a_{j,k}=0$ by (5.7). Moreover, if $a_{j+1,m}\neq 0$, again by (5.7), we have that

$$\frac{a_{j,k}|w_j|}{a_{j+1,m}}\le C,$$

whence we obtain (5.6).

Conversely, given $k\in\mathbb{N}$, there exists $m\in\mathbb{N}$ such that

$$a_{j+1,m}=0\Rightarrow a_{j,k}=0,\quad \forall j\in J$$

and there exists $M\in(0,\infty)$ such that

$$M=\sup_{j\in J}\frac{a_{j,k}|w_j|}{a_{j+1,m}}.$$

Then, for each $j\in J$, we have that $a_{j,k}|w_j|\le Ma_{j+1,m}$.

If $p=[1,\infty)$, then for each $(x_j)_{j\in J}\in\lambda_p(A,J)$, we obtain

$$\begin{aligned}\left\|B_w\left((x_j)_{j\in J}\right)\right\|_k^p &= \sum_{j\in J}|a_{j,k}w_j|^p|x_{j+1}|^p \le \sum_{j\in J}|Ma_{j+1,m}|^p|x_{j+1}|^p\\ &= M^p\sum_{j\in J}|a_{j+1,m}x_{j+1}|^p\\ &= M^p\left\|(x_j)_{j\in J}\right\|_m^p.\end{aligned}$$

If $p = 0$, then for each $(x_j)_{j\in J} \in \lambda_0(A, J)$:

$$\|B_w((x_j)_{j\in J})\|_k = \sup_{j\in J} |a_{j,k} w_j x_{j+1}| \leq M \sup_{j\in J} |a_{j+1,m} x_{j+1}| = M \|(x_j)_{j\in J}\|_m .$$

Therefore, for any $p \in \{0\} \cup [1, \infty)$, B_w is a well-defined bounded operator. □

We refer the reader to the books [24] and [28] for a detailed study of Köthe sequence spaces.

The following results characterize Li-Yorke chaos for unilateral and bilateral weighted backward shifts. We will present the proof for the bilateral case in detail. For the unilateral case, being analogous to the previous one, we will present the proof more briefly.

Theorem 5.11. *Consider a Köthe sequence space $X := \lambda_p(A, \mathbb{Z})$, where $A := (a_{j,k})$ is a Köthe matrix on $\mathbb{Z}$ with nonzero entries and $p \in \{0\} \cup [1, \infty)$. Let $w := (w_n)_{n\in\mathbb{Z}}$ be a sequence of nonzero scalars such that the bilateral weighted backward shift B_w is a well-defined operator on X. Then B_w is Li-Yorke chaotic if and only if the following conditions hold:*

(a') There is an increasing sequence $(n_i)_{i\in\mathbb{N}}$ of positive integers such that

$$\lim_{i\to\infty} a_{-n_i,k} w_{-n_i} \cdots w_{-1} = 0, \quad \forall k \in \mathbb{N}.$$

(b') There exists $k \in \mathbb{N}$ such that

$$\sup\left\{ \frac{a_{i,k} |w_i \cdots w_j|}{a_{j+1,\ell}} : i, j \in \mathbb{Z}, i \leq j \right\} = \infty, \quad \forall \ell \in \mathbb{N}.$$

Proof. Suppose that B_w is Li-Yorke chaotic. We know that property *(a')* holds, since it is equivalent to property (a) in Theorem 5.7. Suppose, by contradiction, that property *(b')* is false. Then, for each $k \in \mathbb{N}$, there exists $\ell_k \in \mathbb{N}$ such that

$$C_k := \sup\left\{ \frac{a_{i,k} |w_i \cdots w_{j'}|}{a_{j'+1,\ell_k}} : i, j' \in \mathbb{Z},\ i \leq j' \right\} < \infty. \tag{5.8}$$

Take $x := (x_j)_{j\in\mathbb{Z}} \in X$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$. If $p \in [1, \infty)$, then

$$\|B_w^n(x)\|_k^p = \sum_{j\in\mathbb{Z}} |a_{j,k} w_j \cdots w_{j+n-1} x_{j+n}|^p \leq \sum_{j\in\mathbb{Z}} |C_k a_{j+n,\ell_k} x_{j+n}|^p = C_k^p \|x\|_{\ell_k}^p . \tag{5.9}$$

Indeed, taking $i := j$ and $j' := j + n - 1$ in (5.8), we obtain that

$$\frac{a_{j,k} |w_j \cdots w_{j+n-1}|}{a_{j+n,\ell_k}} \leq C_k \Rightarrow a_{j,k} |w_j \cdots w_{j+n-1}| \leq C_k a_{j+n,\ell_k},$$

whence the inequality in (5.9) holds. If $p = 0$, then

$$\|B_w^n(x)\|_k = \sup_{j\in\mathbb{Z}} |a_{j,k} w_j \cdots w_{j+n-1} x_{j+n}| \le \sup_{j\in\mathbb{Z}} |C_k a_{j+n,\ell_k} x_{j+n}| = C_k \|x\|_{\ell_k}.$$

In any case, we see that the sequence $(B_w^n(x))_{n\in\mathbb{N}}$ is bounded for all $x \in X$. Hence, B_w does not admit an irregular vector, which contradicts the fact that B_w is Li-Yorke chaotic.

Conversely, suppose that properties *(a')* and *(b')* hold. We shall prove that B_w is Li-Yorke chaotic by applying Theorem 5.7. Since (a) of Theorem 5.7 is equivalent to property *(a')*, it remains to show that property (b) of the Theorem 5.7 is also true. For this purpose, fix $k \in \mathbb{N}$ as in *(b')*. For each $n \in \mathbb{N}$, the continuity of $(B_w)^n$ implies the existence of a constant $C_n \in (0, \infty)$ and an integer $r_n \in \mathbb{N}$ such that

$$\|B_w^n(x)\|_k \le C_n \|x\|_{r_n} \quad \forall x \in X. \tag{5.10}$$

Now, for each $n \in \mathbb{N}$, consider

$$E_n := \sup\left\{ \frac{a_{i,k} |w_i \cdots w_j|}{a_{j+1,r_n}} : i, j \in \mathbb{Z},\ j - i = n - 1 \right\}.$$

Applying (5.10) to the canonical vectors e_j, we see that

$$E_n \le C_n,\ \forall n \in \mathbb{N}. \tag{5.11}$$

Indeed, for an arbitrary $n \in \mathbb{N}$, since $B_w^n\left((x_j)_{j\in\mathbb{Z}}\right) = (w_j \cdots w_{j+n-1} x_{j+n})_{j\in\mathbb{Z}}$, we have that

$$B_w^n(e_{j'}) = w_{j'-n} \cdots w_{j'-1} e_{j'-n}.$$

Hence, for any $p \in \{0\} \cup [1, \infty)$:

$$\|B_w^n(e_{j'})\|_k \le C_n \|e_{j'}\|_{r_n} = C_n\ a_{j',r_n},\ \forall j' \in \mathbb{Z}. \tag{5.12}$$

But

$$\|B_w^n(e_{j'})\|_k = |w_{j'-n} \cdots w_{j'-1}| \|e_{j'-n}\|_k = a_{j'-n,k} |w_{j'-n} \cdots w_{j'-1}|,\ \forall j' \in \mathbb{Z}.$$

Thus, by (5.12), we have that

$$a_{j'-n,k} |w_{j'-n} \cdots w_{j'-1}| \le C_n\ a_{j',r_n},$$

whence

$$\frac{a_{j'-n,k} |w_{j'-n} \cdots w_{j'-1}|}{a_{j',r_n}} \le C_n,\ \forall j' \in \mathbb{Z}. \tag{5.13}$$

Taking $i := j' - n$ and $j := j' - 1$ in (5.13), we obtain that $j - i = (j'-1) - (j'-n) = n - 1$ and

$$\frac{a_{i,k}\,|w_i \cdots w_j|}{a_{j+1,r_n}} \leq C_n.$$

By getting all $j' \in \mathbb{Z}$, the formulas $i := j' - n$ and $j := j' - 1$ gives all $i, j \in \mathbb{Z}$ satisfying $j - i = n - 1$. Hence, (5.11) holds.

Without loss of generality, we can assume that the sequence $(r_n)_{n\in\mathbb{N}}$ is strictly increasing. We shall construct a sequence $(t_s)_{s\in\mathbb{N}}$ of positive integers and two sequences $(i_s)_{s\in\mathbb{N}}$ and $(j_s)_{s\in\mathbb{N}}$ of integers such that the following properties hold:

(α) The sequence $(t_s)_{s\in\mathbb{N}}$ is strictly increasing.

(β) For every $s \in \mathbb{N}$, we have that $i_s \leq j_s$ and $a_{i_s,k}\,|w_{i_s} \cdots w_{j_s}| > 2^s a_{j_s+1,t_s}$.

(γ) The sequence $(n_s)_{s\in\mathbb{N}}$, where $n_s := j_s - i_s + 1$, is strictly increasing.

(δ) $j_s \neq j_t$ whenever $s \neq t$.

We begin by putting $t_1 := r_1$ and by choosing $i_1, j_1 \in \mathbb{Z}$ such that

$$i_1 \leq j_1 \quad \text{and} \quad \frac{a_{i_1,k}\,|w_{i_1} \cdots w_{j_1}|}{a_{j_1+1,t_1}} > 2.$$

The equality in *(b')* for $\ell := t_1$ guarantees the existence of such a pair i_1, j_1. Suppose that $t_1, \cdots, t_s$, $i_1, \cdots, i_s$ and $j_1, \cdots, j_s$ have already been chosen with the desired properties. Let

$$C := \max\{C_1, \ldots, C_{n_s+1}\}$$

and choose $t_{s+1} \in \mathbb{N}$ such that

$$t_{s+1} > \max\{t_s, r_{n_s+1}\}.$$

For each $i, j \in \mathbb{Z}$ such that $j - i \in \{0, 1, \cdots, n_s\}$, we have that $N := j - i + 1 \in \{1, \ldots, n_s + 1\}$ and it follows from (5.11) that

$$\frac{a_{i,k}\,|w_i \cdots w_j|}{a_{j+1,r_N}} \leq E_N \leq C_N \leq C. \tag{5.14}$$

Since $t_{s+1} > r_{n_s+1} \geq r_N$ (because $N \leq n_s + 1$ and $(r_n)_n$ is strictly increasing), we have that

$$\frac{a_{i,k}\,|w_i \cdots w_j|}{a_{j+1,t_{s+1}}} \leq \frac{a_{i,k}\,|w_i \cdots w_j|}{a_{j+1,r_N}}.$$

Thus, by (5.14),

$$\sup\left\{\frac{a_{i,k}\left|w_i\cdots w_j\right|}{a_{j+1,t_{s+1}}} : i,j\in\mathbb{Z},\ 0\le j-i\le n_s\right\}\le C. \tag{5.15}$$

Now, we have to consider two possibilities.

Case 1. There exists $j_0\in\mathbb{Z}$ such that

$$\sup\left\{\frac{a_{i,k}\left|w_i\cdots w_{j_0}\right|}{a_{j_0+1,t_{s+1}}} : i\in\mathbb{Z},\ i\le j_0\right\}=\infty. \tag{5.16}$$

Then, (5.16) holds with j instead of j_0 for every $j\ge j_0$. Hence, there exist $i_{s+1},j_{s+1}\in\mathbb{Z}$ such that $j_{s+1}>\max\{j_1,\cdots,j_s\}$, $i_{s+1}\le j_{s+1}$ and

$$\frac{a_{i_{s+1},k}\left|w_{i_{s+1}}\cdots w_{j_{s+1}}\right|}{a_{j_{s+1}+1,t_{s+1}}}>\max\left\{C,2^{s+1}\right\}. \tag{5.17}$$

Thus, (5.15) implies that $n_{s+1}=j_{s+1}-i_{s+1}+1>n_s$. In fact, if $j_{s+1}-i_{s+1}+1\le n_s$, then $0\le j_{s+1}-i_{s+1}\le n_s-1<n_s$. Therefore, by (5.15), we would have to

$$\frac{a_{i_{s+1},k}\left|w_{i_{s+1}}\cdots w_{j_{s+1}}\right|}{a_{j_{s+1}+1,t_{s+1}}}\le C,$$

which contradicts (5.17). Hence, $n_{s+1}=j_{s+1}-i_{s+1}+1>n_s$, i.e., $(n_s)_{s\in\mathbb{N}}$ is strictly increasing.

Case 2. For all $j\in\mathbb{Z}$,

$$K_j:=\sup\left\{\frac{a_{i,k}\left|w_i\cdots w_j\right|}{a_{j+1,t_{s+1}}} : i\in\mathbb{Z},\ i\le j\right\}<\infty. \tag{5.18}$$

In this case, we apply the equality in *(b')* for $\ell:=t_{s+1}$ to obtain a pair $i_{s+1},j_{s+1}\in\mathbb{Z}$ such that $i_{s+1}\le j_{s+1}$ and

$$\frac{a_{i_{s+1},k}\left|w_{i_{s+1}}\cdots w_{j_{s+1}}\right|}{a_{j_{s+1}+1,t_{s+1}}}>\max\left\{C,K_{j_1},\ldots,K_{j_s},2^{s+1}\right\}. \tag{5.19}$$

If $j_{s+1}\in\{j_1,\ldots,j_s\}$, then

$$\frac{a_{i_{s+1},k}\left|w_{i_{s+1}}\cdots w_{j_{s+1}}\right|}{a_{j_{s+1}+1,t_{s+1}}}\le K_{j_m}<\infty,$$

for some $m\in\{1,\ldots,s\}$, which contradicts (5.19). Therefore, we have $j_{s+1}\notin\{j_1,\ldots,j_s\}$. Moreover, $n_{s+1}>n_s$. Indeed, if $n_{s+1}=j_{s+1}-i_{s+1}+1\le n_s$, then $0\le j_{s+1}-i_{s+1}<n_s$. Hence, by (5.15), we would have to

$$\frac{a_{i_{s+1},k}\left|w_{i_{s+1}}\cdots w_{j_{s+1}}\right|}{a_{j_{s+1}+1,t_{s+1}}}\le C,$$

which contradicts (5.19). Thus, $(n_s)_{s\in\mathbb{N}}$ is strictly increasing.

This completes the construction of the sequences satisfying (α)-(δ).

Now, define

$$x_{j_s+1} := \frac{1}{2^s a_{j_s+1,t_s}} \ (s \in \mathbb{N}) \ \text{ and } \ x_j := 0 \ \text{ if } \ j \in \mathbb{Z} \setminus \{j_s + 1 : s \in \mathbb{N}\}.$$

Property (δ) ensures that the sequence $x := (x_j)_{j\in\mathbb{Z}}$ is well-defined. Given $\ell \in \mathbb{N}$, property (α) implies the existence of an integer $s_0 \in \mathbb{N}$ such that $t_s \geq \ell$ whenever $s > s_0$ (indeed, because the sequence $(t_s)_{s\in\mathbb{N}}$ is increasing and so, $s > s_0 \Rightarrow t_s > t_{s_0}$). Therefore, for any $q \in [1, \infty)$,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} |a_{j,\ell} x_j|^q &= \sum_{s=1}^{\infty} |a_{j_s+1,\ell} x_{j_s+1}|^q \leq \sum_{s=1}^{s_0} |a_{j_s+1,\ell} x_{j_s+1}|^q + \sum_{s=s_0+1}^{\infty} |a_{j_s+1,t_s} x_{j_s+1}|^q \\ &= \sum_{s=1}^{s_0} |a_{j_s+1,\ell} x_{j_s+1}|^q + \sum_{s=s_0+1}^{\infty} \left(\frac{1}{2^s}\right)^q < \infty.\end{aligned}$$

If $p \in [1, \infty)$, we take $q := p$ and obtain that $x \in X$. If $p = 0$, we take $q := 1$ and obtain that

$$\sum_{j\in\mathbb{Z}} |a_{j,\ell} x_j| < \infty, \quad \forall \ell \in \mathbb{N}.$$

Since this implies that

$$\lim_{|j|\to\infty} |a_{j,\ell} x_j| = 0, \quad \forall \ell \in \mathbb{N},$$

we also conclude that $x \in X$. Hence, in any case, $x \in X$.

By property (β), for any $s \in \mathbb{N}$, we have that $i_s \leq j_s$ and

$$\frac{a_{i_s,k} |w_{i_s} \cdots w_{j_s}|}{2^s a_{j_s+1,t_s}} > 1. \tag{5.20}$$

Thus,

$$\begin{aligned}\|B_w^{n_s}(x)\|_k &= \|(w_j \cdots w_{j+n_s-1} x_{j+n_s})_{j\in\mathbb{Z}}\|_k \geq |a_{i_s,k} w_{i_s} \cdots w_{i_s+n_s-1} x_{i_s+n_s}| \\ &= |a_{i_s,k} w_{i_s} \cdots w_{j_s} x_{j_s+1}| \\ &= \frac{a_{i_s,k} |w_{i_s} \cdots w_{j_s}|}{2^s a_{j_s+1,t_s}} > 1, \text{ by (5.20).}\end{aligned}$$

In view of property (γ), we conclude that $B_w^n(x) \nrightarrow 0$ as $n \to \infty$. Thus, by Theorem 5.7, B_w is Li-Yorke chaotic. $\square$

The next result characterizes Li-Yorke chaos for unilateral weighted backward shifts.

Theorem 5.12. *Consider a Köthe sequence space $X := \lambda_p(A, \mathbb{N})$, where $A := (a_{j,k})$ is a Köthe matrix on $\mathbb{N}$ with nonzero entries and $p \in \{0\} \cup [1, \infty)$. Let $w := (w_n)_{n\in\mathbb{N}}$ be a sequence of nonzero scalars such that the unilateral weighted backward shift B_w is a well-defined operator on X. Then B_w is Li-Yorke chaotic if and only if there exists $k \in \mathbb{N}$ such that*

$$\sup\left\{\frac{a_{i,k}\,|w_i \cdots w_j|}{a_{j+1,\ell}} : i, j \in \mathbb{N}, i \leq j\right\} = \infty, \quad \forall \ell \in \mathbb{N}. \tag{5.21}$$

Proof. Assume that B_w is Li-Yorke chaotic. Suppose, by contradiction, that (5.21) fails for every $k \in \mathbb{N}$. This means that, for for each $k \in \mathbb{N}$, there exists $\ell_k \in \mathbb{N}$ such that

$$E_k := \sup\left\{\frac{a_{i,k}\,|w_i \cdots w_j|}{a_{j+1,\ell_k}} : i, j \in \mathbb{N}, i \leq j\right\} < \infty.$$

Take $x := (x_j)_{j\in\mathbb{N}} \in X$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$. If $p \in [1, \infty)$, then

$$\|(B_w)^n(x)\|_k^p = \sum_{j=1}^{\infty} |a_{j,k} w_j \cdots w_{n+j-1} x_{n+j}|^p \leq \sum_{j=1}^{\infty} |E_k a_{j+n,\ell_k} x_{j+n}|^p \leq E_k^p \,\|x\|_{\ell_k}^p.$$

If $p = 0$, then

$$\|(B_w)^n(x)\|_k = \sup_{j\in\mathbb{N}} |a_{j,k} w_j \cdots w_{n+j-1} x_{j+n}| \leq \sup_{j\in\mathbb{N}} |E_k a_{j+n,\ell_k} x_{j+n}| \leq E_k \,\|x\|_{\ell_k}.$$

In any case, we see that the sequence $((B_w)^n(x))_{n\in\mathbb{N}}$ is bounded for all $x \in X$. Therefore, B_w does not admit an irregular vector, which contradicts the fact that B_w is Li-Yorke chaotic.

Conversely, suppose that there exists $k \in \mathbb{N}$ such that (5.21) holds. We shall construct a vector $x \in X$ such that

$$(B_w)^n(x) \not\to 0 \quad \text{as } n \to \infty. \tag{5.22}$$

By Corollary 5.4, this implies that B_w is Li-Yorke chaotic. For each $n \in \mathbb{N}$, the continuity of $(B_w)^n$ implies the existence of a constant $C_n \in (0, \infty)$ and an integer $r_n \in \mathbb{N}$ such that

$$\|(B_w)^n(x)\|_k \leq C_n \|x\|_{r_n}, \ \forall x \in X. \tag{5.23}$$

Moreover, we may assume that the sequence $(r_n)_{n\in\mathbb{N}}$ is strictly increasing. By applying (5.23) to the canonical vectors e_j ($j \in \mathbb{N}$), we see that

$$\sup\left\{\frac{a_{i,k}|w_i\cdots w_j|}{a_{j+1,r_n}} : i,j\in\mathbb{N}, j-i=n-1\right\} \le C_n,\ \forall n\in\mathbb{N}. \tag{5.24}$$

We shall construct sequences $(t_s)_{s\in\mathbb{N}}$, $(i_s)_{s\in\mathbb{N}}$ and $(j_s)_{s\in\mathbb{N}}$ of positive integers such that the following properties hold:

(α) The sequence $(t_s)_{s\in\mathbb{N}}$ is strictly increasing.

(β) For every $s \in \mathbb{N}$, we have that $i_s \le j_s$ and $a_{i_s,k}|w_{i_s}\cdots w_{j_s}| > 2^s a_{j_s+1,t_s}$.

(γ) The sequence $(n_s)_{s\in\mathbb{N}}$, where $n_s := j_s - i_s + 1$, is strictly increasing.

(δ) $j_s \neq j_t$ whenever $s \neq t$.

We begin by putting $t_1 := r_1$ and by choosing $i_1, j_1 \in \mathbb{N}$ such that

$$i_1 \le j_1 \quad \text{and} \quad \frac{a_{i_1,k}|w_{i_1}\cdots w_{j_1}|}{a_{j_1+1,t_1}} > 2.$$

The equality in (5.21) for $\ell := t_1$ guarantees the existence of such a pair i_1, j_1. Suppose that $t_1,\ldots,t_s$, $i_1,\ldots,i_s$ and $j_1,\ldots,j_s$ have already been chosen with the desired properties. Let

$$C := \max\{C_1,\ldots,C_{n_s+1}\}$$

and choose $t_{s+1} \in \mathbb{N}$ such that

$$t_{s+1} > \max\{t_s, r_{n_s+1}\}.$$

It follows from (5.24) that

$$\sup\left\{\frac{a_{i,k}|w_i\cdots w_j|}{a_{j+1,t_{s+1}}} : i,j\in\mathbb{N}, 0\le j-i\le n_s\right\} \le C. \tag{5.25}$$

For each $j \in \mathbb{N}$, let

$$K_j := \max_{1\le i\le j} \frac{a_{i,k}|w_i\cdots w_j|}{a_{j+1,t_{s+1}}}.$$

By applying the equality in (5.21) for $\ell := t_{s+1}$, we obtain a pair $i_{s+1}, j_{s+1} \in \mathbb{N}$ such that

$$i_{s+1} \le j_{s+1} \quad \text{and} \quad \frac{a_{i_{s+1},k}|w_{i_{s+1}}\cdots w_{j_{s+1}}|}{a_{j_{s+1}+1,t_{s+1}}} > \max\{C, K_{j_1},\ldots,K_{j_s}, 2^{s+1}\}.$$

Hence, by the definition of the K_j's, $j_{s+1} \notin \{j_1, \ldots, j_s\}$. Moreover, (5.25) implies that $n_{s+1} = j_{s+1} - i_{s+1} + 1 > n_s$. This completes the construction of the sequences satisfying (α)–(δ). Now, define

$$x_{j_s+1} := \frac{1}{2^s a_{j_s+1,t_s}} \quad (s \in \mathbb{N}) \quad \text{and} \quad x_j := 0 \ \text{ if } j \in \mathbb{N}\backslash\{j_s + 1 : s \in \mathbb{N}\}.$$

Property (δ) ensures that the sequence $x := (x_j)_{j\in\mathbb{N}}$ is well-defined. Given $\ell \in \mathbb{N}$, property (α) implies the existence of an integer $s_0 \in \mathbb{N}$ such that $t_s \geq \ell$ whenever $s > s_0$. Therefore, for any $q \in [1, \infty)$,

$$\begin{aligned}\sum_{j=1}^{\infty} |a_{j,\ell}\, x_j|^q = \sum_{s=1}^{\infty} |a_{j_s+1,\ell}\, x_{j_s+1}|^q &\leq \sum_{s=1}^{s_0} |a_{j_s+1,\ell}\, x_{j_s+1}|^q + \sum_{s=s_0+1}^{\infty} |a_{j_s+1,t_s} x_{j_s+1}|^q \\ &= \sum_{s=1}^{s_0} |a_{j_s+1,\ell}\, x_{j_s+1}|^q + \sum_{s=s_0+1}^{\infty} \left(\frac{1}{2^s}\right)^q < \infty.\end{aligned}$$

If $p \in [1, \infty)$, we take $q := p$ and obtain that $x \in X$. If $p = 0$, we take $q := 1$ and obtain that

$$\sum_{j=1}^{\infty} |a_{j,\ell}\, x_j| < \infty, \ \forall \ell \in \mathbb{N}.$$

Hence,

$$\lim_{j\to\infty} |a_{j,\ell}\, x_j| = 0, \ \forall \ell \in \mathbb{N},$$

proving that $x \in X$. Thus, in any case, $x \in X$. By property (β), for any $s \in \mathbb{N}$, we have that $i_s \leq j_s$ and

$$\frac{a_{i_s,k}|w_{i_s} \cdots w_{j_s}|}{2^s a_{j_s+1,t_s}} > 1.$$

Thus, for every $s \in \mathbb{N}$,

$$\begin{aligned}\|(B_w)^{n_s}(x)\|_k = \|(w_j \cdots w_{j+n_s-1} x_{j+n_s})_{j\in\mathbb{N}}\|_k &\geq |a_{i_s,k} w_{i_s} \cdots w_{i_s+n_s-1} x_{i_s+n_s}| \\ &= |a_{i_s,k} w_{i_s} \cdots w_{j_s} x_{j_s+1}| \\ &= \frac{a_{i_s,k}|w_{i_s} \cdots w_{j_s}|}{2^s a_{j_s+1,t_s}} > 1.\end{aligned}$$

In view of property (γ), we conclude that (5.22) holds, as it was to be shown. □

For unilateral weighted backward shifts B_w on Köthe sequence spaces, several properties that are equivalent to B_w being Li-Yorke chaotic were obtained in Theorem 1 of [33] and Theorem 3.3 of [16]. Nevertheless, no characterization based solely on the entries of the Köthe matrix and the weights of the shift were known before. Theorem 5.12 fills this gap in the case of Köthe sequence spaces defined by a Köthe matrix with nonzero entries. Moreover, Theorem 5.11 gives the corresponding result in the case of bilateral shifts.

Example 5.13. Let $J := \mathbb{N}$ or $\mathbb{Z}$. If $a_{j,k} := 1$ for all $j \in J$ and $k \in \mathbb{N}$, then

$$\lambda_0(A, J) = c_0(J) \quad \text{and} \quad \lambda_p(A, J) = \ell^p(J), \text{ for } p \in [1, \infty).$$

Indeed,

$$\begin{aligned} \lambda_0(A, J) &= \left\{ (x_j)_{j\in J} \in \mathbb{K}^J : \lim_{j\in J, |j|\to\infty} a_{j,k} x_j = 0, \ \forall k \in \mathbb{N} \right\} \\ &= \left\{ (x_j)_{j\in J} \in \mathbb{K}^J : \lim_{j\in J, |j|\to\infty} x_j = 0, \ \forall k \in \mathbb{N} \right\} \\ &= c_0(J). \end{aligned}$$

And

$$\begin{aligned} \lambda_p(A, J) &= \left\{ (x_j)_{j\in J} \in \mathbb{K}^J : \sum_{j\in J} |a_{j,k} x_j|^p < \infty, \ \forall k \in \mathbb{N} \right\} \\ &= \left\{ (x_j)_{j\in J} \in \mathbb{K}^J : \sum_{j\in J} |x_j|^p < \infty, \ \forall k \in \mathbb{N} \right\} \\ &= \ell^p(J), \end{aligned}$$

whose seminorms are given by

$$\|x\|_k := \left(\sum_{j\in J} |a_{j,k} x_j|^p \right)^{1/p} = \left(\sum_{j\in J} |x_j|^p \right)^{1/p} = \|x\|_p, \ \forall x := (x_j)_{j\in J} \text{ and } k \in \mathbb{N}.$$

In this case, Theorems 5.11 and 5.12 recover Proposition 3.10 and Corollaries 3.11, 4.12 and 4.13 in the case of nonzero weights. This provides another approach to obtain the characterizations of the Li-Yorke chaotic weighted shifts on the classical Banach sequence spaces c_0 and ℓ^p.

Example 5.14. Let $J := \mathbb{N}$ or $\mathbb{Z}$. Let $a_{j,k} := (|j| + 1)^k$ for all $j \in J$ and $k \in \mathbb{N}$, and consider the Köthe sequence space $\lambda_1(A, J)$, with $A := (a_{j,k})_{j\in J, k\in\mathbb{N}}$. As an illustration, let us exhibit a weighted shift B_w on $\lambda_1(A, \mathbb{Z})$ which is Li-Yorke chaotic but not hypercyclic. Consider the weight sequence $w := (w_n)_{n\in\mathbb{Z}}$, where $w_{-n} := \frac{1}{2}$, for all $n \geq 0$, and

$$(w_n)_{n\geq 1} := \left(\frac{1}{2}, 2, \frac{1}{2}, \frac{1}{2}, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, \ldots \right),$$

a concatenation of blocks of the form $\left(\frac{1}{2}, \ldots, \frac{1}{2}, 2, \ldots, 2\right)$ with n numbers $\frac{1}{2}$ followed by n numbers 2, for $n \geq 1$. Since (5.6) holds, the bilateral weighted backward shift B_w is a well-defined operator on $\lambda_1(A, \mathbb{Z})$. Indeed, since $a_{j,k} := (|j|+1)^k$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have that $a_{j,k} \neq 0$, for all $j \in \mathbb{Z}$ and $k \in \mathbb{N}$. If (5.6) does not hold, there exists a $k_0 \in \mathbb{N}$ such that

$$\sup_{j\in\mathbb{Z}} \frac{a_{j,k_0}|w_j|}{a_{j+1,m}} = \infty, \quad \forall m \in \mathbb{N}. \tag{5.26}$$

But $|w_j| \leq 2$, for all $j \in \mathbb{Z}$. Thus,

$$\frac{a_{j,k_0}|w_j|}{a_{j+1,m}} \leq 2\frac{a_{j,k_0}}{a_{j+1,m}} = 2\frac{(|j|+1)^{k_0}}{(|j+1|+1)^m}, \quad \forall j \in \mathbb{Z} \text{ and } \forall m \in \mathbb{N}. \tag{5.27}$$

Taking $m = k_0$ in (5.27), it follows that

$$\frac{(|j|+1)^{k_0}}{(|j+1|+1)^{k_0}} = \frac{\left(1+\dfrac{1}{|j|}\right)^{k_0}}{\left(\left|\dfrac{j+1}{j}\right| + \dfrac{1}{|j|}\right)^{k_0}} = \frac{\left(1+\dfrac{1}{|j|}\right)^{k_0}}{\left(\left|1+\dfrac{1}{j}\right| + \dfrac{1}{|j|}\right)^{k_0}} \to 1 \text{ as } |j| \to \infty.$$

Hence, if $m = k_0$, then (5.27) implies that

$$\sup_{j\in\mathbb{Z}} \frac{a_{j,k_0}|w_j|}{a_{j+1,k_0}} < \infty,$$

which contradicts (5.26). Therefore, B_w satisfies (5.6) and it is a well-defined operator.

Now, consider $(n_i)_{i\in\mathbb{N}} := (n)_{n\in\mathbb{N}}$. Fix any $k \in \mathbb{N}$. Then

$$\begin{aligned} \lim_{i\to\infty} a_{-n_i,k} w_{-n_i} \cdots w_{-1} &= \lim_{n\to\infty} a_{-n,k} w_{-n} \cdots w_{-1} = \lim_{n\to\infty} (|-n|+1)^k w_{-n} \cdots w_{-1} \\ &= \lim_{n\to\infty} \frac{(n+1)^k}{2^n} = 0, \end{aligned}$$

which satisfies *(a')* of Theorem 5.11.

Moreover, for each $\ell \in \mathbb{N}$, we have that

$$\sup_{i\leq j} \frac{a_{i,k}|w_i \cdots w_j|}{a_{j+1,\ell}} = \sup_{i\leq j} \frac{(|i|+1)^k |w_i \cdots w_j|}{(|j+1|+1)^\ell} \geq \sup_{i\leq j} \frac{|w_i \cdots w_j|}{(|j+1|+1)^\ell}, \quad \forall k \in \mathbb{N}. \tag{5.28}$$

For each $n \in \mathbb{N}$, let $i := n^2+1$ and $j := n^2+n$ in (5.28). Hence,

$$\sup_{i\leq j} \frac{a_{i,k}|w_i \cdots w_j|}{a_{j+1,\ell}} \geq \sup_{n\in\mathbb{N}} \frac{|w_{n^2+1} \cdots w_{n^2+n}|}{(|n^2+n+1|+1)^\ell} = \sup_{n\in\mathbb{N}} \frac{2^n}{(|n^2+n+1|+1)^\ell} = \infty, \quad \forall \ell \in \mathbb{N}. \tag{5.29}$$

Therefore, by (5.29), it follows that *(b')* in Theorem 5.11 holds. Thus, B_w is Li-Yorke chaotic.

On the other hand, let us recall the following theorem of [22]:

Theorem 5.15. *([22], Theorem 4.13) Let X be a Fréchet sequence space over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Suppose that the weighted backward shift B_w is an operator on X. The following assertions are equivalent:*

(i) B_w is hypercyclic;

(ii) There is an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that, for any $j \in \mathbb{Z}$,

$$\left(\prod_{\nu=j-n_k+1}^{j} w_\nu\right) e_{j-n_k} \to 0 \quad \text{and} \quad \left(\prod_{\nu=j+1}^{j+n_k} w_\nu\right)^{-1} e_{j+n_k} \to 0$$

in X as $k \to \infty$.

By adapting Theorem 5.15 to our context, we have that B_w is hypercyclic on $\lambda_1(A,\mathbb{Z})$ if and only if there exists an increasing sequence $(n_j)_{j\in\mathbb{N}}$ of positive integers such that, for any $\ell \in \mathbb{Z}$, we have that

$$\left(\prod_{\nu=\ell-n_j}^{\ell-1} w_\nu\right) e_{\ell-n_j} \to 0 \ \text{ and } \ \left(\prod_{\nu=\ell}^{\ell+n_j-1} w_\nu\right)^{-1} e_{\ell+n_j} \to 0 \tag{5.30}$$

in $\lambda_1(A,\mathbb{Z})$ as $j \to \infty$. However, in our example, for each $n \in \mathbb{N}$ and $\ell = 1$, we have that

$$\begin{aligned}
\left\| \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_{1+n} \right\|_1 = \left\| \frac{e_{n+1}}{w_1 \cdots w_n} \right\|_1 &= \frac{1}{|w_1 \cdots w_n|} \|e_{n+1}\|_1 \\
&= \frac{1}{|w_1 \cdots w_n|} |a_{n+1,1}| \\
&= \frac{1}{|w_1 \cdots w_n|} |(n+1+1)^1| \\
&= \frac{n+2}{w_1 \cdots w_n} \\
&\geq n+2, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Therefore, the second condition in (5.30) fails for $\ell = 1$, whence B_w is not hypercyclic.

Remark 5.16. For $J := \mathbb{N}$ or $\mathbb{Z}$, the space $\lambda_1(A, J)$ is also known as the *space of rapidly decreasing sequences on* J and it is denoted by $s(J)$, which is a classical example of a non-normable Fréchet sequence space. For more information on the space $s(J)$, we refer to references [24] and [28].

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