Instituto Alberto Luiz Coimbra de
Pós-Graduação e Pesquisa de Engenharia
COPPE
UFRJ

ANALYSIS OF THE DYNAMICS OF DUOPOLY MODELS UNDER STATE FEEDBACK ADVERTISING POLICIES WITH SWITCHING

Rolando Cuevas Núñez

Tese de Doutorado apresentada ao Programa de Pós-graduação em Engenharia Elétrica, COPPE, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Engenharia Elétrica.

Orientador: Amit Bhaya

Rio de Janeiro
Março de 2017

ANALYSIS OF THE DYNAMICS OF DUOPOLY MODELS UNDER STATE FEEDBACK ADVERTISING POLICIES WITH SWITCHING

Rolando Cuevas Núñez

TESE SUBMETIDA AO CORPO DOCENTE DO INSTITUTO ALBERTO LUIZ COIMBRA DE PÓS-GRADUAÇÃO E PESQUISA DE ENGENHARIA (COPPE) DA UNIVERSIDADE FEDERAL DO RIO DE JANEIRO COMO PARTE DOS REQUISITOS NECESSÁRIOS PARA A OBTENÇÃO DO GRAU DE DOUTOR EM CIÊNCIAS EM ENGENHARIA ELÉTRICA.

Examinada por:

Prof. Amit Bhaya, Ph.D.

Prof. André Barreira da Silva Rocha, Ph.D.

Prof. Tiago Roux de Oliveira, D. Sc.

Prof. Alessandro Jacoud Peixoto, D. Sc.

Prof. Eugenius Kaszkurewicz, D. Sc.

RIO DE JANEIRO, RJ – BRASIL
MARÇO DE 2017

Cuevas Núñez, Rolando

Analysis of the dynamics of duopoly models under state feedback advertising policies with switching/Rolando Cuevas Núñez. – Rio de Janeiro: UFRJ/COPPE, 2017.

XX, 121 p.: il.; 29,7cm.

Orientador: Amit Bhaya

Tese (doutorado) – UFRJ/COPPE/Programa de Engenharia Elétrica, 2017.

Referências Bibliográficas: p. 78 – 82.

1. Duopoly. 2. Vidale-Wolfe model. 3. Lotka-Volterra model. 4. Advertising. 5. Switching control. I. Bhaya, Amit. II. Universidade Federal do Rio de Janeiro, COPPE, Programa de Engenharia Elétrica. III. Título.

To my parents

Acknowledgements

I want to thank Prof. Bhaya for his patience and help.
Many thanks to CAPES that gave me financial support necessary to complete this work.
I'm very thankful to the NACAD laboratory's staff for giving a place to work during this years.
Finally, I greatly appreciate the support of my friends from UFRJ and FPUNA-PY during these years.

Resumo da Tese apresentada à COPPE/UFRJ como parte dos requisitos necessários para a obtenção do grau de Doutor em Ciências (D.Sc.)

ANÁLISE DA DINÂMICA DE MODELOS DE DUOPOLIO SOB POLÍTICAS DE RETROALIMENTAÇÃO DE ESTADO CHAVEADAS

Rolando Cuevas Núñez

Março/2017

Orientador: Amit Bhaya

Programa: Engenharia Elétrica

Esta tese analisa a utilização de políticas competitivas de publicidade em uma classe proposta de modelos dinâmicos de duopólio, inspirados no modelo clássico Vidale-Wolfe. Uma suposição básica é de que a empresa com uma menor participação no mercado chaveia para um controle que utiliza um esforço extra para alcançar a firma concorrente. Investigam-se padrões de dinâmica dos modelos propostos, sujeitos a políticas de publicidade competitivas baseadas nas variáveis de estado observadas, calculando-se um índice de desempenho associado. Mostra-se como utilizar este índice em conjunção com a análise qualitativa da dinâmica para prever o resultado de um determinado cenário, para eventual tomada de decisão.

Abstract of Thesis presented to COPPE/UFRJ as a partial fulfillment of the requirements for the degree of Doctor of Science (D.Sc.)

ANALYSIS OF THE DYNAMICS OF DUOPOLY MODELS UNDER STATE FEEDBACK ADVERTISING POLICIES WITH SWITCHING

Rolando Cuevas Núñez

March/2017

Advisor: Amit Bhaya

Department: Electrical Engineering

This thesis investigates the effect of competitive advertising policies in a proposed class of duopoly dynamic models, inspired on the classical Vidale-Wolfe model. A basic assumption is that the firm with a lower market share than its target market share level switches on an extra control effort in its attempt to reach and surpass its competitor. Dynamical patterns of the proposed models, subject to competitive advertising policies which are based on the observed state variable, are investigated. An associated performance index is calculated. This index in conjunction with the qualitative analysis of the dynamics should guide the management decision on advertising spending.

Contents

List of Figures **xii**

List of Tables **xviii**

List of Symbols **xx**

1 Introduction **1**

1.1 Structure of the thesis 2
1.2 Literature review 3
1.2.1 Dynamic models of response to advertising in duopolies 3
1.2.2 Dynamic duopoly models involving price and advertising effort . 4
1.2.3 Dynamic duopoly models involving only advertising effort (non-price models) 5
1.2.4 Differential games of advertising 7
1.2.5 Duopoly differential games with dynamics derived from Vidale-Wolfe or Lanchester models 9
1.2.6 Characteristics of differential game solutions 10

2 Proposed approach to study advertising duopoly dynamics **13**

2.1 Outline of the proposed approach 13
2.2 Proposed unified model of advertising dynamics 15
2.3 Associated Performance Index 15
2.4 Motivation for using state dependent control 16
2.5 Objectives of the firms 18
2.6 Classes of Control Functions used by each firm 18
2.7 Objective of this thesis 19

3 Dynamics of a modification to the Vidale-Wolfe-Deal model under feedback control **20**

3.1 Introduction 20
3.2 Main result for model VWDsC 23

3.2.1 The state variables (market shares) x_1, x_2 are nondecreasing along the trajectories . . . 25
3.2.2 Equilibria in U, their location and stability properties . . . 25
3.2.3 Types of separatrices . . . 26
3.2.4 Reaching, sliding and crossing behaviors with regard to the equal share line Σ . . . 27
3.3 Designing advertising policies: a pictorial version of the main results for model VWDsC . . . 27
3.4 Simulation of different scenarios . . . 29
3.5 Evaluating outcomes of the proposed policies using a performance index in model VWDsC . . . 32
3.6 Remarks on model VWDsC . . . 33

4 Modification of Vidale-Wolfe-Deal model with Predatory Advertising 35
4.1 Introduction . . . 35
4.2 Equilibrium points of model VWDsP . . . 36
4.3 Dynamics of trajectories reaching the equal market share line Σ in model VWDsP . . . 38
4.4 Pseudoequilibrium points of model VWDsP . . . 40
4.5 Simulations of model VWDsP . . . 43
4.6 Evaluating outcomes of the proposed policies using a performance index for model VWDsP . . . 47
4.7 Remarks on model VWDsP . . . 50
4.8 Model Vidale-Wolfe-Deal model with 2 switching lines (VWD2s) . . . 51
4.9 Simulations of model VWD2s . . . 52
4.10 Evaluating outcomes of the proposed policies using a performance index for model VWD2s . . . 54
4.11 Remarks on model VWD2s . . . 56

5 Duopoly model based on Lotka-Volterra dynamics with switched predation (LVsP) 57
5.1 Introduction . . . 57
5.2 Equilibria of model LVsP . . . 59
5.3 Dynamics of trajectories reaching the equal number of clients line Σ in model LVsP . . . 60
5.4 Simulations of different scenarios of model LVsP . . . 62
5.5 Evaluating outcomes of the used policies using a performance index for model LVsP . . . 71
5.6 Remarks on model LVsP . . . 74

6 Conclusions and Future Work **76**

Bibliography **78**

Appendices **83**

A Filippov solutions on the discontinuity boundary **84**

B Algorithm for obtaining separatrices of model VWDsC **87**

C Demonstrations of properties of model VWDsC **90**

D Numerical considerations on simulating Odes with discontinuous RHS (right-hand sides) **97**

E Review of duopoly models **101**

E.1 K. R. Deal, 1979 [1] 101
E.1.1 Model derived from 101
E.1.2 Formulation 101
E.1.3 Solution 102
E.2 J. Case, 1979 [2] 102
E.2.1 Model derived from 102
E.2.2 Formulation 102
E.2.3 Solution 103
E.3 G. Sorger, 1989 [2] 103
E.3.1 Model derived from 103
E.3.2 Formulation 103
E.3.3 Solution 103
E.4 P. K. Chintagunta, N. J. Vilcassim, 1992 [3] 103
E.4.1 Model derived from 103
E.4.2 Formulation 104
E.4.2.1 Open-loop 104
E.4.2.2 Closed-loop 105
E.4.3 Solution 105
E.5 G. Erickson, 1992 [4] 105
E.5.1 Model derived from 105
E.5.2 Formulation 105
E.5.2.1 Case approach to Lanchester game 106
E.5.3 Lanchester differential game 107
E.5.4 Closed-loop solution 107
E.5.4.1 Open-loop solution 108

E.5.4.2 Conclusions . . . 108
E.5.5 Solution . . . 108
E.6 G. A. Fruchter, S. Kalish, 1997 [5] . . . 108
E.6.1 Model derived from . . . 108
E.6.2 Formulation . . . 109
E.6.3 Solution . . . 109
E.6.3.1 Open-loop . . . 109
E.6.3.2 Closed-loop . . . 110
E.7 Q. Wang, Z. Wu, 2001 [6] . . . 111
E.7.1 Model derived from . . . 111
E.7.2 Formulation . . . 111
E.7.3 Solution . . . 111
E.8 R. Jarrar, G. Martín-Herrán, G. Zaccour, 2004 [7] . . . 112
E.8.1 Derived from . . . 112
E.8.2 Formulation . . . 112
E.8.3 Solution . . . 112
E.9 F. M. Bass, A. Krishnamoorthy, A. Prasad, S. Sethi, 2005 [8] . . . 114
E.9.1 Model derived from . . . 114
E.9.2 Formulation . . . 114
E.9.3 Solution . . . 115
E.10 Q. Wang, Z. Wu, 2007 [9] . . . 116
E.10.1 Model derived from . . . 116
E.10.2 Model . . . 116
E.10.3 Solution . . . 117
E.11 A. Krishnamoorthy, A. Prasad, S. P. Sethi, 2010 [10] . . . 117
E.11.1 Model derived from . . . 117
E.11.2 Formulation . . . 117
E.11.3 Solution . . . 118
E.12 S. Jorgensen, G. Martín-Herrán, G. Zaccour, 2010 [11] . . . 119
E.12.1 Leitmann-Schmitendorf model . . . 119
E.12.2 Tractability of calculating the equilibrium . . . 119
E.12.3 Linear and state-redundant differential games . . . 119
E.12.4 Linear state games . . . 120
E.12.5 State-redundant games . . . 121

List of Figures

2.1 Control $u(t)$ applied to achieve the sales goal of at least M^*. The continuous case is shown in (a) the control $u(t) = k(M^* - S(t) > 0$ when current sales $S(t) < M^*$ and $u(t) = 0$ when $M^* < S(t)$. The discontinuous case is shown in (b) where the control is a switching control $u(t) = M_{left}$ when $S(t) < M^*$ and $u(t) = M_{right}$ when $M^* < S(t)$. 17

2.2 A speedometer for indicator *Sales in millions of $* using a standard coloring for the status of a metric: *red=In trouble* (from 0 to 22), *yellow=Attention needed* (from 22 to 44), and *green=Normal* (from 44 to 70). 17

3.1 The upper region U, the lower region L, the switching line $\Sigma = \{\mathbf{x} : H(\mathbf{x}) = x_2 - x_1 = 0\}$, the segment Sat defined by $1 - x_1 - x_2 = 0$. The segment Sat^U (resp. Sat^L) is the portion of Sat that lies in region U (resp. L). $\mathcal{T}$ is the triangle defined by the points $(1, 0)$, $(0, 1)$, $(0, 0)$. The point marked E is the equal share saturation point, i.e. where both firms have half of the total market. In this figure the sliding end is $s_{end} < 0.5$ and there is one separatrix $\varphi_2^L = \varphi_1^L$ 22

3.2 *Possible outcomes for firm F_2, which has been chosen, without loss of generality, to have a bigger initial market share*. The sliding end point s_{end} is marked by a dot. (a) RS: With initial conditions that lie below φ_1^U, F_2 loses its initial advantage, but ends with equal market share at saturation, (b) $RSRet$: For some trajectories firm F_2's initial advantage reduces to an equal share, but at saturation F_2 has more than half of the market, (c) RSC: Firm F_2 loses its initial advantage, has a period of equal share and at saturation line ends with less than half of the market, (d) RC: Firm F_2 loses its initial advantage and ends, at saturation line, with less than half of the market, without experiencing a period of equal share. 24

3.3 Types of separatrices: The superscript identifies the field used in the calculation and the subscript identifies the firm to which the separatrix applies. (a) There are two separatrices φ_1^U and φ_2^L. The trajectories that start in the white region between φ_1^U and φ_2^L are such that both firms end up with half of the market share in the long run. The cases depicted in (b) and (c) have only one separatrix, in case (b) the separatrix is $\varphi_2^U = \varphi_1^U$, in case (c) the separatrix is $\varphi_2^L = \varphi_1^L$. 26

3.4 The gray regions of U contain the initial conditions for which: (a) F_1 attains 50% of the saturated market, (b) F_1 surpasses the saturated market share of F_2. 27
3.5 The gray regions of U contain the initial conditions for which both firms end with an equal saturated market share in the long term. In (b) these initial conditions lie on the separatrix $\varphi_2^U = \varphi_1^U$ and the region collapses to a single curve (the separatrix) 28
3.6 The gray regions of U contain the initial conditions in U for which the firm F_1 end with a saturated market share smaller than the saturated market share of F_2. 28
3.7 Insufficient effort by initially disadvantaged firm F_1. Phase plane of model (3.3)-(3.6) with parameters $k_1 = 1$, $c_1 = 0.3$, $k_2 = 2$, $c_2 = 0.3$. The highlighted gray trajectory τ starts in U and is such that firm F_1 reaches the equal market share, remains on it but finally firm F_2 ends with more than half of the saturated market share. This corresponds to Theorem 3.2.1 2. 30
3.8 If $s_{end} = 0.5$, there are separatrices φ_1^U and φ_2^L. All trajectories τ with initial conditions below φ_1^U and above φ_2^L end at the saturated equal market share E, i.e. firm F_2 finally ends with half of the saturated market share. The firm F_1 does a bigger constant effort $c_1 > c_2$ but a smaller constant rate $k_1 < k_2$ when compared to the previous scenario. The phase plane of model (3.3)-(3.6) corresponds to parameters $k_1 = 1, c_1 = 2, k_2 = 2, c_2 = 0.3$. The highlighted gray trajectory τ is such that it reaches the equal market share Σ and remains on it until the saturated equal market share point E. This corresponds to Theorem 3.2.1 1. 31
3.9 Firm F_2 loses its initial advantage. The phase plane of model (3.3)-(3.6) corresponds to parameters $k_1 = 3.1$, $c_1 = 1.8$, $k_2 = 2$, $c_2 = 0.3$. All trajectories below the separatrix $\varphi_2^U = \varphi_1^U$ that reach the equal market share line on the sliding segment Σ_s, remain on it and then they cross into region L, with firm F_1 ending with more than half of the saturated market share (see Theorem 3.2.1 3a). The trajectories that reach the equal market share line on the crossing segment Σ_c directly cross into region L (see Theorem 3.2.1 3b). The highlighted gray trajectory is such that it reaches the equal market share line Σ, remains on it for some time and then it crosses into region L where firm F_1 ends up with more than half of saturated the market share. 32
3.10 F_2 spends more on advertising, i.e. it makes bigger efforts k_2, c_2 when compared to F_1. With a sales revenue coefficient $r = 2$, both firms have negative index values in the dark gray region and Firm F_2 has negative index values and firm F_1 has positive index values in the light gray region. 33
3.11 With a bigger sales revenue coefficient is $r = 4$ there are trajectories where both firms make a profit, i.e. their indexes are positive, even where their corresponding long term market share is lesss than 50%. 33

4.1 Model VWDsP, scenario 1: Firm F_1 and F_2 spending the same on predatory advertising. 45

4.2 Model VWDsP, scenario 1: Time plots of the market shares $x_1(t)$, $x_2(t)$ with initial conditions $x_1(t) = 0.04$ and $x_2(t) = 0.05$ 45
4.3 Model VWDsP, scenario 1: Signs of the inner products used in the calculation of conditions for crossing and sliding. 45
4.4 Model VWDsP, scenario 2: Firm F_1 doing big spending on advertising when compared to firm F_2 ($c_{21} \approx 7.33c_{12}$). 46
4.5 Model VWDsP, scenario 2: Time plots of the market shares $x_1(t)$, $x_2(t)$ with initial conditions $x_1(t) = 0.04$ and $x_2(t) = 0.05$ 46
4.6 Model VWDsP, scenario 2: Signs of the inner products used in the calculation of conditions for crossing and sliding. 47
4.7 Model VWDsP, scenario 1: Trajectories plotted according to their index values at final time. 48
4.8 Model VWDsP, scenario 1: $J_1(x_1(t), x_2(t), t)$ vs. $J_2(x_1(t), x_2(t), t)$ 48
4.9 Model VWDsP, scenario 1: Value of indices when the initial value $x_1(0) = 0.04$ is fixed. 49
4.10 Model VWDsP, scenario 1: Value of indices when the initial value $x_2(0) = 0.05$ is fixed. 49
4.11 Model VWDsP, scenario 2: Trajectories plotted according to their index values at final time. 50
4.12 Model VWDsP, scenario 2: $J_1(x_1(t), x_2(t), t)$ vs. $J_2(x_1(t), x_2(t), t)$ 50
4.13 Model VWDsP, scenario 2: Value of indices when the initial value $x_1(0) = 0.04$ is fixed. 50
4.14 Model VWDsP, scenario 2: Value of indices when the initial value $x_2(0) = 0.05$ is fixed. 50
4.15 The regions: upper U, middle M and lower L. The switching lines $\Sigma_{UM} = \{\mathbf{x} : H(\mathbf{x}) = x_2 - 4x_1 = 0\}$, $\Sigma_{ML} = \{\mathbf{x} : H(\mathbf{x}) = x_2 - \frac{3}{7}x_1 = 0\}$,. The segment Sat segment which is defined by $1 - x_1 - x_2 = 0$. $\mathcal{T}$ is the interior of the triangle defined by the points $(1, 0)$, $(0, 1)$, $(0, 0)$ 52
4.16 Model VWD2s, scenario 1: Firm F_1 and F_2 spend similar amounts in advertising. . . . 53
4.17 Model VWD2s, scenario 1: Time plots of the market shares $x_1(t)$, $x_2(t)$ with initial conditions $x_1(t) = 0.04$ and $x_2(t) = 0.05$ 53
4.18 Model VWD2s, scenario 2: Firm F_1 spends more than F_2 in advertising. 53
4.19 Model VWD2s, scenario 2: Time plots of the market shares $x_1(t)$, $x_2(t)$ with initial conditions $x_1(t) = 0.04$ and $x_2(t) = 0.05$ 53
4.20 Model VWD2s, scenario 1: Trajectories plotted according to their index values at final time. 54
4.21 Model VWD2s, scenario 1: Value of indices when the initial value $x_1(0) = 0.04$ is fixed. 55
4.22 Model VWD2s, scenario 1: Value of indices when the initial value $x_2(0) = 0.05$ is fixed. 55
4.23 Model VWD2s, scenario 2: Trajectories plotted according to their index values at final time. 55
4.24 Model VWD2s, scenario 2: Value of indices when the initial value $x_1(0) = 0.04$ is fixed. 56
4.25 ModelVWD2s, scenario 2: Value of indices when the initial value $x_2(0) = 0.05$ is fixed. 56

5.1 Model LVsP, scenario 1: Firm F_1 doing little spending on advertising when compared to firm F_2. 63
5.2 Model LVsP, scenario 1: Signs of the inner products used in the calculation of conditions for crossing and sliding. 64
5.3 Model LVsP, scenario 2: Firm F_1 increases its effort. In this scenario both firms coexist in the long term, firm F_2 still ends with a bigger number of clients. 64
5.4 Model LVsP, scenario 2: Signs of the inner products used in the calculation of conditions for crossing and sliding. 65
5.5 Model LVsP, scenario 3: Firm F_1 increases again its effort and this introduces a pseudoequilibrium. Firms F_1 and F_2 coexist with an equal number of clients. The phase plane shows trajectories crossing and then returning to Σ 66
5.6 Model LVsP, scenario 3: Same simulation as in Figure 5.5, but here the $stepsize = 0.0002$ is smaller and this reduces most of the chattering on the equal number of clients line Σ. 66
5.7 Model LVsP, scenario 3: State variable as function of time for a trajectory having crossing and sliding behavior. When the $stepsize = 0.001$ the trajectories have noticeable chattering when it slides. 67
5.8 Model LVsP, scenario 3: Same simulation as in Figure 5.7, but here the $stepsize = 0.0002$ is smaller and this reduces most of the chattering when the trajectory slides. . . 67
5.9 Model LVsP, scenario 3: Signs of the inner products used in the calculation of conditions for crossing and sliding. 67
5.10 Model LVsP, scenario 4: Firm F_1 increases c_{21} and diminishes d_{21}. In the long term, firms F_1 and F_2 coexist with an equal number of clients. The phase plane shows that trajectories slides to a pseudoequilibrium. 68
5.11 Model LVsP, scenario 4: Same simulation as in Figure 5.10, but here the $stepsize = 0.0002$ is smaller and this reduces most of the chattering on the equal number of clients line Σ. 68
5.12 Model LVsP, scenario 4: Signs of the inner products used in the calculation of conditions for crossing and sliding. 69
5.13 Model LVsP, scenario 5: Firm F_1 increases again its effort and this time firm F_1 is the one ending with a bigger number of clients. 70
5.14 Model LVsP, scenario 5: Signs of the inner products used in the calculation of conditions for crossing and sliding. 71
5.15 Model LVsP, scenario 1: Trajectories plotted according to their index values at final time. In this scenario firm F_2 has positive indexes while firm F_1 has negative indexes. 72
5.16 Model LVsP, scenario 1: $J_1(x_1(t), x_2(t), t)$ vs. $J_2(x_1(t), x_2(t), t)$ 72
5.17 Model LVsP, scenario 1: Value of indices when the initial value $x_1(0) = 0.5$ is fixed. . . 72
5.18 Model LVsP, scenario 1: Value of indices when the initial value $x_2(0) = 0.03$ is fixed. . 72

5.19 Model LVsP, scenario 4: Trajectories plotted according to their index values at final time. In this scenario both firms F_1 and F_2 have positive indexes. 73
5.20 Model LVsP, scenario 4: $J_1(x_1(t), x_2(t), t)$ vs. $J_2(x_1(t), x_2(t), t)$ 73
5.21 Model LVsP, scenario 4: Value of indices when the initial value $x_1(0) = 0.5$ is fixed. . . 73
5.22 Model LVsP, scenario 4: Value of indices when the initial value $x_2(0) = 0.03$ is fixed. . 73
5.23 Model LVsP, scenario 5: Trajectories plotted according to their index values at final time. In this scenario firm F_1 has positive indexes while firm F_1 has negative indexes. 74
5.24 Model LVsP, scenario 5: $J_1(x_1(t), x_2(t), t)$ vs. $J_2(x_1(t), x_2(t), t)$ 74
5.25 Model LVsP, scenario 5: Value of indices when the initial value $x_1(0) = 0.5$ is fixed. . . 74
5.26 Model LVsP, scenario 5: Value of indices when the initial value $x_2(0) = 0.03$ is fixed. . 74

A.1 (a) Sample phase portrait showing segments Σ_s and Σ_c. (b) V, Z crossing points, W, B sliding points. B is a pseudoequilibrium point, i.e. $\angle(f^U, f^L) = 180°$ 85

B.1 The fields f^U or f^L approaching the equal market share point E from the left can be as shown. In the case of the dashed line, the slope is less than unity and the field indicates that the trajectory comes from U. In the case of the continuous line, the slope is greater than unity and the field indicates that the trajectory comes from L. 88

C.1 Configurations of fields f^U and f^L for a point belonging to Σ_s: The configuration in (a) is the only possible configuration. The configuration of fields in (b) is impossible. 93
C.2 Suppose that $k_2 > k_1$, $s_{end} < 0.5$ and firm F_1 has an initial lower market share than F_2, then: (a) There is no trajectory τ that reaches the equal market share line on the sliding segment Σ_s, remains on it for some time and then crosses into L, with F_1 ending with a bigger market share than F_2 (b) There is no trajectory τ that reaches the equal market share line on the crossing segment Σ_c and crosses into L, with F_1 ending with a bigger saturated market share than F_2 95

D.1 Simulation ran with Piiroinen, Kuznetsov Matlab software. Trajectories shoul stop on the saturated line Sat, however in the simulation they continue moving upwards. 97
D.2 Trajectory on the phase plane showing more than one crossing. This situation does not comply with Filippov's theory for Odes with discontinuous RHS. 98
D.3 A magnified section of the discontinuity line Σ. For a point that lies in the ϵ-*tube* around Σ and satisfies the sliding condition (A.6), the point is assigned the field (A.2) defined by the Filippov rule for the numerical discretization 98
D.4 Phase plane plotted with simulation data calculated using Berkeley Madonna. 100
D.5 Phase plane plotted with simulation data calculated using the basic ϵ-*tube* algorithm. . . 100
D.6 Zoom of the phase plane nearby the sliding end point s_{end} 100

D.7 Trajectory calculated using the basic ϵ-*tube* algorithm. 100

List of Tables

1.1 Duopoly differential games and their solutions. CLNE= closed-loop Nash equilibrium, OLNE= open-loop Nash equibilibrium, FS= Feedback strategy, i.e. subgame perfect (not depending on initial conditions) 11

4.1 One parameter variation in model VWDsP. Increasing k_1 or c_1 moderately increases the market share value at pseudoequilibrium. On the other hand, increasing the predation effort c_{21} decreases noticeably the market share value at pseudoequilibrium. 44

4.2 Model VWDsP, scenario 1: Equilibrium points of field $\mathbf{f}^L$ with parameters k_1=1.5, c_1=0.7, k_2 =2, c_2 =1, c_{12} =0.15 45

4.3 Model VWDsP, scenario 1: Equilibrium points of field $\mathbf{f}^U$ with parameters $k_1 = 1.5$, $c_1 = 0.7$, $c_{21} = 0.15$, $k_2 = 2$, $c_2 = 1$ 45

4.4 Model VWDsP, scenario 2: Equilibrium points of field $\mathbf{f}^L$ with parameters $k_1 = 1.5$, $c_1 = 0.7$, $k_2 = 2$, $c_2 = 1$, $c_{12} = 0.15$ 47

4.5 Model VWDsP, scenario 2: Equilibrium points of field $\mathbf{f}^U$ with parameters $k_1 = 1.5$, $c_1 = 0.7$, $c_{21} = 1.1$, $k_2 = 2$, $c_2 = 1$ 47

5.1 Model LVsP, scenario 1: Equilibrium points of field $\mathbf{f}^L$ with parameters b_1=6.8, a_{11}=1, d_{21} =1.6, b_2 =10.9, a_{22} =1, c_{12} =5.5 63

5.2 Model LVsP, scenario 1: Equilibrium points of field $\mathbf{f}^U$ with parameters b_1=6.8, a_{11} =1, c_{21} =3.6, b_2 =10.9, a_{22} =1, d_{12} =4.6 63

5.3 Model LVsP, scenario 2: Equilibrium points of field $\mathbf{f}^L$ with parameters b_1=6.8, a_{11}=1, d_{21} =3, b_2 =10.9, a_{22} =1, c_{12} =5.5 65

5.4 Model LVsP, scenario 2: Equilibrium points of field $\mathbf{f}^U$ with parameters b_1=6.8, a_{11} =1, c_{21} =4.4, b_2 =10.9, a_{22} =1, d_{12} =4.6 65

5.5 Model LVsP, scenario 3: Equilibrium points of field $\mathbf{f}^L$ with parameters b_1=6.8, a_{11}=1, d_{21} =4.5, b_2 =10.9, a_{22} =1, c_{12} =5.5 66

5.6 Model LVsP, scenario 3: Equilibrium points of field $\mathbf{f}^U$ with parameters b_1=6.8, a_{11} =1, c_{21} =4.9, b_2 =10.9, a_{22} =1, d_{12} =4.6 66

5.7 Model LVsP, scenario 4: Equilibrium points of field $\mathbf{f}^L$ with parameters b_1=6.8, a_{11}=1, d_{21} =3, b_2 =10.9, a_{22} =1, c_{12} =5.5 68

5.8 Model LVsP, scenario 4: Equilibrium points of field $\mathbf{f}^U$ with parameters b_1=6.8, a_{11} =1, c_{21} =5.5, b_2 =10.9, a_{22} =1, d_{12} =4.6 68

5.9 Model LVsP, scenario 5: Equilibrium points of field $\mathbf{f}^L$ with parameters b_1=6.8, a_{11}=1, d_{21} =6.3, b_2 =10.9, a_{22} =1, c_{12} =5.5 70

5.10 Model LVsP, scenario 5: Equilibrium points of field $\mathbf{f}^U$ with parameters b_1=6.8, a_{11} =1, c_{21} =7.1, b_2 =10.9, a_{22} =1, d_{12} =4.6 70

List of Symbols

[F] Firm, 5

[G] Goodwill, 4

[H] Hamiltonian, 9

[J] Performance index, 8

[M] Saturation of the sales rate, the total potential market size, 5

[R] Revenue coefficient, 8

[$S(t)$] Sales rate at time t, 5

[V] Value function, 109

[β_i] Constant, effectiveness of advertising of F_i. Response constant, 5

[δ] Diminishing or decaying rate, 5

[λ] Costate, 9

[$\mathbf{J}_i(x_1, x_2)$] Jacobian matrix of firm F_i, which is a function of the states x_1, x_2, 25

[i] Firm index $i = 1, 2$, 5

[p] Price, 4

[q] Quantity of products to sell, 4

[r] Discount factor, used to convert future values to present values, 93

[t] Time, 5

[u] Advertising, 4

[u_{ij}] Predatory advertising on F_i made by $F_j, i \neq j, i, j = 1, 2$, 7

[x_i] State variable, market share of firm F_i unless otherwise stated, 7

[z] Other factors, 4

Chapter 1

Introduction

A duopoly refers to a situation in which two companies own all or nearly all of the market for a given product or service. A duopoly is the most basic form of oligopoly, which refers to a market dominated by a small number of companies. A duopoly can have the same impact on the market as a monopoly if the two players collude on prices or output. Collusion results in consumers paying higher prices than they would in a truly competitive market and is illegal under antitrust laws in many countries. For example, Boeing and Airbus have been called a duopoly for their command of the large passenger airplane market. Similarly, Amazon and Apple have been called a duopoly for their dominance in the e-book marketplace [12]. A central feature of oligopoly markets, is competitive interdependence: the decisions of every firm significantly affect the profits of competitors. In contrast, each seller in a perfectly competitive market is so small that it has an imperceptible competitive impact on its rivals [13].

Advertising has been and continues to be an important topic for academic research, in both marketing and management science, as attested by the appearance of recent surveys devoted to the topic in both areas. As Erickson [14] writes "A critical aspect of the advertising budgeting process involves competitive issues – anticipated spending levels of major competitors, effects that competitive advertising may have on the firm's market share, sales, and profit, and the interactive nature of a competitor's advertising with a firm's own. Competition is ignored only at the firm's peril; empirical studies (e.g., Little 1979) have shown quite clearly that competitive advertising can have a direct, and negative, effect on a company's market share."

Since a competitive scenario with strategic decisions being made by two competing firms is usually described by the word duopoly, this term is also used while describing dynamic models of advertising competition. However, it is important to point out here that, in the simplest case of two firms producing an identical product, the normal use of the term duopoly in the economics literature involves the use of product price, implicitly or explicitly. This is the case with the three most popular models of duopoly behavior, namely quantity leadership (Stackelberg), price leadership, simultaneous quantity setting

(Cournot), simultaneous price setting (Bertrand), the terms in parentheses are the names of the economists who introduced these models which now bear their names [15]. In contrast, as we will see, many models of advertising competition focus on advertising effort (as a control) and its effect on market share (controlled variable), without explicitly taking product price into account. This thesis will concentrate on such models, in order to avoid the complications of price dynamics and emphasize the control aspects and strategic interactions.

Specifically, we consider a duopolistic industry in which two firms are engaged in a brand competition using advertising policies which may have a predatory effect on the opponent's brand. If an increase of advertising in one firm causes a decrease in the sales of its rivals, then that advertising can be called predatory, although the firm may gain more than the sum of the losses of the other firms [16, p. 144, chap. 6]. In comparative advertising a brand can be implicitly or explicitly compared two or more products. Although it is legitimate to compare a product favorably against other products, it can be challenged by law in countries such as the US [17, p. 71, chap. 3]. Alternatively, predatory advertising can be an inevitable side effect of market saturation – in other words, if one firm gains clients, it is because the other has lost them.

1.1 Structure of the thesis

In the remainder of this chapter we review models of advertising and models of duopolies that use advertising in the definition of their dynamics. We also include one way of formulating the differential game duopoly approach and the policies obtained for nonprice duopoly models using differential games. This should help the reader contrast the approach proposed in this thesis with differential games applied to duopolies.

In chapter 2, we discuss our approach to the study of advertising policies in a duopoly. In brief, the approach consists in using simple implementable advertising policies in a given model and with them identify initial conditions which lead to a firm fulfilling its target market share. At the same time, we calculate an associated performance index for the dynamical system. The obtained market share and the associated index value should guide management on the decision of adopting or not a given advertising policy. Note that our approach is complementary to differential games approach where a control function that maximizes a performance index is calculated.

In chapters 3, 4 and 5 the proposed approach is applied to different duopoly models of advertising. The models in chapters 3 and 4 are extensions of the models Vidale-Wolfe and Deal. In chapter 3, a complete description of the dynamics for the model VWDsC is given. In chapter 4, more modifications to Vidale-Wolfe are studied. In chapter 5, we consider the model Lotka-Volterra with switched predation which is a model based on the Lotka-Volterra model for competing species.

The final chapter 6 summarizes the contributions and indicates some possible future work.

1.2 Literature review

We will first briefly review the different models and approaches in the literature with regard to competitive advertising in a duopoly. Broadly speaking, there are models which study the dynamics of market shares or sales of the competing firms and those that study a hypothetical quantity called consumer "goodwill" under the effect of different types of advertising. There are approaches which consider only the dynamics of the market share or sales and others that consider the interaction of prices with the market dynamics. Finally, in terms of the mathematical tools used to study the different classes of models, there are two main approaches: (i) using the tools of dynamical systems, and (ii) using differential games.

We first present the different classes of models that have been proposed, followed by a brief overview of differential game approaches as well as a critique of these approaches, leading to our proposal, which is centered on a dynamical systems approach that seeks to model some qualitative and behavioral aspects of complex advertising decision making, in order to capture some of its important aspects. Thus, the possible value of the our approach lies in its adequacy as a model of aggregate advertising decision making, its mathematical tractability and its potential implementability by real decision makers.

1.2.1 Dynamic models of response to advertising in duopolies

One of the earliest mathematical models of advertising was proposed by Dorfman and Steiner [18], they assumed that a firm makes two kinds of choices: the price of its product and the amount of its advertising budget. They related the quantity the firm can sell per unit of time, q, its price, p, and its advertising variable, in this case advertising budget, u, as a function

$$q = f(p, u) \tag{1.1}$$

and used graphical methods and calculus to derive a basic qualitative result stated as follows: *A firm which can influence the demand for its product by advertising will, in order to maximize its profits, choose the advertising budget and price such that the increase in gross revenue resulting from a one dollar increase in advertising expenditure is equal to the ordinary elasticity of demand for the firm's product.* On the other hand, they conclude their article by saying "*There are good grounds for doubting the economic significance of the whole business of writing down profit functions (or drawing curves) and finding points of zero partial derivatives (or graphical points of tangency). Such devices are merely aids to thinking about practical problems and it may be an uneconomical expenditure of ef-*

fort to devote too much ingenuity to developing them. Yet such devices are aids to clear thought and, if sufficiently simple and flexible, they help us find implications, interrelationships, and sometimes contradictions which might escape notice without them. Such aids are particularly needed in the field of ***nonprice competition**" (emphasis ours).*

We will work with models for advertising dynamics in nonprice competition in a duopoly. We continue with a literature review of models of advertising dynamics, briefly reviewing the various classes of models, in accordance with the broad classifications price and nonprice models.

1.2.2 Dynamic duopoly models involving price and advertising effort

In 1962, shortly after publication of the Dorfman-Steiner model, Nerlove and Arrow [19] introduced their model for the goodwill stock of a firm. It is sometimes referred to as a two level model in the sense that another variable of interest is derived from the goodwill. Advertising (the control variable) affects the goodwill G directly and goodwill then determines, through a functional relationship, the main variable of interest q which is the quantity to be produced (and sold) by the firm. The corresponding equations are:

$$\dot{G}(t) = u(t) - \delta G(t) \tag{1.2}$$

$$q = f(p, G, z) \tag{1.3}$$

where G is the goodwill, $u(t)$ is the advertising and δ is a decay factor, the quantity q to produce or sell as a function of the price p, the goodwill G and other factors represented by z.

Finally, we briefly mention the class of Cournot duopoly models with advertising, although since they consider price, they will not be considered further in this proposal. Cournot duopoly [15, p. 507, sec. 27.5] is very important in oligopoly theory but usually it is not considered part of marketing. In a Cournot duopoly, there is a single price affected by the total production of the firms. Cellini and Lambertini, in 2003, used an inverse demand function $p(t) = (res(t) - q(t))^{\frac{1}{\alpha}}$, where $p(t)$ is the price at time t, $res(t)$ is the consumer reservation price, $q(t)$ is the total quantity produced by the firms and the parameter $\alpha \in (0, \infty)$ determines the curvature of the demand. Their assumption is that $res(t)$ can be increased by firms advertising efforts $u_i(t)$. They use a dynamic similar to Nerlove-Arrow $r\dot{e}s(t) = \sum_{i=1}^{n} u_i(t) - \delta res(t), \quad res(0) = res_0 > 0$ [20, p. 130-132, sec. 6.2]. Escobido and Hatano derive a quantity production adjustment process in a Cournot duopoly that is equivalent to a Lotka-Volterra competing species equation [21]. Another type of models is dynamic games with price and advertising, for a discussion see [20, p. 132-141, sec. 6.3].

1.2.3 Dynamic duopoly models involving only advertising effort (nonprice models)

In 1957, Vidale and Wolfe [22] carried out detailed analyses of real firms and advertising data in order to propose their well known eponymous model:

$$\frac{dS(t)}{dt} = \beta u(t)\frac{M - S(t)}{M} - \delta S(t) \tag{1.4}$$

where $S(t)$ is the sales rate at time t, β is a response constant, M is the saturation of the sales rate, $u(t)$ is the advertising rate and δ is a decay constant of the sales.

Another important class of models are derived from the Lanchester model of human warfare [23] which was first applied to advertising competition by Kimball [24], and reapplied to oligopolies by Little [25] in 1979. Little presented the following Lanchester model

$$\dot{S}_1 = \beta_1 u_1 S_2 - \beta_2 u_2 S_1 \tag{1.5}$$
$$\dot{S}_2 = \beta_2 u_2 S_1 - \beta_1 u_1 S_2 \tag{1.6}$$

where S_i is the sales rate of firm F_i, u_i is the advertising rate of F_i, β_i is the advertising effectiveness constant of F_i, the total market sales is $M = S_1 + S_2$. The sales of F_i increases proportionally to its competitor's sales S_j and decreases proportionally to its own sales S_i. The situation is symmetric for the opponent. Notice that the substitutions $S_1 = S$, $S_2 = M - S$, $\beta_1 M = \beta$ and $\beta_2 u_2 = \delta$ in (1.5) result in the Vidale-Wolfe monopoly model, so that the Lanchester model can be seen as a generalization of (1.4) [25].

Little [25] argued that a good model of advertising should include the sales response to advertising expenditures, the carryover effect of past advertising on current sales, as well as the possibility of diminishing returns to cumulative advertising expenditures and generalized the Lanchester model to comply with the response having the diminishing returns property:

$$\dot{S}_1 = \beta_1 u_1^{\alpha_1} S_2 - \beta_2 u_2^{\alpha_2} S_1 \tag{1.7}$$
$$\dot{S}_2 = \beta_2 u_2^{\alpha_2} S_1 - \beta_1 u_1^{\alpha_1} S_2 \tag{1.8}$$

where $0 < \alpha_i < 1$ corresponds to a concave response in u_i and $\alpha_i > 1$ corresponds to an S-shaped response in u_i.

There have been several reviews of models of advertising, one of the first by Sethi in 1977 [26] was mainly on monopolies. Later, reviews by Feichtinger et al. in 1994 [27], Erickson in 1995 [28] focused mainly on duopolies for which open-loop advertsing strategies were computed. In a more recent survey by Huang et al. 2012 [29] although duopolies are mentioned, the main focus is on oligopolies and supply chains, which are in many cases one manufacturer and two resellers. More recent surveys on differential games focus on supply chains [30], [31].

The models for the dynamics of many of the duopolies and of the oligopolies proposed in the literature were derived from Nerlove-Arrow [19], Diffusion [32], Vidale-Wolfe [22] and Lanchester [24] [25]. In 2012, Bhaya and Kaszkurewicz proposed a unified model inspired by the Vidale-Wolfe [33] and a variation of it in [34]. In 2014, Wang et al. proposed a new dynamic model of advertising based on Lotka-Volterra [35]. Our focus is on models that consider the sales response to advertising, we will study the models similar to the models proposed by Bhaya and Kaszkurewicz [33, 34], and the model proposed by Wang et al. [35].

Sales response to advertising are models derived from Vidale-Wolfe or Lanchester models. One of the first was presented by Deal [1] and it is a simplified model of the dynamics of a duopoly, as an extension of the monopoly model of Vidale and Wolfe. We will discuss Deal's model, given below, as a starting point for the introduction of our model later on:

$$\dot{s}_1(t) = -\delta_1 s_1(t) + \beta_1 u_1(t)[M - s_1(t) - s_2(t)]/M \tag{1.9}$$

$$\dot{s}_2(t) = -\delta_2 s_2(t) + \beta_2 u_2(t)[M - s_1(t) - s_2(t)]/M \tag{1.10}$$

where $s_i(t)$ is sales rate for firm F_i at time t, $u_i(t)$ is the advertising expenditure for F_i at time t, δ_i is the sales decay parameter, β_i is the sales response parameter and M the total potential market size. In this model there is an expression for the sales response of each competitor, and the effect of competition is also recognized by the introduction of the coupling term $(M - s_1 - s_2)/M$ in each equation. The coupling term can be interpreted as follows. The direct effect of competitive pressure is that, as the sales to all other market parties increase, the size of the remaining unconquered market decreases and this in turn diminishes the sales effectiveness of successive advertising expenditures for the firm.

Wang et al. [6] combined Vidale-Wolfe and Lanchester models as follows

$$\frac{dx_i}{dt} = \beta_i u_i(1 - x_i) - \beta_j u_j x_i - \delta_i x_i, \quad i, j = 1, 2 \quad i \neq j \quad x_i(0) = x_{i0} \tag{1.11}$$

where the $x_i = s_i/M$ is the market share of firm F_i, the cross term $-\beta_j u_j x_i$ is from the Lanchester model and the terms $\beta_i u_i(1 - x_i) - \delta_i x_i$ from Vidale-Wolfe. The Lanchester model was used by many authors to model the dynamics on the saturated market ($x_2 + x_1 = 1$) [6], Wang et. al lifted that restriction so that the market shares could grow from small initial values to saturation ($0 \leq x_1 + x_2 \leq 1$). In 2014, Hung et al. [36] used a Lotka-Volterra model for the dynamics on the saturated market.

Bhaya and Kaszkurewicz [33] proposed a unifying model derived from Vidale-Wolfe

$$\dot{x}_1 = u_{11}(1 - x_1 - x_2) - u_{12}(x_1 + x_2) \tag{1.12}$$
$$\dot{x}_2 = u_{22}(1 - x_1 - x_2) - u_{21}(x_1 + x_2) \tag{1.13}$$

where u_{ii} is the positive advertising acting on the unconquered market $(1 - x_1 - x_2)$ and $u_{ij}, \quad i, j = 1, 2 \quad i \neq j$ is the predatory advertising, i.e., the advertising of firm j acting on the conquered market $(x_1 + x_2)$.

In an extension of their previous model, Bhaya and Kaszkurewicz [34] assume that the loss of one firm is the gain of the other. The dissertation by Cruz [37] uses a (CLF) Lyapunov Control Function for the model defined by Bhaya and Kaszkurewicz in [34].

Recently Wang et al. proposed an aggregate model for a duopoly. In the first level advertising generates the maximum sales that could be achieved for each firm. In the second level the sales follow a Lotka-Volterra competing species dynamics in which each firm can grow its sales until its corresponding maximum. The sales response to advertising model based on Lotka-Volterra proposed in [35] is

$$\dot{S}_1 = S_1(b_1 - a_{11}S_1 - a_{12}S_2) \tag{1.14}$$
$$\dot{S}_2 = S_2(b_2 - a_{21}S_1 - a_{22}S_2) \tag{1.15}$$

where S_i is the sales of firm i at time t, $b_i = f(u_i)$ is the intrinsic sales growth and u_i is the advertising level of firm i, a_{ii} is the growth restriction of firm i on itself and a_{ij} is the growth restriction on firm i created by firm j. The term having the growth restriction a_{ii} is also known as an overcrowding term in ecology while the term having a_{ij} is referred to as a predation term.

1.2.4 Differential games of advertising

Apart from the model, the other important ingredient required for a mathematical analysis of advertising dynamics is the stipulation of an objective function for a firm. This is a strategic managerial task and not an easy one, because it needs to be time varying, in response to the life cycle of the product, to competitive pressures and so on. In addition, objectives of competing firms may differ. For instance, one firm may be interested only in maximizing its total profit over the fiscal year and its competitor may be interested only in maximizing its market share by the end of the year. Evidently, different objectives result in different advertising strategies. These two objectives could be weighted and additively combined into one as is common in optimal control theory, leading to a performance index. Deal [1] proposed one such performance index, which he called a tempered performance index in order to highlight the fact that the choice of weight can be used to trade off (or temper) market share and profit:

$$J_i = w_i S_i(t_f)/[S_i(t_f) + S_j(t_f)] + \int_{t_0}^{t_f} [R_i S_i(t) - u_i^2(t)] dt \tag{1.16}$$

where i refers to one of the firms and j to the other, S_i is the sales, R_i is the net revenue coefficient, w_i the weighting factor for the performance index, t_0 the initial time of the planning horizon and t_f the final time of the planning horizon.

As Deal points out, his model is a simple formulation of a complex advertising decision, but one that represents the important characteristics of advertising decision making. Furthermore, he states explicitly that the real value of the model lies in its adequacy as a normative model of aggregate advertising decision making and in its approachability for solution by differential games techniques.

When each firm optimizes its performance index subject to the system dynamics, the firms are said to be participating in a differential game. Suppose that the performance index J_i is associated with firm i. In the finite horizon case, the objective has two terms: the profit from time t_0 until t_f plus a salvage value at final time t_f.

$$J_i = \int_{t_0}^{t_f} g_i(x_1, \ldots, x_n, u_1, \ldots, u_n, t) dt + h_i(x_i(t_f), t_f), \quad i = 1, \ldots, n \tag{1.17}$$

or, in the case of infinite horizon, the objective has one term: the profit from time 0 until ∞.

$$J_i = \int_{t_0}^{\infty} g_i(x_1, \ldots, x_n, u_1, \ldots, u_n, t) dt, \quad i = 1, \ldots, n \tag{1.18}$$

The firm i attempts to maximize J_i subject to the dynamics

$$\dot{S}_i = f_i(x_1, \ldots, x_n, u_1, \ldots, u_n, t), \quad x_i(0) = x_{i0}, \quad i = 1, \ldots, n \tag{1.19}$$

where the controls u_i are advertising rates, the state variables x_i are market shares or sales [28].

Assuming that competitors cannot collude and that each of them has complete information on the best strategies of its opponent, or that it can infer them, Nash equilibria are sought. Informally, a Nash equibilibrium is a set of strategies where one player cannot improve its outcome unilaterally. Nash equilibria can be found using Hamilton-Jacobi-Bellman-Isaacs equations, obtained from the objective (1.17) and the constraints (1.19).

The Hamiltonian for the differential game just described is [28]:

$$H_i = g_i + \sum_{j=1}^{n} \lambda_{ij} \dot{x}_j, \quad i = 1, \ldots, n \tag{1.20}$$

where λ_{ij} are called costate variables. The necessary conditions for a Nash equibilibrium are given by

$$\frac{\partial H_i}{\partial u_i} = 0, \quad i = 1, \ldots, n \tag{1.21}$$

and for a given a finite horizon T:

$$\begin{aligned}
\dot{\lambda}_{ij} &= -\frac{\partial H_i}{\partial x_j} - \sum_{k \neq i} \frac{\partial H_i}{\partial u_k} \frac{\partial u_k^*}{\partial x_j}, \quad i, j = 1, \ldots, n \\
\lambda_{ii}(t_f) &= \left. \frac{\partial H_i}{\partial x_i} \right|_{t=t_f}, \quad i, j = 1, \ldots, n \\
\lambda_{ij}(t_f) &= 0, \quad i, j = 1, \ldots, n
\end{aligned} \tag{1.22}$$

where u_k^* are Nash equilibrium strategies. If there is an infinite horizon, the constraints on the costates $\lambda_{ii}(t_f)$, $\lambda_{ij}(t_f)$ are replaced by an assumption that the problem approaches a steady state.

There are two types of Nash equilibria: open-loop and closed-loop. Open-loop equilibrium strategies depend only on time t. Closed-loop strategies depends on states S_i as well as time t. A closed-loop equilibrium that does not depend on initial conditions is called a feedback strategy. Finding closed loop strategies is difficult because the sum of partial derivatives in (1.22) is nonzero in general and the system of differential equations are not solvable in many cases. The closed-loop strategies can be determined only in special cases.

Remark: Even though many of the early models were descriptive, they were conducive to the use of differential game theory in order to derive the associated optimal dynamic advertising policies. In fact, most of the economics and management literature is focused on designing management strategies that optimize some pre-determined performance indices, using tools from differential game theory, to investigate Nash equilibria that represent some possible equilibrium market shares [38–40], and the recent surveys [29], [30], [31]. Several studies also emphasize the appearance of chaos in duopolistic or oligopolistic models (see, e.g., [41, 42] and references therein).

1.2.5 Duopoly differential games with dynamics derived from Vidale-Wolfe or Lanchester models

Table 1.1 lists duopoly differential games that use Vidale-Wolfe or Lanchester dynamics. We can make the following observations from the entries in the table:

- Note that the feedback strategies found in most of the cases use the approach suggested by Case in 1979 [2]. Only Bass et al. [8] and Krishnamoorthy et al. [10] do not cite Case, however they also use the value function approach with infinite horizon and make assumptions about the form of functions. Krishnamoorthy et al. uses a value function V that is linear ($V(S_i, S_j) = k_i(T - S_i - S_j)$ where $i, j = 1, 2$ $i \neq j$) and two types of demand function, one linear and the other isoelastic. Bass et al. uses a value function V that is linear ($V_i(S_1, S_2) = \alpha_i + \rho_i S_i + \gamma_i S_{3-i}$ where $i = 1, 2$ and $\alpha_i, \rho_i, \gamma_i \in \mathbb{R}$).

- In most of the cases the open-loop solution is computed numerically.
- The last entry in Table 1.1, corresponding to Jorgensen et al. [11], is not directly related to Lanchester or Vidale-Wolfe. They bring attention to state-redundant differential games which have the property that the open-loop solution and the closed-loop solution are the same. The study of differential games with this property began with the work of Leitmann-Schmittendorf.

1.2.6 Characteristics of differential game solutions

- The usability of a solution of a differential game depends on its type of equilibrium strategy, i.e., closed-loop or open-loop strategy.
  - A closed-loop strategy that is also a feedback strategy is preferred so that the competitors can adjust their strategies as the state changes. A closed-loop strategy must satisfy (1.19), (1.21) and (1.22), i.e. a system of partial differential equations where the state equations have initial conditions and the costate equations have final conditions.
  - Open-loop strategies are viable only if competitors decide their strategy at the beginning of the interaction and do not make any adjustment thereafter. In the case of open-loop strategies, they must satisfy (1.19), (1.21) and (1.22). However, in this case the summation in (1.22) is zero because the strategy depends only on time not on the states and open-loop equilibria can be computed. Also, if the time horizon is infinite the terminal time horizon conditions should be replaced by steady state assumptions.
- It is assumed that each competitor has full knowlegdge of the competitive interaction and the profit structure of the other competitors [14, p. 7, chap. 2].
- Solving differential games with either open-loop or closed-loop Nash equilibrium strategies is difficult [9]. In 1995, Erickson [28] stated that without some development in the area of partial differential equations closed-loop solutions for oligopolies will continue being difficult to obtain. In 2004, Jarrar et al.[7] stated that in many cases researchers simplify the HJB equations to make them analytically tractable and find a solution.

Published by	Dynamics	Type of solution	Remarks
Deal 1979 section E.1, [1]	V-W	OLNE numerical. Finite horizon.	
Case 1979 section [E.2], [2]	Lanchester	CLNE FS analytical. Infinite horizon	Zero discount factor
Sorger 1989 section [E.3], [2]	Lanchester	OLNE, CLNE FS analytical. Finite, infinite horizon	Nonzero discount factor. Square root of market share (dim. returns). For finite horizon does qual. analysis of system of diff. eq.
Chintagunta et al. 1992 section E.4, [3]	Lanchester	CLNE FS analytical. OLNE numerical. Infinite horizon	Square root on advertising (dim. returns). Zero discount factor. Empirical data test.
Erickson 1992 section E.5, [4]	Lanchester	CLNE FS analytical. OLNE numerical. Infinite horizon	Square root on advertising (dim. returns). Zero discount factor. Empirical data test.
Fruchter et al. 1997 section E.6, [5]	Lanchester	CLNE numerical, OLNE numerical. Infinite horizon	Solves numerically an IBVP (initial value boundary problem) instead of a TBVP (two boundary). Nonzero discount factor. Time dependent CLNE proportional to OLNE.
Wang et al. 2001 section E.7, [6]	Lanchester and V-W	CLNE numerical. OLNE numerical. Finite horizon	No discount factor. Empirical data test.
Jarrar et al. 2004 section E.8, [7]	Lanchester	CLNE numerical. Infinite horizon	Nonzero discount factor. Numerical sol. of differential equations of value function.
Bass et al. 2005 section E.9, [8]	Lanchester. Brand adv., generic adv.	CLNE FS symbolic. Infinite horizon	Solves numerically a system of algebraic eq. to get the symbolic values. Nonzero discount.
Wang et al. 2007 section E.10, [9]	Lanchester.	CLNE numerical. OLNE numerical. Infinite horizon	Empirical study. Square root of advertising in the dynamics is not always suitable for the problem.
Krishnamoorthy et al. 2010 section E.11, [10]	V-W. Price and advertising	CLNE FS analytical.Infinite horizon	Value function is linear on the state. Two forms of demand function.
Jorgensen et al. 2010 section E.12, [11]	State-Redundant games.	Analytical	For state-redundant games and some linear-state games the OLNE solution is the same as the CLNE FS solution. For instance, Leitmann-Schmittendorf.

Table 1.1: Duopoly differential games and their solutions. CLNE= closed-loop Nash equilibrium, OLNE= open-loop Nash equibilibrium, FS= Feedback strategy, i.e. subgame perfect (not depending on initial conditions)

- The use of numerical algorithms for solving competitive advertising is widespread, only special cases can be solved analytically. See Table 1.1.

- Even though the closed-loop solution and in particular the feedback strategy solution, which depends only on the state, is more appealing to decision makers of the firms, in many cases it is the open-loop solution that is calculated because it is more tractable [11].

- As a remark, note that when facing the tradeoff between a closed-loop solution and the tractability of the open-loop solutions, researchers have looked for problem structures where both solutions are the same, i.e. the open-loop solution is a closed-loop solution [11]. These are very special rather crafted cases.

Chapter 2

Proposed approach to study advertising duopoly dynamics

In this chapter, we will put together the observations regarding the types of models and approaches available and propose an approach that allows some flexibility in modeling different types of strategic behavior by the competing firms and, in order to do so, examines the consequences of adopting such strategies, rather than attempting to compute them as optimizers of some performance criterion.

2.1 Outline of the proposed approach

Our perspective is to investigate market share dynamics of some nonprice duopoly models subject to a competitive advertising policy. Once the possible market share outcomes have been characterized, they are also evaluated with respect to a performance index. The primary concern is not with maximizing a performance index, rather the objective for firm F_i is to take action to try to prevent its current market share falling below its target market share. This approach can be regarded as being complementary to the differential games approach, which starts out by optimizing a performance index, through the calculation of an optimal advertising effort (which is often difficult to calculate and implement).
As we recall from the previous section, the *problems with finding the optimal control for a differential game* are as follows:

- It is generally not analytically computable.
- In most cases it is not in feedback form for a finite horizon.
- The approaches taken so far are to make simplifications or assumptions with regard to the dynamics in order to obtain an analytical solution or a numerical solution.
- In general both firms need complete information (state and parameters) of the system. However, information of a competitor may not be readily available.

We propose an *alternative approach* in which:

- Firm F_i takes action to try to reach at least its target market share.
- A performance index is used to evaluate the advantage of different advertising controls, but not to design them, as in the differential game approach.
- A simple and naturally implementable policy is used. It is based on switching control that relies only on the observed state of the system, but in which each firm does not use private or privileged information of its competitor in order to design its policy.

In this context, we will make the following *assumptions* in order to model the behavior of firms:

A1. Both firms have complete access to current market share information (i.e., to states x_1, x_2).

A2. The total market capacity is normalized to unity (applicable when the state variable is the market share of the firm).

A3. The market environment is purely competitive, i.e. there is no collusion or market division between the firms.

A4. A firm changes its advertising expenditure only when it perceives that it has a smaller market share than its target market share.

A5. The change in the previous assumption occurs in a predetermined manner, by switching on an additional positive advertising effort directed at regaining lost market share (and possibly surpassing the competitor's market share).

Given these assumptions, our *focus is on analyzing the resulting dynamics and identifying initial market shares which achieve the following goal*:

G The goal of a firm is to reach or surpass its target market share. The firm with an initial market share that is smaller than its target applies an additional advertising effort to reach its target. Once a firm surpasses its target market share, it stops making its additional advertising expenditure.

Evidently, if the goal of surpassing a target share is unachievable, then it is of interest to know if the long term outcome for the firm is its target market share.

2.2 Proposed unified model of advertising dynamics

This section first describes a general model which contains, as special cases, the models of interest previously discussed. We propose a unified model (UM) that can generate different models with appropriate parameter choices:

$$\dot{x}_1 = u_{11}(x_1, x_2)(1 - x_1 - x_2) - v_{12}(x_1, x_2) + v_{21}(x_1, x_2) \tag{2.1}$$

$$\dot{x}_2 = u_{22}(x_1, x_2)(1 - x_1 - x_2) - w_{21}(x_1, x_2) + w_{12}(x_1, x_2) \tag{2.2}$$

where $u_{11}(x_1, x_2), u_{22}(x_1, x_2), v_{12}(x_1, x_2), v_{21}(x_1, x_2), w_{12}, w_{21}(x_1, x_2)$ are the terms to be defined, and the states x_1, x_2 are the market shares of firms F_1, F_2 unless otherwise specified.

The UM model generalizes the model in [34]. Some models generated from (2.1)-(2.2) are listed below:

- The Vidale-Wolfe model without decay terms. The values for the terms in (2.1)-(2.2) are:

$$u_{11} = k_1, \qquad v_{12} = v_{21} = 0 \tag{2.3}$$

$$u_{22} = k_2, \qquad w_{21} = w_{12} = 0 \tag{2.4}$$

- The Lanchester model. The values for the terms in (2.1)-(2.2) are:

$$u_{11} = 0, \qquad v_{12} = v_{21} = k_1 x_2 \tag{2.5}$$

$$u_{22} = 0, \qquad w_{21} = w_{12} = k_2 x_1 \tag{2.6}$$

- The Lotka-Volterra model. The values for the terms in (2.1)-(2.2) are:

$$u_{11} = b_1 x_1, \qquad v_{12} = ((a_{11} - b_1)x_1 + (a_{12} - b_1)x_1)x_1, v_{21} = 0 \tag{2.7}$$

$$u_{22} = b_2 x_2, \qquad w_{21} = ((a_{22} - b_2)x_2 + (a_{21} - b_2)x_1)x_2, w_{12} = 0 \tag{2.8}$$

2.3 Associated Performance Index

Often, *profit* is also a criterion of interest. Deal [1] proposed a performance index which uses market share and profit (see equation (1.16)), and we use the following simplified version which corresponds to the index used by Sethi in [40, p. 196, chap. 7] without the discount factor, since a finite horizon problem is being considered:

$$J_i(r, x_i(t), u_i(t), t) = \int_{t_0}^{t_f} [r x_i(t) - u_i(t)] dt \tag{2.9}$$

where i refers to firm i, r is the maximum sales revenue factor of the market, t_0 the initial time of the planning horizon and t_f the final time of the planning horizon. We use J_i as an abbreviation for $J_i(r, x_i(t), u_i(t), t)$. Note that a firm makes profits when its index $J(x_i)$ is positive, whereas it makes a loss when its index $J(x_i)$ is negative. The term $rx_i(t)$ represents the income of firm F_i, so that the factor r converts market share x_i to a money unit, i.e. thousand of dollars. The factor r can be estimated as the product of the maximum number of items that can be sold in the market multiplied by the average price of the product.

2.4 Motivation for using state dependent control

Consider the following system which is the simplified Vidale-Wolfe model for a monopoly firm F without the decay term.

$$\dot{S}(t) = u(t)(M - S(t)) \tag{2.10}$$

where S is the sales, M is total sales the market can support, $u(t)$ is the advertising effort or control. Suppose that F can sell at most an amount $M^* < M$; this could happen because it has a limited production given its current factory capability. Since the term $(M - S(t))$ on the right hand side of (2.10) is always positive for all possible $S(t)$, it follows that, in order for $S(t)$ to increase, we should have $u(t) > 0$ whenever $S(t) \in [0, M^*]$. When $S(t)$ exceeds M^*, it is reasonable to suppose that spends zero advertising effort, i.e., $u(t) = 0$ for $S(t) \geq M^*$. One particular choice of continuous control that achieves this is $k(M^* - S(t)$ for $S(t) \in [0, M^*]$ and $u(t) = 0$ for $S(t) \geq M^*$. Another, possibly more realistic strategy recognizes that, even when a given maximum sales level has been achieved, advertising effort should merely be reduced and not extinguished, in order to avoid losing the sales level it has achieved, due to ongoing externalities. The two strategies just described are depicted in Figures 2.1 (a) and (b), with the former being continuous and the latter discontinuous.

Additional motivation for decision makers to use switching control comes from the concept of performance dashboards from [43, p. 4-5, chap. 1]. A performance dashboard communicates strategic objectives and enables business people to measure, monitor, and manage the key activities and processes needed to achieve their goals. To achieve all this, a performance dashboard provides three main sets of functionality:

- Monitor critical business processes and activities using metrics that trigger alerts when performance falls below predefined targets.
- Analyze the root cause of problems by exploring relevant and timely information from multiple perspectives at various levels of detail.

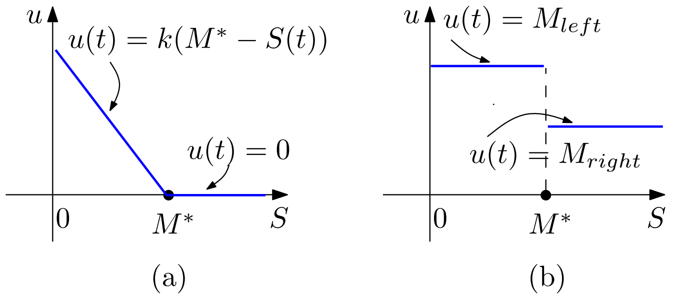


Figure 2.1: Control $u(t)$ applied to achieve the sales goal of at least M^*. The continuous case is shown in (a) the control $u(t) = k(M^* - S(t) > 0$ when current sales $S(t) < M^*$ and $u(t) = 0$ when $M^* < S(t)$. The discontinuous case is shown in (b) where the control is a switching control $u(t) = M_{left}$ when $S(t) < M^*$ and $u(t) = M_{right}$ when $M^* < S(t)$.

- Manage people and processes to improve decisions, optimize performance, and steer the organization in the right direction.

Performance dashboards provide benefits such as actionable information delivered in a timely fashion that lets users take action to fix a problem [43, p. 15, chap. 1].

The goal of the dashboard is to present organized data to the decision maker in an easy-to-understand format. In addition to providing the data of the current metric, the dashboard provides values of other metrics, at comparable times, that would help the decision maker in the understanding of the current metric. A dashboard is more than simple reporting, a dashboard is also interactive. [44, p. 402, chap. 10]. Figure 2.2 shows a speedometer which is one way of displaying part of a dashboard.

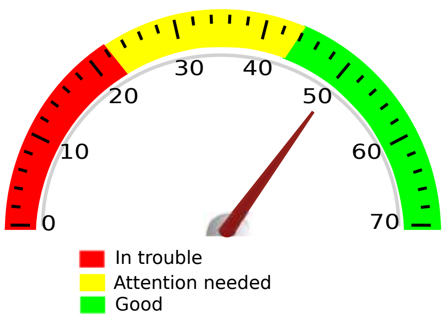


Figure 2.2: A speedometer for indicator *Sales in millions of \$* using a standard coloring for the status of a metric: *red=In trouble* (from 0 to 22), *yellow=Attention needed* (from 22 to 44), and *green=Normal* (from 44 to 70).

Following the dashboard usage in business management, one can see that an switching control strategy can be regarded as a corrective action when, for instance, the state of the metric *market share* or *sales* changes its status from green (normal) to yellow (attention needed).

2.5 Objectives of the firms

The goal of firm F_i is to have, at the saturated market, at least its target market share $target_i$. The target of a firm defines the switching line. This line goes from the origin to the target market share of the firm on the saturated market share line. This switching line specifies the change of controls of the firm: When a firm perceives that its market share is below its target share, that firm turns on an extra control. Depending on the value of $target_1 + target_2$ the possibilities are:

1. The sum $target_1 + target_2 = 1$. In this case, both firms have the same switching line. For instance, if both firms want to have at least with 0.5 of the market share in the long run then the switching line is the equal share line $x_2 - x_1 = 0$.

2. The sum $target_1 + target_2 > 1$. In this case, there are two switching lines, one for each firm. Even though in a duopoly the market shares of the two firms comprise the total market, it is highly probable, given the competition between the firms, that each firm wants a bigger market share than the other firm. For instance, suppose that firm F_1 desires 0.6 of the market share and F_2 desires 0.7 of the final market share. The switching line for F_1 is $x_2 - \frac{0.4}{0.6}x_1 = 0$ and F_1 achieves its goal when $x_1 > \frac{0.6}{0.4}x_2$. The switching line for F_2 is $x_2 - \frac{0.7}{0.3}x_1 = 0$ and F_2 achieves its goal of having at least 0.7 of the market share when $x_2 > \frac{0.7}{0.3}x_1$.

3. The sum $target_1 + target_2 < 1$. In this case, there are two switching lines, one for each firm. Even though this case seems to have no conflict, depending on the initial conditions and the parameters of the system, a firm might or might not reach its target market share.

2.6 Classes of Control Functions used by each firm

The control laws to be used are simple laws. A policy $u = kx$ which is state dependent with gain parameter $k > 0$, is said to be a ***constant rate control***, and if $u = k > 0$ (not dependent on state), then it is called a ***constant effort control***. In general, the controls we will use are constant effort, constant rate or a sum of them.

Specifically we propose the use of switching control or variable structure control [45]. Each firm uses the state x to test if its target market share is currently fulfilled. For example, the firm F_i could use a constant effort control while its goal is achieved, but uses a constant effort plus a constant rate control when its goal is not fulfilled. In other words, F_i applies its additional constant rate control to try to fulfill its goal. For instance, if the goal of each firm is to have at least 0.5 of the total market at steady state, the switching control would be

$$u_i(x_1, x_2) = \begin{cases} k_i x_i + c_i & \text{if} \quad x_i < x_j \\ k_i x_i & \text{if} \quad x_i > x_j \end{cases} \qquad i = 1, 2; \; j \neq i \tag{2.11}$$

Note that, in order to define a dynamical system subject to control (2.11) on the discontinuity line $x_2 - x_1 = 0$, we will use the Filippov rule (see Appendix A, [46]).

2.7 Objective of this thesis

In addition to proposing modifications of advertising models, this thesis focuses on answering the following question, that follows naturally from the assumptions made above:

- Is it possible to identify initial market shares and advertising effort parameters for which the firm with an initial market share below its target will (respectively, will not) be able to attain it?

Also, we want to define a reasonable performance index that will allow a classification of advertising strategies.

Chapter 3

Dynamics of a modification to the Vidale-Wolfe-Deal model under feedback control

3.1 Introduction

We chose to work with the Vidale-Wolfe model [22] because it directly relates market share with advertising and it was tested against real world data. Deal [1] was the first to use the Vidale-Wolfe model for the duopoly case. In addition it has been an inspiration for many models [29].

Before presenting the model Vidale-Wolfe-Deal with switching control (VWDsC), we give some notation related to second order dynamical systems with discontinuous right-hand sides. For general second order models (i.e. having a state vector $\mathbf{x} = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$), an ODE with one discontinuity boundary Σ and *lower and upper vector fields* $\mathbf{f}^L = [\mathrm{f}_1^L(\mathbf{x}), \mathrm{f}_2^L(\mathbf{x})]^T$ and $\mathbf{f}^U = [\mathrm{f}_1^U(\mathbf{x}), \mathrm{f}_2^U(\mathbf{x})]^T$ on either side of Σ is written as:

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{f}^L(\mathbf{x}), & \mathbf{x} \in L \\ \mathbf{f}^U(\mathbf{x}), & \mathbf{x} \in U \end{cases} \tag{3.1}$$

where the regions L, U are separated by $\Sigma = \{\mathbf{x} : H(\mathbf{x}) = 0\}$. *$L$ corresponds to the lower region where* $H(\mathbf{x}) < 0$ and *U to the upper region where* $H(\mathbf{x}) > 0$.

Next, we proceed to the basic control policy which is used in the model VWDsC:

$$u_i(x_1, x_2) = \begin{cases} k_i x_i + c_i & \text{if} \quad x_i < x_j \\ k_i x_i & \text{if} \quad x_i > x_j \end{cases} \tag{3.2}$$

where $i = 1, 2$ and $j \neq i$. Policy (3.2) can be interpreted as follows: if firm F_i is leading the market (i.e., $x_i > x_j$), it uses a constant effort advertising policy ($k_i x_i$); however at the moment at which the competing firm F_j becomes the leader (exceeds the market share of firm F_i), firm F_i switches to an additional (constant rate) effort (c_i) in its attempt to avoid falling behind its competitor.

In order to proceed, some terminology is introduced. The policy (3.2) is one that switches from constant effort to constant effort plus constant rate and vice-versa, whenever *the equal market share line* $x_1 = x_2$ is crossed. Since the switching policy is discontinuous on this boundary, the equal share line is also referred to as *discontinuity boundary* and denoted by the symbol Σ.

Using this previous notation and the policies 3.2 as u_{ii} and $v_{ij} = w_{ij} = 0$ with $i, j = 1, 2$ in model UM (2.1 - 2.2), the equations that define the model VWDsC are as follows:
The upper field $\mathbf{f}^U$, which defines the dynamics in the region U, is given by:

$$\dot{x}_1 = (k_1 x_1 + c_1)(1 - x_1 - x_2) \tag{3.3}$$

$$\dot{x}_2 = k_2 x_2 (1 - x_1 - x_2) \tag{3.4}$$

The lower field $\mathbf{f}^L$, which defines the dynamics in the region L, is given by:

$$\dot{x}_1 = k_1 x_1 (1 - x_1 - x_2) \tag{3.5}$$

$$\dot{x}_2 = (k_2 x_2 + c_2)(1 - x_1 - x_2) \tag{3.6}$$

On Σ, the dynamics is defined following the Filippov rule which is based, roughly speaking, on a convex combination of the lower and upper fields (see Appendix A, [46, p. 50-52, chap. 2] for mathematical details).
Remark: The policy described above belongs to the class of switching policies, in which the switching is based on the observations of the states (market shares). In fact, many other choices of control laws are possible for models derived from the Vidale-Wolfe model and represent different hypotheses about the behavior of competing firms (see, for example, [14, 29, 42, 47, 48]).

Figure 3.1 identifies the regions and the relevant lines for model (3.3)-(3.6). The discontinuity boundary Σ is the line $x_2 = x_1$, also written as $H(\mathbf{x}) = x_2 - x_1 = 0$. The line $1 - x_1 - x_2 = 0$ corresponds to *a saturated market*, and is denoted as Sat. The segment Sat^U (resp. Sat^L) corresponds to firm F_2 (resp. F_1) ending with a bigger saturated market share. The point E is the equal share saturation point, i.e. where both firms have half of the total saturated market. The triangle $\mathcal{T}$, defined as $\{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1\}$, is *the set of all feasible states*. The region U is the set $\{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1 \ \wedge \ x_2 > x_1\}$ where the dynamics (3.3)-(3.4) corresponding to f^U is active. The region L is the set $\{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1 \ \wedge \ x_1 > x_2\}$ where the dynamics (3.5)-(3.6) corresponding to f^L is active.

Given the definition of the switching controls, it is clear that, if a trajectory attains the equal market share line Σ at a point below the equal saturated market share E (see Figure 3.1), then both firms switch on their extra-advertising efforts. Depending on the relative magnitude of their efforts, a resultant motion, called *sliding*, may occur for some *segment* of Σ. The endpoint of this sliding motion, marked by a dot in Figure 3.1, is $S_{end} = (s_{end}, s_{end})$. For brevity, we will use only one coordinate when referring to points on the equal share line Σ, i.e. we will use s_{end} when referring to the sliding end point.

A *separatrix* is a trajectory which separates obvious distinct regions in the phase plane [49, p. 111, chap. 3]. Recall that the goal of each firm is to obtain at least 50% of the saturated market share. For each firm F_i, there is a trajectory φ such that it separates the phase plane into two regions: starting in one of the regions it achieves its goal in the long term, while starting in the other it does not. In model (3.3)-(3.6), we call separatrix the trajectory φ which separates regions (of initial conditions) which correspond to different outcomes (e.g. Firm F_1 has a larger long term market share than Firm F_2, or vice-versa).

Observe that separatrices are associated to a vector field as well as to a firm and depending on the parameters of the system, a given firm can have a separatrix derived from f^U or from f^L, hence we use the notation φ_i^Y to denote the separatrix for firm F_i, $i = 1, 2$ obtained using the dynamics of the field f^Y, $Y = U, L$. Figure 3.1) shows the case where there is one separatrix in region L, other cases will be shown in section 3.2.3.

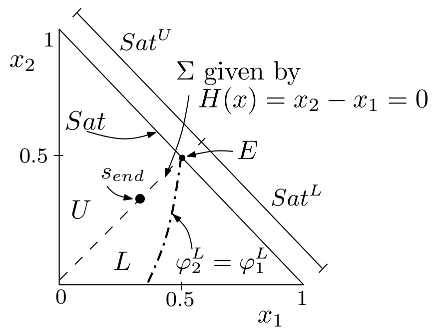


Figure 3.1: The upper region U, the lower region L, the switching line $\Sigma = \{\mathbf{x} : H(\mathbf{x}) = x_2 - x_1 = 0\}$, the segment Sat defined by $1 - x_1 - x_2 = 0$. The segment Sat^U (resp. Sat^L) is the portion of Sat that lies in region U (resp. L). $\mathcal{T}$ is the triangle defined by the points $(1, 0)$, $(0, 1)$, $(0, 0)$. The point marked E is the equal share saturation point, i.e. where both firms have half of the total market. In this figure the sliding end is $s_{end} < 0.5$ and there is one separatrix $\varphi_2^L = \varphi_1^L$

The set of initial shares ($\mathcal{T}$) is partitioned into regions by separatrices. In the cases where there is only one separatrix, one of the firms does not fulfill its goal of at least 50% of the market share even when it begins with a bigger initial market share. For instance, if the separatrix is $\varphi_2^U = \varphi_1^U$ (see Fig. 3.2(c) and (d)) and the initial market share $x_{2_0} > x_{1_0}$ is below φ_2^U, then firm F_2 ends with less than half when market saturation is reached. Given this kind of behavior, the evolution of the market shares after the equal market share is reached until market saturation is attained has to be described.

A trajectory that departs from an initial condition in U or L (i.e. not on Σ) and attains Σ is said to have reached Σ, and the period in which this occurs is known as the *reaching* *phase*. Once the trajectory from U (resp. L) reaches Σ there are the following possibilities (formal mathematical definitions of *sliding* and *crossing* are given in Appendix A):

1. The trajectory *crosses* from U into L (resp. from L into U)
2. The trajectory moves along Σ, which is known as *sliding*, and then crosses from U into L (resp. from L into U) finally ending on Sat^L (resp. Sat^U).
3. The trajectory *slides* on Σ up to a certain point, and then returns to U (resp. to L) finally ending on Sat^U (resp. Sat^L).
4. The trajectory *slides* on Σ until it attains the saturated equal market share point E.

For brevity we will refer case 1 as reaching and crossing (RC), case 2 as reaching, sliding and crossing (RSC), case 3 as reaching, sliding and returning ($RSRet$) and case 4 as (RS). This cases are shown in Figure 3.2.

3.2 Main result for model VWDsC

The main result for model VWDsC is a complete description of market share dynamics. Theorem 3.2.1 characterizes the market share dynamics having initial conditions $x_2(0) > x_1(0)$, i.e. trajectories τ beginning in U, which attain with the equal share line Σ. The case for τ beginning in L, i.e. $x_1(0) > x_2(0)$ is analogous and it is omitted for brevity. Other possibilities, completing the description of the market share dynamics, are given in remarks 1 and 2, which follows Theorem 3.2.1.

Theorem 3.2.1 (Characterization of market share dynamics for model (3.3)-(3.6) with controls (3.2)). *Suppose that τ is a trajectory that starts in region U, i.e. firm F_1 starts out with a smaller market share $x_1(0) < x_2(0)$ and that the parameters k_1, c_1, k_2, c_2 are known, then the trajectories reaching the equal market share line Σ can have one of the following behaviors:*

1. *If the sliding end point $s_{end} = 0.5$ then any trajectory τ with initial conditions in region U and below separatrix φ_1^U reaches the equal share line and slides until the saturated equal share point E on the saturated market line (see Figure 3.2a)*
2. *If the sliding end point $s_{end} < 0.5$ and $k_2 > k_1$, then the separatrix is $\varphi_2^L = \varphi_1^L$ and some trajectories beginning in region U reach the equal share line Σ, move on it until the point s_{end} and return to U finally ending on the saturated market Sat with firm F_2 having more than 50% of the market share.(see Figure 3.2b)*
3. *If the sliding end point $s_{end} < 0.5$ and $k_1 > k_2$, then the separatrix is $\varphi_2^U = \varphi_1^U$ and the trajectories beginning in region U and below φ_2^U reach the equal share line Σ,*

(a) If the trajectories reach the equal market share on the sliding segment Σ_s, they stay on the equal market share until the point s_{end} and then cross into region L finally ending on the saturated market Sat with firm F_1 having more than 50% of the market share. (see Figure 3.2c)

(b) If the trajectories reach the equal market share line on the crossing segment Σ_c, they cross into region L finally ending on the saturated market Sat with firm F_1 having more than 50% of the market share.(see Figure 3.2d)

Proof. Proof of Theorem 3.2.1 given in Appendix C *Demonstrations of properties of model VWDsC.* □

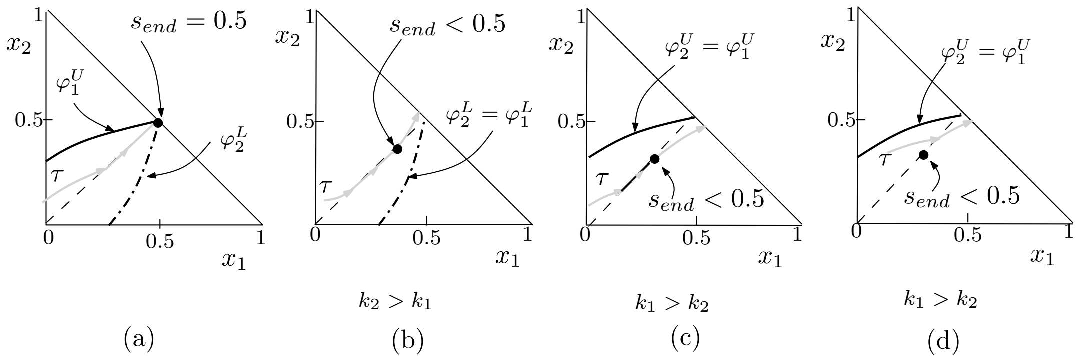


Figure 3.2: *Possible outcomes for firm F_2, which has been chosen, without loss of generality, to have a bigger initial market share.* The sliding end point s_{end} is marked by a dot. (a) RS: With initial conditions that lie below φ_1^U, F_2 loses its initial advantage, but ends with equal market share at saturation, (b) $RSRet$: For some trajectories firm F_2's initial advantage reduces to an equal share, but at saturation F_2 has more than half of the market, (c) RSC: Firm F_2 loses its initial advantage, has a period of equal share and at saturation line ends with less than half of the market, (d) RC: Firm F_2 loses its initial advantage and ends, at saturation line, with less than half of the market, without experiencing a period of equal share.

Remarks:

1. If firm F_2's separatrix is φ_2^U, then trajectories beginning above φ_2^U do not attain the equal share line Σ and they lead to firm F_2 ending with more than 50% of the market share (see Figure 3.2a, 3.2c and 3.2d).

2. If firm F_2's separatrix is φ_2^L (see Figure 3.2b), then any trajectory beginning in U, whether it attains Σ or not, leads to firm F_2 ending with more than 50% of the market share. This last case applies also to trajectories beginning in L and above φ_2^L.

3. When the firms spend a period of time on the equal share line, i.e. sliding occurs, both firms turn on their extra effort which leads to a larger advertising cost.

4. There are trajectories which cannot occur, when parameters have a given relation: When $k_2 > k_1$ (resp. $k_1 > k_2$) it is not possible that firm F_1 (respectively F_2) ends

with a larger saturated market share. See Propositions C.0.5 and C.0.6 in Appendix C.

All the proofs are detailed in Appendix C *Demonstrations of properties of model VWDsC*.

3.2.1 The state variables (market shares) x_1, x_2 are nondecreasing along the trajectories

For all trajectories originating in the interior of the triangle $\mathcal{T}$, equations (3.3)-(3.6) imply that the state derivatives $\dot{x}_1, \dot{x}_2$ are always nonnegative, this implies that they are non-decreasing. The variables x_1, x_2 always increase, except at Sat where x_1, x_2 remain unchanged, see section 3.2.2 next.

3.2.2 Equilibria in U, their location and stability properties

We observe that the set of points $\{(a, 1-a) : a \in [0, 0.5]\}$ is an equilibrium set for the dynamics in U defined by f^U. This is because the term $(1 - x_1 - x_2)$ is a factor in the expression defining f^U, implying that the vector field f^U vanishes on the set $\{(a, 1-a) : a \in [0, 0.5]\}$ defined by the upper half of the line Sat.

If the market is unsaturated (i.e., $1 - x_1 - x_2 > 0$), the only equilibrium point for (3.3), (3.4) is $x_1 = -c_1/k_1$, $x_2 = 0$. Under the assumptions that k_i, c_i are positive for $i = 1, 2$, this means that this equilibrium point lies on the negative segment of the x_1 axis and is outside the first quadrant. In other words, it is a virtual equilibrium, in the sense that it is outside of the feasible region $\mathcal{T}$. Also, it is an unstable node.

Behavior of points on Sat

The Jacobian matrix $\mathbf{J}_1(x_1, x_2)$ for (3.3), (3.4) is given by:

$$\begin{bmatrix} -2k_1x_1 + k_1 - c_1 - k_1x_2 & -c_1 - k_1x_1 \\ -k_2x_2 & -2k_2x_2 - k_2x_1 + k_2 \end{bmatrix} \tag{3.7}$$

Substituting the coordinates of a point $(a, 1-a)$ on Sat in (3.7) we have $\mathbf{J}_1(a, 1-a)$

$$\begin{bmatrix} -k_1a - c_1 & -k_1a - c_1 \\ -k_2(1-a) & -k_2(1-a) \end{bmatrix} = -\mathbf{u}\mathbb{1}^\intercal, \tag{3.8}$$

where $\mathbf{u} = (k_1a + c_1, k_2(1-a)) \in \mathbb{R}^2$ and $\mathbb{1} = (1, 1) \in \mathbb{R}^2$. Abbreviating $\mathbf{J}_1(a, 1-a)$ by $\mathbf{J}_1$, the equations $\mathbf{J}_1\mathbf{u} = -(\mathbf{u}\mathbb{1}^\intercal)\mathbf{u}$ and $\mathbf{J}_1\mathbf{w} = 0\mathbf{w}$, for $\mathbf{w} \in \mathbb{1}^\perp$ ($\mathbb{1}^\perp = \{\mathbf{v} : \mathbf{v}\mathbb{1}^\intercal = 0\}$) show that $-\mathbb{1}^\intercal\mathbf{u}$ is a negative eigenvalue associated with the direction of the eigenvector $\mathbf{u}$, and 0 (zero) is an eigenvalue associated with an eigenvector in $\mathbb{1}^\perp$ (the eigenvector $\mathbf{w}$ can be chosen as $(1, -1)$).

Since for $a \in [0, 0.5)$ and any choices of k_1, c_1, k_2 (all positive), the eigenvector $\mathbf{u}$ points towards Sat and furthermore the eigenvector $(1, -1)$ is parallel to Sat, we can conclude that trajectories in the interior of $\mathcal{T}$ will approach Sat. Once a point on Sat is attained, the trajectory stops evolving ($\dot{x}_1 = \dot{x}_2 = 0$). In this sense, Sat is a continuum of points that attract trajectories originating in $\mathcal{T}$, and behave like equilibria. The points on Sat are attractive but a perturbation that changes the state from point P_1 on Sat to another point P_2 on Sat will stay on P_2.

3.2.3 Types of separatrices

Separatrices associated with the regions U and L are defined as follows: φ^U (resp. φ^L) is a trajectory confined to U (resp. L) starting at an initial condition on the x_2 (resp. x_1) axis and ending at point E. At least one separatrix will exist because a separatrix is a special trajectory that reaches Sat at the equal market share saturated market E and one of the firms or both could reach the equal share saturated market E.

Figure 3.3a shows the case in which two separatrices, denoted φ_1^U and φ_2^L occur (recall that the subscript indicates the firm to which the separatrix applies). When both separatrices exist, the region in between them (shown in white in Figure 3.3a) is a region where the firms tie, i.e. each firm ends up with half the saturated market share. There are parameter values for which only one separatrix exists $\varphi_1^U = \varphi_2^U$ (resp. $\varphi_1^L = \varphi_2^L$) and it applies to both firms. This is shown in Figure 3.3b (resp. Figure 3.3c). Firm F_2(resp. F_1) achieves its goal above (resp. below) its separatrix.

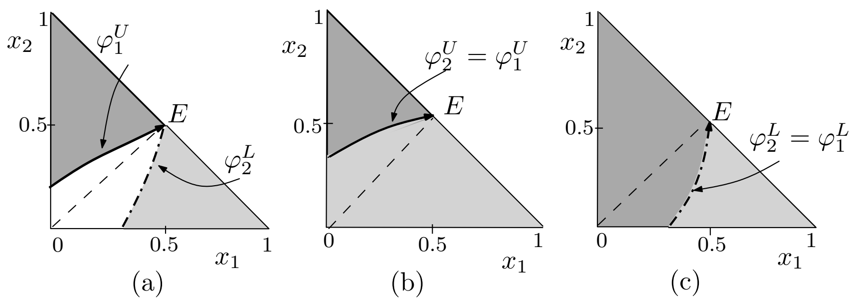


Figure 3.3: Types of separatrices: The superscript identifies the field used in the calculation and the subscript identifies the firm to which the separatrix applies. (a) There are two separatrices φ_1^U and φ_2^L. The trajectories that start in the white region between φ_1^U and φ_2^L are such that both firms end up with half of the market share in the long run. The cases depicted in (b) and (c) have only one separatrix, in case (b) the separatrix is $\varphi_2^U = \varphi_1^U$, in case (c) the separatrix is $\varphi_2^L = \varphi_1^L$.

If only one separatrix exists then it applies to both firms (see Figure 3.3b and 3.3c). If two separatrices exist, the separatrix for firm F_1 is φ_1^U and the separatrix for firm F_2 is φ_2^L (see Figure 3.3a). The mathematical details needed to calculate the separatrix of a firm are given in Appendix B.

3.2.4 Reaching, sliding and crossing behaviors with regard to the equal share line Σ

If only one separatrix $\varphi_2^U = \varphi_1^U$ (resp. $\varphi_2^L = \varphi_1^L$) exists, then trajectories beginning in U (resp. in L) and below the separatrix φ_2^U (resp. above φ_1^L) lead to firm F_2 (resp. F_1) not attaining its goal of at least 50% of the saturated market share. These trajectories, on reaching the equal share line Σ could slide on Σ and then cross it into L, or could directly cross Σ into L. If there are two separatrices φ_2^L and φ_1^U, trajectories beginning in the region in between them reach Σ and then slide on it until the saturated equal market share point E. Thus, it remains to find additional conditions on the policy parameters (k_i, c_i) that result in these different types of possible behavior. Notice that it is necessary to define the segment $\Sigma_s \subset \Sigma$ on which sliding occurs. Similarly, it is necessary to define a segment $\Sigma_c \subset \Sigma$ on which crossing occurs.

The details about reaching, sliding and crossing the equal share line Σ are given in Appendix C *Demonstrations of properties of model VWDsC*.

3.3 Designing advertising policies: a pictorial version of the main results for model VWDsC

To facilitate the use of the results given in section 3.2, this section provides a pictorial version of the main results, identifying regions of the phase plane (i.e. initial conditions) that lead to different outcomes for the long term market shares of the competing firms. In order to do this, we assume, without loss of generality, that firm F_1 starts out with a lower initial market share (i.e. $x_1(0)$ is in U).

Figure 3.4 shows regions of all initial conditions in U for which firm F_1 can attain or surpass the market share of firm F_2. In Figure 3.4a firm F_1 will attain 50% of the saturated market share while in Figure 3.4b firm F_1 will end with more than 50% of the saturated market share. If the separatrix is $\varphi_2^L = \varphi_1^L$, then there are no initial conditions in U for which F_1 attains at least 50% of the saturated market. For this reason the case $\varphi_2^L = \varphi_1^L$ is not included in Figure 3.4.

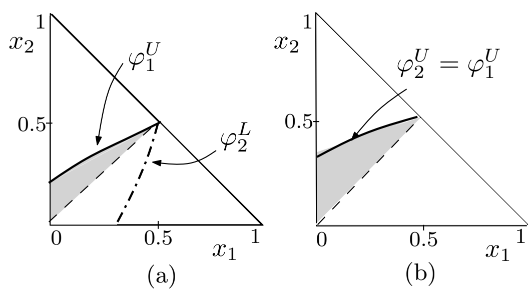


Figure 3.4: The gray regions of U contain the initial conditions for which: (a) F_1 attains 50% of the saturated market, (b) F_1 surpasses the saturated market share of F_2.

Figure 3.5a shows a gray region for which F_1 attains equal market share in the long term. In Figure 3.5b, only initial conditions on the separatrix, shown in gray, lead to both firms ending with 50% of the saturated market share. If the separatrix is $\varphi_2^L = \varphi_1^L$, then there are no initial conditions in U for which F_1 attains 50% of the saturated market. For this reason the case $\varphi_2^L = \varphi_1^L$ is not included in Figure 3.5.

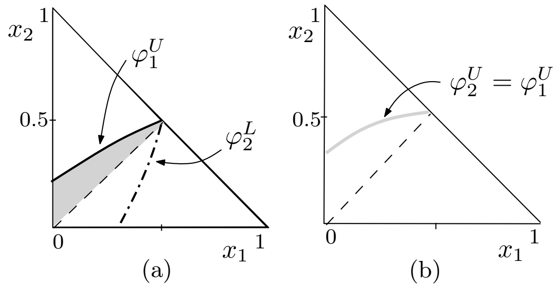


Figure 3.5: The gray regions of U contain the initial conditions for which both firms end with an equal saturated market share in the long term. In (b) these initial conditions lie on the separatrix $\varphi_2^U = \varphi_1^U$ and the region collapses to a single curve (the separatrix)

Figure 3.6 shows gray regions of initial conditions for which F_1 can never attain the saturated market share of its competitor, firm F_2.

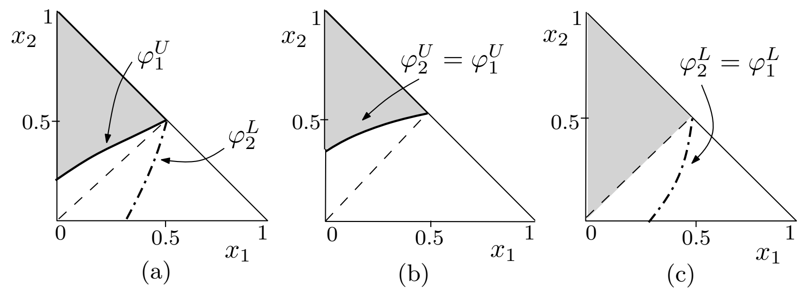


Figure 3.6: The gray regions of U contain the initial conditions in U for which the firm F_1 end with a saturated market share smaller than the saturated market share of F_2.

In summary, supposing that firm F_1 has a lower initial market share, the strategy determined by Theorem 3.2.1 is to use a constant rate $k_1 > k_2$. With this choice, the separatrix φ_1^U exists and if the initial market shares are below it, then firm F_1 is guaranteed 50% of the saturated market share (see Figure 3.4a) or even more than 50% of the market share (see Figure 3.4b). In this case, firm F_1 should also calculate its performance index to evaluate if its investment plans are better aligned with having 50% of the saturated market share or having more than 50% of it (the calculation of a performance index will be discussed in the next section). If the initial conditions are above the separatrix φ_1^U, then firm F_1 will end with less than 50% of the saturated market share, not attaining its goal. Again, the question of whether to enter the market or not, should be decided using a performance index.

3.4 Simulation of different scenarios

Next, we present simulations with different parameters displayed on the phase plane. Without loss of generality, the scenarios are commented on from the viewpoint of firm F_2. Recall that the goal of each firm is to end with at least 50% of the saturated market share. We assume that F_2 is already in the market so that its initial market share satisfies $x_2(0) > x_1(0)$. Also, suppose that the parameters k_2, c_2, k_1, c_1 are known. The entering firm F_1, i.e. F_1 enters the market, also wants to get at least half of the saturated market. The simulations were done using the software Berkeley Madonna [51].

Notice that in all scenarios firm F_2 achieves its goal of having at least half of the market for any trajectory τ is above its separatrix, which can be φ_2^U or φ_2^L. Firm F_1 achieves its goal for any trajectory τ below its separatrix, which can be φ_1^U or φ_1^L. If there are two separatrices, the region in between φ_1^U and φ_2^L is a draw region where each firm ends with half of the saturated market.

In the first scenario, shown in Figure 3.7, we investigate if there is a trajectory such that firm F_1 reaches the equal market share line and finally ends with more than half of the saturated market share when $k_1 < k_2$, i.e. when F_1 makes an insufficient effort. The answer is no. If $k_1 < k_2$ there is no trajectory that reaches the equal market share line, remains on it and finally ends with more than half of the saturated market share (Proposition C.0.5) neither a trajectory that reaches the equal market share line on the crossing segment and finally ends with more than half of the saturated market share (Proposition C.0.6). However, if $k_2 > k_1$ and $c1/(k_2 - k_1) < 0.5$, there are trajectories τ that reach the equal market share Σ, remains on it and then return to U, so that finally F_2 ends with more than half of the saturated market. This corresponds to Theorem 3.2.1 2 with $s_{end} < 0.5$ and is highlighted in gray color in Figure 3.7.

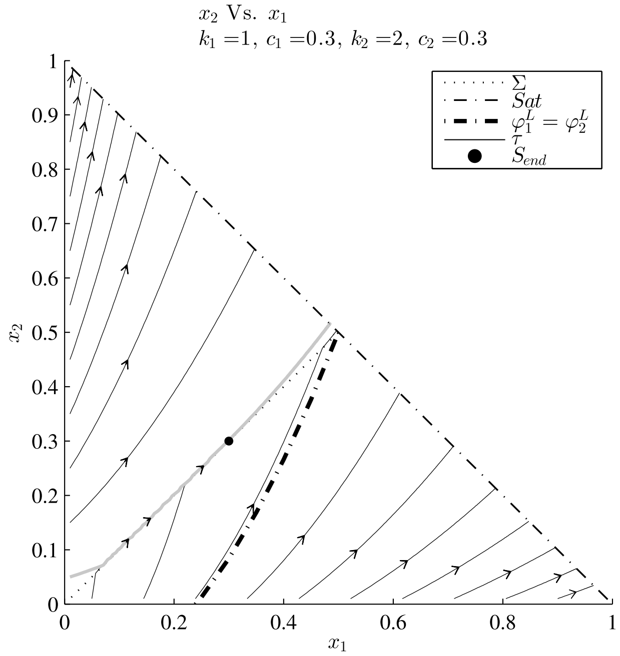


Figure 3.7: Insufficient effort by initially disadvantaged firm F_1. Phase plane of model (3.3)-(3.6) with parameters $k_1 = 1$, $c_1 = 0.3$, $k_2 = 2$, $c_2 = 0.3$. The highlighted gray trajectory τ starts in U and is such that firm F_1 reaches the equal market share, remains on it but finally firm F_2 ends with more than half of the saturated market share. This corresponds to Theorem 3.2.1 2.

In the next scenario, shown in Figure 3.8, if firm F_1 increases its the constant effort $c_1 > c_2$ but keeps its constant rate $k_1 < k_2$, does firm F_1 ends with more than half of the saturated market share? If the sliding end point $s_{end} = 0.5$ there are two separatrices φ_1^U and φ_2^L. The trajectories with initial conditions below φ_1^U and above φ_2^L reach the equal market share line, remains on it and finally reach the saturated equal market share point $E = 0.5$, so that finally firms F_2 and F_1 end with half of the saturated market share. For instance, when the parameters are $k_1 = 1, c_1 = 2, k_2 = 2, c_2 = 0.3$, trajectories that reach Σ slide until 0.5 as shown in Figure 3.8. These cases correspond to Theorem 3.2.1 1 where firm F_2 retains half of the saturated market share.

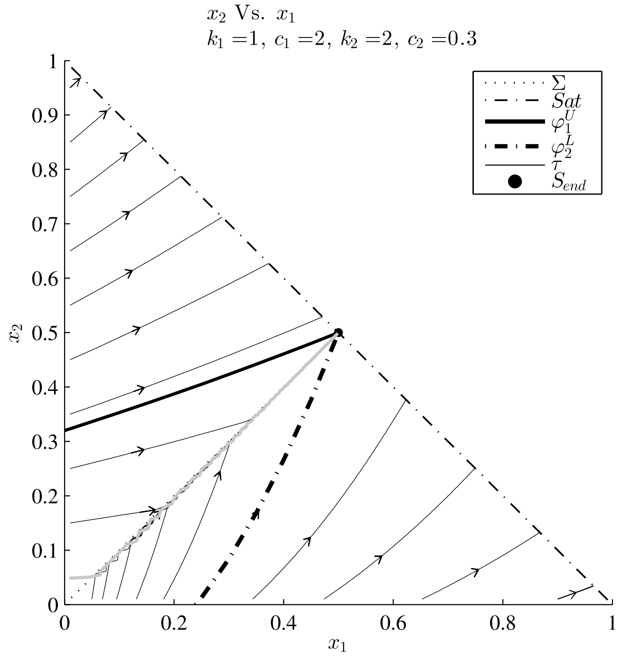


Figure 3.8: If $s_{end} = 0.5$, there are separatrices φ_1^U and φ_2^L. All trajectories τ with initial conditions below φ_1^U and above φ_2^L end at the saturated equal market share E, i.e. firm F_2 finally ends with half of the saturated market share. The firm F_1 does a bigger constant effort $c_1 > c_2$ but a smaller constant rate $k_1 < k_2$ when compared to the previous scenario. The phase plane of model (3.3)-(3.6) corresponds to parameters $k_1 = 1, c_1 = 2, k_2 = 2, c_2 = 0.3$. The highlighted gray trajectory τ is such that it reaches the equal market share Σ and remains on it until the saturated equal market share point E. This corresponds to Theorem 3.2.1 1.

In the last scenario, shown in Figure 3.9, if firm F_1 increases its constant rate $k_1 > k_2$, does F_1 ends with more than half of the saturated market share? If the sliding end coordinate $s_{end} < 0.5$, the answer is yes, firm F_1 can end with more than half of the saturated market share. Figure 3.9 shows trajectories sliding on Σ and then crossing into region L (this corresponds to Theorem 3.2.1 3a) and trajectories directly crossing into region L (this corresponds to Theorem 3.2.1 3b). Note that if firm F_2 had used a constant effort $c_2 = 0.8$ it would have attained half of the market share (this corresponds to Theorem 3.2.1 1). With a constant effort $c_2 = 0.8$, all trajectories that attain the equal market share line Σ stay on it until the saturated equal market share point E.

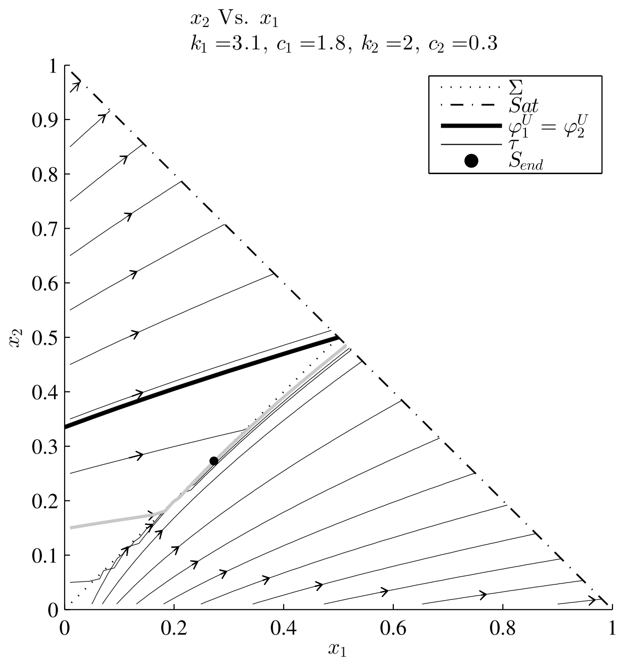


Figure 3.9: Firm F_2 loses its initial advantage. The phase plane of model (3.3)-(3.6) corresponds to parameters $k_1 = 3.1$, $c_1 = 1.8$, $k_2 = 2$, $c_2 = 0.3$. All trajectories below the separatrix $\varphi_2^U = \varphi_1^U$ that reach the equal market share line on the sliding segment Σ_s, remain on it and then they cross into region L, with firm F_1 ending with more than half of the saturated market share (see Theorem 3.2.1 3a). The trajectories that reach the equal market share line on the crossing segment Σ_c directly cross into region L (see Theorem 3.2.1 3b). The highlighted gray trajectory is such that it reaches the equal market share line Σ, remains on it for some time and then it crosses into region L where firm F_1 ends up with more than half of saturated the market share.

3.5 Evaluating outcomes of the proposed policies using a performance index in model VWDsC

Market share is one viewpoint of the interaction between the two firms. We use the index (2.9) with the control functions given in in (3.2) to measure the performance of each firm. A firm makes profits when its index $J(x_i)$ is positive, whereas it makes a loss when its index $J(x_i)$ is negative. The simulations were done using the software Berkeley Madonna [51].

Figure 3.10 shows a situation in which there is only one separatrix $\varphi_2^L = \varphi_1^L$ which means that firm F_2 has a larger region (above the separatrix) of initial conditions for which it has a long term market share greater or equal to 50%. In figure 3.10 the sales revenue coefficient $r = 2$ is low enough that despite the long term market share advantage of firm F_2, its performance index is always negative for initial conditions on the axis. The light (resp. dark) gray region demarcates sets of initial conditions for which the performance index of firm F_1 is positive (resp. negative). This means that for the chosen (low) sales revenue coefficient, firm F_2 makes a loss despite its larger market share, while firm F_1 operates with a profit despite its lower market share. This suggests that the level of advertising expenditure of firm F_2 is excessive.

Figure 3.11 shows that with a bigger sales revenue coefficient $r = 4$, there are three final outocomes of the indexes: only firm F_2 has a positive index, both firms have positive indexes or only firm F_1 has a positive index. Besides, note that most of the trajectories correspond to both firms making profit even though in several of those trajectories a firm does not fulfill its target market share at saturated market.

Thus, the value of the sales revenue coefficient is decisive for assessing profitability with respect to market share. In fact this analysis should be carried out to test any desired choice of parameters for different sales revenue coefficients in order to design the appropriate advertising policies of type (3.2).

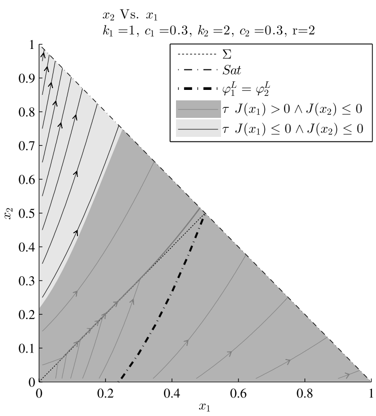


Figure 3.10: F_2 spends more on advertising, i.e. it makes bigger efforts k_2, c_2 when compared to F_1. With a sales revenue coefficient $r = 2$, both firms have negative index values in the dark gray region and Firm F_2 has negative index values and firm F_1 has positive index values in the light gray region.

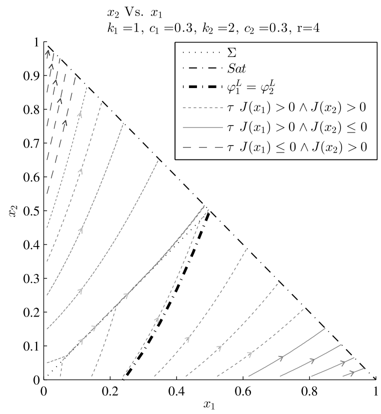


Figure 3.11: With a bigger sales revenue coefficient is $r = 4$ there are trajectories where both firms make a profit, i.e. their indexes are positive, even where their corresponding long term market share is lesss than 50%.

3.6 Remarks on model VWDsC

- The entering firm F_1 needs its constant effort $k_1 > k_2$ to obtain more than half of the long term market share.
- The entering firm F_1 needs its constant effort $k_1 = k_2$ to obtain half of the long term market share.
- Not using the discount rate, which brings profit future values to present values is a mathematical simplification. However, in the simulations the system reaches its steady state in a small number of time units so that it can be considered a finite time horizon, so that it is unnecessary to consider all profits in terms of present value.

- The performance index identifies which trajectories make a profit, thus the index and the saturated market share values can be used to make advertising expenditure decisions. For example, if the revenue factor of the market is low, then high spending on advertising could lead to big market share but low profit.

Chapter 4

Modification of Vidale-Wolfe-Deal model with Predatory Advertising

4.1 Introduction

We study a modification of the Vidale-Wolfe-Deal model that has switched predation, this model will be called VWDsP. We assume that both firms want to attain at least half of the market share. In this model each firm uses the same control u_{ii} in regions U and L:

$$u_{11} = k_1 x_1 + c_1 \tag{4.1}$$

$$u_{22} = k_2 x_2 + c_2 \tag{4.2}$$

The switching control is on the predatory advertising. A firm F_i turns on its predatory advertising when it is losing i.e., when $x_i < x_j$. The predatory advertising by firm F_1 is:

$$w_{21} = \begin{cases} c_{21} x_2 & \text{if } x_1 < x_2 \ , \\ 0 & \text{if } x_1 > x_2 \end{cases} \tag{4.3}$$

and the predatory advertising by firm F_2 is:

$$v_{12} = \begin{cases} c_{12} x_1 & \text{if } x_2 < x_1 \ , \\ 0 & \text{if } x_2 > x_1 \end{cases} \tag{4.4}$$

We replace (4.1), (4.1), (4.3) and (4.4) in model (2.1-2.2) to get the VWDsP model: The upper field f^U, which defines the dynamics in the region above Σ, is

$$\dot{x}_1 = (k_1 x_1 + c_1)(1 - x_1 - x_2) \tag{4.5}$$

$$\dot{x}_2 = (k_2 x_2 + c_2)(1 - x_1 - x_2) - c_{21} x_2 \tag{4.6}$$

when $x_2 > x_1$

The lower field f^L, which defines the dynamics in the region below Σ, is

$$\dot{x}_1 = (k_1x_1 + c_1)x_1(1 - x_1 - x_2) - c_{12}x_1 \tag{4.7}$$

$$\dot{x}_2 = (k_2x_2 + c_2)(1 - x_1 - x_2) \tag{4.8}$$

$$\text{when } x_1 > x_2$$

On Σ, the dynamics is defined following the Filippov rule (see Appendix A, [46, p. 50-52, chap. 2]).

4.2 Equilibrium points of model VWDsP

The saturated market line $Sat(1 - x_1 - x_2 = 0)$ is not stable anymore. In region U, $\dot{x}_1$ is zero on Sat but $\dot{x}_2$ is not. In region L, $\dot{x}_1$ is not zero on Sat but $\dot{x}_2$ is. Let a trajectory τ begin in region U and satisfy $1 - x_{1_0} - x_{2_0} > 0$, with this setting the dynamics of the system is nondecreasing for market share x_1 so that τ tries to reach $1 - x_1 - x_2$. However, market share x_2 does not allow τ to reach the saturated market line $1 - x_1 - x_2$ because of the term $-c_{21}x_2$.

In order to calculate the equilibrium points we will work with the field $\mathbf{f}^U$ (the calculation of the equilibria of $\mathbf{f}^L$ is analogous). The system (4.5)-(4.6) has three equilibria:

$$P_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \tag{4.9}$$

The eigenvalues of the Jacobian at equilibrium point P_1 are

$$\lambda_1(P_1) = \frac{-c_1 - c_2 - c_{21} - k_1 - \sqrt{(c_1 + c_2 + c_{21} + k_1)^2 - 4c_{21}(k_1 + c_1)}}{2} \tag{4.10}$$

$$\lambda_2(P_1) = \frac{-c_1 - c_2 - c_{21} - k_1 + \sqrt{(c_1 + c_2 + c_{21} + k_1)^2 - 4c_{21}(k_1 + c_1)}}{2} \tag{4.11}$$

$$P_2 = \begin{bmatrix} -\frac{c_1}{k_1} \\ \frac{c_1k_2 - c_2k_1 - c_{21}k_1 + k_1k_2 + \sqrt{(c_2k_1 + c_1k_2 - c_{21}k_1 + k_1k_2)^2 + 4c_2c_{21}k_1^2}}{2k_1k_2} \end{bmatrix} \tag{4.12}$$

and the eigenvalues of the Jacobian at equilibrium point P_2 are

$$\lambda_1(P_2) = \frac{c_1k_2 + c_2k_1 + c_{21}k_1 + k_1k_2 - \sqrt{(c_2k_1 + c_1k_2 - c_{21}k_1 + k_1k_2)^2 + 4c_2c_{21}k_1^2}}{2k2} \tag{4.13}$$

$$\lambda_2(P_2) = \frac{-\sqrt{(c_2k_1 + c_1k_2 - c_{21}k_1 + k_1k_2)^2 + 4c_2c_{21}k_1^2}}{k_1} \tag{4.14}$$

$$P_3 = \begin{bmatrix} -\frac{c_1}{k_1} \\ \frac{c_1k_2 - c_2k_1 - c_{21}k_1 + k_1k_2 - \sqrt{(c_2k_1 + c_1k_2 - c_{21}k_1 + k_1k_2)^2 + 4c_2c_{21}k_1^2}}{2k_1k_2} \end{bmatrix} \tag{4.15}$$

and the eigenvalues of the Jacobian at equilibrium point P_3 are

$$\lambda_1(P_3) = \frac{c_1k_2 + c_2k_1 + c_{21}k_1 + k_1k_2 + \sqrt{(c_2k_1 + c_1k_2 - c_{21}k_1 + k_1k_2)^2 + 4c_2c_{21}k_1^2}}{2k_2} \tag{4.16}$$

$$\lambda_2(P_3) = \frac{\sqrt{(c_2k_1 + c_1k_2 - c_{21}k_1 + k_1k_2)^2 + 4c_2c_{21}k_1^2}}{k_1} \tag{4.17}$$

Propositions 4.2.1, 4.2.2 and 4.2.3 below refer to trajectories beginning in U with their dynamics defined by (4.5)-(4.6) which are attracted by the virtual stable node P_1 (which is virtual because it lies in region L but corresponds to the field $\mathbf{f}^U$)

Proposition 4.2.1 (Equilibrium point P_1 is stable). *For all positive parameter values Equilibrium point P_1 of system (4.5)-(4.6) is stable and virtual.*

Proof. The equilibrium $P_1 = [1, 0]^T$ corresponds to the dynamics defined by $\mathbf{f}^U$. However, it lies in region L (because $x_1 > x2$) where the field $\mathbf{f}^L$ defines the dynamics, so that P_1 is virtual.

If $(c_1 + c_2 + c_{21} + k_1)^2 - 4c_{21}(k_1 + c_1) > 0$ then we can write

$$\sqrt{(c_1 + c_2 + c_{21} + k_1)^2 - 4c_{21}(k_1 + c_1)} = (c_1 + c_2 + c_{21} + k_1)^2 - 4c_{21}(k_1 + c_1) - \alpha > 0 \tag{4.18}$$

where α is a real positive number. Using 4.18 in the eigenvalues equations (4.10) and (4.11), we obtain

$$\lambda_1(P_1) = -2(c_1 + c_2 + c_{21} + k_1) + \alpha < 0 \tag{4.19}$$

$$\lambda_2(P_1) = -\alpha \tag{4.20}$$

because both eigenvalues are real negative numbers, the equilibrium point P_1 is a stable node in this case.

If$(c_1 + c_2 + c_{21} + k_1)^2 - 4c_{21}(k_1 + c_1) < 0$ then the eigenvalues are a pair of conjugate imaginary numbers with negative real part and the equilibrium point P_1 is a stable focus.

Then the equilibrium point P_1 of system (4.5)-(4.6) is virtual and stable.

□

Proposition 4.2.2 (Condition for equilibrium point P_2 to be a saddle.). *If* $2k_1(c_{21} - \sqrt{c_2c_{21}}) > 0$ *then the equilibrium point P_2 of system (4.5)-(4.6) is a saddle.*

Proof. The eigenvalue $\lambda_2(P_2)$, given in (4.14), is negative because it is the negative of the square root of the sum of two squared numbers. In order to analyze' the eigenvalue

$\lambda_1(P_2)$, given in (4.13), we use the triangle inequality to obtain

$$\sqrt{(c_2k_1 + c_1k_2 - c_{21}k_1 + k_1k_2)^2 + 4c_2c_{21}k_1^2} < (c_2k_1 + c_1k_2 - c_{21}k_1 + k_1k_2) + 2k_1\sqrt{c_2c_{21}} \tag{4.21}$$

substituting (4.21) into (4.13) leads to

$$\lambda_1(P_2) > 2k_1(c_{21} - \sqrt{c_2c_{21}}) \tag{4.22}$$

because the LHS of (4.22) is positive by hypothesis, $\lambda_1(P_2) > 0$. Because $\lambda_1(P_2) > 0$ and $\lambda_2(P_2) < 0$, the equilibrium point P_2 is a saddle. □

Remark: If $2k_1(c_{21} - \sqrt{c_2c_{21}}) < 0$, *the eigenvalue* $\lambda_1(P_2)$ *could still be positive. It is the condition* $\lambda_1(P_2) > 0$ *and* $\lambda_2(P_2) < 0$ *that makes the equilibrium point* P_2 *a saddle.*

Proposition 4.2.3 (Equilibrium point P_3 is an unstable node). *For all positive parameter values Equilibrium point P_3 of system (4.5)-(4.6) is unstable.*

Proof. Because all the system parameters are positive numbers and the expressions of the eigenvalues corresponding to point P_3 (4.16) and (4.17), the eigenvalues $\lambda_1(P_3) > 0$ and $\lambda_2(P_3) > 0$. This implies that the equilibrium point P_3 is an unstable node. □

4.3 Dynamics of trajectories reaching the equal market share line Σ in model VWDsP

Determining the crossing Σ_c and sliding Σ_s sets: To describe trajectories that reach the equal market share line Σ, we need to determine the crossing set Σ_c and the sliding set Σ_s. To calculate these sets, we use the crossing condition (4.23). (See Appendix A).

$$\langle \nabla H(\mathbf{w}), \mathbf{f}^U(\mathbf{w})\rangle \langle \nabla H(\mathbf{w}), \mathbf{f}^L(\mathbf{w})\rangle > 0 \tag{4.23}$$

where $\mathbf{w} \in \Sigma$, i.e. $\mathbf{w} = (w, w)$ for model (4.5)-(4.6).

An interpretation for condition (4.23) is that each inner product projects the corresponding field onto $\nabla H(\mathbf{w})$, which is the normal to Σ. The convention is that a positive (resp. negative) sign indicates that the projection points into region U (respectively L). Then, one way for obtaining the crossing set Σ_c is to calculate where $\langle \nabla H(\mathbf{w}), \mathbf{f}^U(\mathbf{w})\rangle$ and $\langle \nabla H(\mathbf{w}), \mathbf{f}^L(\mathbf{w})\rangle$ have the same sign. The advantage of analyzing each product separately is that it simplifies the algebra.

If the system parameters are known, using the point $\mathbf{w} = (w, w)$ as argument for the inner products in (4.23) defines the scalar functions

$$g_U(w) := \langle \nabla H(\mathbf{w}), \mathbf{f}^U(\mathbf{w}) \rangle \tag{4.24}$$

$$g_L(w) := \langle \nabla H(\mathbf{w}), \mathbf{f}^L(\mathbf{w}) \rangle \tag{4.25}$$

In the case of model (4.5)-(4.8), the equal market share line Σ is described by $H(\mathbf{x}) = x_2 - x_1 = 0$ and its normal is $\nabla H(\mathbf{x}) = [-1 \quad 1]^T$, with this information the corresponding scalar functions are:

$$g_U(w) = (2k_1 - 2k_2)w^2 + (2c_1 - 2c_2 - c_{21} - k_1 + k_2)w - (c_1 - c_2) \tag{4.26}$$

$$g_L(w) = (2k_1 - 2k_2)w^2 + (2c_1 - 2c_2 + c_{12} - k_1 + k_2)w - (c_1 - c_2) \tag{4.27}$$

Since the right hand sides of (4.26)-(4.27) are parabolas in w, $\text{sign}(g_U(w))$ and $\text{sign}(g_L(w))$ can be determined by calculus. Once the signs are calculated, the sets can be combined to obtain the crossing set Σ_c, which is comprised of the points where the values of parabolas have the same sign, and sliding set Σ_s, which is the complement of Σ_c, i.e. the points where the values of parabolas w are zero or have different signs. The zeros of (4.26) are:

$$v_1 = \frac{-\beta + \sqrt{\beta^2 + 8(k_1 - k_2)(c_1 - c_2)}}{4(k_1 - k_2)} \tag{4.28}$$

$$v_2 = \frac{-\beta - \sqrt{\beta^2 + 8(k_1 - k_2)(c_1 - c_2)}}{4(k_1 - k_2)} \tag{4.29}$$

where $\beta = (2c_1 - 2c_2 - c_{21} - k_1 + k_2)$ and the zeros of (4.27) are:

$$w_1 = \frac{-\gamma + \sqrt{\gamma^2 + 8(k_1 - k_2)(c_1 - c_2)}}{4(k_1 - k_2)} \tag{4.30}$$

$$w_2 = \frac{-\gamma - \sqrt{\gamma^2 + 8(k_1 - k_2)(c_1 - c_2)}}{4(k_1 - k_2)} \tag{4.31}$$

where $\gamma = (2c_1 - 2c_2 + c_{12} - k_1 + k_2)$

The second derivatives of (4.26) and (4.27) are:

$$g_U(w)'' = 4(k_1 - k_2) \tag{4.32}$$

$$g_L(w)'' = 4(k_1 - k_2) \tag{4.33}$$

We describe the process of determining the crossing set Σ_c and the sliding set Σ_s the with the following algorithm:

Algorithm for calculating the crossing Σ_c and the sliding set Σ_s

1. For each field $F = U, L$ determine the sign of $g_F(w)$. The signs are calculated

using the fact that (4.26) and (4.27) are parabolas on the variable w and thus their second derivatives are constants.

- Let the w_1, w_2 be the zeros of $g_F(w)$
- If $\text{sign}(g_F(w)'') < 0$, then $g_F(w)$ has a maximum
  - If w_1, w_2 are real numbers, $\text{sign}(g_F(w))$ determined as follows:

$$\text{sign}(g_F(w)) = \begin{cases} 1 & \text{if } w \text{ is between its zeros } w_1,\ w_2 \\ 0 & \text{if } w \text{ is any of its zeros } w_1,\ w_2 \\ -1 & \text{for any other value of } w \end{cases} \tag{4.34}$$

  - If w_1, w_2 are conjugate imaginary numbers, then $\text{sign}(g_F(w)) = -1$.
- If $\text{sign}(g_F(w)'') > 0$, then $g_F(w)$ has a minimum
  - If w_1, w_2 are real numbers, $\text{sign}(g_F(w))$ is determined as follows:

$$\text{sign}(g_F(w)) = \begin{cases} -1 & \text{if } w \text{ is between } w_1,\ w_2 \\ 0 & \text{if } w = w_1 \text{ or } w = w_2 \\ 1 & \text{for any other value of } w \end{cases} \tag{4.35}$$

  - If w_1, w_2 are conjugate imaginary numbers, then $\text{sign}(g_F(w)) = 1$.

2. Combine the signs of the fields obtained in step 1 to calculate the sets

- the crossing set is

$$\Sigma_c = \{w | \text{sign}(g_U(w))\text{sign}(g_L(w)) > 0\} \tag{4.36}$$

- the sliding set is the complement of Σ_c, i.e. its elements are all w for which the projections do not have the same sign.

4.4 Pseudoequilibrium points of model VWDsP

A pseudoequilibrium point is an equilibrium that may come into existence because of the use of switching control (See Appendix A). The pseudoequilibrium candidates are obtained using the fact that at a pseudoequilibrium the fields must be aligned and have opposite directions. Thus for the system (4.5)-(4.8) it must hold that

$$\frac{(k_1x_1 + c_1)(1 - x_1 - x_2)}{(k_1x_1 + c_1)(1 - x_1 - x_2) - c_{12}x_1} = \frac{(k_2x_2 + c_2)(1 - x_1 - x_2) - c_{21}x_2}{(k_2x_2 + c_2)(1 - x_1 - x_2)} < 0 \tag{4.37}$$

considering that the pseudoequilibrium is on Σ where $x_1 = x_2 = w^*$ and after some algebraic operations, (4.37) can be rewritten as

$$\frac{1}{1 - \frac{c_{12}w^*}{(k_1w^*+c_1)(1-2w^*)}} = 1 - \frac{c_{21}w^*}{(k_2w^* + c_2)(1 - 2w^*)} < 0 \tag{4.38}$$

The pseudoequilibrium candidates for model (4.5), (4.8), which were obtained using the proportion in (4.38), are points on Σ having coordinates w^*:

$$w_1^* = \frac{-(A - B) - \sqrt{(A - B)^2 + 4B(2c_1c_{21} + 2c_{12}c_2)}}{4B} \tag{4.39}$$

$$w_2^* = \frac{-(A - B) + \sqrt{(A - B)^2 + 4B(2c_1c_{21} + 2c_{12}c_2)}}{4B} \tag{4.40}$$

$$w_3^* = 0 \tag{4.41}$$

where $A = 2c_1c_{21} + 2c_{12}c_2 + c_{12}c_{21}$ and $B = c_{21}k_1 + c_{12}k_2$.

Notice that for w_3^* the proportion is positive, so that it *cannot be a pseudoequilibrium.*

Proposition 4.4.1 (Feasibility interval for a pseudoequilibrium point). *If w^* is a pseudoequilibrium of model (4.5)-(4.8) then the following inequalities are satisfied*

(i) A pseudoequilibrium can occur in the interval

$$0 < w^* < 1/2$$

(ii) The relation of the predation effort with the other control efforts at a pseudoequilibrium is

$$c_{21}w^* > (k_2w^* + c_2)(1 - 2w^*)$$
$$c_{12}w^* > (k_1w^* + c_1)(1 - 2w^*)$$

Proof. Given that the parameters $k_1, c_1, k_2, c_2, c_{12}, c_{21}$ are positive, the condition $(1 - 2w^*) > 0$ must hold to fulfill (4.38). This leads to $w^* < 1/2$. Since we are only interested in positive values for the states x_1, x_2, item (i) is proved and the interval where a pseudoequilibrium can occur is

$$0 < w < 1/2 \tag{4.42}$$

Also, for the proportion in (4.38) to be negative, the following conditions must hold

$$c_{21}w^* > (k_2w^* + c_2)(1 - 2w^*)$$
$$c_{12}w^* > (k_1w^* + c_1)(1 - 2w^*)$$

and item (ii) is proved. $\square$

Proposition 4.4.2 (Signs of pseudoequilibrium candidates). *The pseudoequibilibrium candidates w_1^* given by (4.39) and w_2^* given by (4.40) have the following properties:*

(i) w_1^* *is a negative number.*

(ii) w_2^* *is a positive number.*

Proof. By multiplying $w_1^* w_2^*$ we get

$$w_1^* w_2^* = -4B(2c_1 c_{21} + 2c_{12} c_2) \tag{4.43}$$

which is a negative number because the parameters of model (4.5)-(4.8) are all positive numbers. Then, one factor has to be positive and the other negative.

Because the square root is a nondecreasing function we can write

$$\sqrt{(A-B)^2 + 4B(2c_1 c_{21} + 2c_{12} c_2)} = (A-B) + \alpha \tag{4.44}$$

where $\alpha > 0$. Substituting (4.44) into (4.40)

$$w_2^* = \alpha \tag{4.45}$$

Because $\alpha > 0$, w_2^* is a positive number and (ii) is proved.

Since $w_1^* w_2^* < 0$ by (4.43) and $w_2^* > 0$, w_1^* is a negative number and (i) is proved. □

Given that the states have to be positive, the only possible pseudoequilibrium for model (4.5), (4.8) is $w_2^* > 0$.

Proposition 4.4.3 (Pseudoequilibrium feasability satisfied by w_2^*). *If the pseudoequilibrium candidates w_1^* and w_2^*, resp. defined in (4.39) and (4.40), satisfy*

$$w_1^* + w_2^* > 0 \tag{4.46}$$

then $w_2^* < \frac{1}{2}$.

Proof. Adding $w_1^* + w_2^*$ we get

$$w_1^* + w_2^* = \frac{1}{2} - \frac{A}{2B} \tag{4.47}$$

Using the hypothesis $w_1^* + w_2^* > 0$ in the previous we obtain

$$\frac{A}{B} < 1 \tag{4.48}$$

Rewriting w_2^* in (4.40) we obtain

$$w_2^* = \frac{-(A-B) + \sqrt{(A-B)^2 + 4B(A - c21c12)}}{4B} \tag{4.49}$$

Since the terms A, B and the parameters are nonnegative we can write

$$w_2^* < \frac{-(A-B)+\sqrt{(A-B)^2+4BA}}{4B} \tag{4.50}$$

Applying the triangular inequality to the RHS of the previous inequality and simplifying terms we obtain

$$w_2^* < \frac{-(A-B)+\sqrt{(A-B)^2+4BA}}{4B} < \frac{1}{2}\sqrt{\frac{A}{B}} \tag{4.51}$$

Using (4.48) in (4.51)

$$w_2^* < \frac{-(A-B)+\sqrt{(A-B)^2+4BA}}{4B} < \frac{1}{2}\sqrt{\frac{A}{B}} < \frac{1}{2} \tag{4.52}$$

so that $w_2^* < \frac{1}{2}$ and w_2^* satisfies Proposition 4.4.1.

□

Lemma 4.4.4 (Pseudoequilibrium of Model VWDsP). *Trajectories of the system (4.5)-(4.8) end at a pseudoequilibrium point.*

4.5 Simulations of model VWDsP

We have done several simulations of the model (4.5)-(4.8) and the trajectories ended at a pseudoequilibrium point on the equal market share line Σ. Depending on the parameters, a trajectory that reaches Σ may slide on it, cross it or have both behaviors at different time values.

Table 4.1 shows different parameter values and the corresponding pseudoequilibrium. Notice how increasing parameter k_1 (lines *1-5*) and parameter c_1 (lines *6-9*) increases the market share value attained at a the pseudoequilibrium. However, increasing the predation c_{21} done by firm F_1 on firm F_2 (lines *10-13*) decreases noticeably the market share value attained at the pseudoequilibrium.

We also ran simulations with the parameters $k_2, c_2, c_{21}, k_1, c_1, c_{12}$ taking a permutation of the values $0.25, 1.25, 2.5, 3.75, 5$ and all ended at a pseudoequilibrium point on the equal share line Σ below $(.5, .5)$ and above $(0, 0)$ so that the trajectories of model (4.5)-(4.8) end at a pseudoequilibrium on the equal market share line independent of the efforts done by the firms.

Next, we present the qualitative analysis of two scenarios: in the first, both firms F_1 and F_2 do the same predation effort. In the second scenario, firm F_1 does a predation effort that is bigger than the effort of firm F_2.

The first scenario of model (4.5)-(4.8) has the parameters $k_1 = 1.5$, $c_1 = 0.7$, $c_{21} = 0.15$, $k_2 = 2$, $c_2 = 1$, $c_{12} = 0.15$. Firm F_1 and F_2 apply the same predation effort.

#	k_1	c_1	c_{21}	k_2	c_2	c_{12}	Pseudoequilibrium
1	0.2	0.1	0.1	1	0.5	0.4	(0.476757, 0.476757)
2	1	0.1	0.1	1	0.5	0.4	(0.478694, 0.478694)
3	2	0.1	0.1	1	0.5	0.4	(0.480711, 0.480711)
4	3	0.1	0.1	1	0.5	0.4	(0.482384, 0.482384)
5	4	0.1	0.1	1	0.5	0.4	(0.483793, 0.483793)
6	0.2	0.5	0.1	1	0.5	0.4	(0.478772, 0.478772)
7	0.2	1	0.1	1	0.5	0.4	(0.480841, 0.480841)
8	0.2	2	0.1	1	0.5	0.4	(0.483955, 0.483955)
9	0.2	3.1	0.1	1	0.5	0.4	(0.486381, 0.486381)
10	0.2	0.1	0.4	1	0.5	0.4	(0.423564, 0.423564)
11	0.2	0.1	1	1	0.5	0.4	(0.36038, 0.36038)
12	0.2	0.1	2	1	0.5	0.4	(0.309017, 0.309017)
13	0.2	0.1	3	1	0.5	0.4	(0.283095, 0.283095)

Table 4.1: One parameter variation in model VWDsP. Increasing k_1 or c_1 moderately increases the market share value at pseudoequilibrium. On the other hand, increasing the predation effort c_{21} decreases noticeably the market share value at pseudoequilibrium.

The corresponding phase plane is shown in Figure 4.1 where one can see all trajectories go to a pseudoequilibrium on the equal market share line. The equilibrium points of the fields are shown in Tables 4.2 and 4.3, note that field $\mathbf{f}^L$ has a virtual stable equilibrium in region U, respectively field $\mathbf{f}^U$ has a virtual stable equilibrium in region L.

Dynamics of a trajectory on the equal market share line Σ

- Crossing and sliding sets are determined using the algorithm 4.3. The projections for determining the sets are $\langle \nabla H(\mathbf{w}), \mathbf{f}^U(\mathbf{w}) \rangle = -w^2 - 0.25w + 0.3$, that has the zeros $w_1 = -0.68681$ and $w_2 = 0.43681$, and $\langle \nabla H(\mathbf{w}), \mathbf{f}^L(\mathbf{w}) \rangle = -w^2 + 0.05w + 0.3$, that has the zeros $w_1 = -0.52329$ and $w_2 = 0.57329$. Because we are interested only w values that are nonnegative and below saturation ($w = 0.5$), the crossing set is $\Sigma_c = [0, 0.43681)$ (where trajectories cross into region U) and the sliding set is $\Sigma_s = [0.43681, 0.5]$. Figure 4.3 shows the the plots of the signs of the projections and their products which are used by the algorithm for determining the segments on the equal share line.

- One of the pseudoequilibrium ratios corresponding to pseudoequilibrium condition for scenario 1 is $-\frac{1.0\left(1.05\,w^3+0.0075\,w^2-0.255\,w\right)}{12.0\,w^4+0.2\,w^3-5.8\,w^2-0.05\,w+0.7}$. Scenario 1 has a pseudoequilibrium $w_2^* = 0.48925$ because its field component ratio is negative with numerator value of 0.03084 and denominator value of -0.04255

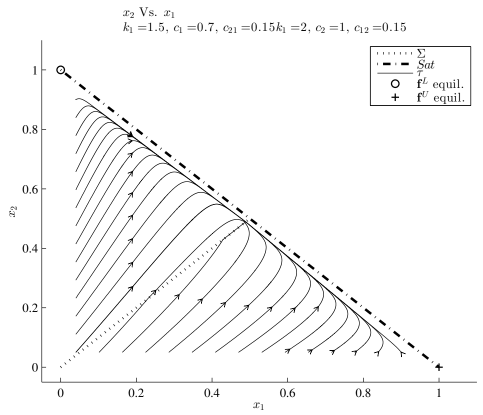


Figure 4.1: Model VWDsP, scenario 1: Firm F_1 and F_2 spending the same on predatory advertising.

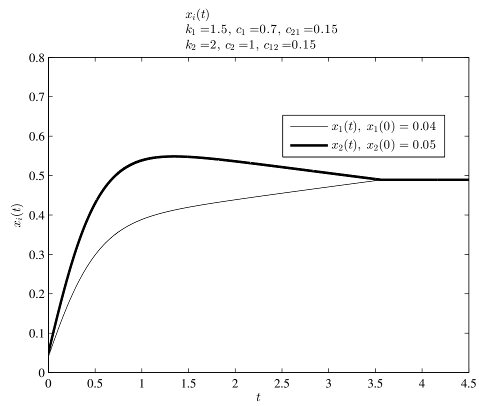


Figure 4.2: Model VWDsP, scenario 1: Time plots of the market shares $x_1(t)$, $x_2(t)$ with initial conditions $x_1(t) = 0.04$ and $x_2(t) = \ 0.05$

$\mathbf{f}^L$	$\mathbf{x}$	$\mathbf{Eigs}(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -3.7293 \\ -0.12066 \end{bmatrix}$	Stable node
	$\begin{bmatrix} 1.42467 \\ -0.5 \end{bmatrix}$	$\begin{bmatrix} -2.874 \\ 0.15065 \end{bmatrix}$	Saddle
	$\begin{bmatrix} -0.491341 \\ -0.5 \end{bmatrix}$	$\begin{bmatrix} 3.9827 \\ 2.874 \end{bmatrix}$	Unstable node

Table 4.2: Model VWDsP, scenario 1: Equilibrium points of field $\mathbf{f}^L$ with parameters k_1=1.5, c_1=0.7, k_2 =2, c_2 =1, c_{12} =0.15

$\mathbf{f}^U$	$\mathbf{x}$	$\mathbf{Eigs}(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -3.2484 \\ -0.10159 \end{bmatrix}$	Stable node
	$\begin{bmatrix} -0.466667 \\ 1.41129 \end{bmatrix}$	$\begin{bmatrix} -3.8618 \\ 0.08307 \end{bmatrix}$	Saddle
	$\begin{bmatrix} -0.466667 \\ -0.51962 \end{bmatrix}$	$\begin{bmatrix} 3.8618 \\ 2.9794 \end{bmatrix}$	Unstable node

Table 4.3: Model VWDsP, scenario 1: Equilibrium points of field $\mathbf{f}^U$ with parameters $k_1 = 1.5$, $c_1 = 0.7$, $c_{21} = 0.15$, $k_2 = 2$, $c_2 = 1$

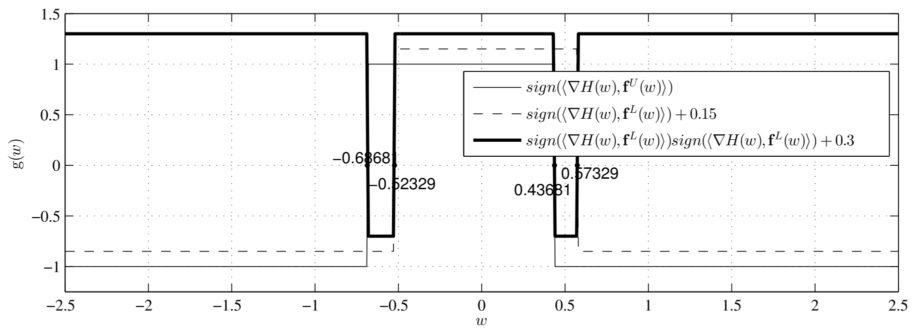


Figure 4.3: Model VWDsP, scenario 1: Signs of the inner products used in the calculation of conditions for crossing and sliding.

In the second scenario of model (4.5)-(4.8) firm F_1 does a bigger predation effort than the predation effort of firm F_2, i.e. $c_{21} \approx 7.33 c_{12}$. This scenario exemplifies the situation where large predation effort (by firm F_1) or small predation effort (by firm F_2) leads to a pseudoequilibrium on the equal market share line. The parameters in this scenario are: $k_1 = 1.5$, $c_1 = 0.7$, $c_{21} = 1.1$, $k_2 = 2$, $c_2 = 1$, $c_{12} = 0.15$. Figure 4.4 shows the phase plane and the equilibrium points of the fields are shown in Tables 4.4 and 4.5, note that field $\mathbf{f}^L$ has a virtual stable equilibrium in region U and field $\mathbf{f}^U$ has a virtual stable equilibrium in region L.

Dynamics of a trajectory on the equal market share line Σ

- Crossing and sliding sets are determined using the algorithm 4.3. The projections for determining the sets are $\langle \nabla H(\mathbf{w}), \mathbf{f}^U(\mathbf{w}) \rangle = -1.0\, w^2 - 1.2\, w + 0.3$, that has the zeros $w_1 = -1.4124$ and $w_2 = 0.2124$, and $\langle \nabla H(\mathbf{w}), \mathbf{f}^L(\mathbf{w}) \rangle = -1.0\, w^2 + 0.05\, w + 0.3$, that has the zeros $w_1 = -0.52329$ and $w_2 = 0.57329$. Because we are interested only in w values which are nonnegative and below saturation ($w = 0.5$) the crossing set is $\Sigma_c = [0, 0.2124)$ (where trajectories cross into region U) and the sliding set is $\Sigma_s = [0.2124, 0.5]$. Figure 4.6 shows the the plots of the signs of the projections and their products which are used by the algorithm for determining the segments on the equal share line.

- One of the pseudoequilibrium ratios corresponding to pseudoequilibrium condition for scenario 2 is $-\frac{1.0\left(3.9w^3+0.055w^2-0.92w\right)}{12w^4+0.2w^3-5.8w^2-0.05w+0.7}$. Scenario 2 has a pseudoequilibrium $w_2^* = 47869$ because its field component ratio is negative with numerator value of 0.06043 and denominator value of -0.01137.

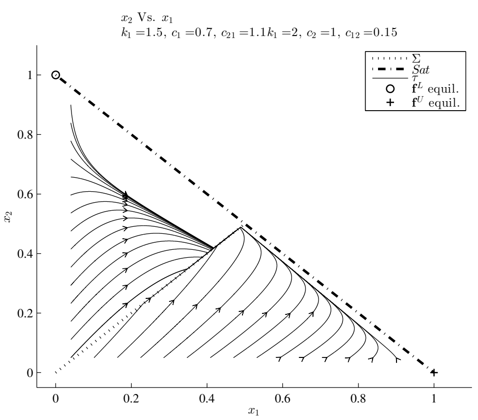


Figure 4.4: Model VWDsP, scenario 2: Firm F_1 doing big spending on advertising when compared to firm F_2 ($c_{21} \approx 7.33 c_{12}$).

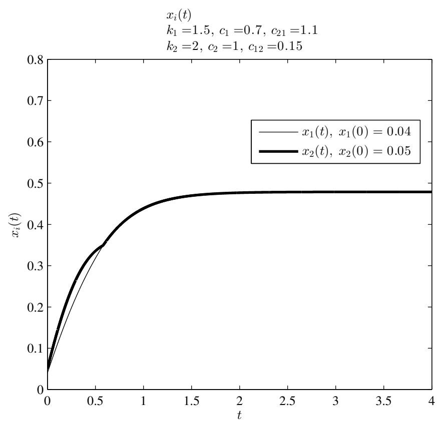


Figure 4.5: Model VWDsP, scenario 2: Time plots of the market shares $x_1(t)$, $x_2(t)$ with initial conditions $x_1(t) = 0.04$ and $x_2(t) = \ 0.05$

$\mathbf{f}^L$	$\mathbf{x}$	$\mathbf{Eigs}(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}-3.7293\\-0.12066\end{bmatrix}$	Stable node
	$\begin{bmatrix}1.42467\\-0.5\end{bmatrix}$	$\begin{bmatrix}-2.874\\0.15065\end{bmatrix}$	Saddle
	$\begin{bmatrix}-0.491341\\-0.5\end{bmatrix}$	$\begin{bmatrix}3.9827\\2.874\end{bmatrix}$	Unstable node

Table 4.4: Model VWDsP, scenario 2: Equilibrium points of field $\mathbf{f}^L$ with parameters $k_1 = 1.5$, $c_1 = 0.7$, $k_2 = 2$, $c_2 = 1$, $c_{12} = 0.15$

$\mathbf{f}^U$	$\mathbf{x}$	$\mathbf{Eigs}(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}-3.6341\\-0.66592\end{bmatrix}$	Stable node
	$\begin{bmatrix}-0.466667\\1.08966\end{bmatrix}$	$\begin{bmatrix}-3.5253\\0.56551\end{bmatrix}$	Saddle
	$\begin{bmatrix}-0.466667\\-0.672993\end{bmatrix}$	$\begin{bmatrix}3.5253\\3.2095\end{bmatrix}$	Unstable node

Table 4.5: Model VWDsP, scenario 2: Equilibrium points of field $\mathbf{f}^U$ with parameters $k_1 = 1.5$, $c_1 = 0.7$, $c_{21} = 1.1$, $k_2 = 2$, $c_2 = 1$

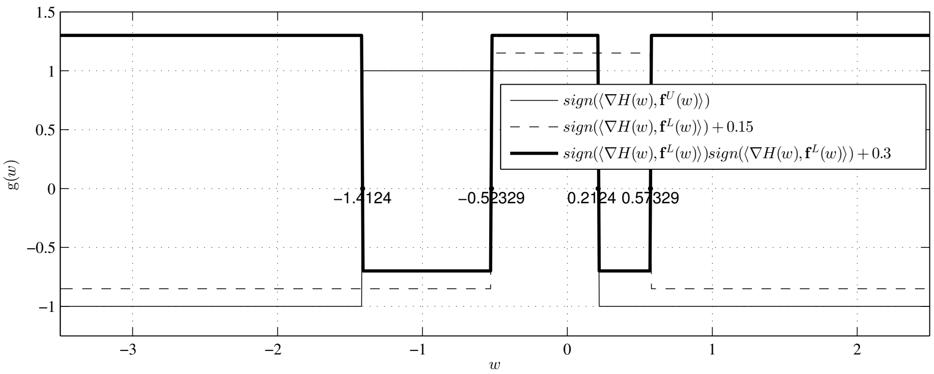


Figure 4.6: Model VWDsP, scenario 2: Signs of the inner products used in the calculation of conditions for crossing and sliding.

4.6 Evaluating outcomes of the proposed policies using a performance index for model VWDsP

We use the index (2.9). Note that a firm makes a profit when its index J_i is positive, whereas it makes a loss when its index is negative.

The market share $x_i(t)$ evolution is given by the system (4.5), (4.8) and the control functions $u_i(t)$ used for the index are liste below. The control function for each firm in the upper field f^U are

$$u_1 = (k_1 x_1 + c_1) \tag{4.53}$$

$$u_2 = (k_2 x_2 + c_2) - c_{21} \tag{4.54}$$

when $x_2 > x_1$

The control function for each firm in the lower field f^L are

$$u_1 = (k_1 x_1 + c_1) - c_{12} \tag{4.55}$$
$$u_2 = (k_2 x_2 + c_2) \tag{4.56}$$
$$\text{when } x_1 > x_2$$

On Σ, both firms turn on their extra effort

$$u_1 = (k_1 x_1 + c_1) - c_{12} \tag{4.57}$$
$$u_2 = (k_2 x_2 + c_2) - c_{21} \tag{4.58}$$
$$\text{when } x_1 = x_2$$

The scenarios were simulated using the software Matlab. *The first scenario* of model (4.5)-(4.8) has the parameters $k_1 = 1.5$, $c_1 = 0.7$, $c_{21} = 0.15$, $k_2 = 2$, $c_2 = 1$, $c_{12} = 0.15$ and $r = 4.5$ where both firms do the same predation effort, i.e. $c_{21} = c_{12}$. In this setting both firms end at a pseudoequilibrium point below 0.5 on the equal market share line Σ.

Even if both firms end on the equal market share line, a firm needs to identify on which trajectories it makes a profit. Figure 4.7 shows the trajectories in the phase plane according to their index value at final time, these indexes were calculated using a revenue factor $r = 4.5$. Only firm F_2 makes a profit on the solid line trajectories, both firms make a profit on the dashed line trajectories and only firm F_1 makes a profit on the dashed-dotted line trajectories, firm F_1 profits in more trajectories because it is doing smaller efforts ($k_1 < k_2$ and $c_1 < c_2$). Figure 4.8 shows $J_2 Vs. J_1$ to help identify compromise solutions, i.e. conditions that end in positive J_i and those values cannot improve anymore.

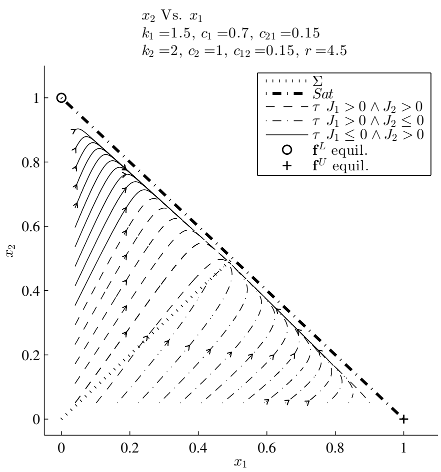


Figure 4.7: Model VWDsP, scenario 1: Trajectories plotted according to their index values at final time.

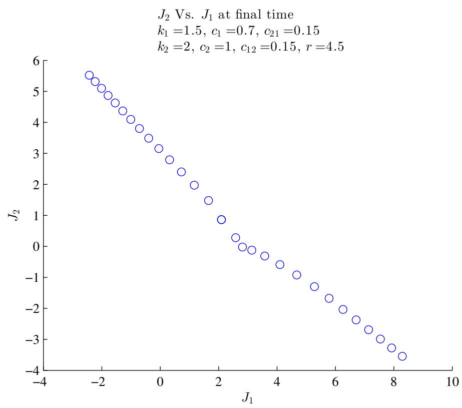


Figure 4.8: Model VWDsP, scenario 1: $J_1(x_1(t), x_2(t), t)$ vs. $J_2(x_1(t), x_2(t), t)$

Figures 4.9 and 4.10 show the index values when the market share of the entering firm ($x_i < x_j$) is fixed and the market share of the other firm is increasing. From these figures one can see that the firm that begins with a smaller market share has in most of the cases a smaller and decreasing performance index value.

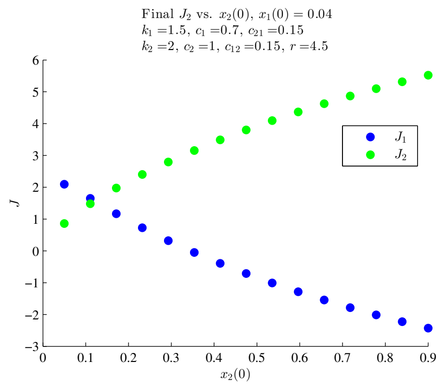


Figure 4.9: Model VWDsP, scenario 1: Value of indices when the initial value $x_1(0) = 0.04$ is fixed.

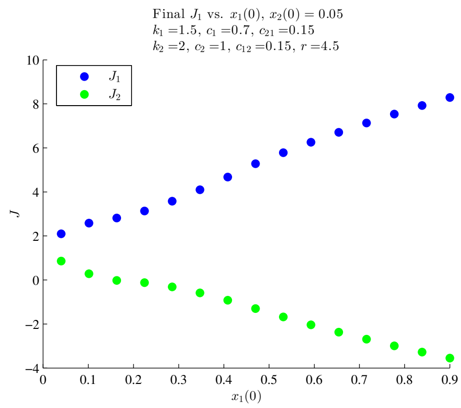


Figure 4.10: Model VWDsP, scenario 1: Value of indices when the initial value $x_2(0) = 0.05$ is fixed.

The second scenario of model VWDsP has firm F_1 with a bigger predation effort $c_{21} = 1.1$.

A firm needs to identify on which trajectories it makes a profit regardless of its final market share, which in this case is on the equal market share. Figure 4.11 shows the trajectories in the phase plane according to their index value at final time, these indexes were calculated using a revenue factor $r = 4.5$. Only firm F_2 makes a profit on the solid line trajectories, both firms make a loss on the dotted line trajectories and only firm F_1 makes a profit on the dashed-dotted line trajectories, firm F_1 profits less than in the first scenario because now it is spending far more in predatory advertising. Figure 4.12 shows $J_2 Vs. J_1$ to help identify compromise solutions, i.e. conditions that end in positive J_i and those values cannot improve anymore.

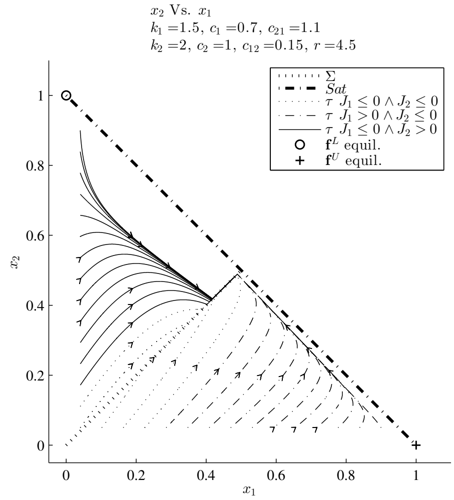


Figure 4.11: Model VWDsP, scenario 2: Trajectories plotted according to their index values at final time.

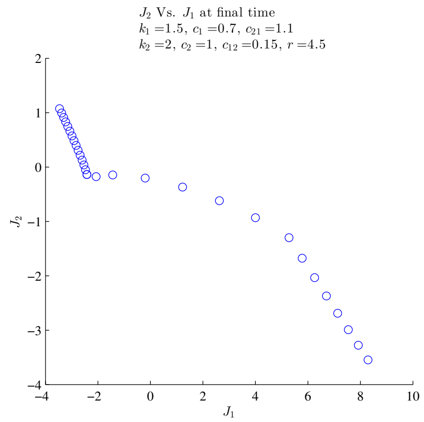


Figure 4.12: Model VWDsP, scenario 2: $J_1(x_1(t), x_2(t), t)$ vs. $J_2(x_1(t), x_2(t), t)$

Figures 4.13 and 4.14 show the index values when the market share of the entering firm ($x_i < x_j$) is fixed and the market share of the other firm is increasing. From these figures one can see that the firm that begins with a smaller market share has in most of the cases a smaller and decreasing performance index value.

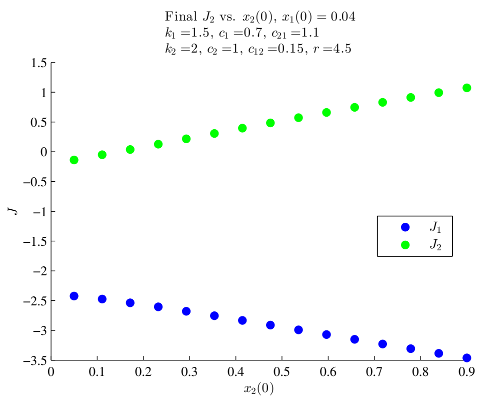


Figure 4.13: Model VWDsP, scenario 2: Value of indices when the initial value $x_1(0) = 0.04$ is fixed.

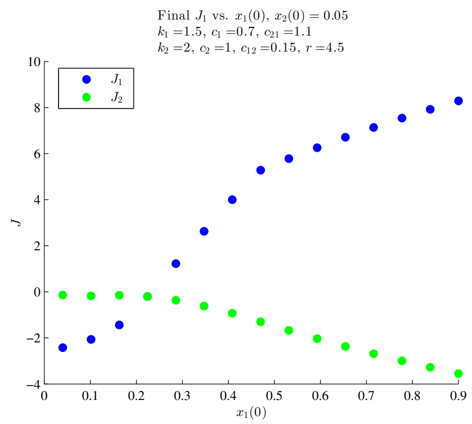


Figure 4.14: Model VWDsP, scenario 2: Value of indices when the initial value $x_2(0) = 0.05$ is fixed.

4.7 Remarks on model VWDsP

- The saturated market is not reached, i.e. the long run equal market share is below the saturated equal market share.
- As the predation effort u_{ij} increases, the steady state market share decreases, see

Table 4.1.

- Even though it may seem reasonable for a firm F_i to turn on a predatory advertising spending u_{ji} when its target market is below its goal ($x_i < x_j$), from the analysis of model VWDsP we see that this leads to both firms reaching an equal market share in the long run independent of the efforts they might apply, and this may not be reasonable since it would mean that a firm spending very little on advertising would end with the same market share as a firm spending big amounts. Further analysis by the decision maker would be needed to decide whether this could happen: for example, in some markets with unsatisfied buyers, frequently changing from one firm to the other.

4.8 Model Vidale-Wolfe-Deal model with 2 switching lines (VWD2s)

A model which is more realistic in terms of target market shares is the model VWD2s. In this model, the target market shares of the firms are different, firm F_1 desires a long term market share $target_1$, and F_2 desires a long term market share $target_2$ which satisfy $target_1 + target_2 > 1$. In this setting, each F_i has its own switching line.

The model VWD2s is as follows:

1. F_2 fulfills its $target_2$ in the region U wich is above Σ_{UM}, which is defined by the line $x_2 - \frac{target_2}{1-target_2}x_1 = 0$. The upper field $\mathbf{f}^U$ operating in U is Σ_{UM} , is

$$\dot{x}_1 = (k_1x_1 + c_1)(1 - x_1 - x_2) \tag{4.59}$$

$$\dot{x}_2 = k_2x_2(1 - x_1 - x_2) \tag{4.60}$$

$$\text{when } x_2 > \frac{target_2}{1 - target_2}x_1$$

2. The region M is where neither firm obtains its target market share. It is below Σ_{UM} and above Σ_{ML}, which is defined by the line $x_2 - \frac{1-target_1}{target_1}x_1 = 0$. The middle field $\mathbf{f}^M$ operating in M is

$$\dot{x}_1 = (k_1x_1 + c_1)(1 - x_1 - x_2) \tag{4.61}$$

$$\dot{x}_2 = (k_2x_2 + c_2)(1 - x_1 - x_2) \tag{4.62}$$

$$\text{when } x_2 < \frac{target_2}{1 - target_2}x_1 \text{ and } x_2 > \frac{1 - target_1}{target_1}x_1$$

3. The region L is where firm F_1 obtains its target market share. It is below Σ_{ML},

which is the line $x_2 - \frac{1-target_1}{target_1}x_1 = 0$. The lower field $\mathbf{f}^L$ operating in L is

$$\dot{x}_1 = k_1 x_1(1 - x_1 - x_2) \tag{4.63}$$

$$\dot{x}_2 = (k_2 x_2 + c_2)(1 - x_1 - x_2) \tag{4.64}$$

$$\text{when } x_2 < \frac{1-target_1}{target_1}x_1$$

4. On the switching lines Σ_{UM} and Σ_{ML} the dynamics is defined following the Filippov rule which is based, roughly speaking, on a convex combination of the lower and upper fields with respect to the corresponding switching line (see Appendix A, [46, p. 50-52, chap. 2] for mathematical details).

Figure 4.15 identifies the regions and the relevant lines for model VWD2s which is defined by (4.59)-(4.64). It uses $target_2 = 0.8$ and $target_1 = 0.7$. Σ_{UM} is the line $x_2 - \frac{0.8}{0.2}x_1 = 0$ and Σ_{ML} is the line $x_2 - \frac{3}{7}x_1 = 0$. The line $1 - x_1 - x_2 = 0$ is the full market line which corresponds to the saturated market, which will be called Sat. The interior of the triangle $\mathcal{T}$, defined as $(1, 0)$, $(0, 1)$, $(0, 0)$ is the set of all feasible states. The region U is the set $\{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1 \wedge x_2 > 4x_1 \wedge x2 > \frac{3}{7}x_1\}$ where the dynamics corresponding to f^U is acting. The region M is the set $\{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1 \wedge x_2 < 4x_1 \wedge x2 > \frac{3}{7}x_1\}$ where the dynamics corresponding to f^M is acting. The region L is the set $\{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1 \wedge x_2 < 4x_1 \wedge x2 < \frac{3}{7}x_1\}$ where the dynamics corresponding to f^L is acting.

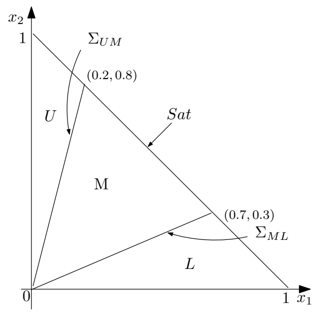


Figure 4.15: The regions: upper U, middle M and lower L. The switching lines $\Sigma_{UM} = \{\mathbf{x} : H(\mathbf{x}) = x_2 - 4x_1 = 0\}$, $\Sigma_{ML} = \{\mathbf{x} : H(\mathbf{x}) = x_2 - \frac{3}{7}x_1 = 0\}$,. The segment Sat segment which is defined by $1 - x_1 - x_2 = 0$. $\mathcal{T}$ is the interior of the triangle defined by the points $(1, 0)$, $(0, 1)$, $(0, 0)$

4.9 Simulations of model VWD2s

The scenarios were simulated using the software Matlab. *The first scenario* of model (4.59)-(4.64) has the parameters $k_1 = 1.5$, $c_1 = 0.7$, $c_{21} = 0.15$, $k_2 = 2$, $c_2 = 1$, $c_{12} = 0.15$ and $r = 4.5$ where the control efforts of the firms are not very different and the outcome in terms of final index values is divided evenly between the firms.

In the *first scenario*, shown in Figures 4.16 and 4.17, the firms do similar control efforts and the long run market share values are evenly distributed on the saturated market share. In the switching lines Σ_{UM} and Σ_{ML} crossing is the behavior observed.

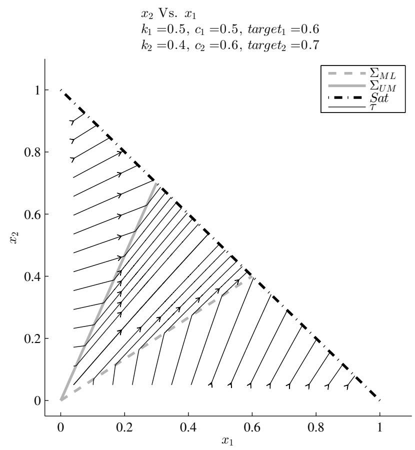


Figure 4.16: Model VWD2s, scenario 1: Firm F_1 and F_2 spend similar amounts in advertising.

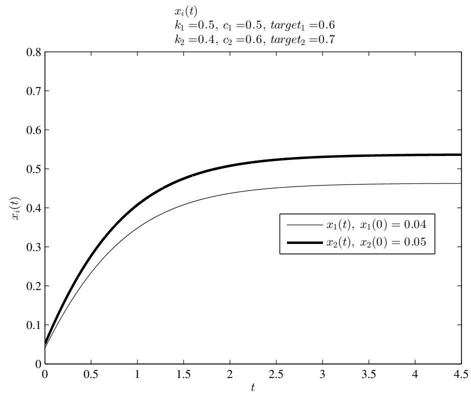


Figure 4.17: Model VWD2s, scenario 1: Time plots of the market shares $x_1(t)$, $x_2(t)$ with initial conditions $x_1(t) = 0.04$ and $x_2(t) = 0.05$

In the *second scenario*, shown in Figures 4.18 and 4.19, firm F_1 does bigger control efforts than firm F_2 and most of the trajectories go down where firm F_1 has a bigger long run market share. In the switching lines crossing and sliding are the behaviors observed.

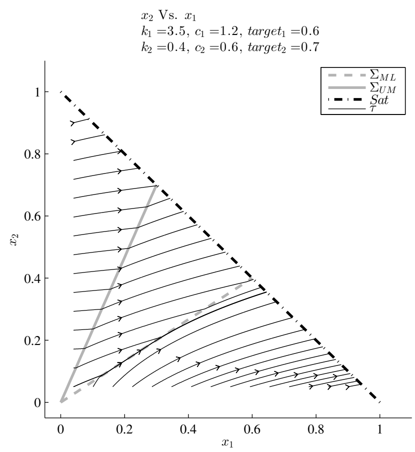


Figure 4.18: Model VWD2s, scenario 2: Firm F_1 spends more than F_2 in advertising.

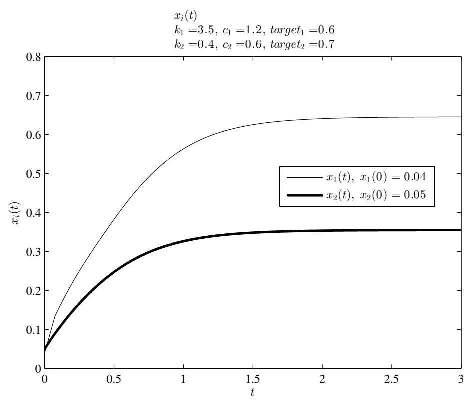


Figure 4.19: Model VWD2s, scenario 2: Time plots of the market shares $x_1(t)$, $x_2(t)$ with initial conditions $x_1(t) = 0.04$ and $x_2(t) = 0.05$

4.10 Evaluating outcomes of the proposed policies using a performance index for model VWD2s

We use the index (2.9). Note that a firm makes a profit when its index J_i is positive, whereas it makes a loss when its index is negative.

In the *first scenario*, shown in Figures 4.20-4.22, the control efforts of the firms are not very different and the outcome in terms of final index values is divided evenly between the firms. Figure 4.20 shows the trajectories in the phase plane according to their index value at final time, these indexes were calculated using a revenue factor $r = 3.5$. Only firm F_2 makes a profit on the solid line trajectories, both firms make a profit on the dashed line trajectories and only firm F_1 makes a profit on the dashed-dotted line trajectories, firm F_1 profits in more trajectories because it is doing smaller efforts ($k_1 < k_2$ and $c_1 < c_2$).

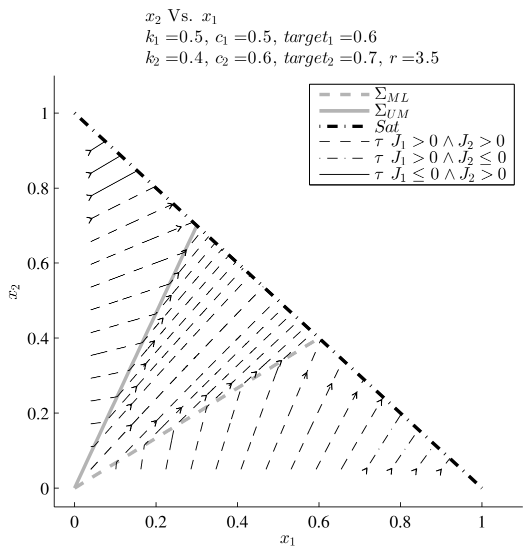


Figure 4.20: Model VWD2s, scenario 1: Trajectories plotted according to their index values at final time.

Figures 4.21 and 4.22 show the index values when the market share of the entering firm ($x_i < x_j$) is fixed and the market share of the other firm is increasing. From these figures one can see that the firm that begins with a smaller market share has in most of the cases a smaller and decreasing performance index value.

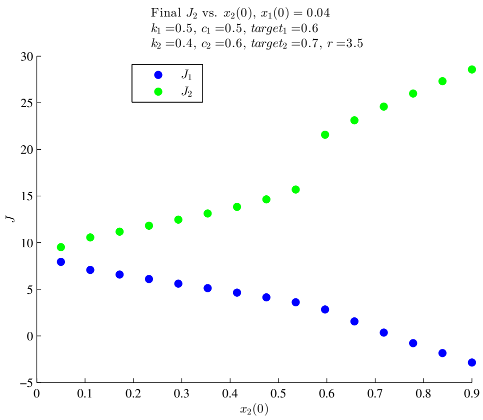


Figure 4.21: Model VWD2s, scenario 1: Value of indices when the initial value $x_1(0) = 0.04$ is fixed.

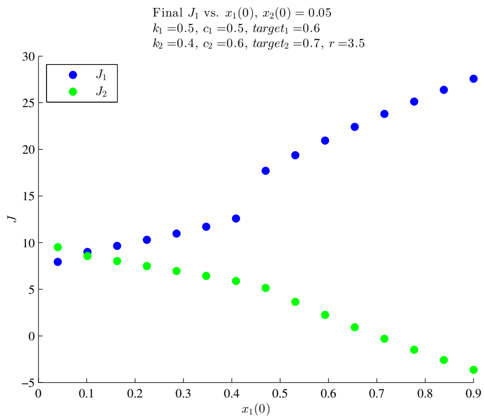


Figure 4.22: Model VWD2s, scenario 1: Value of indices when the initial value $x_2(0) = 0.05$ is fixed.

In the *second scenario*, shown in Figures 4.23-4.25, firm F_1 does a bigger control effort than firm F_2. In this setting more trajectories go down, i.e. where F_1 end with a bigger market share in the long run. Figure 4.23 shows the trajectories in the phase plane according to their index value at final time, these indexes were calculated using a revenue factor $r = 3.5$. Only firm F_2 makes a profit on the solid line trajectories, and both firms make a loss on the dotted line trajectories. Firm F_1 does not make a profit because its big spending on advertising does not generate profit for the market revenue factor $r = 3.5$.

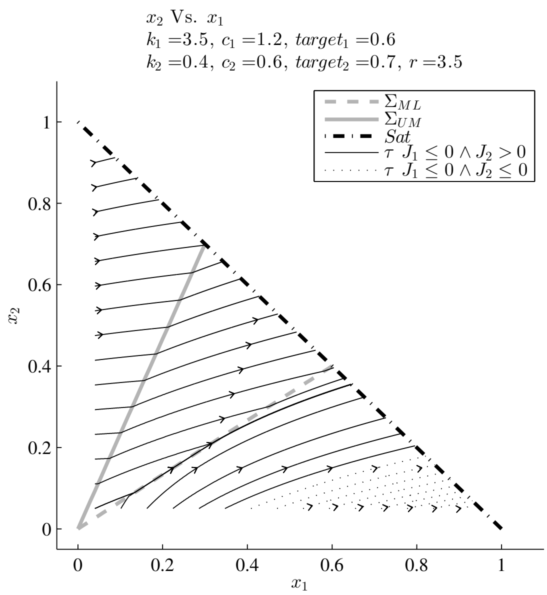


Figure 4.23: Model VWD2s, scenario 2: Trajectories plotted according to their index values at final time.

Figures 4.24 and 4.25 show the index values when the market share of the entering firm ($x_i < x_j$) is fixed and the market share of the other firm is increasing. From these figures one can see that the firm that begins with a smaller market share has in most of the cases a smaller and decreasing performance index value.

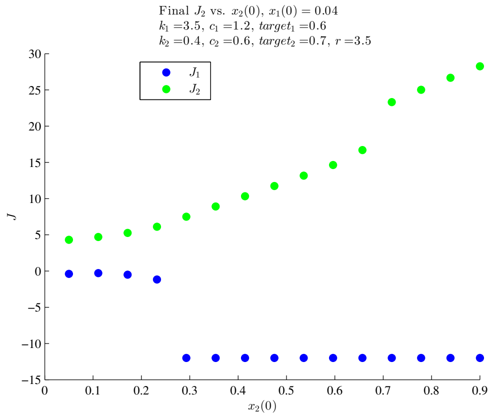


Figure 4.24: Model VWD2s, scenario 2: Value of indices when the initial value $x_1(0) = 0.04$ is fixed.

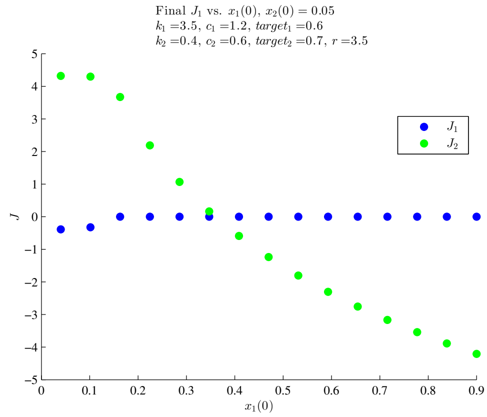


Figure 4.25: ModelVWD2s, scenario 2: Value of indices when the initial value $x_2(0) = 0.05$ is fixed.

4.11 Remarks on model VWD2s

- Model VWD2s extends model VWDsC and the analysis done for model VWDsC applies to model VWD2s on each discontinuity line, for firm F_2 (resp. F_1)the line of interest is Σ_{UM} (resp. Σ_{ML}).
  - The saturated market share line is a set of attractive points.
  - The state variables are nondecreasing, they increase while they are below the saturated market share Sat and remain unchanged once they reach Sat.
  - Trajectories that reach the switching line Σ_{UM} (or Σ_{ML}) have the same behavior as in model VWDsC, the difference is the slope of the switching lines.
  - The model has no pseudoequilibrium point (neither on Σ_{UM} nor on Σ_{ML}).
  - Each firm has a separatrix, in the case of firm F_2 (resp. firm F_1) a separatrix is calculated using its target saturated market share point $(1 - target_2, target_2)$ (resp. $(target_1, 1 - target_1)$) which is where Σ_{UM} (resp. Σ_{ML}) and Sat intersect. A separatrix for firm F_2 (resp. F_1) may be φ^U or φ^M (resp. φ^M or φ^L), i.e. obtained with fields $\mathbf{f}^U$ or $\mathbf{f}^M$ (resp. $\mathbf{f}^M$ or $\mathbf{f}^L$).
  - Firm F_2 (resp. F_1) fulfills its target market share above (resp. below) its separatrix.
- The difference is that a separatrix may span more than one region: for firm F_2 (resp. F_1) it may span region U or regions M and L (resp. regions M and U or region L).

Chapter 5

Duopoly model based on Lotka-Volterra dynamics with switched predation (LVsP)

5.1 Introduction

Wang et al. [35] proposed a duopoly model with Lotka-Volterra competing species dynamics where the state variable is sales:

$$\dot{x}_1 = x_1(b_1 - a_{11}x_1) - a_{12}x_1x_2 \tag{5.1}$$

$$\dot{x}_2 = x_2(b_2 - a_{22}x_2) - a_{21}x_1x_2 \tag{5.2}$$

in this model, all the coefficients a_{ij}, b_i are assumed nonnegative and it is stated that $b_i = f(advertising)$.

Wang et al. [35] showed that model (5.1-5.2) fulfills the desired properties of advertising function [25], but there is no exact hint on how to obtain the coefficient b_i as a function of advertising. Because of this we consider this model theoretical. However, recent papers use Lotka-Volterra models and this indicates that they should be explored. In 2016, Marasco et al. [52] used a nonautonomous Lotka-Volterra system to model market share dynamics that depend on utility functions of the competing firms. Because of the special functional forms they impose, Marasco et al. can obtain an analytical solution to the system. In 2014, Hung et al. [36] modeled the saturated market dynamics using a Lotka-Volterra model.

In analogy with the biological models, the value b_i/a_{ii} could be thought of as a carrying capacity (i.e., market saturation level) for firm i and the so-called "interaction" term $a_{ij}x_ix_j$ as a term describing predatory advertising. Note that if the state variable to be used is market share, the Lotka-Volterra model does not result in invariance of total market share (always equal to one), ensuring instead that each market share variable is below or equal to a given upper bound (via the carrying capacity of each firm). One way for model (5.1-5.2) to enforce the invariance of total saturated market share is setting $b_1 = a_{11} = a_{12} = b_2 = a_{22} = a_{21} = 1$, but this reduces the model (5.1-5.2) to Vidale-Wolfe-Deal's model where $u_i = x_i$.

The fact that the Lotka-Volterra model cannot enforce the saturated market restriction led to the choice, in this chapter, of the number of clients as the state variable instead of market share. In this case the predatory term $a_{ij}x_ix_j$, i.e. advertising, removes clients from one firm. In the context of a duopoly for an essential good, it is also reasonable to argue that clients removed from one firm go to the other firm and, motivated by this observation, we propose the following duopoly model with Lotka-Volterra type dynamics, together with controlled predatory terms:

$$\dot{x}_1 = x_1(b_1 - a_{11}x_1) - u_{12}x_1x_2 + u_{21}x_1x_2 \tag{5.3}$$

$$\dot{x}_2 = x_2(b_2 - a_{22}x_2) + u_{12}x_1x_2 - u_{21}x_1x_2 \tag{5.4}$$

where b_i, a_{ii} defines the saturation level for firm i, u_{ij} is a control coefficient determining predatory advertising by firm j on firm i (i.e, firm j capturing clients of firm i). In what follows, it will be postulated that the advertising effort or expenditure by a firm modifies the values of the control coefficients u_{ij} depending on its perception of its client base.

The policy u_{ij} is a predatory advertising effort made by firm F_j on firm F_i and it is as follows:

$$u_{ij} = \begin{cases} c_{ij} & \text{if } x_j < x_i \ , \\ d_{ij} & \text{if } x_j > x_i \end{cases} \tag{5.5}$$

where $i, j = 1, 2$, $i \neq j$ and the policy $c_{i,j} > d_{i,j}$, i.e. firm F_j applies a bigger predatory advertising constant effort on firm F_i clients when $x_j < x_i$, this is when F_j has a smaller number of clients.

After using the policies (5.5) and some algebra in model (5.3-5.4) its fields are obtained:

The upper field $\mathbf{f}^U$, which defines the dynamics in region U, is given by

$$\dot{x_1} = x_1(b_1 - a_{11}x_1 + (-d_{12} + c_{21})x_2) \tag{5.6}$$

$$\dot{x_2} = x_2(b_2 - (-d_{12} + c_{21})x_1 - a_{22}x_2) \tag{5.7}$$

The lower field $\mathbf{f}^L$, which defines the dynamics in region L, is given by

$$\dot{x_1} = x_1(b_1 - a_{11}x_1 + (-c_{12} + d_{21})x_2) \tag{5.8}$$

$$\dot{x_2} = x_2(b_2 - (-c_{12} + d_{21})x_1 - a_{22}x_2) \tag{5.9}$$

On Σ, the dynamics is defined following the Filippov rule (see Appendix A, [46, p. 50-52, chap. 2]). Note that all coefficients $b_i, a_{ii}, c_{ij}, d_{ij}$ $(i, j = 1, 2,\ i \neq j)$ are real nonnegative numbers.

5.2 Equilibria of model LVsP

We analyze the equilibrium points in field $\mathbf{f}^U$ (the analysis of field $\mathbf{f}^L$ is analogous). The equilibria are the zeros of the RHS (right hand side) of system (5.6), (5.7). In order to find if a equilibrium point is stable or not, the jacobian of the system is used:

$$\begin{bmatrix} -2a_{11}x_1 + b_1 + (c_{21} - d_{12})x_2 & (c_{21} - d_{12})x_1 \\ x_2(-c_{21} + d_{12}) & b_2 - (c_{21} - d_{12})x_1 - 2a_{22}x_2 \end{bmatrix} \tag{5.10}$$

The four equilibria points of the model (5.6), (5.7) are:

1. Equilibrium point $P_1 = (0, 0)$ and its corresponding jacobian matrix $J(P_1) = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$ with the eigenvalues readily on the diagonal, so that P_1 is an unstable node because $b_i > 0$, $i = 1, 2$.

2. Equilibrium point $P_2 = (0, b_2/a_{22})$ and its corresponding jacobian matrix $J(P_2) = \begin{bmatrix} b_1 + \frac{(c_{21}-d_{12})b_2}{a_{22}} & 0 \\ \frac{b_2*(-c_{21}+d_{12})}{a_{22}} & -b_2 \end{bmatrix}$ with the eigenvalues on the diagonal.

3. Equilibrium point $P_3 = (b_1/a_{11}, 0)$ and its corresponding jacobian matrix $J(P_3) = \begin{bmatrix} -b1 & \frac{(c_{21}-d_{12})b_1}{a_{11}} \\ 0 & b_2 - \frac{(c_{21}-d_{12})b_1}{a_{11}} \end{bmatrix}$ with the eigenvalues on the diagonal.

4. Equilibrium point $P_4 = \left(\frac{a_{22}b_1+b_2c_{21}-b_2d_{12}}{a_{11}a_{22}+c_{21}^2-2c_{21}d_{12}+d_{12}^2}, \frac{a_{11}b_2-b_1c_{21}+b_1d_{12}}{a_{11}a_{22}+c_{21}^2-2c_{21}d_{12}+d_{12}^2}\right)$ and its corresponding jacobian matrix is

$$\begin{bmatrix} -\frac{2a_{11}(a_{22}b_1+b_2\beta)}{a_{11}a_{22}+\beta^2} + \frac{\beta(a_{11}b_2-b_1\beta)}{a_{11}a_{22}+\beta^2} + b_1 & \frac{\beta(a_{22}b_1+b_2\beta)}{(a_{11}a_{22}+\beta^2)} \\ -\frac{\beta(a_{11}b_2-b_1\beta)}{a_{11}a_{22}+\beta^2} & -\frac{2a_{22}(a_{11}b_2-b_1\beta)}{a_{11}a_{22}+\beta^2} - \frac{\beta(a_{22}b_1+b_2\beta)}{a_{11}a_{22}+\beta^2} + b_2 \end{bmatrix}$$

   where $\beta = c_{21} - d_{12}$.

The type of the equilibrium points P_2, P_3 and P_4 depends on parameter values, for instance using the parameters $a_{11} = 1$, $a_{22} = 1$, $b_1 = 10.9$, $b_2 = 6.8$, $c_{12} = 7.1$, $c_{21} = 5.5$, $d_{12} = 6.3$, $d_{21} = 4.6$ the equilibrium points P_2, P_3 and P_4 are respectively saddle, saddle and stable focus.

5.3 Dynamics of trajectories reaching the equal number of clients line Σ in model LVsP

Determining the crossing Σ_c and sliding Σ_s sets: To describe a trajectory that reaches the equal number of clients Σ, we need to determine if that trajectory reaches the equal number of clients Σ in the crossing set Σ_c or in the sliding set Σ_s. For this we will use the Algorithm 4.3.

In the case of model (5.6)-(5.8), the equal number of clients line Σ is described by $H(\mathbf{x}) = x_2 - x_1 = 0$ and its normal is $\nabla H(\mathbf{x}) = [-1 \quad 1]^T$, and the following scalar functions are parabolas, then we can use Algorithm 4.3.

$$g_U(w) := \langle \nabla H(\mathbf{w}), \mathbf{f}^U(\mathbf{w}) \rangle = (a_{11} - a_{22} - 2c_{21} + 2d_{12})\, w^2 + (b_2 - b_1)\, w \qquad (5.11)$$

$$g_L(w) := \langle \nabla H(\mathbf{w}), \mathbf{f}^L(\mathbf{w}) \rangle = (a_{11} - a_{22} + 2c_{12} - 2d_{21})\, w^2 + (b_2 - b_1)\, w \qquad (5.12)$$

where $\mathbf{w} = (w, w)$ which is a point on the equal number of clients line Σ.

Algorithm 4.3 calculates sign$(g_U(w))$ and sign$(g_L(w))$ using their respective zeros and second derivatives. Once the signs of the values of the points in each parabola are calculated, the sets can be combined to obtain Σ_c, the crossing set that is comprised of the points where the values of parabolas have the same sign, and Σ_s, the sliding set that is comprised of the points where the values of the parabolas have opposite sign. To apply Algorithm 4.3 the zeros needed are:

$$\text{Zeros of } g_U(w): \quad w_1^U = 0 \qquad w_2^U = \frac{b_1 - b_2}{a_{11} - a_{22} - 2c_{21} + 2d_{12}} \qquad (5.13)$$

$$\text{Zeros of } g_L(w): \quad w_1^L = 0 \qquad w_2^L = \frac{b_1 - b_2}{a_{11} - a_{22} + 2c_{12} - 2d_{21}} \qquad (5.14)$$

The needed second derivatives are:

$$g_U(w)'' = 2a_{11} - 2a_{22} - 4c_{21} + 4d_{12} \qquad (5.15)$$

$$g_L(w)'' = 2a_{11} - 2a_{22} + 4c_{12} - 4d_{21} \qquad (5.16)$$

Next we enunciate a proposition which for the given case exemplifies the usage of the fact that the $g_U(w)$ and $g_L(w)$ are parabolas. (For the the calculations Algorithm 4.3 is used and it takes into account other cases)

Proposition 5.3.1 (Calculating the crossing set Σ_c and sliding set Σ_s). *Suppose that*

(i) $b_1 < b_2$

(ii) $g_U(w)'' < 0$

(iii) $g_L(w)'' > 0$

then for nonnegative w values

- *the crossing set is $\Sigma_c = \{w|w \in (0, w_2^U)\}$*
- *the sliding set is $\Sigma_s = \{w|w \in [w_2^U, \infty)\}$*

Proof. First we find the sign of w_2^U. Comparing (5.13) and (5.15) we find that the denominator of w_2^U is $2g_U(w)''$ then by assumption (*ii*) the denominator of w_2^U is negative. The numerator of w_2^U is negative by assumption (*i*). Because of this we have

$$w_2^U > 0 \tag{5.17}$$

The function $g_U(w)$ is a parabola on variable w by (5.11) and by assumption (*ii*) its second derivative is negative, then it has a maximum. From this and (5.17) we have

$$\text{sign}(g_U(w)) = \begin{cases} -1 & \text{if } w \in (-\infty, 0) \\ 1 & \text{if } w \in (0, w_2^U) \\ -1 & \text{if } w \in (w_2^U, \infty) \end{cases} \tag{5.18}$$

Now we find the sign of w_2^L. Comparing (5.14) and (5.16) we find that the denominator of w_2^L is $2g_L(w)''$ then by assumption (*ii*) the denominator of w_2^U is positive. The numerator of w_2^L is negative by assumption (*i*). Because of this we have

$$w_2^L < 0 \tag{5.19}$$

The function $g_L(w)$ (5.12) is a parabola and by assumption (*iii*) its second derivative is positive, then it has a minimum. From this and (5.19) we have

$$\text{sign}(g_L(w)) = \begin{cases} 1 & \text{if } w \in (-\infty, w_2^L) \\ -1 & \text{if } w \in (w_2^L, 0) \\ 1 & \text{if } w \in (0, \infty) \end{cases} \tag{5.20}$$

By the crossing condition (A.5) (in Appendix A) and the definitions (5.11)-(5.12), the crossing set Σ_c is comprised of the values w where $\text{sign}(g_U(w)) = \text{sign}(g_L(w)) \neq 0$ and the sliding set Σ_s is its complement. Combining this and the results in (5.18) and (5.20), and considering we are interested in the nonnegative values of w, the sets for crossing and sliding on the equal number of clients line Σ are

$$\Sigma_c = \{w|w \in (0, w_2^U)\} \tag{5.21}$$

$$\Sigma_s = \{w|w \in [w_2^U, \infty)\} \tag{5.22}$$

□

Determining pseudoequilibrium points: A pseudoequilibrium point is an equilibrium that may come into existence because of the use of switching control (See Appendix A). The pseudoequilibrium candidates are obtained using the fact that at a pseudoequilibrium the fields must be aligned and have opposite directions. The pseudoequilibrium candidates for model (5.6), (5.9) are:

$$w_1^* = 0 \tag{5.23}$$

$$w_2^* = \frac{b_1 + b_2}{a_{11} + a_{22}} \tag{5.24}$$

Note that

- the pseudoequilibrium candidates obtained are independent of the predation coefficients $c_{ij},\ d_{ij}$ where $i \neq j$ and $i, j = 1, 2$.
- the candidate $w_1^* = 0$ is not stable because the point $(0, 0)$ is unstable for both fields $\mathbf{f}^U$ and $\mathbf{f}^L$.
- the candidate w_2^* can be a pseudoequilibrium only if $w_2^* \in \Sigma_s$.

5.4 Simulations of different scenarios of model LVsP

The first scenario of model (5.6)-(5.9) has the parameters $b_1 = 6.8$, $a_{11} = 1$, $c_{21} = 3.6$, $d_{21} = 1.6$, $b_2 = 10.9$, $a_{22} = 1$, $c_{12} = 5.5$, $d_{12} = 4.6$. In this case, the efforts c_{21}, d_{21} are small when compared to those of firm F_2. The corresponding phase plane is shown in Figure 5.1 where one can see that firm F_2 ends with all the clients and firm F_1 is extincted from the market. The equilibrium points of the fields are shown in Tables 5.1 and 5.2, note that both fields have stable equilibrium points in region U. The stable equilibrium of field $\mathbf{f}^L$ is virtual but it still attracts trajectories beginning in L to U. It is virtual because it corresponds to the dynamics of field $\mathbf{f}^L$ but lies in region U where the dynamics is given by field $\mathbf{f}^U$.

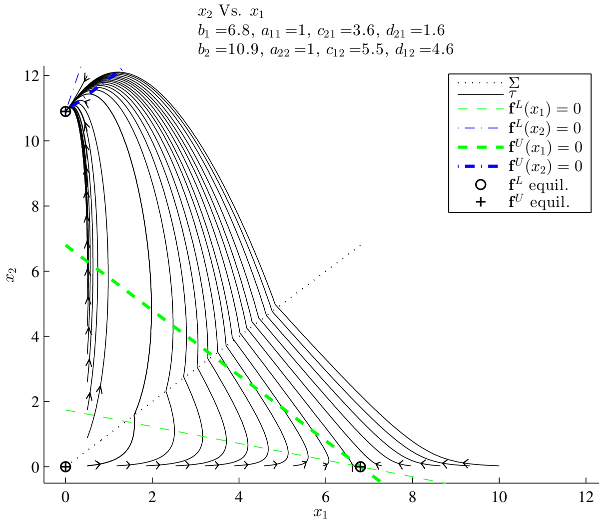


Figure 5.1: Model LVsP, scenario 1: Firm F_1 doing little spending on advertising when compared to firm F_2.

$\mathbf{f}^L$	$\mathbf{x}$	$\mathbf{Eigs}(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}10.9\\6.8\end{bmatrix}$	Unstable node
	$\begin{bmatrix}6.8\\0\end{bmatrix}$	$\begin{bmatrix}37.42\\-6.8\end{bmatrix}$	Saddle
	$\begin{bmatrix}0\\10.9\end{bmatrix}$	$\begin{bmatrix}-35.71\\-10.9\end{bmatrix}$	Stable node
	$\begin{bmatrix}-2.20296\\2.30845\end{bmatrix}$	$\begin{bmatrix}-9.1323\\9.0268\end{bmatrix}$	Saddle

Table 5.1: Model LVsP, scenario 1: Equilibrium points of field $\mathbf{f}^L$ with parameters b_1=6.8, a_{11}=1, d_{21} =1.6, b_2 =10.9, a_{22} =1, c_{12} =5.5

$\mathbf{f}^U$	$\mathbf{x}$	$\mathbf{Eigs}(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}10.9\\6.8\end{bmatrix}$	Unstable node
	$\begin{bmatrix}0\\10.9\end{bmatrix}$	$\begin{bmatrix}-10.9\\-4.1\end{bmatrix}$	Stable node
	$\begin{bmatrix}6.8\\0\end{bmatrix}$	$\begin{bmatrix}17.7\\-6.8\end{bmatrix}$	Saddle
	$\begin{bmatrix}-2.05\\8.85\end{bmatrix}$	$\begin{bmatrix}-10.317\\3.517\end{bmatrix}$	Saddle

Table 5.2: Model LVsP, scenario 1: Equilibrium points of field $\mathbf{f}^U$ with parameters b_1=6.8, a_{11} =1, c_{21} =3.6, b_2 =10.9, a_{22} =1, d_{12} =4.6

Dynamics of a trajectory on the equal number of clients line Σ

- Crossing and sliding sets are determined using the algorithm 4.3 for their calculation. The projections for determining the sets are $\langle \nabla H(\mathbf{w}), \mathbf{f}^U(\mathbf{w}) \rangle = 2.0\, w^2 + 4.1\, w$, that has the zeros $w_1 = 0.0$ and $w_2 = -2.05$, and $\langle \nabla H(\mathbf{w}), \mathbf{f}^L(\mathbf{w}) \rangle = 7.8\, w^2 + 4.1\, w$, that has the zeros $w_1 = 0.0$ and $w_2 = -0.5256$. Figure 5.2 shows the the plots of the signs of the projections and their products to help visualize the procedure. Note that we are interested only in nonnegative w values, then the crossing set is $\Sigma_c = (0, \infty)$, and all the trajectories that reach Σ cross into U.
- There is no pseudoequilibrium since candidate $w_2^* = 8.85$ belongs to the crossing set Σ_c.

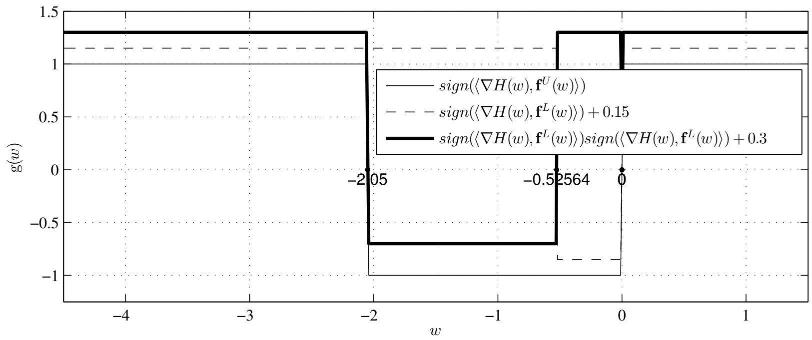


Figure 5.2: Model LVsP, scenario 1: Signs of the inner products used in the calculation of conditions for crossing and sliding.

The second scenario of model (5.6)-(5.9) has the parameters $b_1 = 6.8$, $a_{11} = 1$, $c_{21} = 4.4$, $d_{21} = 3$, $b_2 = 10.9$, $a_{22} = 1$, $c_{12} = 5.5$, $d_{12} = 4.6$. In this case firm F_1 does more effort than in the previous scenario, but it is not enough for beating the number of clients of firm F_2. Firm F_2 ends with bigger number of clients but now firm F_1 is not extincted from the market. The corresponding phase plane is shown in Figure 5.3. The equilibrium points of the fields are shown in Tables 5.3 and 5.4, note that both fields have stable equilibrium points in region U. In the case of field $\mathbf{f}^L$ this equilibrium is virtual but still attracts trajectories beginning in L to U.

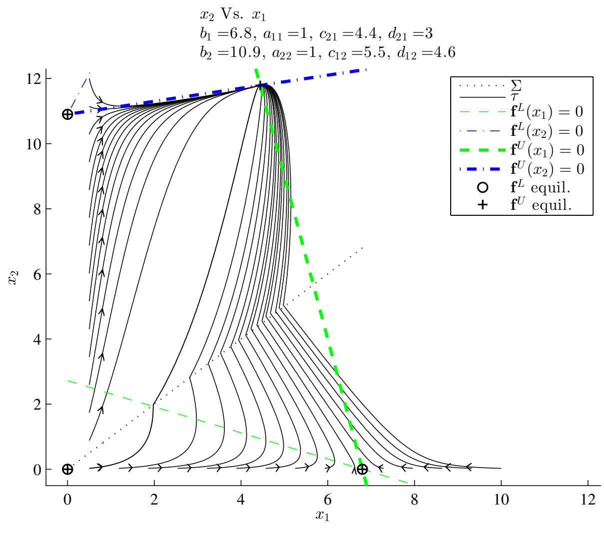


Figure 5.3: Model LVsP, scenario 2: Firm F_1 increases its effort. In this scenario both firms coexist in the long term, firm F_2 still ends with a bigger number of clients.

- Crossing and sliding sets are determined using the algorithm 4.3 for their calculation. The projections for determining the sets are $\langle \nabla H(\mathbf{w}), \mathbf{f}^U(\mathbf{w}) \rangle = 0.4\, w^2 + 4.1\, w$, that has the zeros $w_1 = -10.25$ and $w_2 = 0.0$, and $\langle \nabla H(\mathbf{w}), \mathbf{f}^L(\mathbf{w}) \rangle = 5.0\, w^2 + 4.1\, w$, that has the zeros $w_1 = -0.82$ and $w_2 = 0.0$. Figure 5.4 shows the the plots of the signs of the projections and their products. Note that we are

$\mathbf{f}^L$	$\mathbf{x}$	$\mathbf{Eigs}(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}10.9\\6.8\end{bmatrix}$	Unstable node
	$\begin{bmatrix}6.8\\0\end{bmatrix}$	$\begin{bmatrix}27.9\\-6.8\end{bmatrix}$	Saddle
	$\begin{bmatrix}0\\10.9\end{bmatrix}$	$\begin{bmatrix}-20.45\\-10.9\end{bmatrix}$	Stable node
	$\begin{bmatrix}-2.82069\\3.84828\end{bmatrix}$	$\begin{bmatrix}-9.3998\\8.3722\end{bmatrix}$	Saddle

Table 5.3: Model LVsP, scenario 2: Equilibrium points of field $\mathbf{f}^L$ with parameters b_1=6.8, a_{11}=1, d_{21} =3, b_2 =10.9, a_{22} =1, c_{12} =5.5

$\mathbf{f}^U$	$\mathbf{x}$	$\mathbf{Eigs}(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}10.9\\6.8\end{bmatrix}$	Unstable node
	$\begin{bmatrix}0\\10.9\end{bmatrix}$	$\begin{bmatrix}-10.9\\4.62\end{bmatrix}$	Saddle
	$\begin{bmatrix}6.8\\0\end{bmatrix}$	$\begin{bmatrix}12.26\\-6.8\end{bmatrix}$	Saddle
	$\begin{bmatrix}4.44231\\11.7885\end{bmatrix}$	$\begin{bmatrix}-11.4913\\-4.7395\end{bmatrix}$	Stable node

Table 5.4: Model LVsP, scenario 2: Equilibrium points of field $\mathbf{f}^U$ with parameters b_1=6.8, a_{11} =1, c_{21} =4.4, b_2 =10.9, a_{22} =1, d_{12} =4.6

interested only in nonnegative w values, then the crossing set is $\Sigma_c = (0, \infty)$, and all the trajectories that reach Σ cross into U.

- There is no pseudoequilibrium since candidate $w_2^* = 8.85$ belongs to the crossing set Σ_c.

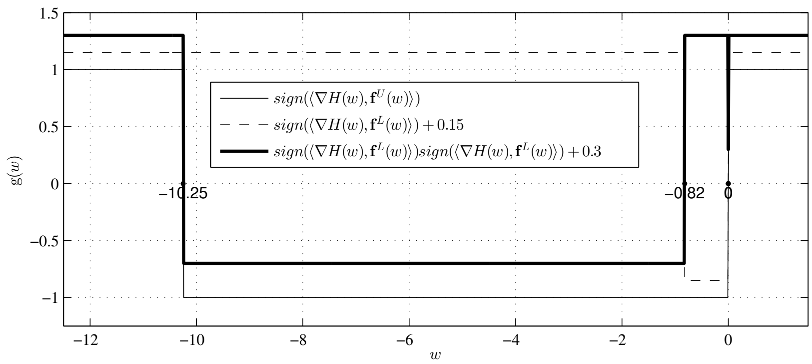


Figure 5.4: Model LVsP, scenario 2: Signs of the inner products used in the calculation of conditions for crossing and sliding.

The third scenario of model (5.6)-(5.9) has the parameters $b_1 = 6.8$, $a_{11} = 1$, $c_{21} = 4.9$, $d_{21} = 4.5$, $b_2 = 10.9$, $a_{22} = 1$, $c_{12} = 5.5$, $d_{12} = 4.6$. In this scenario, firm F_1 again increases its efforts and now manages to end with the same number of clients of firm F_2. Some trajectories starting in region L crosses into region U but after some time return to the equal number of clients line Σ to end at a pseudoequilibrium point. The corresponding phase plane is shown in Figures 5.5 and 5.5. The only difference between these phase planes is that the step size in Figure 5.5 is smaller and this results in the reduction of the chattering observed in the sliding trajectories. Figures 5.7 and 5.8 that shows the time plot $x_i(t)$ of a trajectory that has crossing and sliding behaviors, again the trajectories are the same what is different between this figures is the setpsize, which is smaller in the figure without noticeable chattering. The equilibrium points of the fields are shown in

Tables 5.5 and 5.6, note that in this case both fields have stable virtual equilibrium points: the virtual equilibrium of $\mathbf{f}^L$ is $\mathbf{x} = [0 \;\; 10.9]^T$ and the virtual equilibrium of field $\mathbf{f}^U$ is $\mathbf{x} = [9.23853 \;\; 8.12844]^T$. The trajectories beginning in region L(resp. U) are attracted to region U (resp. L) by the virtual equilibrium of field $\mathbf{f}^L$ (resp. $\mathbf{f}^U$).

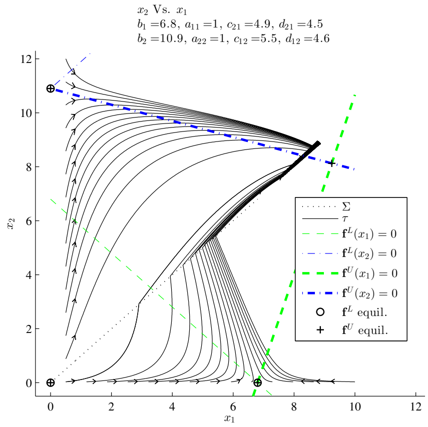


Figure 5.5: Model LVsP, scenario 3: Firm F_1 increases again its effort and this introduces a pseudoequilibrium. Firms F_1 and F_2 coexist with an equal number of clients. The phase plane shows trajectories crossing and then returning to Σ

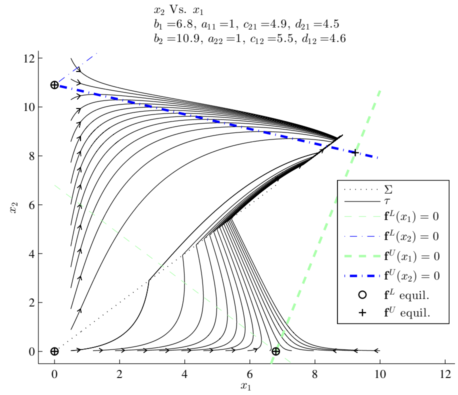


Figure 5.6: Model LVsP, scenario 3: Same simulation as in Figure 5.5, but here the $stepsize = 0.0002$ is smaller and this reduces most of the chattering on the equal number of clients line Σ.

$\mathbf{f}^L$	$\mathbf{x}$	$\mathbf{Eigs}(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}10.9\\6.8\end{bmatrix}$	Unstable node
	$\begin{bmatrix}6.8\\0\end{bmatrix}$	$\begin{bmatrix}17.7\\-6.8\end{bmatrix}$	Saddle
	$\begin{bmatrix}0\\10.9\end{bmatrix}$	$\begin{bmatrix}-10.9\\-4.1\end{bmatrix}$	Stable node
	$\begin{bmatrix}-2.05\\8.85\end{bmatrix}$	$\begin{bmatrix}-10.317\\3.517\end{bmatrix}$	Saddle

Table 5.5: Model LVsP, scenario 3: Equilibrium points of field $\mathbf{f}^L$ with parameters b_1=6.8, a_{11}=1, d_{21} =4.5, b_2 =10.9, a_{22} =1, c_{12} =5.5

$\mathbf{f}^U$	$\mathbf{x}$	$\mathbf{Eigs}(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}10.9\\6.8\end{bmatrix}$	Unstable node
	$\begin{bmatrix}0\\10.9\end{bmatrix}$	$\begin{bmatrix}-10.9\\10.07\end{bmatrix}$	Saddle
	$\begin{bmatrix}6.8\\0\end{bmatrix}$	$\begin{bmatrix}8.86\\-6.8\end{bmatrix}$	Saddle
	$\begin{bmatrix}9.23853\\8.12844\end{bmatrix}$	$\begin{bmatrix}-8.6835-2.5398i\\-8.6835+2.5398i\end{bmatrix}$	Stable focus

Table 5.6: Model LVsP, scenario 3: Equilibrium points of field $\mathbf{f}^U$ with parameters b_1=6.8, a_{11} =1, c_{21} =4.9, b_2 =10.9, a_{22} =1, d_{12} =4.6

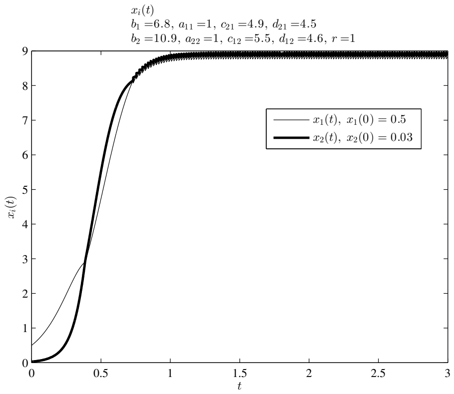


Figure 5.7: Model LVsP, scenario 3: State variable as function of time for a trajectory having crossing and sliding behavior. When the $stepsize = 0.001$ the trajectories have noticeable chattering when it slides.

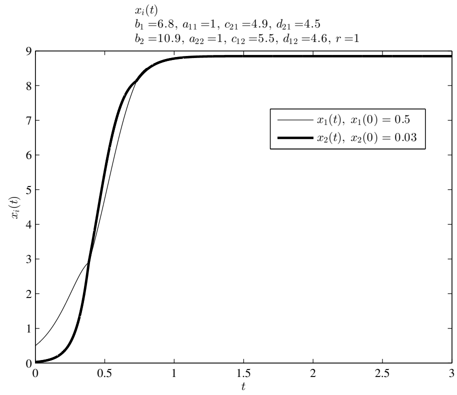


Figure 5.8: Model LVsP, scenario 3: Same simulation as in Figure 5.7, but here the $stepsize = 0.0002$ is smaller and this reduces most of the chattering when the trajectory slides.

- Crossing and sliding sets are determined using the algorithm 4.3 for their calculation. The projections for determining the sets are $\langle \nabla H(\mathbf{w}), \mathbf{f}^U(\mathbf{w}) \rangle = 4.1\,w - 0.6\,w^2$, that has the zeros $w_1 = 0.0$ and $w_2 = 6.83$, and $\langle \nabla H(\mathbf{w}), \mathbf{f}^L(\mathbf{w}) \rangle = 2.0\,w^2 + 4.1\,w$, that has the zeros $w_2 = 0.0$ and $w_2 = -2.05$. Figure 5.9 shows the the plots of the signs of the projections and their products. Note that we are interested only in nonnegative w values, then the crossing set is Σ_c interval $(0, 6.8333)$ and the sliding set is $\Sigma_s = [6.8333, \infty)$. Trajectories that reach Σ_c cross into U and trajectories that reach Σ_s slide until a pseudoequilibrium point.

- There is a pseudoequilibrium point $w_2^* = 8.85$ that belongs to the sliding set Σ_s.

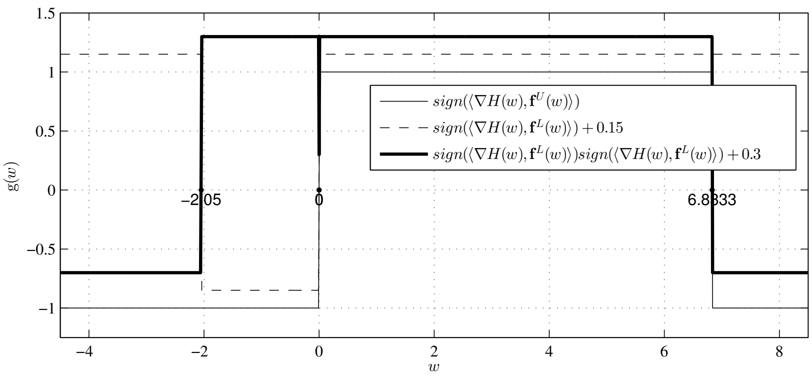


Figure 5.9: Model LVsP, scenario 3: Signs of the inner products used in the calculation of conditions for crossing and sliding.

The fourth scenario of model (5.6)-(5.9) has the parameters $b_1 = 6.8$, $a_{11} = 1$, $c_{21} = 5.5$, $d_{21} = 3$, $b_2 = 10.9$, $a_{22} = 1$, $c_{12} = 5.5$, $d_{12} = 4.6$. The corresponding phase plane is shown in Figure 5.10 and 5.11. The only difference between these phase planes is that the step size in Figure 5.11 is smaller and this results in the reduction of the chattering observed in the sliding trajectories. When compared to the third scenario, firm F_1 increases its effort $c_{21} = 5 > 4.9$ but decreases $d_{21} = 3 < 4.5$, with these efforts firm F_1 manages to end with the same number of clients of firm F_2. But now the observed trajectories only slides on Σ until they reach a pseudoequilibrium point. The equilibrium points of the fields are shown in Tables 5.7 and 5.8, note that in this case both fields have stable virtual equilibrium points: the virtual equilibrium of $\mathbf{f}^L$ is $\mathbf{x} = [0\ \ 10.9]^T$ and the virtual equilibrium of field $\mathbf{f}^U$ is $\mathbf{x} = [9.1768\ \ 2.64088]^T$. So that the trajectories beginning in region L(resp. U) are attracted to region U (resp. L).

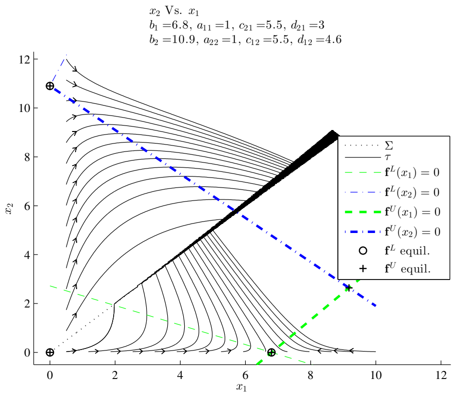


Figure 5.10: Model LVsP, scenario 4: Firm F_1 increases c_{21} and diminishes d_{21}. In the long term, firms F_1 and F_2 coexist with an equal number of clients. The phase plane shows that trajectories slides to a pseudoequilibrium.

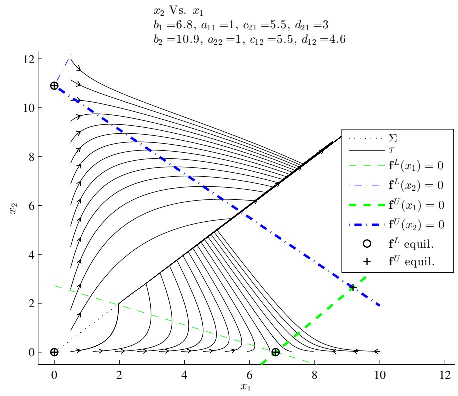


Figure 5.11: Model LVsP, scenario 4: Same simulation as in Figure 5.10, but here the *stepsize* = 0.0002 is smaller and this reduces most of the chattering on the equal number of clients line Σ.

$\mathbf{f}^L$	$\mathbf{x}$	$\mathbf{Eigs}(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}10.9\\6.8\end{bmatrix}$	Unstable node
	$\begin{bmatrix}6.8\\0\end{bmatrix}$	$\begin{bmatrix}27.9\\-6.8\end{bmatrix}$	Saddle
	$\begin{bmatrix}0\\10.9\end{bmatrix}$	$\begin{bmatrix}-20.45\\-10.9\end{bmatrix}$	Stable node
	$\begin{bmatrix}-2.82069\\3.84828\end{bmatrix}$	$\begin{bmatrix}-9.3998\\8.3722\end{bmatrix}$	Saddle

Table 5.7: Model LVsP, scenario 4: Equilibrium points of field $\mathbf{f}^L$ with parameters b_1=6.8, a_{11}=1, d_{21} =3, b_2 =10.9, a_{22} =1, c_{12} =5.5

$\mathbf{f}^U$	$\mathbf{x}$	$\mathbf{Eigs}(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}10.9\\6.8\end{bmatrix}$	Unstable node
	$\begin{bmatrix}0\\10.9\end{bmatrix}$	$\begin{bmatrix}16.61\\-10.9\end{bmatrix}$	Saddle
	$\begin{bmatrix}6.8\\0\end{bmatrix}$	$\begin{bmatrix}-6.8\\4.78\end{bmatrix}$	Saddle
	$\begin{bmatrix}9.1768\\2.64088\end{bmatrix}$	$\begin{bmatrix}-5.9088-2.9918i\\-5.9088+2.9918i\end{bmatrix}$	Stable focus

Table 5.8: Model LVsP, scenario 4: Equilibrium points of field $\mathbf{f}^U$ with parameters b_1=6.8, a_{11} =1, c_{21} =5.5, b_2 =10.9, a_{22} =1, d_{12} =4.6

- Crossing and sliding sets are determined using the algorithm 4.3 for their calculation. The projections for determining the sets are $\langle\nabla H(\mathbf{w}), \mathbf{f}^U(\mathbf{w})\rangle = 4.1\,w - 1.8\,w^2$, that has the zeros $w_1 = 0.0$ and $w_2 = 2.278$, and $\langle\nabla H(\mathbf{w}), \mathbf{f}^L(\mathbf{w})\rangle =$

$5.0\,w^2 + 4.1\,w$, that has the zeros $w_1 = -0.82$ and $w_2 = 0.0$. Figure 5.12 shows the the plots of the signs of the projections and their products. Note that we are interested only in nonnegative w values, then the crossing set is $\Sigma_c = (0, 2.2778)$and the sliding set is $\Sigma_s = [2.2778, \infty)$. Trajectories that reach Σ_c cross into U and trajectories that reach Σ_s slide until a pseudoequilibrium point.

- There is a pseudoequilibrium point $w_2^* = 8.85$ that belongs to the sliding set Σ_s.

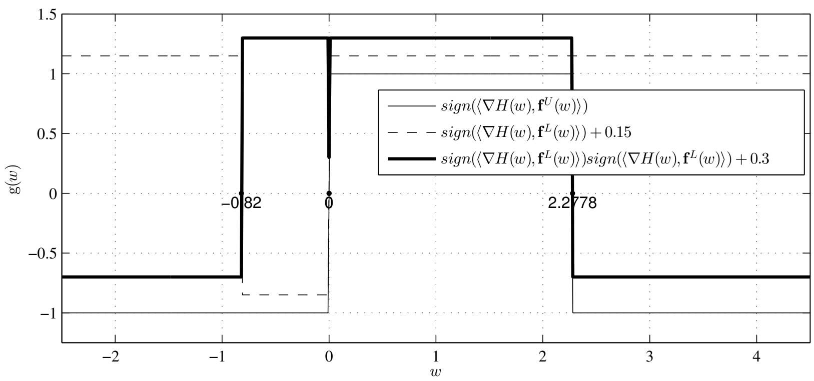


Figure 5.12: Model LVsP, scenario 4: Signs of the inner products used in the calculation of conditions for crossing and sliding.

Finally, the fifth scenario of model (5.6)-(5.9) has the parameters $b_1 = 6.8$, $a_{11} = 1$, $c_{21} = 7.1$, $d_{21} = 6.3$, $b_2 = 10.9$, $a_{22} = 1$, $c_{12} = 5.5$, $d_{12} = 4.6$. When compared to the fourth scenario, firm F_1 increases its efforts $c_{21} = 7.1 > 5$ and $d_{21} = 6.3 > 3$, with these efforts firm F_1 ends with a bigger number of clients than firm F_2. The corresponding phase plane is shown in Figure 5.13 where the observed trajectories cross into L. The equilibrium points of the fields are shown in Tables 5.9 and 5.10, note that in this case both fields have stable equilibrium points in region L. In the case of field $\mathbf{f}^U$ it is a stable virtual equilibrium and the trajectories beginning in region U are attracted to region L.

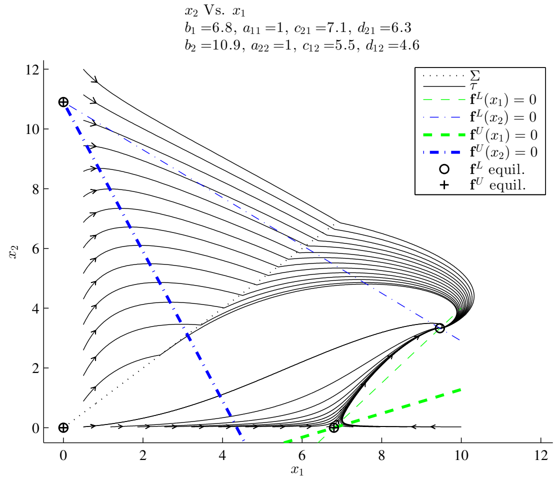


Figure 5.13: Model LVsP, scenario 5: Firm F_1 increases again its effort and this time firm F_1 is the one ending with a bigger number of clients.

$\mathbf{f}^L$	$\mathbf{x}$	**Eigs**$(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}10.9\\6.8\end{bmatrix}$	Unstable node
	$\begin{bmatrix}6.8\\0\end{bmatrix}$	$\begin{bmatrix}-6.8\\5.46\end{bmatrix}$	Saddle
	$\begin{bmatrix}0\\10.9\end{bmatrix}$	$\begin{bmatrix}15.52\\-10.9\end{bmatrix}$	Saddle
	$\begin{bmatrix}9.46341\\3.32927\end{bmatrix}$	$\begin{bmatrix}-6.3963-3.2798i\\-6.3963+3.2798i\end{bmatrix}$	Stable focus

Table 5.9: Model LVsP, scenario 5: Equilibrium points of field $\mathbf{f}^L$ with parameters b_1=6.8, a_{11}=1, d_{21} =6.3, b_2 =10.9, a_{22} =1, c_{12} =5.5

$\mathbf{f}^U$	$\mathbf{x}$	**Eigs**$(\mathbf{J}(\mathbf{x}))$	Type
	$\begin{bmatrix}0\\0\end{bmatrix}$	$\begin{bmatrix}10.9\\6.8\end{bmatrix}$	Unstable node
	$\begin{bmatrix}0\\10.9\end{bmatrix}$	$\begin{bmatrix}34.05\\-10.9\end{bmatrix}$	Saddle
	$\begin{bmatrix}6.8\\0\end{bmatrix}$	$\begin{bmatrix}-6.8\\-6.1\end{bmatrix}$	Stable node
	$\begin{bmatrix}4.69655\\-0.841379\end{bmatrix}$	$\begin{bmatrix}-7.6166\\3.7614\end{bmatrix}$	Saddle

Table 5.10: Model LVsP, scenario 5: Equilibrium points of field $\mathbf{f}^U$ with parameters b_1=6.8, a_{11} =1, c_{21} =7.1, b_2 =10.9, a_{22} =1, d_{12} =4.6

- Crossing and sliding sets are determined using the algorithm 4.3 for their calculation. The projections for determining the sets are $\langle \nabla H(\mathbf{w}), \mathbf{f}^U(\mathbf{w})\rangle = 4.1\,w - 5.0\,w^2$, that has the zeros $w_1 = 0.0$ and $w_2 = 0.82$, and $\langle \nabla H(\mathbf{w}), \mathbf{f}^L(\mathbf{w})\rangle = 4.1\,w - 1.6\,w^2$, that has the zeros $w_1 = 0.0$ and $w_2 = 2.5625$. Figure 5.14 shows the the plots of the signs of the projections and their products. Note that we are interested only in nonnegative w values, then the crossing set is $\Sigma_c = (0, 0.82) \cup (2.5625, \infty)$ and the sliding set is $\Sigma_s = [0.82, 2.5625]$. Trajectories that reach Σ_c in the interval $(0, 0.82)$ cross into U and trajectories that reach Σ_c in the interval $(2.5625, \infty)$ cross into L.

- There is no pseudoequilibrium point because the $w_2^* = 8.85$ belongs to the crossing set Σ_c.

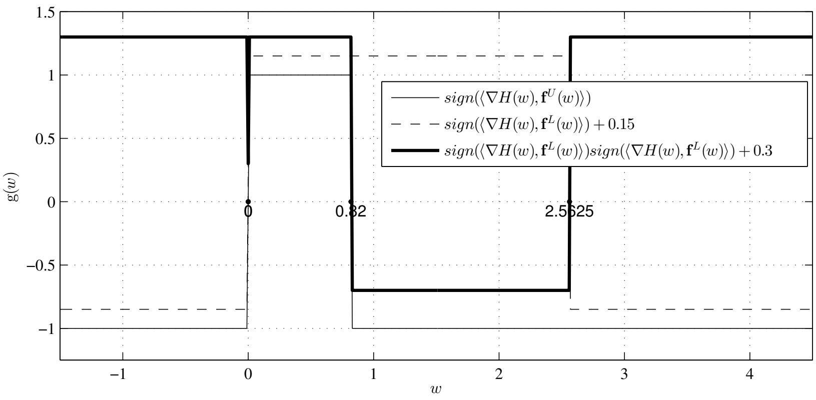


Figure 5.14: Model LVsP, scenario 5: Signs of the inner products used in the calculation of conditions for crossing and sliding.

5.5 Evaluating outcomes of the used policies using a performance index for model LVsP

We use the index (2.9). Note that a firm makes a profit when its index J_i is positive, whereas it makes a loss when its index is negative. The number of clients $x_i(t)$ evolution is given by the system (5.6), (5.7) and the controls $u_i(t)$ are given in (5.5).

The scenarios were simulated using the software Matlab. In the *first scenario*, the controls $c_{21} = 3.6$ and $d_{12} = 1.6$ actioned by firm F_1 are small when compared to the controls $c_{21} = 5.5$ and $d_{21} = 4.6$ of firm F_2. In this setting firm F_2 ends with more clients than firm F_1, actually firm F_1 ends without clients. Figure 5.15 shows the trajectories in the phase plane according to their index value at final time, these indexes were calculated using a revenue factor $r = 1$ and firm F_2 is the only firm that makes a profit. There is little variation of the index values as can be seen in Figures 5.17 and 5.18. The small increases of the index values are more noticeable in Figure 5.16 where the indexes are plotted against each other.

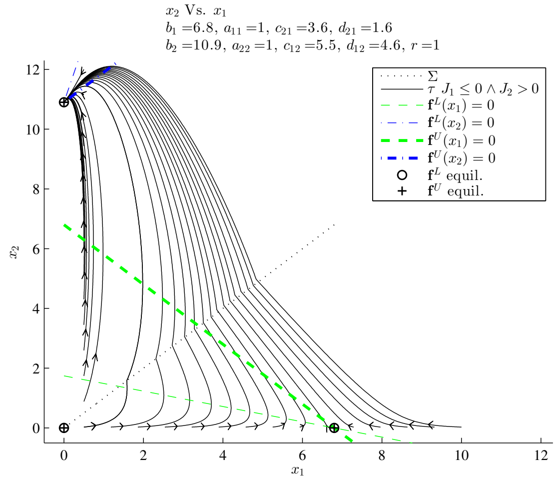


Figure 5.15: Model LVsP, scenario 1: Trajectories plotted according to their index values at final time. In this scenario firm F_2 has positive indexes while firm F_1 has negative indexes.

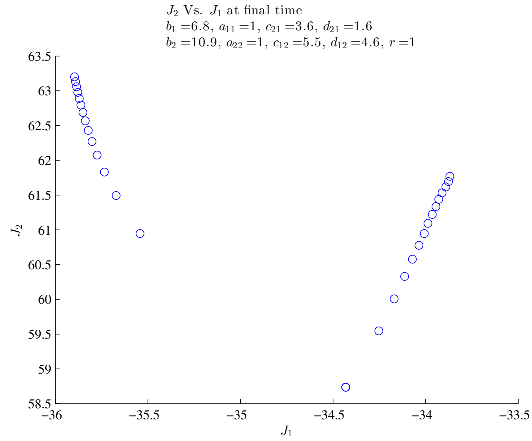


Figure 5.16: Model LVsP, scenario 1: $J_1(x_1(t), x_2(t), t)$ vs. $J_2(x_1(t), x_2(t), t)$

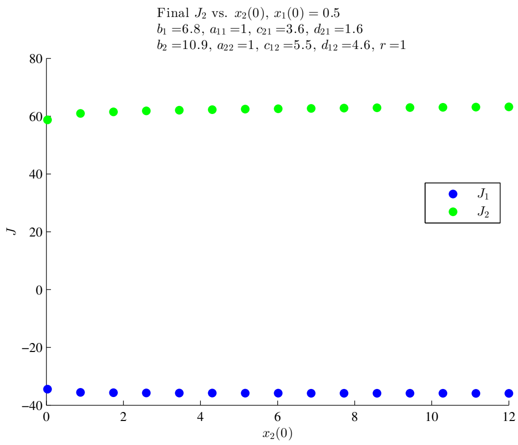


Figure 5.17: Model LVsP, scenario 1: Value of indices when the initial value $x_1(0) = 0.5$ is fixed.

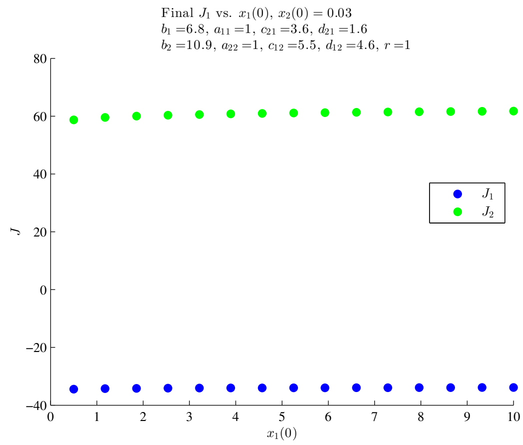


Figure 5.18: Model LVsP, scenario 1: Value of indices when the initial value $x_2(0) = 0.03$ is fixed.

In the next scenario, the controls $c_{21} = 5.5$ and $d_{12} = 3$ actioned by firm F_1 are similar in magnitude when compared to the controls $c_{21} = 5.5$ and $d_{21} = 4.6$ of firm F_2. Now, both firms end with the same number of clients. Figure 5.19 shows the trajectories according to their indexes at final time, these indexes were calculated using a revenue factor $r = 1$ and both firms make a profit. The firm that begins with a smaller number of clients makes a smaller profit: (a) In Figure 5.21 the initial conditions are $x_1(0) = 0.5$ and $x_2(0) \in [0.03, 12]$, the index J_2 grows as $x_2(0)$ grows and it is bigger than J_1 when $x_2(0) > x_1(0)$. (b) In Figure 5.22 the initial conditions are $x_2(0) = 0.03$ and $x_1(0) \in [0, 10]$, the index J_1 increases as $x_1(0)$ increases and it is bigger than J_2 when $x_1(0) > x_2(0)$.

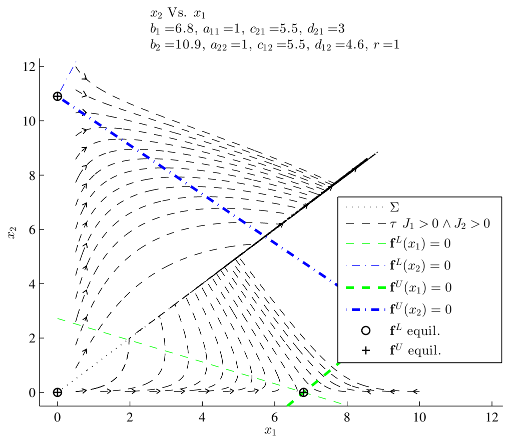


Figure 5.19: Model LVsP, scenario 4: Trajectories plotted according to their index values at final time. In this scenario both firms F_1 and F_2 have positive indexes.

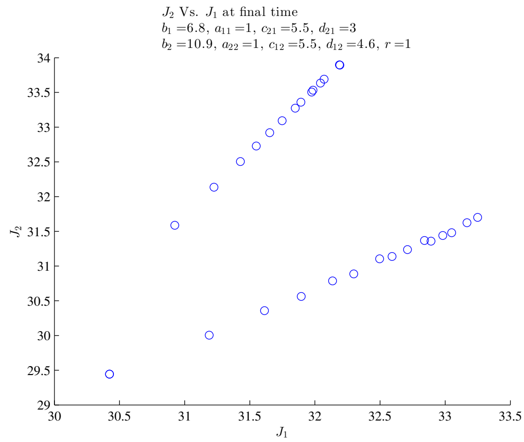


Figure 5.20: Model LVsP, scenario 4: $J_1(x_1(t), x_2(t), t)$ vs. $J_2(x_1(t), x_2(t), t)$

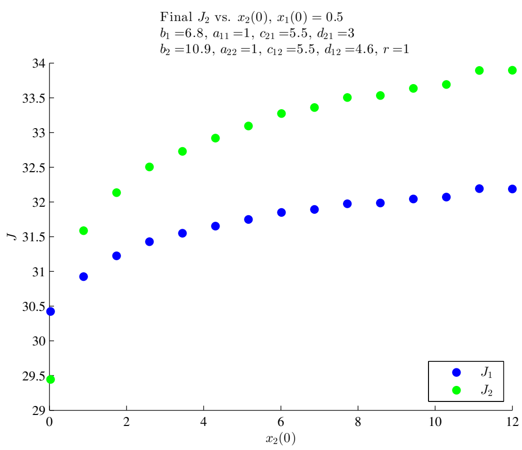


Figure 5.21: Model LVsP, scenario 4: Value of indices when the initial value $x_1(0) = 0.5$ is fixed.

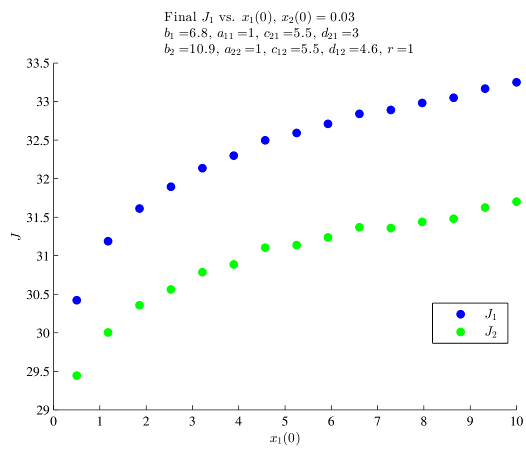


Figure 5.22: Model LVsP, scenario 4: Value of indices when the initial value $x_2(0) = 0.03$ is fixed.

In the last scenario, the controls $c_{21} = 7.1$ and $d_{12} = 6.3$ actioned by firm F_1 are bigger in magnitude when compared to the controls $c_{21} = 5.5$ and $d_{21} = 4.6$ of firm F_2. In this situation firms F_1 and F_2 coexist in the long term, with firm F_1 having more clients than firm F_2. Figure 5.23 shows the the trajectories in the phase plane according to their index value at final time, these indexes were calculated using a revenue factor $r = 1$. Firm F_1 is the only firm that makes profit, even though firms coexist in the long term. In Figure 5.25 the index J_2 increases as x_2 increases but the growth is small and the index remains negative. In Figure 5.26 the index J_1 increases as x_1 increases. When the index values are plotted against each other, i.e. J_2 vs. J_1, as in Figure 5.24 the indexes do not seem to create a front although their values have rather small variations (see Figures 5.25 and 5.26).

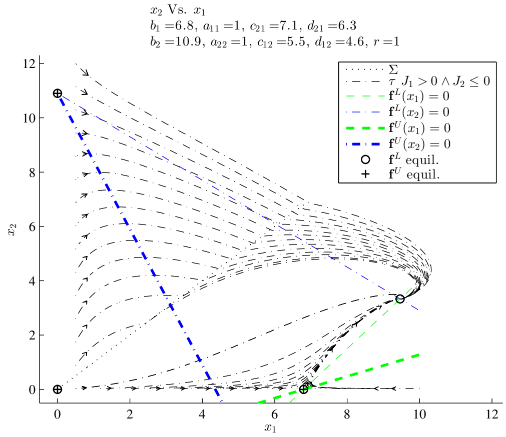


Figure 5.23: Model LVsP, scenario 5: Trajectories plotted according to their index values at final time. In this scenario firm F_1 has positive indexes while firm F_1 has negative indexes.

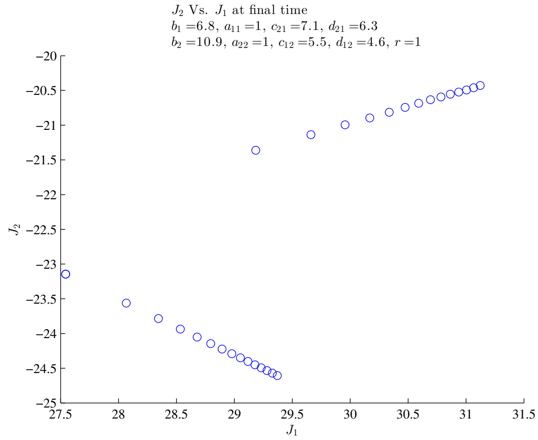


Figure 5.24: Model LVsP, scenario 5: $J_1(x_1(t), x_2(t), t)$ vs. $J_2(x_1(t), x_2(t), t)$

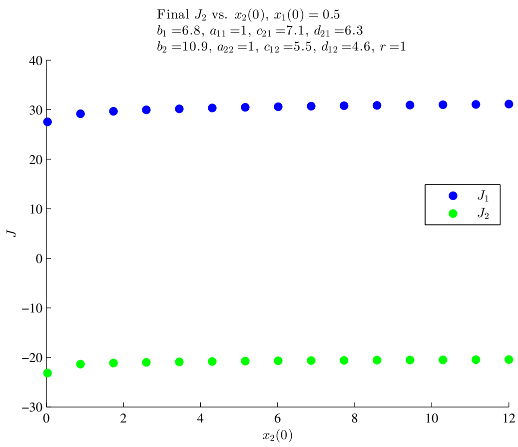


Figure 5.25: Model LVsP, scenario 5: Value of indices when the initial value $x_1(0) = 0.5$ is fixed.

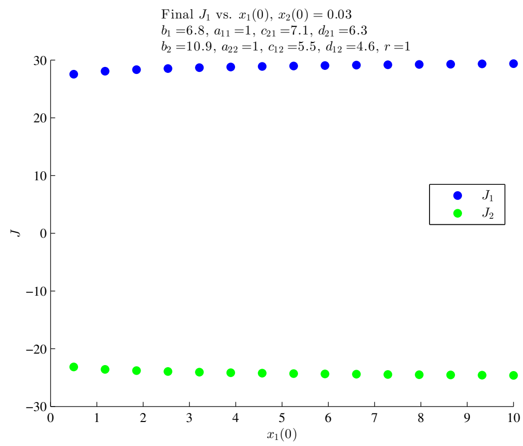


Figure 5.26: Model LVsP, scenario 5: Value of indices when the initial value $x_2(0) = 0.03$ is fixed.

5.6 Remarks on model LVsP

- The entering firm, i.e. the firm with a smaller initial market share, can introduce a pseudoequilibrium on the equal number of clients line Σ using an appropriate control value. Using a control value larger than this, the entering firm can achieve a larger number of clients than its competitor.

- The firm decides which advertising control efforts to apply using its performance index in a finite horizon, i.e. the firm evaluates the costs and benefits it obtains with different advertising efforts.

- If the state variable is market share, the Lokta-Volterra model does not result in invariance of total market share (always equal to one), ensuring instead that each market share variable is below or equal to a given upper bound. This fact led to the choice, in this chapter, of the number of clients as the state variable instead of market share.

Chapter 6

Conclusions and Future Work

This thesis focuses on the analysis of qualitative behavior of the dynamics of models, making use of simple policies, whereas the emphasis in the existing literature has been mainly on the optimization of quadratic (or related) performance indices, as well as on the differential game-theoretic aspects of such models.

The relatively simple choices of the switching behavior, variable structure control theory made possible a complete qualitative analysis, for any choice of initial condition, and for several different strategies and scenarios. This is to be contrasted with a specific trajectory produced by optimal control approaches. In fact, this possibility of complete qualitative analysis is sometimes referred to as a flight simulator mode [53] and should, in our opinion, be considered a strength of the proposed approach.
The contributions of this thesis are as follows:

- Proposal of the models VWD2s (Vidale-Wolfe-Deal with two switching lines) and LVsP (Lotka-Volterra with switched predation and conservation).
- The qualitative analysis of the proposed models, under switching control based on measurement of each firm's market share. The proposed switching control, unlike those resulting from optimal control/differential game approaches, is simple and thus implementable.
- Use of a performance index to identify trajectories that make a profit at saturated market in conjunction with the qualitative analysis of the market share dynamics to aid in the choice of advertising policies.

Some possible future works are listed:

- Developing a user friendly interface for the pieces of software elaborated for this thesis, in order to simulate the dynamics of a duopoly subject to switching policies.
- Discussing whether firms can switch between distinct models, because it has been assumed that both firms follow the same model. However, in line with the switching

behavior that the firms are allowed to have, this leaves open the analysis of what happens if the disadvantaged firm decides to change its strategy (e.g., switch to a different one) and a game theoretic analysis would be appropriate in this case.

- Using delays in the estimates of market shares (which may introduce chaos or limit cycles).
- Using a combined logistic function to enforce the saturated market restriction and relating it to a Lotka-Volterra model.
- Using the current value of the associated performance index in the switching decision.

Bibliography

[1] DEAL, K. R. "Optimizing advertising expenditures in a dynamic duopoly", *Operations Research*, v. 27, n. 4, pp. 682–692, 1979.

[2] SORGER, G. "Competitive Dynamic Advertising: A Modification of the Case Game", *Journal of Economic Dynamics and Control*, v. 13, pp. 55–80, 1989.

[3] CHINTAGUNTA, P. K., VILCASSIM, N. J. "An empirical investigation of advertising strategies in a dynamic duopoly", *Management Science*, v. 38, n. 9, pp. 1230–1244, 1992.

[4] ERICKSON, G. M. "Empirical Analysis of Closed-Loop Duopoly Advertising Strategies", *Management Science*, v. 38, n. 12, pp. 1732–1749, 1992.

[5] FRUCHTER, G. E., KALISH, S. "Closed loop advertising strategies in a Duopoly", *Management Science*, v. 43, n. 1, pp. 54 – 63, 1997.

[6] WANG, Q., WU, Z. "A duopolistic model of dynamic competitive advertising", *European Journal of Operational Research*, v. 128, pp. 213 – 226, 2001.

[7] JARRAR, R., MARTÍN-HERRÁN, G., ZACCOUR, G. "Markov Perfect Equilibrium Advertising Strategies of Lanchester Duopoly Model: A technical note", *Management Science*, v. 50, n. 7, pp. 995 – 1000, 2004.

[8] BASS, F. M., KRISHNAMOORTHY, A., PRASAD, A., et al. "Generic and Brand Advertising Strategies in a Dynamic Duopoly", *Marketing Science*, v. 24, n. 4, pp. 556 – 568, 2004.

[9] WANG, Q., WU, Z. "An empirical study on the Lanchester model of combat for competitive advertising decisions", *European Journal of Operational Research*, v. 183, n. 2, pp. 871 – 881, 2007. ISSN: 0377-2217.

[10] KRISHNAMOORTHY, A., PRASAD, A., SETHI, S. P. "Optimal pricing and advertising in a durable-good duopoly", *European Journal of Operational Research*, v. 200, pp. 486–497, 2010.

[11] JORGENSEN, S., MARTÍN-HERRÁN, G., ZACCOUR, G. "The Leitmann-Schmitendorf advertising differential game", *Applied Mathematics and Computation*, v. 217, pp. 1110 – 1116, 2010.

[12] "Duopoly definition". `http://www.investopedia.com/terms/d/duopoly.asp`, 2015. Accessed: 2015-02-09.

[13] BESANKO, D., BRAEUTIGAM, R. R. *Microeconomics*. 5th ed. New Jersey, USA, Wiley, 2014. ISBN: 9781118572276.

[14] ERICKSON, G. *Dynamic Models of Advertising Competition*. 2nd ed. , Kluwer Academics Publishers, 2003. ISBN: 1402072678.

[15] VARIAN, H. R. *Intermediate Microeconomics. A Modern Approach*. 8th ed. , W. W. Norton & Company, 2010. ISBN: 9780393934243.

[16] FRIEDMAN, J. W. *Oligopoly Theory*. 1st ed. Cambridge UK, Cambridge University Press, 1983.

[17] MORIARTY, S., MITCHELL, N., WELLS, W. *Advertising & IMC. Principles and Practice*. 9th ed. New Jersey, USA, Prentice Hall, 2012.

[18] DORFMAN, R., STEINER, P. O. "Optimal advertising and optimal equality", *American Economic Review*, v. 44, pp. 826–836, 1954.

[19] NERLOVE, M., ARROW, K. J. "An operations research study of sales response to advertising", *Economica*, v. 29, pp. 129–142, 1962.

[20] JØRGENSEN, S., ZACCOUR, G. *Differential Games in Marketing*. 1st ed. New York, USA, Springer Science + Business Media, LLC, 2004. ISBN: 97814613347248.

[21] ESCOBIDO, M. G. O., HATANO, N. "Critical slowing down in a dynamic duopoly", *International Journal of Modern Physics: Conference Series*, v. 36, 2015.

[22] VIDALE, M. L., WOLFE, H. B. "An operations research study of sales response to advertising", *Operations Research*, v. 5, pp. 370–381, 1957.

[23] LANCHESTER, F. W. *Aircraft in warfare*. New York, Appleton, 1916.

[24] KIMBALL, G. E. "Some industrial applications of military operations research methods", *Operations Research*, v. 5, n. 2, pp. 201–204, 1957.

[25] LITTLE, J. D. C. "Aggregate advertising models: the state of the art", *Operations Research*, v. 27, pp. 629–667, 1979.

[26] P.SETHI, S. "Dynamic Optimal Control Models in Advertising: A Survey", *SIAM Review*, v. 19, n. 4, pp. 685–725, 1977.

[27] FEICHTINGER, G., HARTL, R. F., SETHI, S. P. "Dynamic Optimal Control Models in Advertising: Recent Developments", *Management Science*, v. 40, n. 2, pp. 195–226, 12 1994.

[28] ERICKSON, G. M. "Differential games of advertising competition", *European Journal of Operational Research*, v. 83, 1995.

[29] HUANG, J., LENG, M., LIANG, L. "Recent developments in dynamic advertising research", *European Journal of Operational Research*, v. 220, n. 3, pp. 591–609, 2012.

[30] JØRGENSEN, S., ZACCOUR, G. "A survey of game-theoretic models of cooperative advertising", *European Journal of Operational Research*, v. 237, pp. 1–14, 2014.

[31] AUST, G., BUSCHER, U. "Cooperative advertising models in supply chain management: A review", *European Journal of Operational Research*, v. 234, pp. 1–14, 2014.

[32] OZGA, S. A. "Imperfect markets through lack of knowledge", *The Quarterly Jornal of Economics*, v. 74, n. 1, pp. 29–53, 1960.

[33] BHAYA, A., KASZKUREWICZ, E. "A study of duopolistic dynamics with competitive advertising based on state-dependent switching behavior". In: IEEE (Ed.), *Proceedings of the* 51^{st} *IEEE Conference on Decision and Control (CDC)*, pp. 7081–7087, Maui, 12 2012.

[34] BHAYA, A., KASZKUREWICZ, E. "Duopolistic dynamics in markets with competitive advertising and churn". In: *Control Conference (ASCC), 2013 9th Asian*, pp. 1–6, June 2013.

[35] WANG, M., GOU, Q., WU, C., et al. "An aggregate advertising response model based on consumer population dynamics", *Int. J. Applied Management Science,*, v. x, pp. 1 – 17, 2012.

[36] HUNG, H.-C., TSAI, Y.-S., WU, M.-C. "A modified Lotka–Volterra model for competition forecasting in Taiwan's retail industry", *Computers & Industrial Engineering*, v. 77, pp. 70 – 79, 2014. ISSN: 0360-8352.

[37] CRUZ, B. *Control Lyapunov Function Design of Advertising in a Duopoly*. Rio de Janeiro, Master's dissertation. PEE-UFRJ, 2014.

[38] JORGENSEN, S. "A survey of some differential games in advertising", *Journal of Economic Dynamics and Control*, v. 4, pp. 341–369, 1982.

[39] FEICHTINGER, G., JORGENSEN, S. "Differential game model in management science", *European Journal of Operational Research*, v. 14, pp. 137–155, 1983.

[40] SETHI, S. P., THOMPSON, G. L. *Optimal Control Theory: Applications to Management Science and Economics*. 2nd ed. New York, USA, Springer-Verlag, 2000.

[41] BISCHI, G., NAIMZADA, A. "Global Analysis of a Dynamic Duopoly Game with Bounded Rationality". In: Filar, J., Gaitsgory, V., Mizukami, K. (Eds.), *Advances in Dynamic Games and Applications*, v. 5, *Annals of the International Society of Dynamic Games*, Birkhauser, pp. 361–385, Boston, USA, 2000. ISBN: 978-1-4612-7100-0.

[42] BISCHI, G., CHIARELLA, C., KOPEL, M., et al. *Nonlinear Oligopolies: Stability and Bifurcations*. New York, USA, Springer, 2010. ISBN: 9783642021060.

[43] ECKERSON, W. W. *Performance Dashboards. Measuring, Monitoring, and Managing your Business*. 2nd ed. Hoboken, New Jersey, USA, John Wiley & Sons, Inc, 2011.

[44] SAUTER, V. L. *Decision support systems for business intelligence*. 2nd ed. Hoboken, New Jersey, USA, John Wiley & Sons, Inc, 2010.

[45] DECARLO, R., ZAK, S., MATTHEWS, G. "Variable structure control of nonlinear multivariable systems: a tutorial", *Proceedings of the IEEE*, v. 76, n. 3, pp. 212–232, Mar 1988.

[46] FILIPPOV, A. *Differential Equations with Discontinuous Righthand Sides: Control Systems*. Dordrecht, Boston, London, Kluwer Academic Publishers, 1988. ISBN: 9789027726995.

[47] DEAL, K., SETHI, S., THOMPSON, G. "A bilinear-quadratic differential game in advertising". In: *Control Theory in Mathematical Economics*, Decker, Marcel, pp. 91–109, 1979.

[48] QI, J., WANG, D. W. “Optimal Control Strategies for an Advertising Competing Model”, *Systems Engineering - Theory and Practice*, v. 27, n. 1, pp. 39 – 44, 2007. ISSN: 1874-8651.

[49] JORDAN, D., SMITH, P. *Nonlinear Ordinary Diferential Equations: An Introduction for Scientists and Engineers*. 4th ed. Oxford UK, Oxford Press, 2007. ISBN: 0199208255.

[50] KHALIL, H. K. *Nonlinear Systems*. 3rd ed. , Prentice Hall, 2002. ISBN: 0130673897.

[51] MACEY, R. I., FOSTER, G. F. “Berkeley Madonna”. `http://www.berkeleymadonna.com`, 2010.

[52] MARASCO, A., PICUCCI, A., ROMANO, A. “Market share dynamics using Lotka–Volterra models”, *Technological Forecasting and Social Change*, v. 105, pp. 49 – 62, 2016. ISSN: 0040-1625.

[53] MORECROFT, J. *Strategic Modelling and Business Dynamics: A Feedback Systems Approach*. Chichester, John
& Sons, Incs, 2007. ISBN: 9780470012864.

[54] DERCOLE, F., GRAGNANI, A., RINALDI, S. “Bifurcation analysis of piecewise smooth ecological models”, *Theoretical Population Biology*, v. 72, n. 2, pp. 197 – 213, 2007. ISSN: 0040-5809.

[55] PIIROINEN, P. T., KUZNETSOV, Y. A. “An Event-driven Method to Simulate Filippov Systems with Accurate Computing of Sliding Motions”, *ACM Trans. Math. Softw.*, v. 34, n. 3, pp. 13:1–13:24, maio 2008. ISSN: 0098-3500.

[56] BURDEN, R. L., FAIRES, J. D. *Numerical Analysis*. 7th ed. USA, Brook Cole, 2000.

Appendices

Appendix A

Filippov solutions on the discontinuity boundary

Since the model used in this paper presents discontinuities in the vector fields on the equal share line, it is necessary to define the vector field on the discontinuity boundary, so that trajectories that reach it can be continued in an unambiguous manner. We will use the concept of a Filippov solution [46]. We recapitulate Filippov theory for the second order models (state vector $\mathbf{x} = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$) very briefly below, following the very clear presentation in [54]. The ODE with one discontinuity boundary Σ and vector fields $\mathbf{f}^L = [f_1^L(\mathbf{x}), f_2^L(\mathbf{x})]^T$ and $\mathbf{f}^U = [f_1^U(\mathbf{x}), f_2^U(\mathbf{x})]^T$ on either side of Σ is written as:

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{f}^L(\mathbf{x}), & \mathbf{x} \in L \\ \mathbf{f}^U(\mathbf{x}), & \mathbf{x} \in U \end{cases} \tag{A.1}$$

where the regions L, U are separated by Σ. Σ is defined by $H(\mathbf{x}) = 0$. L corresponds to the region where $H(\mathbf{x}) < 0$ and U to the region where $H(\mathbf{x}) > 0$.

According to Filippov [46], solutions of (A.1) can be constructed by concatenating standard solutions in L, U and sliding solutions on Σ, which are obtained using the Filippov convex rule. The *crossing set* $\Sigma_c \subset \Sigma$ is defined as the set of all points $\mathbf{x} \in \Sigma$, at which the two vectors $\mathbf{f}^L(\mathbf{x})$, $\mathbf{f}^U(\mathbf{x})$ have nonzero components of the same sign, transversal to Σ ($\langle \nabla H(\mathbf{x}), \mathbf{f}^U(\mathbf{x})\rangle\langle \nabla H(\mathbf{x}), \mathbf{f}^L(\mathbf{x})\rangle > 0$). By definition, at such a point $\mathbf{x}$, a trajectory of (A.1) reaching x from L (resp. U) crosses Σ and is concatenated with a trajectory entering U (resp. L) from $\mathbf{x}$. The *sliding set* Σ_s is defined as the complement to Σ_c in Σ. The sliding set may contain singular sliding points, i.e. points with zero as component on the normal to Σ. Either both vectors $\mathbf{f}^L(\mathbf{x})$ and $\mathbf{f}^U(\mathbf{x})$ are tangent to Σ, or one of them vanishes while the other is tangent to Σ, or both vanish at a sliding singular point.

The Filippov rule associates the following convex combination $\mathbf{\Phi}(\mathbf{x})$ of the two vectors $\mathbf{f}^L(\mathbf{x})$, $\mathbf{f}^U(\mathbf{x})$ to each nonsingular sliding point $\mathbf{x} \in \Sigma_s$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^2$.

$$\mathbf{\Phi}(\mathbf{x}) = \lambda \mathbf{f}^L(\mathbf{x}) + (1-\lambda)\mathbf{f}^U(\mathbf{x}) \tag{A.2}$$

$$\lambda = \frac{\langle \nabla H(\mathbf{x}), \mathbf{f}^U(\mathbf{x})\rangle}{\langle \nabla H(\mathbf{x}), \mathbf{f}^U(\mathbf{x}) - \mathbf{f}^L(\mathbf{x})\rangle} \tag{A.3}$$

where λ was obtained using the condition that $\mathbf{\Phi}(\mathbf{x})$ lies on the tangent line to Σ at $\mathbf{x}$.

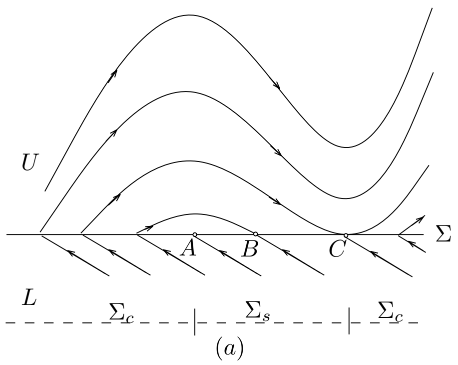
The equation

$$\dot{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x}), \quad \mathbf{x} \in \Sigma_s \tag{A.4}$$

is a scalar differential equation on Σ_s, which is smooth on one-dimensional sliding subintervals of Σ_s. Solutions of this equation are called *sliding solutions*.

Equilibria of (A.4) for which the vectors $\mathbf{f}^L(\mathbf{x})$, $\mathbf{f}^U(\mathbf{x})$ are anticollinear ($\angle(\mathbf{f}^U, \mathbf{f}^L) = 180°$) and transversal to Σ_s in opposite directions ($\langle \nabla H(\mathbf{x}), \mathbf{f}^U(\mathbf{x})\rangle\langle \nabla H(\mathbf{x}), \mathbf{f}^L(\mathbf{x})\rangle < 0$), are called *pseudo-equilibria* of (A.1) and also referred to as quasi-equilibria by Filippov [46]. Thus pseudo-equilibria lie in the interior of sliding segments. A non-singular sliding point $\mathbf{x} \in \Sigma_s$ for which $\mathbf{f}^L = 0, \lambda = 1$ or $\mathbf{f}^U = 0, \lambda = 0$ implies $\mathbf{\Phi} = \mathbf{0}$ and is called a *boundary equilibrium point*. A tangent point T is a point at which the vectors $\mathbf{f}^L, \mathbf{f}^U$ are non-zero, but one of them is tangent to Σ. A sliding segment terminates either at a boundary equilibrium, or at a tangent point.

Phase portraits of Filippov systems can have multiple sliding segments and be rather complex, Figure A.1 (a) is a rather simple phase portrait to exemplify previous definitions. In this example Σ is a line. The depicted area for the given fields has a sliding subsegment Σ_s, and two crossing subsegments Σ_c. Points A and C are boundary equilibrium points. Figure A.1 (b) shows the corresponding fields f^U, f^L at V, W, B, Z. Here V, Z are crossing points, W, B are sliding points. Notice that B is a pseudoequilibrium point.



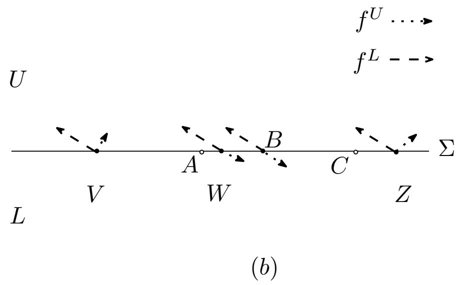


Figure A.1: (a) Sample phase portrait showing segments Σ_s and Σ_c. (b) V, Z crossing points, W, B sliding points. B is a pseudoequilibrium point, i.e. $\angle(f^U, f^L) = 180°$

The conditions for deciding if a trajectory will cross or slide at point $\mathbf{x} \in \Sigma$ or to determine if $\mathbf{x}$ is a pseudo-equilibrium are:

- Crossing ($\mathbf{x} \in \Sigma_c$):

$$\langle \mathbf{f}^L(\mathbf{x}), \nabla H(\mathbf{x}) \rangle \langle \mathbf{f}^U(\mathbf{x}), \nabla H(\mathbf{x}) \rangle > 0 \tag{A.5}$$

- Sliding ($\mathbf{x} \in \Sigma_s$):

$$\langle \mathbf{f}^L(\mathbf{x}), \nabla H(\mathbf{x}) \rangle \langle \mathbf{f}^U(\mathbf{x}), \nabla H(\mathbf{x}) \rangle <= 0 \tag{A.6}$$

- Pseudo-equilibrium : ($\mathbf{x} \in \Sigma_s$) and $\exists\, w \in \mathbb{R}^+$ such that

$$\mathbf{f}^L(\mathbf{x}) = -w\mathbf{f}^U(\mathbf{x}) \tag{A.7}$$

The anticollinearity between $\mathbf{f}^U(\mathbf{x})$ and $\mathbf{f}^L(\mathbf{x})$ is expressed by (A.7). Condition (A.7) is simpler to apply, resulting in equations of lower degree in the components of $\mathbf{x}$, than the equivalent condition $\frac{\langle \mathbf{f}^L(\mathbf{x}), \mathbf{f}^U(\mathbf{x}) \rangle}{(\|\mathbf{f}^L(\mathbf{x})\| \|\mathbf{f}^U(\mathbf{x})\|} = -1$.

Appendix B

Algorithm for obtaining separatrices of model VWDsC

We use the dynamics (3.3), (3.4) (resp. (3.5), (3.6)) to calculate φ^U (resp. φ^L) in the phase plane $x_1 - x_2$ from the corresponding expression for $\frac{dx_2}{dx_1}$. Furthermore, we use $\frac{dx_2}{dx_1}\Big|_E$ as an abbreviation for the limiting expression for $\frac{dx_2}{dx_1}$ as it approaches the saturated equal market share E from region U (resp. from region L).

Thus, in order to derive a condition that determines if a trajectory starting in U will reach the equal share E, we obtain $\frac{dx_2}{dx_1}$ from (3.3), (3.4):

$$\frac{dx_2}{dx_1} = \frac{k_2 x_2}{k_1 x_1 + c_1} \tag{B.1}$$

The solution of (B.1) using the initial conditions (x_{2_0}, x_{1_0}) is

$$x_2 = \frac{x_{2_0}}{(k_1 x_{1_0} + c_1)^{k_2/k_1}} (k_1 x_1 + c_1)^{\frac{k_2}{k_1}}. \tag{B.2}$$

Renaming x_{1_0} as X_1 and x_{2_0} as X_2 and setting $(x_1, x_2) = (0.5, 0.5)$ in (B.2) yields the locus of all initial conditions (X_1, X_2) that end at $(0.5, 0.5)$, i.e. the separatrix φ_U

$$X_2 = 0.5 \frac{(k_1 X_1 + c_1)^{k_2/k_1}}{(k_1 0.5 + c_1)^{\frac{k_2}{k_1}}} \tag{B.3}$$

Analogously, trajectories in L satisfy:

$$\frac{dx_2}{dx_1} = \frac{k_2 x_2 + c_2}{k_1 x_1} \tag{B.4}$$

The solution of (B.4) using the initial conditions (x_{2_0}, x_{1_0}) is

$$x_2 = \frac{k_2 x_{2_0} + c_2}{k_2 (x_{1_0})^{k_2/k_1}} x_1^{\frac{k_2}{k_1}} - \frac{c_2}{k_2} \tag{B.5}$$

and, as before setting $x_{1_0} = X_1$, $x_{2_0} = X_2$ and $(x_1, x_2) = (0.5, 0.5)$ yields the equation for the separatrix φ_L:

$$X_2 = \frac{1}{k_2}\left(\left(\frac{1}{2} + \frac{c_2}{k_2}\right) k_2 (2X_1)^{k_2/k_1} - c_2\right) \tag{B.6}$$

Remark: The *geometrical idea used to decide whether a separatrix exists or not* is shown in Figure B.1. The trajectory defining a separatrix φ_U (resp. φ_L) calculated from (B.3) (resp. (B.6)) must lie in its defining region U (resp. L), which means that its slope at E, calculated from (B.1) (resp. (B.4)) must be less than (resp. greater than) unity, so that it comes from region U and points into the region L (resp. comes from L and points into U) in the neighborhood of E.

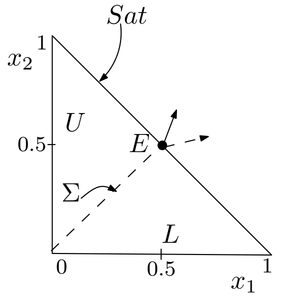


Figure B.1: The fields f^U or f^L approaching the equal market share point E from the left can be as shown. In the case of the dashed line, the slope is less than unity and the field indicates that the trajectory comes from U. In the case of the continuous line, the slope is greater than unity and the field indicates that the trajectory comes from L.

We describe the process of determining the existence of separatrices as the following algorithm:

Algorithm (Existence of separatrices)

1. Calculate the slope $s_{f^U} = \frac{dx_2}{dx_1}\big|_E$ using (B.1) (resp. the slope $s_{f^L} = \frac{dx_2}{dx_1}\big|_E$ using (B.4)).

2. If $s_{f^U} > 1$ (resp. $s_{f^L} < 1$) then the separatrix φ_U (resp. φ_L) does not exist. In this case, the curve (B.3) comes from L, i.e. it partially lies in L (resp. (B.6) comes from U, i.e. it partially lies in U) , but it was derived from the equations of f^U(resp. f^L).

3. For region U: If $s_{f^U} < 1$

   (a) Set $X_1 = 0$ in (B.3) and solve for X_2. This determines the point at which (B.3) intersects the x_2 axis.

   (b) If $X_2 \in [0, 1]$, then

Use $x_{1_0} = X_1$ and $x_{2_0} = X_2$ as the initial conditions in (B.2)($X_1,\ X_2$ from step 3a). The resulting equation defines the separatrix φ_U.

(c) If $X_2 \notin [0,1]$ or $\nexists x_2$, then the separatrix φ^U does not exist.

4. For region L: If $s_{fL} > 1$

(a) Set $X_2 = 0$ in (B.6) and solve for X_1. This determines the point at which (B.6) intersects the x_1 axis.

(b) If $X_1 \in [0,1]$, then

Use $x_{1_0} = X_1$ and $x_{2_0} = X_2$ as the initial conditions in (B.5)($X_1,\ X_2$ from step 4a). The resulting equation defines the separatrix φ^L.

(c) If $X_1 \notin [0,1]$ or $\nexists x_1$, then the separatrix φ^L does not exist.

5. Once φ_U and φ_L are determined, they are associated with each firm:

(a) If only one separatrix exists, then both firms are associated with it. If only φ_U (resp. φ_L) exists, then the separatrix for firm F_2 is φ_2^U (resp. φ_2^L) and the separatrix for firm F_1 is φ_1^U (resp. φ_1^L)

(b) If both separatrices exist, then the separatrix for firm F_2 is φ_2^L and the separatrix for firm F_1 is φ_1^U

Remark: All trajectories with initial conditions above (resp. below) separatrix of firm F_2 (resp. F_1) lead to firm F_2 (resp. F_1) attaining its goal of gaining at least 50% of the saturated market share and if both separatrices exist, then all trajectories originating from initial conditions in the region between them will converge to the saturated equal market share point E. Since a separatrix is just a trajectory in the phase plane that ends at the saturated market line Sat, it follows that trajectories below (resp. above) a separatrix always remain below it (resp. above it). This is because if the trajectories were to intersect at a point that is not an equilibrium there would be more than one value for the derivative and the uniqueness of the derivative would be violated.

Appendix C

Demonstrations of properties of model VWDsC

Theorem 3.2.1 (Characterization of market share dynamics for model (3.3)-(3.6) with controls (3.2)). *Suppose that τ is a trajectory that starts in region U, i.e. firm F_1 starts out with a smaller market share $x_1(0) < x_2(0)$ and that the parameters k_1, c_1, k_2, c_2 are known. then the trajectories reaching the equal market share line Σ can have one of the following behaviors:*

1. *If the sliding end point $s_{end} = 0.5$ then any trajectory τ with initial conditions in region U and below separatrix φ_1^U reaches the equal share line and slides until the saturated equal share point E on the saturated market line (see Figure 3.2a)*

2. *If the sliding end point $s_{end} < 0.5$ and $k_2 > k_1$, then the separatrix is $\varphi_2^L = \varphi_1^L$ and some trajectories beginning in region U reach the equal share line Σ, move on it until the point s_{end} and return to U finally ending on the saturated market Sat with firm F_2 having more than 50% of the market share.(see Figure 3.2b)*

3. *If the sliding end point $s_{end} < 0.5$ and $k_1 > k_2$, then the separatrix is $\varphi_2^U = \varphi_1^U$ and the trajectories beginning in region U and below φ_2^U reach the equal share line Σ,*

   (a) *If the trajectories reach the equal market share on the sliding segment Σ_s, they stay on the equal market share until the point s_{end} and then cross into region L finally ending on the saturated market Sat with firm F_1 having more than 50% of the market share. (see Figure 3.2c)*

   (b) *If the trajectories reach the equal market share line on the crossing segment Σ_c, they cross into region L finally ending on the saturated market Sat with firm F_1 having more than 50% of the market share.(see Figure 3.2d)*

Proof. 1. *If sliding end point $s_{end} = 0.5$ then any trajectory τ with initial conditions in region U and below separatrix φ_1^U reaches the equal share line and slides until* *the saturated equal share point E on the saturated market line.*

When $s_{end} = 0.5$, there are two separatrices. To see this, suppose there is only one separatrix, without loss of generality suppose φ^U does not exist φ^L exists. A s_{end}, the field f^U points from L into U since φ^U does not exist and the field f^L also points from L into U since φ^L exists (see the remark in Appendix B *Algorithm for obtaining separatrices*). Then, this implies that s_{end} is a crossing point which is a contradiction with s_{end} being a sliding point. Then, two separatrices exist and the separatrix corresponding to firm F_1 is φ_1^U. Next, determine X_2 from (B.3) by setting $X_1 = x_1(0)$, if $x_2(0) < X_2$ the trajectory $\tau_{x_{1(0)},x_{2(0)}}$ is below φ_1^U. Because trajectories in the phase plane can only cross at singular points, the trajectory $\tau_{x_{1(0)},x_{2(0)}}$ runs below φ_1^U and reaches Σ_s. By Proposition C.0.7 there is no pseudo-equilibrium point, so that $\tau_{x_{1(0)},x_{2(0)}}$ slides on Σ until the saturated equal market share point E which is an attractive point.

2. *If sliding end point $s_{end} < 0.5$ and $k_2 > k_1$, then the separatrix is $\varphi_2^L = \varphi_1^L$ and some trajectories beginning in region U reach the equal share line Σ, move on it until the point s_{end} and return to U finally ending on the saturated market Sat with firm F_2 having more than 50% of the market share.* When $s_{end} < 0.5$ and $k_2 > k_1$, the separatrix is $\varphi_2^L = \varphi_1^L$: Suppose that separatrix φ^U exists, this implies there is a trajectory τ that reaches Σ_c at a crossing point. But this is a contradiction with Proposition C.0.6, so that the only separatrix is $\varphi_2^L = \varphi_1^L$. Next, determine if $\tau_{x_{1(0)},x_{2(0)}}$, obtained ,intersects with Σ.If it intersects Σ it does it on a sliding point (intersection on a crossing point is impossible by Proposition C.0.6). After reaching Σ, $\tau_{x_{1(0)},x_{2(0)}}$ slides on it and at sliding end it returns to U (since crossing into L is impossible by Proposition C.0.5). By Proposition C.0.4 and Proposition C.0.6, once crossing occurs there is no reaching Σ anymore and $\tau_{x_{1(0)},x_{2(0)}}$ will end in Sat^U with Firm F_2 having more than 50% of the market.

3. *If sliding end point $s_{end} < 0.5$ and $k_1 > k_2$, then the separatrix is $\varphi_2^U = \varphi_1^U$ and the trajectories beginning in region U and below φ_2^U reach the equal share line Σ,* Suppose the separatrix is φ_L, then at the saturated market share f^L points from L into U. But this contradicts Proposition C.0.6 which when applied to the situation $k_1 > k_2$ states that there is no trajectory starting in L and reaching the crossing segment Σ_c. Then the separatrix has to be $\varphi_1^U = \varphi_2^U$.

   (a) *If the trajectories reach the equal market share on the sliding segment Σ_s, they stay on the equal market share until the point s_{end} and then cross into region L finally ending on the saturated market Sat with firm F_1 having more than 50% of the market share.*

   In this case the trajectory $\tau_{x_{1(0)},x_{2(0)}}$ reaches the sliding segment Σ_s and slides on it until s_{end} (by Proposition C.0.7 there is no pseudo-equilibrium point). When it reaches s_{end}, trajectory $\tau_{x_{1(0)},x_{2(0)}}$ crosses into L. To see this, suppose it returns to U, so that that the final saturated market would be in Sat^U above the saturated equal market share E which is the intersection of φ_1^U with Sat, then $\tau_{x_{1(0)},x_{2(0)}}$ would intersect φ_1^U at a point below Sat which is not singular point of the system which is not possible. Then, trajectory $\tau_{x_{1(0)},x_{2(0)}}$ has to end in Sat^L with Firm F_1 having more than 50% of the market.

(b) If the trajectories reach the equal market share line on the crossing segment Σ_c, they cross into region L finally ending on the saturated market Sat with firm F_1 having more than 50% of the market share.

In this case the trajectory $\tau_{x_{1(0)},x_{2(0)}}$ reaches the crossing segment Σ_c. At it, $\tau_{x_{1(0)},x_{2(0)}}$ could cross into L or return to U. Suppose that it returns to U, this leads to the contradiction of $\tau_{x_{1(0)},x_{2(0)}}$ intersecting with $\varphi_1^U = \varphi_2^U$ at a nonsingular point as in (a). Then, $\tau_{x_{1(0)},x_{2(0)}}$ has to cross into L and end in Sat^L with Firm F_1 having more than 50% of the market.

□

Proposition C.0.1 (Characterization of the sliding segment Σ_s). *The sliding segment Σ_s for the model (3.3)-(3.6) is one of the following:*

(i) If $k_2 > k_1$, Σ_s is the segment from 0 to $s_{end} = \min\{0.5, \frac{c_1}{k_2-k_1}\}$.

(ii) If $k_1 > k_2$, Σ_s is the segment from 0 to $s_{end} = \min\{0.5, \frac{c_2}{k_1-k_2}\}$.

(iii) If $k_2 = k_1$, Σ_s is the segment from 0 to $s_{end} = 0.5$.

Proof. We analyze sliding of the model (3.3)-(3.6) using the tangents to a trajectory evaluated in the neighborhood of $\mathbf{x} \in \Sigma$. This leads to simpler algebraic operations than the standard sliding condition (A.6) from Appendix A *Filippov solutions on the discontinuity boundary*. Evaluating that the sliding condition is satisfied using the tangents is equivalent to using the intersection of signs of the projections of the fields onto the normal to Σ, (which is another simplification to solving directly (A.6) and will be used in later chapters). Still, we consider that the use of tangents on the equal share line Σ is straightforward for model (3.3)-(3.6).

Since the coefficients k_1, k_2, c_1, c_2 as well as the variables x_1, x_2 are positive, the value of the tangent is positive. The tangents $\tan_{f^U}(x_1, x_2)$ and $\tan_{f^L}(x_1, x_2)$ where the vector field f^L points into U and simultaneously the vector field f^U points into L, thus satisfy

$$\tan_{f^U}(x_1, x_2) = \frac{k_2 x_2}{k_1 x_1 + c_1} < 1$$
$$\tan_{f^L}(x_1, x_2) = \frac{k_2 x_2 + c_2}{k_1 x_1} > 1$$

for a point $(x_1, x_2) = (a, a)$ on Σ this leads to

$$-c_2 < a(k_2 - k_1) < c_1 \tag{C.1}$$

First, considering the case $k_2 > k_1$ in (C.1) yields the sliding condition

$$\frac{-c_2}{k_2 - k_1} < a < \frac{c_1}{k_2 - k_1} \tag{C.2}$$

From (C.2) and since $a \in \Sigma$ and $a \in \mathcal{T}$, then a belongs to the segment from 0 to $s_{end} = \min\{0.5, \frac{c_1}{k_2-k_1}\}$ so that (i) is proved.
Considering the case $k_2 < k_1$ in (C.1) yields the sliding condition

$$\frac{-c_1}{k_1 - k_2} < a < \frac{c_2}{k_1 - k2} \tag{C.3}$$

From (C.3) and since $a \in \Sigma$ and $a \in \mathcal{T}$, then a belongs to the segment from 0 to $s_{end} = \min\{0.5, \frac{c_2}{k_1-k_2}\}$ so that (ii) is proved.
Third, considering $k_1 = k_2$ in (C.1) yields

$$-c_2 < c_1 \tag{C.4}$$

In this case, all $\mathbf{x} \in \Sigma$ that are not pseudo-equilibrium points are sliding points (see Appendix A *Filippov solutions on the discontinuity boundary*) because (C.4) is always satisfied. From (C.4) and since $a \in \Sigma$ and $a \in \mathcal{T}$, then a belongs to the segment from 0 to 0.5

□

Proposition C.0.2 (Configuration of vector fields $\mathbf{f}^L$, $\mathbf{f}^U$ on the segment Σ_s). *The configuration of the vector fields approaching the sliding segment Σ_s for the model (3.3)-(3.6) is f^U pointing into L and f^L pointing into U*

Proof. Proposition C.0.2 is true because Proposition C.0.1 was derived from that configuration of fields, which is shown in Figure C.1a and the other configuration where f^U is pointing into U and f^L is pointing into L is not possible (Figure C.1b) as shown below.

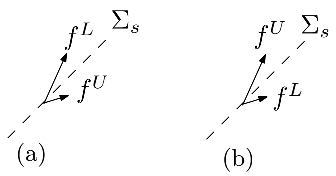


Figure C.1: Configurations of fields f^U and f^L for a point belonging to Σ_s: The configuration in (a) is the only possible configuration. The configuration of fields in (b) is impossible.

For this configuration the tangents $\tan_{f^U}(x_1, x_2)$ and $\tan_{f^L}(x_1, x_2)$ where the vector field f^L points into L and simultaneously the vector field f^U points into U, thus satisfy

$$\tan_{f^U}(x_1, x_2) = \frac{k_2 x_2}{k_1 x_1 + c_1} > 1$$
$$\tan_{f^L}(x_1, x_2) = \frac{k_2 x_2 + c_2}{k_1 x_1} < 1$$

for a point $(x_1, x_2) = (a, a)$ on Σ this yields

$$c_1 < a(k_2 - k_1) < -c_2 \tag{C.5}$$

from (C.5) and considering that the model parameters are positive real numbers a has to be smaller than a negative real number and bigger than a positive real number which is a contradiction, so this configuration of the fields is not possible. In the case $k_2 > k_1$, $a > \frac{c_1}{k_2-k_1}$ and $a < \frac{-c_2}{k_2-k_1}$. In the case $k_1 > k_2$, $a < \frac{-c_1}{k_1-k_2}$ and $a > \frac{c_2}{k_1-k_2}$. □

Proposition C.0.3 (Characterization of the crossing segment Σ_c). *The crossing segment Σ_c for the model (3.3)-(3.6) is one of the following:*

(i) If $k_2 > k_1$ and $\frac{c_1}{k_2-k_1} < 0.5$, Σ_c is the segment from $\frac{c_1}{k_2-k_1}$ to 0.5.

(ii) If $k_1 > k_2$ and $\frac{c_2}{k_1-k_2} < 0.5$, Σ_c is the segment from $\frac{c_2}{k_1-k_2}$ to 0.5.

(iii) If $c_2 = c_1 = 0$ or $k_2 > k_1$ and $c_1 = 0$ or $k_1 > k_2$ and $c_2 = 0$, Σ_c is the segment from 0 to 0.5.

Proof. Since by definition $\Sigma_c \cup \Sigma_s = \Sigma$ and a point $\mathbf{x}$ on Σ belongs either to Σ_s or Σ_c, we can compute $\Sigma_c = \Sigma - \Sigma_s$.

From proposition C.0.1(1), if $k_2 > k_1$, Σ_s is the segment from 0 to s_{end} where $s_{end} = \min\{0.5, \frac{c_1}{k_2-k_1}\}$. Thus, in order to have a crossing segment in $\mathcal{T}$, we must have $\frac{c_1}{k_2-k_1} < 0.5$. In this case Σ_c is the segment from $\frac{c_1}{k_2-k_1}$ to 0.5 and (i) is proved.

From proposition C.0.1(2), if $k_1 > k_2$, Σ_s is the segment from 0 to s_{end} where $s_{end} = \min\{0.5, \frac{c_2}{k_1-k_2}\}$. Thus, in order to have a crossing segment in $\mathcal{T}$, we must have $\frac{c_2}{k_1-k_2} < 0.5$. In this case Σ_c is the segment from $\frac{c_2}{k_1-k_2}$ to 0.5 and (ii) is proved.

Let $c_1 = c_2 = 0$ in (C.1), this leads to $0 < a < 0$, so that $\Sigma_s = \emptyset$ and Σ_c is the segment from 00 to 0.5. If $k_2 > k_1$ and $c_1 = 0$, then $s_{end} = 0$ and Σ_c is the segment from 0 to 0.5. If $k_1 > k_2$ and $c_2 = 0$, then $s_{end} = 0$ and Σ_c is the segment from 0 to 0.5. This proves (iii). □

Proposition C.0.4 (The sliding segment Σ_s is followed by the crossing segment Σ_c). *If the sliding segment Σ_s and the crossing segment Σ_c both exist, Σ_s is followed by Σ_c. Then, once crossing occurs there is no possibility of regaining market share for the firm that lost part of it.*

Proof. Since both the sliding segment Σ_s and the crossing segment Σ_c exist, from propositions C.0.1 and C.0.3, we see that in $\mathcal{T}$, only Σ_s followed by Σ_c can occur. $\square$

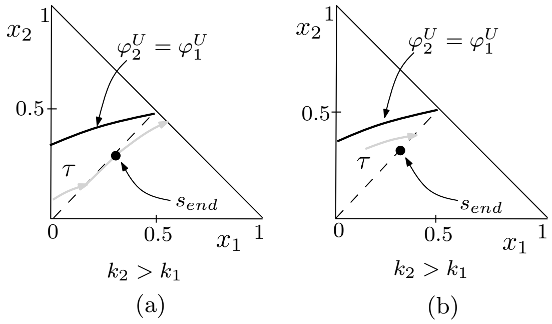


Figure C.2: Suppose that $k_2 > k_1$, $s_{end} < 0.5$ and firm F_1 has an initial lower market share than F_2, then: (a) There is no trajectory τ that reaches the equal market share line on the sliding segment Σ_s, remains on it for some time and then crosses into L, with F_1 ending with a bigger market share than F_2 (b) There is no trajectory τ that reaches the equal market share line on the crossing segment Σ_c and crosses into L, with F_1 ending with a bigger saturated market share than F_2

Proposition C.0.5. *If $k_2 > k_1$ and the sliding end point $s_{end} < 0.5$, then there is no trajectory τ that has initial conditions in U and reaches Σ at a point (v, v) below the sliding end point s_{end}, slides until s_{end} and then crosses into L. (See Figure C.2a)*

Proof. To see this, we obtain an expression for $\tan_{f^L}(c_1/(k_2 - k_1), c_1/(k_2 - k_1))$

$$\tan_{f^L}(c_1/(k_2 - k_1), c_1/(k_2 - k_1)) = k_2/k_1 + (k_2 - k_1)c_2/c_1k_1 \tag{C.6}$$

Since $k_2 > k_1$ and all parameters k_1, c_1, k_2, c_2 are positive (C.6) is always greater than 1, i. e. the condition for crossing into L is never met when $k_2 > k_1$. $\square$

Proposition C.0.6. *If $k_2 > k_1$ and the sliding end point $s_{end} < 0.5$, then there is no trajectory τ that has initial conditions in U and reaches the crossing segment Σ_c. (See Figure C.2b)*

Proof. Consider a disk with center at $(x_1, x_2) \in \Sigma$ and radius δ. Let τ be a trajectory that passes by $(x_1, x_2 = x_1 + \delta)$ in U. If the trajectory τ were to hit Σ in the crossing segment $(\frac{c_1}{k_2 - k_1} < x_2 = x_1 < 0.5)$, the tangent at $(x_1, x_2 = x_1 + \delta)$ has to be less than 1 as $\delta \to 0$ for crossing from U into L

$$\lim_{\delta \to 0^+} \tan_{f^U}(x_1, x_1 + \delta) = \frac{k_2 x_1}{k_1 x_1 + c_1} < 1 \tag{C.7}$$

Let the LHS of (C.7) be rewritten as

$$\frac{1}{k_1/k_2 + c_1/(k_2 x_1)} \tag{C.8}$$

Since the crossing needs to be in Σ_c, we have $x_1 > c_1/(k_2 - k_1)$, so that $c_1 < x_1(k_2 - k_1)$. Using this in (C.8)

$$\frac{1}{k_1/k_2 + c_1/(k_2 x_1)} > \frac{1}{k_1/k_2 + (x_1(k_2 - k_1)/(k_2 x_1))} = 1 \tag{C.9}$$

But (C.9) conflicts with (C.7), so that it is not possible that a trajectory τ beginning in U and satisfying $k_2 > k_1$ to intersect the equal market share on the crossing segment Σ_c. Similarly, for a trajectory τ starting in region L crossing directly into region U can only occur when $k_2 > k_1$. □

The next proposition shows that he model (3.3)-(3.6) does not have pseudoequilibrium points. Pseudo-equilibrium points are the stationary points of the differential equation (A.4), see Appendix A.

Proposition C.0.7 (Nonexistence of pseudo-equilibria). *The model (3.3)-(3.6) does not have any pseudo-equilibrium points.*

Proof. Substituting $x_1 = x_2 = a$ in (3.3)-(3.6), the pseudo-equilibrium condition (A.7) for their existence is

$$\frac{k_1 a + c_1}{k_1 a} = \frac{k_2 a}{k_2 a + c_2} < 0$$

which cannot be satisfied because the parameter values are positive numbers, and point (a, a) belongs to the first quadrant. Hence, the model (3.3), (3.6) does not present any pseudo-equilibrium point. □

Appendix D

Numerical considerations on simulating Odes with discontinuous RHS (right-hand sides)

In the first simulations, the package by Piiroinen and Kuznetzov [55] was used for simulating Odes with discontinuous RHS. However, we had problems with some simulations showing wrong trajectories, trajectories which did not stop at the corresponding equilibrium point. For instance, the simulation of model VWDsC in Figure D.1 shows trajectories continuing upwards on the saturated market line Sat and then stopping in the second quadrant where state variable x_1 is negative. The problem was reported but we received no answer. As of 2016, the package is not available for download anymore.

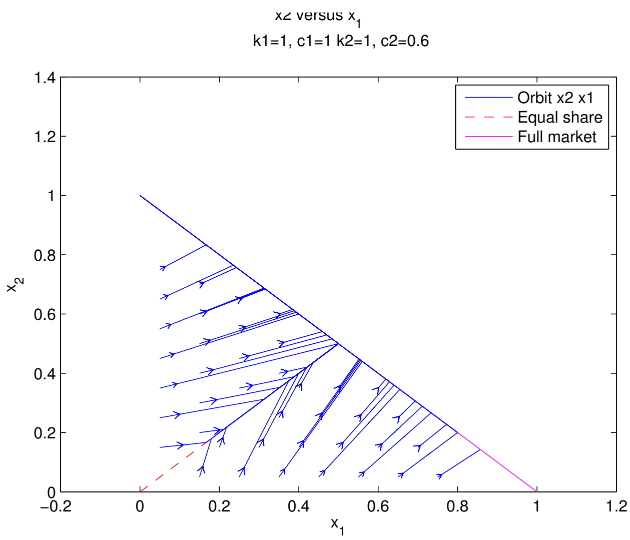


Figure D.1: Simulation ran with Piiroinen, Kuznetsov Matlab software. Trajectories shoul stop on the saturated line Sat, however in the simulation they continue moving upwards.

After that, we used the application Berkeley Madonna [51] for running the simulations. However, we found wrong trajectories model VWDsC like the one shown in Figure

D.2. From the experience accumulated doing simulations with Berkeley Madonna application we inferred that Berkeley Madonna considers the function if -used to implement Odes with discontinuous RHS- a continuous function and in some cases you get a reasonable behavior on the sliding segments and in other it disagrees with the theory of Odes with discontinuous RHS. Also, for more complex dynamics like the Lotka-Volterra derive model defining the dynamics with the function if becomes very complex.

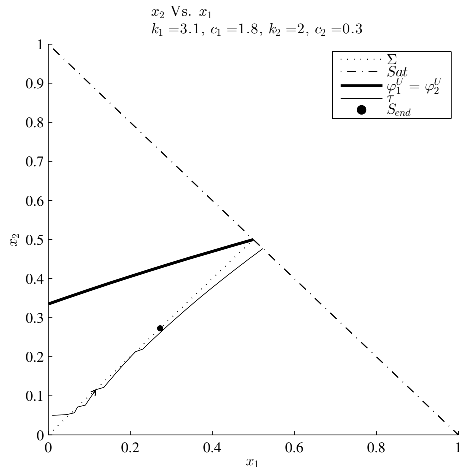


Figure D.2: Trajectory on the phase plane showing more than one crossing. This situation does not comply with Filippov's theory for Odes with discontinuous RHS.

Diminishing the time step in the simulations ran with Berkeley Madonna reduced the chattering in many cases. However, this was not the case for some simulations of the model LVsP presented in Chapter 5 which still had a good amount of chattering.

Because of the previous situations, we implemented a simple discretization which supports the discretization methods Forward-Euler or RK4 (Runge Kutta 4) and uses the Filippov Rule for the field value on the sliding segments. The result are obviously better in smoothing the chattering on the sliding set Σ_s because the field defined by the Filippov rule on the sliding segment is tangent to the discontinuity line. However, since we are working with finite precision arithmetic when doing simulations on computers, any point which lies inside an ϵ-*tube* around the switching line and satisfies the sliding condition (A.6) is assigned the field (A.2) corresponding to the Filippov rule for sliding dynamics. See Figure D.3.

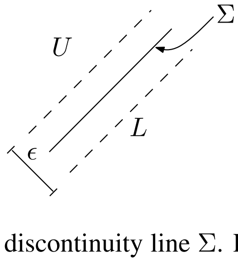


Figure D.3: A magnified section of the discontinuity line Σ. For a point that lies in the ϵ-*tube* around Σ and satisfies the sliding condition (A.6), the point is assigned the field (A.2) defined by the Filippov rule for the numerical discretization

Consider an autonomous Ode System given in vector form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{D.1}$$

For instance if $\mathbf{x} \in \mathrm{R}^2$, $\mathbf{x} = [x_1, x_2]^T$ and $\mathbf{f}(\mathbf{x}) = [f_1(x_1, x_2), f_2(x_1, x_2)]^T$
The ***Forward Euler*** discretization [56, p. 257, chap. 5] calculates the next time value for x using the first order Taylor approximation:

$$\mathbf{x}(t+h) = \mathbf{x}(t) + h\mathbf{f}(\mathbf{x}(t)) \tag{D.2}$$

The fourth term ***Runge-Kutta (RK4)*** [56, p. 277-278, chap. 5] is more complex and requires more operations. It calculates the next time value for x using the following steps:

1. calculate $\mathbf{k_1} = h\mathbf{f}(\mathbf{x}(t))$
2. calculate $\mathbf{k_2} = h\mathbf{f}(\mathbf{x}(t) + 1/2\mathbf{k_1})$
3. calculate $\mathbf{k_3} = h\mathbf{f}(\mathbf{x}(t) + 1/2\mathbf{k_2})$
4. calculate $\mathbf{k_4} = h\mathbf{f}(\mathbf{x}(t) + \mathbf{k_3})$
5. calculate
$$\mathbf{x}(t+h) = \mathbf{x}(t) + 1/6(\mathbf{k_1} + 2\mathbf{k_2} + 2\mathbf{k_3} + \mathbf{k_4}) \tag{D.3}$$

A basic ϵ-tube algorithm for calculating $\mathbf{x}$ for the next time step

1. Determine the field $\mathbf{f}$ to be used for calculating $\mathbf{x}(t+h)$
   - if ($\mathbf{x}(t) \in \epsilon$-*tube*) and $\mathbf{x}(t) \in \Sigma_s$) , then the field to be used is $\mathbf{f}(\mathbf{x}(t)) = \mathbf{\Phi}(\mathbf{x}(t))$ with $\mathbf{\Phi}(\mathbf{x})$ defined in (A.2).
   - if($x_2(t) > x_1(t) + \epsilon$), then the field to be used is $\mathbf{f}(\mathbf{x}(t)) = \mathbf{f}^U(\mathbf{x}(t))$
   - if($x_1(t) > x_2(t) - \epsilon$), then the field to be used is $\mathbf{f}(\mathbf{x}(t)) = \mathbf{f}^L(\mathbf{x}(t))$
2. Calculate $\mathbf{x}(t+h)$ using the field $\mathbf{f}$ in the selected discretization method, i.e. in (D.2) for Forward Euler or (D.3) for RK4.

We show phase planes from simulations done with Berkeley Madonna (Figure D.4) and with the basic ϵ-*tube* algorithm (Figure D.5). Obviously there is a trade-off between the size of ϵ-*tube* and the discretization step size. One would like to use a thin ϵ-*tube* but one also needs that the simulation solution falls inside the ϵ-*tube* and this implies using a small step size. The thinner the tube, the smaller the step size.

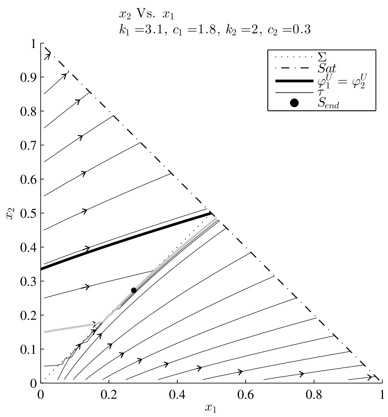


Figure D.4: Phase plane plotted with simulation data calculated using Berkeley Madonna.

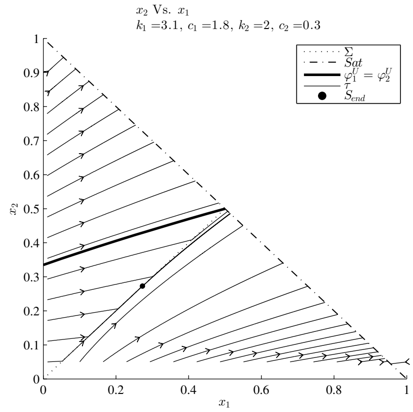


Figure D.5: Phase plane plotted with simulation data calculated using the basic ϵ-*tube* algorithm.

Figure D.6 shows that there are small errors that can be seen when doing a zoom near the sliding end point s_{end} when in the phase plane of Figure D.5, this is because of finite precision arithmetic and the use of the ϵ-*tube*. The zoom also shows that taking into account the numerical errors, the sliding segment is effectively s_{end}. Figure D.7 corresponds to the same trajectory shown in Figure D.2.

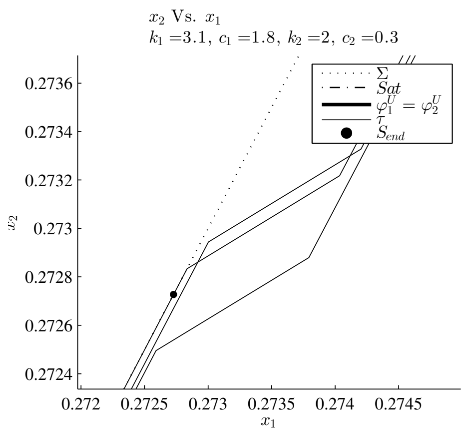


Figure D.6: Zoom of the phase plane nearby the sliding end point s_{end}

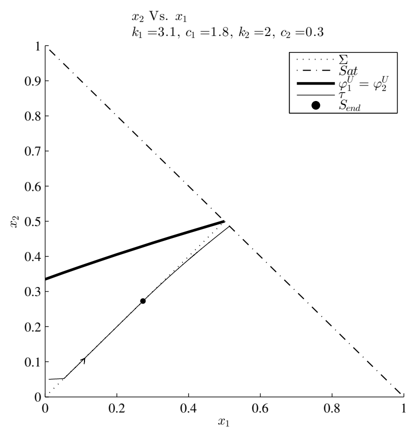


Figure D.7: Trajectory calculated using the basic ϵ-*tube* algorithm.

Remark: We assume that for a point that lies in the ϵ-*tube* the firm, whose target market share is represented by Σ, has its extra effort turned on when calculating its associated performance index J. The justification is that the firm wants more market share than its target market share even though they accept at least their target market share.

Appendix E

Review of duopoly models

E.1 K. R. Deal, 1979 [1]

E.1.1 Model derived from

Vidale-Wolfe.

E.1.2 Formulation

The objectives J_1, J_2 of firms F_1, F_2 are

$$\max_{u_1} J_1 = w_1 \frac{S_1(t_f)}{S_1(t_f) + S_2(t_f)} + \int_{t_0}^{t_f} (c_1 S_1(t) - u_1^2(t))dt \tag{E.1}$$

$$\max_{u_2} J_2 = w_2 \frac{S_2(t_f)}{S_1(t_f) + S_2(t_f)} + \int_{t_0}^{t_f} (c_2 S_2(t) - u_2^2(t))dt \tag{E.2}$$

and the system dynamics is

$$\dot{S}_1(t) = \beta_1 u_1(t) \frac{[M - S_1(t) - S_2(t)]}{M} - \delta_1 S_1(t) \tag{E.3}$$

$$\dot{S}_2(t) = \beta_2 u_2(t) \frac{[M - S_1(t) - S_2(t)]}{M} - \delta_2 S_2(t) \tag{E.4}$$

where $x_i(t)$ is the sales of firm F_i, $u_i(t)$ is the advertising expenditure of F_i, δ_i is the sales decay parameter, β_i is the sales response parameter, M is the total sales potential. In the objective function, w_i is the weight that is given to the final market proportion obtained, c_i is the net revenue coefficient, u_i^2 is the advertising cost and $c_i x_i(t)$ is the revenue for F_i. *Assumption:* the size of the potential market is fixed at M and this size is not changed by advertising.

E.1.3 Solution

Solves the open-loop Nash equilibrium using a numerical method.

E.2 J. Case, 1979 [2]

E.2.1 Model derived from

Lanchester. Its game formulation/solution is later used by others like Erickson, Chintagunta.

E.2.2 Formulation

$$\text{Firm 1} \max_{u_1} = \int_0^\infty e^{-r_1 t}(g_1 x(t) - \frac{c_1}{2}u_1^2(t)dt \qquad x_1(t) = x(t) \tag{E.5}$$

$$\text{Firm 2} \max_{u_2} = \int_0^\infty e^{-r_2 t}(g_2(1 - x(t)) - \frac{c_2}{2}u_2^2(t)dt \qquad x_2(t) = 1 - x(t) \tag{E.6}$$

subject to

$$\dot{x}(t) = u_1(t)(1 - x(t)) - u_2(t)x(t) \tag{E.7}$$

where

$i = 1, 2$ number of the firm

t time

$x_1(t) = x(t)$ market share of firm 1 at time t

$x_2(t) = 1 - x(t)$ market share of firm 2 at time t

$u_i(t)$ advertising of firm i at time t (control variable)

$r \geq 0$ discount factor

$g_i > 0$ constant unit margin of firm i

$c_i > 0$ constant cost parameter of firm i

$x_0 \in [0, 1]$ initial market share of firm 1 at $t = 0$

E.2.3 Solution

The infinite horizon planning has an analytical closed-loop solution.

Case (1979) derives conditions which a pair of feedback strategies $(u_1(x), u_2(x))$ has to satisfy, if it constitutes a Nash equilibrium for the above game. Moreover, he calculates explicit expressions for $u_i(x)$, $i = 1, 2$, in the case of no discounting (i.e., $r = 0$).

E.3 G. Sorger, 1989 [2]

E.3.1 Model derived from

Uses a model derived from Lanchester. It is a A modification of the Case Game.

E.3.2 Formulation

$$\max_{u_i} = \int_0^{t_f} e^{-r_i t}(g_i x_i(t) - \frac{c_i}{2}u_i^2(t))dt \quad i = 1, 2, x_1(t) = x(t)x_2(t) = 1 - x(t) \tag{E.8}$$

subject to

$$\dot{x}(t) = u_1(t)\sqrt{1 - x(t)} - u_2(t)\sqrt{x(t)} \tag{E.9}$$

where $i = 1, 2$ is the firm index, F_i is the firm i, $x_1(t) = x(t)$ and $x_2(t) = 1 - x(t)$ are the market shares of F_1 and F_2 at time t, $u_i(t)$ is the advertising rate of F_i, $r_i \geq 0$ is the constant discount factor of F_i, g_i is the constant margin of F_i, $c_i > 0$ is the constant cost of F_i Total market potential is constant. Sorger uses the square root of the the market share in the dynamics equation to obtain a market share with a concave response so that it has dimishing returns, in contrast to Little in 1979 who used an exponent on the advertising effort to obtain a concave or an S-shaped response.

E.3.3 Solution

Finite time horizon, Open-Loop, Closed-loop) derives differential equation on the costate. Sorger does a qualitative analysis on that equation.

Infinite time horizon has analytical solution for Open-loop and Closed-loop.

E.4 P. K. Chintagunta, N. J. Vilcassim, 1992 [3]

E.4.1 Model derived from

Lanchester.

Uses econometric estimation of response and differential games. Estimates parameters k_1 and k_2 using a regression.

E.4.2 Formulation

$$\dot{x} = \beta_1\sqrt{u_1}(1-x) - \beta_2\sqrt{u_2}x \tag{E.10}$$

$x_1 = x$ is the market share of firm 1, $x_2 = 1 - x$ is the market share of firm 2.

Discrete equation

$$x_{t+1} - x_t = \beta_1\sqrt{u_{1,t+1}}(1-x_t) - \beta_2\sqrt{u_{2,t+1}}x_t \tag{E.11}$$

So that players need to know the previous period market share M. The problem to be solved is

$$\max_{u_i} \pi_i = \int_0^\infty e^{-rt}(g_i x_i S_{inst} - u_i)dt \tag{E.12}$$

subject to

$$\dot{x}_i = \beta_i\sqrt{u_i}(1-x_i) - \beta_j\sqrt{u_j}x_i \quad i,j = 1,2 \quad i \neq j \tag{E.13}$$

where x_i is the market share of firm i and $x_2 = 1 - x_1$, u_i are the advertising controls. r is the discount factor, g_i is the net contribution as a fraction of dollar sales for firm i, g_i is constant and independent of sales. S_{inst} is the instantaneous total sales in the market in dollars at time t. r is the same for both firms. Parameters β_i have to be empirically estimated. Firms cannot collude, so that a Nash equilibrium is appropriate.

E.4.2.1 Open-loop

To obtain the OLNE (open-loop Nash equilibrium), maximize the Hamiltonian

$$\max H_{u_i} = (g_i x_i S_{inst} - u_i) + \lambda_i(\beta_i\sqrt{u_i}(1-x_i) - \beta_j\sqrt{u_j}x_i) \quad i,j = 1,2 \quad i \neq j \tag{E.14}$$

where λ_i are the costates.

The necessary conditions for the OLNE are

$$\frac{\partial H_i}{\partial u_i} = 0, \quad \dot{x}_i = \frac{\partial H_i}{\partial \lambda_i} \quad \dot{\lambda}_i = r\lambda_i - \frac{\partial H_i}{x_i} \quad i = 1,2 \tag{E.15}$$

After simplifications, the above system of equations is transformed in

$$u_i = \left[\frac{\lambda_i \beta_i (1-x_i)}{2}\right]^2 \tag{E.16}$$

$$\dot{M}_i = \beta_i\sqrt{u_i}(1-x_i) - \beta_j\sqrt{u_j}x_i \quad i,j = 1,2 \quad i \neq j \tag{E.17}$$

$$\dot{\lambda}_i = g_i S_{inst} + \lambda_i(r + \beta_i\sqrt{u_i} + \beta_j\sqrt{u_j} \quad i,j = 1,2 \quad i \neq j \tag{E.18}$$

The system of equations given by (E.16), (E.17) and (E.18) does not have a closed form solution. Chintagunta and Vilcassim *solved it numerically* using an algorithm proposed by Deal in 1979.

E.4.2.2 Closed-loop

They apply the Two-Player Game of Case (1979). The assumption made is that the discount factor r is strictly positive and is close to zero in magnitude.

The advertising controls obtained are

$$u_1 = \frac{1}{3}[2TR^2 - S + 2(T^2R^4 - STR^2)] \tag{E.19}$$

$$u_2 = \frac{1}{3}\left[\frac{2S}{R^2} - T + 2\left(T^2 + \frac{S^2}{R^4} - \frac{ST}{R^2}\right)^{\frac{1}{2}}\right] \tag{E.20}$$

where $R = \frac{k_2}{k_2}\frac{x}{1-x}$, $S = g_1 S_{inst} x$ and $T = g_2 S_{inst}(1-x)$.

Parameters β_1, β_2 in (E.10) were estimated using ordinary least squares and the historical datasets for Coke and Pepsi advertising expenditures.

E.4.3 Solution

- Closed-loop, analytical solution for infinite time horizon. Solution as in Case(1979).
- Open-loop, numerical solution.
- Solutions obtained are also analyzed using empirical data.

E.5 G. Erickson, 1992 [4]

E.5.1 Model derived from

Lanchester.

E.5.2 Formulation

Each firm seek to maximize its discounted profit

$$\max \int_0^\infty e^{-rt} h_i(x, u_1, u_2) \tag{E.21}$$

dynamics is given by

$$\dot{x} = f(x, u_1, u_2), \quad x(0)\ given \tag{E.22}$$

also $0 \geq x \leq 1$ and $u_i > 0$

Firms cannot cooperate, so that it is reasonable that they search for Nash equilibria. *A Nash equilibrium* is a pair of strategies, one for each competitor, which has the property that no competitor would like unilaterally to change its strategy (Moorthy 1985). In a Nash equilibrium, each strategy is a competitor's best strategy, given the strategies of its rival, where "best" means maximizing the profit integral.

There are *two kinds of Nash equilibria* that can be pursued: *open-loop*, where advertising is a function of time, $u_i = u_i(t, x(0))$, in which $x(0)$ is the starting value for market share, and *closed-loop*, where advertising is a function of time and the current state of the system $u_i = u_i(t, x, x(O))$. Unfortunately, open-loop and closed-loop equilibria are generally different. The most frequently used approach in differential games has been to develop open-loop equilibria, primarily because they are easier to compute (Case 1979).

Open-loop equilibria are by definition time consistent, in that if at some intermediate point the competitors are asked to reconsider their strategies they would refuse to change them (Fershtman 1987a). Open-loop strategies are not subgame perfect, they change if initial conditions change. To be subgame perfect, an equilibrium must not depend upon initial conditions. Specifically, strategies $u_i(t, x)$ that depend upon current values of state vaiiables as well as time and that do not depend upon initial conditions are termed feedback strategies (Fershtman 1 987b).

In general, closed-loop equilibria have been difficult to obtain, since they tend to involve partial differential equations (Starr and Ho 1969; Fershtman 1987a).

E.5.2.1 Case approach to Lanchester game

An approach by Case (1979), however, offers hope for a class of problems, those involving a single state variable, for which only ordinary, and not partial, differential equations are required. In particular, Case's approach can be applied to competitive situations involving duopolistic competition for market share.

Case (1979, pp. 210-215) considers what he terms perfect equilibria, which are time invariant (stationary) functions of state variables. This definition of "perfect" equilibria differs from what is elsewhere defined as "perfect" (e.g., Friedman 1986). CASE, JAMES H., Economics and the Competitive Process, New York University Press, New York, 1979.

Procedure by Case

- Define the Hamiltonians

$$H_i = h_i(x, u_1, u_2) + \lambda_i f(x, u_1, u_2) \tag{E.23}$$

  where λ_i is the costate and each firm maximizes its H_i

- Now determine $u_i(x, \lambda_1, \lambda_2)$ and $A_2(x, \lambda_1, \lambda_2)$ that form a Nash equilibrium for the auxiliary game

$$\max H_i, \quad i = 1, 2 \tag{E.24}$$

Define the Hamilton-Jacobi equations

$$\begin{aligned} h_i(x, \hat{u}_1(x, V_1^{'}(x), V_2^{'}(x)), \hat{u}_2(x, V_1^{'}(x), V_2^{'}(x))) \\ + V_i^{'} f(x, V_1^{'}(x), V_2^{'}(x)), \hat{u}_2(x, V_1^{'}(x), V_2^{'}(x))) \\ = rV_i(x) + c_i \quad i = 1, 2 \end{aligned} \tag{E.25}$$

where the c_i are arbitrary real numbers and r is the discount factor, the $V_i(M)$ are the value functions for firm i, they are the discounted profit for firm i on an optimal advertising path. The costate and value functions are related by $\lambda_i = V_i^{'}(x)$, for different starting levels x. If the system of ordinary differential equations defined by (E.25) can be solved for $V_1(x)$ and $V_2(x)$, a perfect equibilibrium is derived:

$$u_i(M) = \hat{u}_i(x, V_1^{'}(x), V_2^{'}(x)) \tag{E.26}$$

An equilibrium (E.26), forms an optimal strategy for each firm i and for any inital value x. Because c_i are arbitrary, there are infinite number of perfect advertising equilibria.

E.5.3 Lanchester differential game

The objective for each firm

$$\max_{u_i} \int_0^\infty e^{-rt}(g_i x_i - u_i)dt, \quad i = 1, 2 \tag{E.27}$$

subject to

$$\dot{x}_i = \beta_i u_i^{\alpha_i}(1 - x_i) - \beta_j u_j^{\alpha_j} x_i, \quad i, j = 1, 2 \quad i \neq j \tag{E.28}$$

where u_i is the advertising effort of firm i, r is the discount factor, g_i is the gross profit rate of firm i, x_i is the market share of firm i and $x_1 = x$, $x_2 = 1 - x$, β_i is a constant for firm i.

E.5.4 Closed-loop solution

Analytical solutions can be derived only for $r = 0$. The following relations result and they define u_i implicitly in term of x.

$$g_i x_i + \frac{1 - \alpha_j}{\alpha_i} u_i - \frac{\beta_{3-i}}{\alpha_i \beta_i} u_i^{1-\alpha_i} u_{3-i}^{\alpha_{3-i} \frac{x_i}{x_{3-i}}}, \quad i = 1, 2 \tag{E.29}$$

Explicit forms for u_i when $\alpha_i = 0.5, \quad i = 1, 2$ in (E.29) are given below

$$u_i(x) = \frac{2\frac{C_i^2}{D_i^2}E_{3-i} - E_i + 2\sqrt{E_i^2 - \frac{C_i^2}{D_i^2}E_iE_{3-i} + \frac{C_i^4}{D_i^4}E_{3-i}^2}}{3} \qquad i = 1, 2 \tag{E.30}$$

where

$$C_i = \frac{x_i}{x_{3-i}}, \quad D_i = \frac{\beta_i}{\beta_{3-i}}, \quad E_i = g_i x - c_i, \quad i = 1, 2 \tag{E.31}$$

Note that x and u_i are simultaneuos in different equations and also nonlinear.

E.5.4.1 Open-loop solution

The following differential equations are derived from the necessary conditions of the open-loop strategies

$$\dot{u}_i = 2u_i\left(r + \frac{\beta_{3-i}\sqrt{u_{3-i}}}{x_{3-i}} - \frac{g_i\beta_i x_{3-i}}{2\sqrt{u_i}}\right), \quad i = 1, 2 \tag{E.32}$$

After estimating parameters, the system (E.32) is solved numerically.

E.5.4.2 Conclusions

For two cases: Soda drinks (Coke, Pepsi) and Beer drinks (Miller, Anheuser-Busch) they estimate parameters for the model and conclude that closed-loop perfect equilibrium fits better the historical datasets

E.5.5 Solution

- Closed-loop, analytical solution for infinite time horizon. Solution as in Case(1979).
- Open-loop, numerical solution.
- Solutions obtained are also analyzed using empirical data.

E.6 G. A. Fruchter, S. Kalish, 1997 [5]

E.6.1 Model derived from

Lanchester.

E.6.2 Formulation

Two firms compete using a system dynamics described using Lanchester model they determine the optimal advertising strategy for maximum discounted profits. New mathematical approach. Discounted profit for each firm

$$\max_{u_k} J_k(u_1, u_2) = \int_0^\infty [g_k x_k(t) - c_k u_k^2(t)] e^{-rt} dt, \quad k = 1, 2 \tag{E.33}$$

subject to

$$\dot{x}(t) = \beta_1 u_1(t)(1 - x(t)) - \beta_2 u_2(t) x(t) \tag{E.34}$$

where $x_1 + x_2 = 1$ and x_k is the fraction of the market that belongs to firm k. The constant g_k is the gross profit rate of firm k, r is the discount rate and c_k is the effectiveness of advertising buying power, usually $c_1 = c_2 = 1$. *The variables* $u_1,\ u_2$ *are the controls based on advertising, in particular* $u_k,\ k = 1, 2$ *is the square root of the advertising expenditures of firm* k. When the current state is also the initial state, the computed optimal controls become a *feedback strategy* which depends on (x, t) . The effectiveness of the combat of firm k is measured by $\beta_k u_k(t)$.

E.6.3 Solution

Numerical solution. They transformed the problem into an initial value problem. Closed-loop solution depends on time and state.

E.6.3.1 Open-loop

The first order necessary conditions using the variational approach as in Bryson and Ho 1975 on the differential game are

$$\dot{\lambda}_k(t) = r\lambda_k(t) + (\beta_1 u_1(t) + \beta_2 u_2(t))\lambda_k(t) + (-1)^k g_k, \quad k = 1, 2 \quad \lim_{t\to\infty} \lambda_k(t) e^{-rt} = 0 \tag{E.35}$$

$$\dot{u}_k(t) = \frac{(-1)^{k+1}}{2} c_k^{-1} \beta_k \lambda_k(t)(1 - x_k(t)), \quad k = 1, 2 \tag{E.36}$$

Substituting (E.36) in (E.35) and (E.34) they obtain the following two-point boundary value problem TPBVP value obtained

$$\dot{x} = \frac{1}{2}[c_1^{-1}\beta_1^2(1 - x)^2\lambda_1 - c_2^{-1}\beta_2^2(x)^2\lambda_2], \quad x(0) = x_0 \tag{E.37}$$

$$\dot{\lambda}_k = r\lambda_k + \frac{1}{2}[c_1^{-1}\lambda_1\lambda_k\beta_1^2(1 - x) - c_2^{-1}\beta_2^2\lambda_2\lambda_k x] - (-1)^{k+1} g_k, \quad \lim_{t\to\infty} \lambda_k(t) e^{-rt} = 0 \tag{E.38}$$

where $k = 1, 2$. In their lemma 1 they let $\lambda_k(t) = (-1)^{k+1} g_k e^{rt}\Phi(t)$, $k = 1, 2$, also λ_k satisfies (E.37) and (E.38). Using this assumptions $\Phi(t)$ satisfies a backward differential equation

$$\Phi^{'}(t) = \frac{1}{2}[c_1^{-1}\beta_1^2 g_1(1 - x^P) - c_2^{-1}\beta_2^2 g_2 x^P]e^{rt}\Phi^2(t) - e^{-rt} \quad \lim_{t\to\infty}\Phi(t) = 0 \qquad \text{(E.39)}$$

Note the change in notation x was replaced by x^P to denote x during planning.

Let

$$\Phi(t)e^{rt} = \psi(x^P) \qquad \text{(E.40)}$$

to get a new backward differential equation for $\psi(x^P)$

$$\begin{aligned}\psi^{'}(x^P)\psi(x^P)[c_1^{-1}\beta_1^2 g_1(1 - x^P)^2 - c_2^{-1}\beta_2^2 g_2(x^P)^2] - [c_1^{-1}\beta_1^2 g_1(1 - x^P)\\ - c_2^{-1}\beta_2^2 g_2 x^P]\psi^2(x^P) = 2r\psi(x^P), \quad \lim_{t\to\infty}\psi(x^P)e^{-rt} = 0 \quad \text{(E.41)}\end{aligned}$$

where

The TPBVP (E.37), (E.38) can be transformed in the following IVP (initial value problem)

$$\dot{x}^P = \frac{1}{2}[c_1^{-1}\beta_1^2(1 - x^P)^2 g_1 - c_2^{-1}\beta_2^2(x^P)^2 g_2]e^{rt}\Phi(t), \quad x(0) = x_0 \qquad \text{(E.42)}$$

$$\dot{\Phi}(t) = \frac{1}{2}[c_1^{-1}\beta_1^2(1 - x^P)g_1 - c_2^{-1}\beta_2^2 x^P g_2]e^{rt}\Phi^2(t) - e^{-rt}, \quad \Phi(0) = \psi(x_0) \qquad \text{(E.43)}$$

where $\psi(x_0)$ is obtained from the backward equation (E.41)

The control for the open-loop problem is

$$u_k^{OL} = \frac{1}{2}c_k^{-1}\beta_k g_k\Phi(t)e^{rt}(1 - x_k^P \quad k = 1, 2 \qquad \text{(E.44)}$$

where $x_1^P = x^P$ and $x_2^P = 1 - x^P$ and x^P satisfies (E.42) and (E.43).

E.6.3.2 Closed-loop

The closed loop is time variant and depends linearly on the actual market share. Time variant coefficients incorporate the discount factor. Its computation requires the solution of a backward differential equation and a set of two nonlinear differential equations for an initial value problem. *Let the close-loop strategies be*

$$u_k^* = \frac{1}{2}c_k^{-1}\beta_k g_k\Phi(t)e^{rt}(1 - x_k), \quad k = 1, 2 \qquad \text{(E.45)}$$

The closed-loop advertising expenditures are proportional to the open-loop advertising expenditures and to the square of the competitor's actual market share. From (E.44) and (E.45), we get

$$u_k^* = u_k^{OL}\frac{1-x_k}{1-x_k^p}, \quad k=1,2 \tag{E.46}$$

E.7 Q. Wang, Z. Wu, 2001 [6]

E.7.1 Model derived from

Lanchester and Vidale-Wolfe.

E.7.2 Formulation

Model

$$\max_{u_i\geq 0} J_i = \int_0^{t_f}(Mp_ix_i - u_i^2)dt + g_ix_i(t_f) \tag{E.47}$$

subject to

$$\frac{dx_i}{dt} = \beta_iu_i(1-x_i) - \beta_ju_jx_i - \delta_ix_i, \quad i,j=1,2 \quad i\neq j \quad x_i(0)=x_{i0} \tag{E.48}$$

where x_i is the market share of firm i, u_i is a variant of the advertising expenditure of firm i, β_i are response constants to the advertising of firm i, δ_i is a decay constant, p_i is the unit price at which firm i sells, M is the total market and g_i is a valuation of the market share of firm i at time T. The market share satisfies $0\leq x_1+x_2\leq 1$ and the total market is normalized to 1, *which is a subtle difference from other models and allows for market growth until saturation.* In other models, $x_1+x_2=1$. The model does not use any discount factor.

E.7.3 Solution

Numerical algorithm for open-loop and closed-loop Nash equilibrium solutions.

The solutions they obtain are numerical, for OLNE and CLNE. i.e., they solve the differential equations for the necessary conditions numerically. Algorithm starts estimating an initial value for each auxiliary variable $\lambda_i(0)$ and $\phi_i(0)$ (for CLNE only). The necessary conditions of Nash optimality are then solved forward in time by using the fourth order Range-Kutta method. The values of auxiliary variables at terminal time T, $\lambda_i(T)$ and $\phi_i(T)$, are with target values g_i and 0. If they are within predetermined ranges a solution is found. Otherwise, the differential equations are solved again with the initial values $\lambda_i(0)$ adjusted by δ_i and $\phi_i(0)$ by σ_i (See expressions for δ_i and σ_i in the paper) They say their model is better for the datasets tested.

E.8 R. Jarrar, G. Martín-Herrán, G. Zaccour, 2004 [7]

E.8.1 Derived from

Lanchester.

E.8.2 Formulation

$$\max_{u_1} \int_0^\infty e^{-rt}(g_1 x - u_1)dt \tag{E.49}$$
$$\max_{u_2} \int_0^\infty e^{-rt}(g_2(1-x) - u_2)dt \tag{E.50}$$

subject to

$$\dot{x}(t) = \beta_1 \sqrt{u_1(t)}(1-x(t)) - \beta_2 \sqrt{u_2(t)}x(t), \quad x(0) = x_0 \tag{E.51}$$

where $x_1(t) = x(t)$ is the market share of firm 1 at time t and $x_2(t) = 1 - x(t)$ is the market share of firm 2 at time t, β_i, $i = 1, 2$ is a constant that denotes the advertising effectiveness of firm i, $u_i(t)$ is the advertising budget of firm i, g_i is the gross margin per point of market share for firm i and r is the discount rate.

From page 996: *However, in an infinite-horizon setting, the assumption of a zero-discount rate lacks conceptual appeal.*

E.8.3 Solution

They proposed a numerical algorithm for solving Markow perfect equilibrium for the Lanchester duopoly model where the discount factor can be nonzero.

Discussion

The standard sufficient condition to determine a stationary MPNE (Markov Perfect Nash Equilibrium) is to find bounded and continuously differentiable functions $V_i(x)$, $i = 1, 2$, satisfying, for all x, $0 \geq x \leq 1$, the Hamilton-Jacobi-Bellman (HJB) equations

$$rV_1(x) = \max_{a_1}[g_1 x - a_1^2 + V_1^{'}(x)(\beta_1 a_1(t)(1-x(t)) - \beta_2 a_2(t)x(t))] \tag{E.52}$$
$$rV_2(x) = \max_{a_2}[g_2(1-x) - a_2^2 + V_2^{'}(x)(\beta_1 a_1(t)(1-x(t)) - \beta_2 a_2(t)x(t))] \tag{E.53}$$

where $a_i^2 = u_i$ the change is for computations purposes, $V_i(x)$ is the value function of firm i and $V_i^{'}(x)$ its derivative. u_i^2 is used for computations.

The usual algorithm for obtaining MPNE strategies is:

- Assume interior solutions and take derivatives of (E.52), (E.53) with respect to a_1,

a_2, respectively, and equate to zero

$$a_1 = \frac{\beta_1}{2}(1-x)V_1^{'}(x), \quad a_2 = -\frac{\beta_2}{2}xV_2^{'}(x) \tag{E.54}$$

For $M \in (0,1)$ these equations can be rewritten as

$$V_1^{'}(M) = \frac{2a_1}{\beta_1(1-x)}, \quad V_2^{'}(x) = -\frac{2a_2}{\beta_2 x} \tag{E.55}$$

- Substitute for ui from E.54) in (E.52), (E.53) to obtain two differential equations involving $V_i(x)$, $V_i^{'}(x)$ $i = 1, 2$, and x.
- Postulate a functional form for $V_i(x)$, $i = 1, 2$ —for example, a polynomial of a certain degree—and determine its coefficients by identification.

The main difficulty is that one cannot find such a functional form (at least nobody has found it yet); hence, two options are possible:

- The first is to obtain these value functions numerically.
- The second is to somehow simplify the HJB equations so that they become analytically tractable. The literature has adopted this last approach.
  For instance, to solve the HJB equations and derive a stationary MPNE, Chintagunta and Vilcassim (1992) use an equivalent system defined in terms of the advertising variables a_1 and a_2 rather than the value functions. Indeed, they use (E.55)) to replace $V_i(x)$ by their expressions in terms of a_i in the HJB equations. Assuming that the discount rate $r = 0$, then the HJB differential equations system is reduced to a simple system of two algebraic equations with two unknown variables, a_1 and a_2.

Jarrar et al. use a system of differential equations on V_i. They substitute a_i from from E.54) in (E.52), (E.53) to obtain

$$rV_1(x) = g_1 x + \left(\frac{\beta_1}{2}(1-x)V_1^{'}(x)\right)^2 + \frac{\beta_2^2}{2}x^2 V_1^{'}(x)V_2^{'}(x) \tag{E.56}$$

$$rV_2(x) = g_2(1-x) + \left(\frac{\beta_2}{2}xV_2^{'}(x)\right)^2 + \frac{\beta_1^2}{2}(1-x)^2 V_1^{'}(x)V_2^{'}(x) \tag{E.57}$$

after algebraic computations they obtain a system of $V_i^{'}$ on the RHS in terms of V_i on the LHS, which they solve numerically with ODE45 available in Matlab.

E.9 F. M. Bass, A. Krishnamoorthy, A. Prasad, S. Sethi, 2005 [8]

E.9.1 Model derived from

Lanchester. It includes brand and generic advertising.

E.9.2 Formulation

Change in primary demand Q is given by

$$\frac{dQ}{dt} = \dot{S}_1(t) + \dot{S}_2(t) = k_1 a_1(t) + k_2 a_2(t) \tag{E.58}$$

where S_i is the rate of change of the sales of F_i (firm i), a_i is the generic advertising of F_i, k_i is the effectiveness of that advertising and $i = 1, 2$.

The generic advertising effect on the sales of firm i is

$$\dot{S}_{i,g}(t) = \theta_i(k_1 a_1(t) + k_2 a_2(t)) \tag{E.59}$$

The brand advertising effect on the sales of firm i, which is a Lanchester model, is given by

$$\dot{S}_{i,b}(t) = \beta_i u_i(t)\sqrt{M(t) - S_i(t)} - \beta_j u_j(t)\sqrt{S_i(t)}, \quad i, j = 1, 2 \quad i \neq j \tag{E.60}$$

where u_i is the advertising of F_i, β_i is the effectiveness of the advertising of F_i.

The total change of sales is $\dot{S}_i = \dot{S}_{i,g}(t) + \dot{S}_{i,b}(t)$, i.e., variation due to generic plus variation due to brand advertising.

$$\begin{aligned}\dot{S}_i(t) = \beta_i u_i(t)\sqrt{M(t) - S_i(t)} - \beta_j u_j(t)\sqrt{S_i(t)} \\ + \theta_i(k_1 a_1(t) + k_2 a_2(t)), \quad S_i(0) = S_{i0} \quad i, j = 1, 2 \quad i \neq j\end{aligned} \tag{E.61}$$

The controls of F_i are its brand advertising $u_i(t)$ and its generic advertising $a_i(t)$. The discounted profit maximization problem is

$$V_i(S_1, S_2) = \max_{u_i(t), a_i(t), p_i(t)} \int_0^\infty e^{-r_i t}\left((1 - b_i p_i(t) + d_i p_{3-i}(t))p_i(t)S_i(t) - C(u_i(t), a_i(t))\right)dt, \tag{E.62}$$

where r_i is the discount rate, p_i is the price charged, b_i and d_i are demand parameters, the factor $(1 - b_i p_i(t) + d_i p_{3-i}(t))$ of $p_i(t)S_i(t$ is interpreted as the reduction in sales due to price competition and $C(u_i(t), a_i(t))$ is the total advertising expenditure of firm F_i. The expenditure in advertising is quadratic

$$C(u_i(t), a_i(t)) = \frac{c_i}{2}((a_i(t))^2 + (u_i(t))^2) \tag{E.63}$$

E.9.3 Solution

They obtain closed loop solutions.They obtain perfect Markov Nash equilibrium, i.e., they use an infinite time horizon. Work with symmetric and asymmetric firms. Symmetric is when the two firms have the same parameters, asymmetric otherwise. For firm i $i = 1, 2$ the optimal decisions are:

- Brand advertising

$$u_i^*(t) = \frac{\beta_i}{c_i}(\rho_i - \gamma_i)\sqrt{S_{3-i}(t)} \tag{E.64}$$

- Generic advertising

$$a_i^*(t) = \frac{k_i}{c_i}(\theta_i\rho_i - \theta_{3-i}\gamma_i) \tag{E.65}$$

- Price

$$p_i^*(t) = \frac{d_i + 2b_{3-i}}{4b_1b_2 - d_1d_2} \tag{E.66}$$

- Value function

$$V_i(S_1, S_2) = \alpha_i + \rho_i S_i + \gamma_i S_{3-i} \tag{E.67}$$

where $\alpha_1, \alpha_2, \rho_1, \rho_2, \gamma_1$ and γ_2 solve the systems of six equations

$$\begin{aligned} r_i\alpha_i - \frac{k_i^2}{2c_i}(\theta_i\rho_i + \rho_{3-i}\gamma_i)^2 \\ - \frac{k_{3-i}^2}{c_{3-i}}(\theta_1\rho_1 + \theta_2\gamma_1)(\theta_2\rho_2 + \theta_1\gamma_2) = 0 \quad i = 1, 2 \end{aligned} \tag{E.68}$$

$$r_i\beta_i - m_i + \frac{\beta_{3-i}^2}{c_{3-i}}(\rho_1 - \gamma_1)(\rho_2 - \gamma_2) = 0 \quad i = 1, 2 \tag{E.69}$$

$$r_i\gamma_i - \frac{\beta_i^2}{2c_i}(\rho_i + \gamma_i)^2 = 0 \quad i = 1, 2 \tag{E.70}$$

Krishnamoorthy et al. conclude that the impact of generic brand advertising is limited and that brand advertising has more effects on the market shares in the long run.

E.10 Q. Wang, Z. Wu, 2007 [9]

E.10.1 Model derived from

Lanchester.

Empirical study of Lanchester. Issues considered are: i) specification of market share response model ii) effect of inflation iii) performance of competitive strategies. It is shown a) square root in market share response equation is often inappropriate b) market share variations are more responsive to current advertising c) closed-loop Nash equilibrium strategies are better than open-loop strategies for maximizing profit d) general perfect equilibria

E.10.2 Model

$$\max_{A_i \geq 0} J_i = \int_0^{t_f} e^{-rt}(S_{inst} pf_i x_i - u_i)dt + g_i x_i(t_f) \tag{E.71}$$

subject to

$$\frac{dx_i}{dt} = \beta_i(u_i(t))^{\alpha_i}(1 - x_i) - \beta_j(u_j(t))^{\alpha_j} x_i, \quad i, j = 1, 2 \quad i \neq j \tag{E.72}$$

where S_{inst} is the instantaneous total sales in the market in dollars at time t, r is the discount rate, x_i is the market share of firm i, pf_i is the unit profit margin of firm i, g_i is the valuation of the ending market in present value, $i = 1, 2$ and the planning interval is $[0, T]$, the adverstising expenditure of firm i is u_i. Perfect equilibria by Case uses an infinite planning horizon.

Little 1979, used $0 < \alpha_i < 1$ *to obtain a sales response that is concave from a Lanchester model.*

Difference with 2001, the exponent α_i, α_j *on the advertising controls. Also discount factor.*

Remark:

The introduction of the power function generalizes the Lanchester model but makes more complex. To overcome this, later studies use the square root function $\alpha_i = 0.5$ for mathematical tractability without empirical validation (Deal 1979, Sorger 1989, Chintagunta Vilcassim 1992)

CLNE solutions allow decision makers to react to current situations and are more appropriate for marketing problems. However, general CLNE solutions have been rarely discussed. Erickson 1992 and Chintagunta-Vilcassim 1992 use perfect equilibria closed-loop Nash equilibrium PECLNE solution developed by Case 1979. PECLNE is a special case of CLNE. PECLNE depends on states alone and they are obtained using an infinite time horizon. Empirical studies by Erickson 1992 and Chintagunta-Vilcassim 1992 showed that they fit better the data analyzed than OLNE.

With the present state of knowledge, solving OLNE or CLNE is difficult. Numerical algorithms have been developed to obtain general OLNE or CLNE solutions to dynamic competitive advertising models (Deal 1979, Wang Wu 2001)

E.10.3 Solution

They solve open-loop Nash equilibrium (OLNE), closed-loop Nash equilibrium (CLNE), perfect equilibria closed-loop Nash equilibria (PECLNE) using the algorithm in Wang, Wu 2001, which is a numerical algorithm.

In their empirical studies they use nonzero discount factor and found that

- market share variations are more responsive to current advertising
- CLNE are better competitive advertising strategies for firms maximizing profits
- PECLNE equilibrium strategies are usually not good competitive advertising strategies for firms maximizing profits
- square roots of advertising in the system dynamic are not suitable for all industries

E.11 A. Krishnamoorthy, A. Prasad, S. P. Sethi, 2010 [10]

E.11.1 Model derived from

Vidale-Wolfe. The model combines price and advertising for a duopoly differential game.

E.11.2 Formulation

$$\max_{u_1(t),p_1(t)} = \int_0^\infty e^{-r_1 t}((p_1 - m_1)\dot{s}_1(t) - \frac{c_1}{2}u_1^2(t))dt \tag{E.73}$$

$$\max_{u_2(t),p_2(t)} = \int_0^\infty e^{-r_2 t}((p_2 - m_2)\dot{s}_2(t) - \frac{c_2}{2}u_2^2(t))dt \tag{E.74}$$

subject to

$$\dot{s}_1(t) = \rho_1 u_1(t)\sqrt{M - S_1(t) - S_2(t)}D_1(p_1(t)) \tag{E.75}$$

$$\dot{s}_2(t) = \rho_2 u_2(t)\sqrt{M - S_1(t) - S_2(t)}D_2(p_2(t)) \tag{E.76}$$

- $S_i(t)$ is the cumulative sales of firm i at time t
- M is the market potential

- $u_i(t)$ is the advertising effort of firm i at time t
- $p_i(t)$ is the price of firm i at time t
- c_i is the coefficient of advertising cost of firm i
- β_i is the effectiveness of advertising of firm i
- m_i is the marginal cost of production of firm i
- α_i is the demand intercep of firm i (linear demand)
- ϕ_i is the price sensitivity of firm i (linear demand)
- η_i is the price elasticity of firm i (isoelastic demand)
- r_i discount factor of firm i
- $V_i(s_i, s_j)$ is the value function for firm i when its cumulative sales is s_i and the cumulative sales of its competitor is s_j
- D_i is the demand function for firm i

E.11.3 Solution

Closed-loop solution using the assumption of a constant value for the derivative of the value function and two forms for the demand function: linear and isoelastic. The feedback strategies they find are:

$$p_i^*(s_i, s_j) = \frac{1}{2}\left(\frac{\alpha_i}{\phi_i} + m_i - \frac{\partial V_i}{\partial S_i}\right) \tag{E.77}$$

$$u_i^*(s_i, s_j) = \frac{\beta_i}{4c_i\phi_i}\left(\alpha_i + \phi_i\left(\frac{\partial V_i}{\partial S_i} - m_i\right)\right)^2 \sqrt{M - S_i - S_j} \tag{E.78}$$

- They assume a constant value for the derivative of the value function and two forms for the demand function: linear and isoelastic.
- The solution is analytical in term of constants, which are calculated from a system of equations.
- They obtain an analytical solution of the infinite horizon formulation.
- The symmetric case, when all parameters are the same for both firms $c_i = c_j = c, r_i = r_j = r, q_i = q_j = q, m_i = m_j = m, \alpha_i = \alpha_j = \alpha$, and $\phi_i = \phi_j = \phi$ is solved analytically.
- The case where firms can have different the parameter values are obtained numerically.

E.12 S. Jorgensen, G. Martín-Herrán, G. Zaccour, 2010 [11]

E.12.1 Leitmann-Schmitendorf model

The Leitmann-Schmitendorf model has a decay like Vidale-Wolfe and a cross term from Lanchester. Its system dynamics is

$$\dot{x}_i(t) = u_i(t) - \frac{c_i}{2}u_i^2 - k_i u_j(t)x_i(t) - \delta_i x_i(t) \tag{E.79}$$

where $x(t)$ is the sales rate at time t, c_i, δ_i, k_i are positive parameters. The effect of advertising is $u_i(t) - \frac{c_i}{2}u_i^2$, it has a decay like Vidale-Wolfe $\delta_i x_i(t)$ and a Lanchester term $k_i u_j(t)$. The objective functional is

$$J_i(u_i) = \int_0^{t_f} (p_i x_i(t) - u_i(t))dt \tag{E.80}$$

where $p_i > 0$ is the price per unit sold an u_i is the rate of advertising expenditure.

E.12.2 Tractability of calculating the equilibrium

Feedback Nash equilibrium allows players to take decisions throughout the game in contrast to open-loop equilibrium where the players choose their strategies at the beginning without changing it during the game. However, open-loop Nash equilibrium has been popular in applications because it is generally much easier to compute. *Facing the trade-off between strategic appeal of feedback equilibrium and the tractability of open-loop equilibrium, researchers have looked for game structures where an open-loop equilibrium is subgame perfect, in an attempt to get the best of the two equilibrium types. The Leitmann-Smittendorf differential game belongs to the class of State-Redundant differential games. In this class of games, the open-loop Nash equilibrium is subgame perfect.*

E.12.3 Linear and state-redundant differential games

To define the notion of state redundancy, first consider an N player differential game played on the interval $[t\_0, T]$. Let the control $u_i(t) \in U_i \subseteq \mathcal{R}^m$ and the state $x_i(t) \in X \subseteq \mathcal{R}^p$. The state equations describing the motion are

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0 \tag{E.81}$$

where $u(t) = (u_1(t), u_2(T), \ldots, u_N(t)$. The payoff functional is

$$J_i(u, t_0, x_0) = \int_{t_0}^{t_f} g_i(x(t), u(t), t)dt + S_i(x(t_f), t_f) \tag{E.82}$$

where function g_i is player i's instantaneous payoff and S_i. The Hamiltonian of player i is

$$H_i(x, u, \lambda_i, t) = g_i(x, u, t) + \lambda_i f(x, u, t) \tag{E.83}$$

where $\lambda_i(t)$ is a $p-$dimensional vector of costate variables. Candidates for Nash equilibrium have to satisfy the following necessary conditions

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0 \tag{E.84}$$

$$\dot{\lambda}_i(t) = -\frac{\partial H_i(x(t), u(t), \lambda_i(t), t)}{\partial x}, \quad \lambda_i(t_f) = \frac{\partial S_i(x(t_f), t_f)}{\partial x(t_f)} \tag{E.85}$$

$$\frac{\partial H_i(x(t), u(t), \lambda_i(t), t)}{\partial u_i} = 0, \quad i = 1, \ldots, N \tag{E.86}$$

E.12.4 Linear state games

Linear state games were introduced by Dockner et al. as the *state-separable*. The state-separability conditions are:

$$\frac{\partial^2 H_i}{\partial u_i \partial x} = 0, \text{ for } \frac{\partial H_i}{\partial u_i} = 0, \quad \frac{\partial^2 H_i}{\partial x^2} = 0, \quad \frac{\partial^2 S_i}{\partial x^2} = 0 \tag{E.87}$$

for a weaker version of the condition $\frac{\partial^2 H_i}{\partial u_i \partial x} = 0$ see Dockner et al.

Cleamhaut and Wan introduced the trilinear game, which is a kind of linear state game, it is the first model where the open-loop Nash equilibrium is subgame perfect.A game is called trilinear when the Hamiltonians are linear in state and costate variables as well as functions of the control variables.

There are many applications of linear state games in economics and management: Breton et al., Jorgensen and Zaccour, Jorgensen et al., Martín-Herrán and Zaccour, Viscolani and Zaccour, Cellini and Lambertini.

Exponential games are also a subclass of linear state games. In this class the objective is

$$\int_{t_0}^{t_f} g_i(u(t), t) e^{-mu_i x(t)} dt \tag{E.88}$$

where $\mu_i \in \mathcal{R}^p$ are constants. The state equations are

$$\dot{x}(t) = f(u(t), t), \quad x(t_0) = x_0 \tag{E.89}$$

Note that the state dynamics depend only on the controls and the state variables enter the objective functions in the exponent. Dockner et. al used the transformation $y_i(t) = e^{-mu_i x(t)}$. Jorgensen let the control variables enter the into the dynamics and the objective in exponential form and uses the same procedure as Leitmann-Schmitendorf to show that the OLNE is a FNE which is subgame perfect. Yeung extends the game in Jorgensen to a setting where the Hamiltonians are not required to be linear in the state.

E.12.5 State-redundant games

Leitmann-Schmitendorf initiated the state-redundant differential games. A differential game is state-redundant if the following holds: *If after the substitution for the solution of the costate, obtained from equation (E.85), in the Hamiltonian-maximization conditions (E.86), these hamiltonian equations are independent of the state variables and of their initial values, then the game is state-redundant* Note that state-redundant games are state-separable, but state-separable games are not necessarily state-redundant games. Leitmann-Schmitendorf models and Feichtinger model are examples of state-redundant games.