ANALYSIS OF THE DYNAMICS OF DUOPOLY MODELS UNDER STATE FEEDBACK ADVERTISING POLICIES WITH SWITCHING

Rolando Cuevas Núñez


Orientador: Amit Bhaya

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ANALYSIS OF THE DYNAMICS OF DUOPOLY MODELS UNDER STATE FEEDBACK ADVERTISING POLICIES WITH SWITCHING

Rolando Cuevas Núñez

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Examinada por:

Prof. Amit Bhaya, Ph.D.

Prof. André Barreira da Silva Rocha, Ph.D.

Prof. Tiago Roux de Oliveira, D. Sc.

Prof. Alessandro Jacoud Peixoto, D. Sc.

Prof. Eugenius Kaszkurewicz, D. Sc.

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Cuevas Núñez, Rolando


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ANÁLISE DA DINÂMICA DE MODELOS DE DUOPOLIO SOB POLÍTICAS DE RETROALIMENTAÇÃO DE ESTADO CHAVEADAS

Rolando Cuevas Núñez

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Orientador: Amit Bhaya
Programa: Engenharia Elétrica

Esta tese analisa a utilização de políticas competitivas de publicidade em uma classe proposta de modelos dinâmicos de duopólio, inspirados no modelo clássico Vidale-Wolfe. Uma suposição básica é de que a empresa com uma menor participação no mercado chaveia para um controle que utiliza um esforço extra para alcançar a firma concorrente. Investigam-se padrões de dinâmica dos modelos propostos, sujeitos a políticas de publicidade competitivas baseadas nas variáveis de estado observadas, calculando-se um índice de desempenho associado. Mostra-se como utilizar este índice em conjunção com a análise qualitativa da dinâmica para prever o resultado de um determinado cenário, para eventual tomada de decisão.
This thesis investigates the effect of competitive advertising policies in a proposed class of duopoly dynamic models, inspired on the classical Vidale-Wolfe model. A basic assumption is that the firm with a lower market share than its target market share level switches on an extra control effort in its attempt to reach and surpass its competitor. Dynamical patterns of the proposed models, subject to competitive advertising policies which are based on the observed state variable, are investigated. An associated performance index is calculated. This index in conjunction with the qualitative analysis of the dynamics should guide the management decision on advertising spending.
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B.1 The fields $f_U$ or $f_L$ approaching the equal market share point $E$ from the left can be as shown. In the case of the dashed line, the slope is less than unity and the field indicates that the trajectory comes from $U$. In the case of the continuous line, the slope is greater than unity and the field indicates that the trajectory comes from $L$.

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List of Symbols

[F] Firm. 5
[G] Goodwill. 4
[H] Hamiltonian. 9
[J] Performance index. 8
[M] Saturation of the sales rate, the total potential market size. 5
[R] Revenue coefficient. 8
[S(t)] Sales rate at time t. 5
[V] Value function. 109
[β_i] Constant, effectiveness of advertising of F_i. Response constant. 5
[δ] Diminishing or decaying rate. 5
[λ] Costate. 9
[J_i(x_1, x_2)] Jacobian matrix of firm F_i, which is a function of the states x_1, x_2. 25
[i] Firm index i = 1, 2, 5
[p] Price. 4
[q] Quantity of products to sell. 4
[r] Discount factor, used to convert future values to present values. 93
[t] Time. 5
[u] Advertising. 4
[u_{ij}] Predatory advertising on F_i made by F_j, i ≠ j, i, j = 1, 2. 7
[x_i] State variable, market share of firm F_i unless otherwise stated. 7
[z] Other factors. 4
Chapter 1

Introduction

A duopoly refers to a situation in which two companies own all or nearly all of the market for a given product or service. A duopoly is the most basic form of oligopoly, which refers to a market dominated by a small number of companies. A duopoly can have the same impact on the market as a monopoly if the two players collude on prices or output. Collusion results in consumers paying higher prices than they would in a truly competitive market and is illegal under antitrust laws in many countries. For example, Boeing and Airbus have been called a duopoly for their command of the large passenger airplane market. Similarly, Amazon and Apple have been called a duopoly for their dominance in the e-book marketplace [12]. A central feature of oligopoly markets, is competitive interdependence: the decisions of every firm significantly affect the profits of competitors. In contrast, each seller in a perfectly competitive market is so small that it has an imperceptible competitive impact on its rivals [13].

Advertising has been and continues to be an important topic for academic research, in both marketing and management science, as attested by the appearance of recent surveys devoted to the topic in both areas. As Erickson [14] writes “A critical aspect of the advertising budgeting process involves competitive issues – anticipated spending levels of major competitors, effects that competitive advertising may have on the firm’s market share, sales, and profit, and the interactive nature of a competitor’s advertising with a firm’s own. Competition is ignored only at the firm’s peril; empirical studies (e.g., Little 1979) have shown quite clearly that competitive advertising can have a direct, and negative, effect on a company’s market share.”

Since a competitive scenario with strategic decisions being made by two competing firms is usually described by the word duopoly, this term is also used while describing dynamic models of advertising competition. However, it is important to point out here that, in the simplest case of two firms producing an identical product, the normal use of the term duopoly in the economics literature involves the use of product price, implicitly or explicitly. This is the case with the three most popular models of duopoly behavior, namely quantity leadership (Stackelberg), price leadership, simultaneous quantity setting.
(Cournot), simultaneous price setting (Bertrand), the terms in parentheses are the names of the economists who introduced these models which now bear their names [15]. In contrast, as we will see, many models of advertising competition focus on advertising effort (as a control) and its effect on market share (controlled variable), without explicitly taking product price into account. This thesis will concentrate on such models, in order to avoid the complications of price dynamics and emphasize the control aspects and strategic interactions.

Specifically, we consider a duopolistic industry in which two firms are engaged in a brand competition using advertising policies which may have a predatory effect on the opponent’s brand. If an increase of advertising in one firm causes a decrease in the sales of its rivals, then that advertising can be called predatory, although the firm may gain more than the sum of the losses of the other firms [16, p. 144, chap. 6]. In comparative advertising a brand can be implicitly or explicitly compared two or more products. Although it is legitimate to compare a product favorably against other products, it can be challenged by law in countries such as the US [17, p. 71, chap. 3]. Alternatively, predatory advertising can be an inevitable side effect of market saturation – in other words, if one firm gains clients, it is because the other has lost them.

1.1 Structure of the thesis

In the remainder of this chapter we review models of advertising and models of duopolies that use advertising in the definition of their dynamics. We also include one way of formulating the differential game duopoly approach and the policies obtained for nonprice duopoly models using differential games. This should help the reader contrast the approach proposed in this thesis with differential games applied to duopolies.

In chapter 2, we discuss our approach to the study of advertising policies in a duopoly. In brief, the approach consists in using simple implementable advertising policies in a given model and with them identify initial conditions which lead to a firm fulfilling its target market share. At the same time, we calculate an associated performance index for the dynamical system. The obtained market share and the associated index value should guide management on the decision of adopting or not a given advertising policy. Note that our approach is complementary to differential games approach where a control function that maximizes a performance index is calculated.

In chapters 3, 4 and 5 the proposed approach is applied to different duopoly models of advertising. The models in chapters 3 and 4 are extensions of the models Vidale-Wolfe and Deal. In chapter 3, a complete description of the dynamics for the model VWDsC is given. In chapter 4, more modifications to Vidale-Wolfe are studied. In chapter 5, we consider the model Lotka-Volterra with switched predation which is a model based on the Lotka-Volterra model for competing species.
The final chapter 6 summarizes the contributions and indicates some possible future work.

1.2 Literature review

We will first briefly review the different models and approaches in the literature with regard to competitive advertising in a duopoly. Broadly speaking, there are models which study the dynamics of market shares or sales of the competing firms and those that study a hypothetical quantity called consumer “goodwill” under the effect of different types of advertising. There are approaches which consider only the dynamics of the market share or sales and others that consider the interaction of prices with the market dynamics. Finally, in terms of the mathematical tools used to study the different classes of models, there are two main approaches: (i) using the tools of dynamical systems, and (ii) using differential games.

We first present the different classes of models that have been proposed, followed by a brief overview of differential game approaches as well as a critique of these approaches, leading to our proposal, which is centered on a dynamical systems approach that seeks to model some qualitative and behavioral aspects of complex advertising decision making, in order to capture some of its important aspects. Thus, the possible value of the our approach lies in its adequacy as a model of aggregate advertising decision making, its mathematical tractability and its potential implementability by real decision makers.

1.2.1 Dynamic models of response to advertising in duopolies

One of the earliest mathematical models of advertising was proposed by Dorfman and Steiner [18], they assumed that a firm makes two kinds of choices: the price of its product and the amount of its advertising budget. They related the quantity the firm can sell per unit of time, $q$, its price, $p$, and its advertising variable, in this case advertising budget, $u$, as a function

$$q = f(p, u)$$

and used graphical methods and calculus to derive a basic qualitative result stated as follows: A firm which can influence the demand for its product by advertising will, in order to maximize its profits, choose the advertising budget and price such that the increase in gross revenue resulting from a one dollar increase in advertising expenditure is equal to the ordinary elasticity of demand for the firm’s product. On the other hand, they conclude their article by saying “There are good grounds for doubting the economic significance of the whole business of writing down profit functions (or drawing curves) and finding points of zero partial derivatives (or graphical points of tangency). Such devices are merely aids to thinking about practical problems and it may be an uneconomical expenditure of ef-
fort to devote too much ingenuity to developing them. Yet such devices are aids to clear thought and, if sufficiently simple and flexible, they help us find implications, interrelationships, and sometimes contradictions which might escape notice without them. Such aids are particularly needed in the field of nonprice competition” (emphasis ours).

We will work with models for advertising dynamics in nonprice competition in a duopoly. We continue with a literature review of models of advertising dynamics, briefly reviewing the various classes of models, in accordance with the broad classifications price and nonprice models.

1.2.2 Dynamic duopoly models involving price and advertising effort

In 1962, shortly after publication of the Dorfman-Steiner model, Nerlove and Arrow [19] introduced their model for the goodwill stock of a firm. It is sometimes referred to as a two level model in the sense that another variable of interest is derived from the goodwill. Advertising (the control variable) affects the goodwill $G$ directly and goodwill then determines, through a functional relationship, the main variable of interest $q$ which is the quantity to be produced (and sold) by the firm. The corresponding equations are:

\[ \dot{G}(t) = u(t) - \delta G(t) \]  
\[ q = f(p, G, z) \]

Where $G$ is the goodwill, $u(t)$ is the advertising and $\delta$ is a decay factor, the quantity $q$ to produce or sell as a function of the price $p$, the goodwill $G$ and other factors represented by $z$.

Finally, we briefly mention the class of Cournot duopoly models with advertising, although since they consider price, they will not be considered further in this proposal. Cournot duopoly [15, p. 507, sec. 27.5] is very important in oligopoly theory but usually it is not considered part of marketing. In a Cournot duopoly, there is a single price affected by the total production of the firms. Cellini and Lambertini, in 2003, used an inverse demand function $p(t) = (res(t) - q(t))^{\frac{1}{\alpha}}$ where $p(t)$ is the price at time $t$, $res(t)$ is the consumer reservation price, $q(t)$ is the total quantity produced by the firms and the parameter $\alpha \in (0, \infty)$ determines the curvature of the demand. Their assumption is that $res(t)$ can be increased by firms advertising efforts $u_i(t)$. They use a dynamic similar to Nerlove-Arrow $\dot{res}(t) = \sum_{i=1}^{n} u_i(t) - \delta res(t)$, $res(0) = res_0 > 0$ [20, p. 130-132, sec. 6.2]. Escobido and Hatano derive a quantity production adjustment process in a Cournot duopoly that is equivalent to a Lotka-Volterra competing species equation [21]. Another type of models is dynamic games with price and advertising, for a discussion see [20, p. 132-141, sec. 6.3].
1.2.3 Dynamic duopoly models involving only advertising effort (nonprice models)

In 1957, Vidale and Wolfe [22] carried out detailed analyses of real firms and advertising data in order to propose their well known eponymous model:

\[
\frac{dS(t)}{dt} = \beta u(t) \frac{M - S(t)}{M} - \delta S(t) \tag{1.4}
\]

where \(S(t)\) is the sales rate at time \(t\), \(\beta\) is a response constant, \(M\) is the saturation of the sales rate, \(u(t)\) is the advertising rate and \(\delta\) is a decay constant of the sales.

Another important class of models are derived from the Lanchester model of human warfare [23] which was first applied to advertising competition by Kimball [24], and reapplied to oligopolies by Little [25] in 1979. Little presented the following Lanchester model

\[
\begin{align*}
\dot{S}_1 &= \beta_1 u_1 S_2 - \beta_2 u_2 S_1 \tag{1.5} \\
\dot{S}_2 &= \beta_2 u_2 S_1 - \beta_1 u_1 S_2 \tag{1.6}
\end{align*}
\]

where \(S_i\) is the sales rate of firm \(F_i\), \(u_i\) is the advertising rate of \(F_i\), \(\beta_i\) is the advertising effectiveness constant of \(F_i\), the total market sales is \(M = S_1 + S_2\). The sales of \(F_i\) increases proportionally to its competitor’s sales \(S_j\) and decreases proportionally to its own sales \(S_i\). The situation is symmetric for the opponent. Notice that the substitutions \(S_1 = S\), \(S_2 = M - S\), \(\beta_1 M = \beta\) and \(\beta_2 u_2 = \delta\) in (1.5) result in the Vidale-Wolfe monopoly model, so that the Lanchester model can be seen as a generalization of (1.4) [25].

Little [25] argued that a good model of advertising should include the sales response to advertising expenditures, the carryover effect of past advertising on current sales, as well as the possibility of diminishing returns to cumulative advertising expenditures and generalized the Lanchester model to comply with the response having the diminishing returns property:

\[
\begin{align*}
\dot{S}_1 &= \beta_1 u_1^{\alpha_1} S_2 - \beta_2 u_2^{\alpha_2} S_1 \tag{1.7} \\
\dot{S}_2 &= \beta_2 u_2^{\alpha_2} S_1 - \beta_1 u_1^{\alpha_1} S_2 \tag{1.8}
\end{align*}
\]

where \(0 < \alpha_i < 1\) corresponds to a concave response in \(u_i\) and \(\alpha_i > 1\) corresponds to an S-shaped response in \(u_i\).

There have been several reviews of models of advertising, one of the first by Sethi in 1977 [26] was mainly on monopolies. Later, reviews by Feichtinger et al. in 1994 [27], Erickson in 1995 [28] focused mainly on duopolies for which open-loop advertising strategies were computed. In a more recent survey by Huang et al. 2012 [29] although
duopolies are mentioned, the main focus is on oligopolies and supply chains, which are in many cases one manufacturer and two resellers. More recent surveys on differential games focus on supply chains [30], [31].

The models for the dynamics of many of the duopolies and of the oligopolies proposed in the literature were derived from Nerlove-Arrow [19], Diffusion [32], Vidale-Wolfe [22] and Lanchester [24] [25]. In 2012, Bhaya and Kaszkurewicz proposed a unified model inspired by the Vidale-Wolfe [33] and a variation of it in [34]. In 2014, Wang et al. proposed a new dynamic model of advertising based on Lotka-Volterra [35]. Our focus is on models that consider the sales response to advertising, we will study the models similar to the models proposed by Bhaya and Kaszkurewicz [33, 34], and the model proposed by Wang et al. [35].

Sales response to advertising are models derived from Vidale-Wolfe or Lanchester models. One of the first was presented by Deal [1] and it is a simplified model of the dynamics of a duopoly, as an extension of the monopoly model of Vidale and Wolfe. We will discuss Deal’s model, given below, as a starting point for the introduction of our model later on:

\[
\begin{align*}
\dot{s}_1(t) &= -\delta_1 s_1(t) + \beta_1 u_1(t)[M - s_1(t) - s_2(t)]/M \\
\dot{s}_2(t) &= -\delta_2 s_2(t) + \beta_2 u_2(t)[M - s_1(t) - s_2(t)]/M
\end{align*}
\]

(1.9)

(1.10)

where \( s_i(t) \) is sales rate for firm \( F_i \) at time \( t \), \( u_i(t) \) is the advertising expenditure for \( F_i \) at time \( t \), \( \delta_i \) is the sales decay parameter, \( \beta_i \) is the sales response parameter and \( M \) the total potential market size. In this model there is an expression for the sales response of each competitor, and the effect of competition is also recognized by the introduction of the coupling term \( (M - s_1 - s_2)/M \) in each equation. The coupling term can be interpreted as follows. The direct effect of competitive pressure is that, as the sales to all other market parties increase, the size of the remaining unconquered market decreases and this in turn diminishes the sales effectiveness of successive advertising expenditures for the firm.

Wang et al. [6] combined Vidale-Wolfe and Lanchester models as follows

\[
\frac{dx_i}{dt} = \beta_i u_i(1 - x_i) - \beta_j u_j x_i - \delta_i x_i, \quad i, j = 1, 2 \quad i \neq j \quad x_i(0) = x_{i0}
\]

(1.11)

where \( x_i = s_i/M \) is the market share of firm \( F_i \), the cross term \(-\beta_j u_j x_i\) is from the Lanchester model and the terms \( \beta_i u_i(1 - x_i) - \delta_i x_i \) from Vidale-Wolfe. The Lanchester model was used by many authors to model the dynamics on the saturated market \((x_2 + x_1 = 1)\) [6]. Wang et. al lifted that restriction so that the market shares could grow from small initial values to saturation \((0 \leq x_1 + x_2 \leq 1)\). In 2014, Hung et al. [36] used a Lotka-Volterra model for the dynamics on the saturated market.
Bhya and Kaszkurewicz [33] proposed a unifying model derived from Vidale-Wolfe

\[ \dot{x}_1 = u_{11}(1 - x_1 - x_2) - u_{12}(x_1 + x_2) \]  
\[ \dot{x}_2 = u_{22}(1 - x_1 - x_2) - u_{21}(x_1 + x_2) \]

where \( u_{ii} \) is the positive advertising acting on the unconquered market \((1 - x_1 - x_2)\) and \( u_{ij}, \ i, j = 1, 2 \ i \neq j \) is the predatory advertising, i.e., the advertising of firm \( j \) acting on the conquered market \((x_1 + x_2)\).

In an extension of their previous model, Bhaya and Kaszkurewicz [34] assume that the loss of one firm is the gain of the other. The dissertation by Cruz [37] uses a (CLF) Lyapunov Control Function for the model defined by Bhaya and Kaszkurewicz in [34].

Recently Wang et al. proposed an aggregate model for a duopoly. In the first level advertising generates the maximum sales that could be achieved for each firm. In the second level the sales follow a Lotka-Volterra competing species dynamics in which each firm can grow its sales until its corresponding maximum. The sales response to advertising model based on Lotka-Volterra proposed in [35] is

\[ \dot{S}_1 = S_1(b_1 - a_{11}S_1 - a_{12}S_2) \]  
\[ \dot{S}_2 = S_2(b_2 - a_{21}S_1 - a_{22}S_2) \]

where \( S_i \) is the sales of firm \( i \) at time \( t \), \( b_i = f(u_i) \) is the intrinsic sales growth and \( u_i \) is the advertising level of firm \( i \), \( a_{ii} \) is the growth restriction of firm \( i \) on itself and \( a_{ij} \) is the growth restriction on firm \( i \) created by firm \( j \). The term having the growth restriction \( a_{ii} \) is also known as an overcrowding term in ecology while the term having \( a_{ij} \) is referred to as a predation term.

1.2.4 Differential games of advertising

Apart from the model, the other important ingredient required for a mathematical analysis of advertising dynamics is the stipulation of an objective function for a firm. This is a strategic managerial task and not an easy one, because it needs to be time varying, in response to the life cycle of the product, to competitive pressures and so on. In addition, objectives of competing firms may differ. For instance, one firm may be interested only in maximizing its total profit over the fiscal year and its competitor may be interested only in maximizing its market share by the end of the year. Evidently, different objectives result in different advertising strategies. These two objectives could be weighted and additively combined into one as is common in optimal control theory, leading to a performance index. Deal [1] proposed one such performance index, which he called a tempered performance index in order to highlight the fact that the choice of weight can
be used to trade off (or temper) market share and profit:

\[ J_i = w_i S_i(t_f)/[S_i(t_f) + S_j(t_f)] + \int_{t_0}^{t_f} [R_i S_i(t) - u_i^2(t)] dt \] (1.16)

where \( i \) refers to one of the firms and \( j \) to the other, \( S_i \) is the sales, \( R_i \) is the net revenue coefficient, \( w_i \) the weighting factor for the performance index, \( t_0 \) the initial time of the planning horizon and \( t_f \) the final time of the planning horizon.

As Deal points out, his model is a simple formulation of a complex advertising decision, but one that represents the important characteristics of advertising decision making. Furthermore, he states explicitly that the real value of the model lies in its adequacy as a normative model of aggregate advertising decision making and in its approachability for solution by differential games techniques.

When each firm optimizes its performance index subject to the system dynamics, the firms are said to be participating in a differential game. Suppose that the performance index \( J_i \) is associated with firm \( i \). In the finite horizon case, the objective has two terms: the profit from time \( t_0 \) until \( t_f \) plus a salvage value at final time \( t_f \).

\[ J_i = \int_{t_0}^{t_f} g_i(x_1, \ldots, x_n, u_1, \ldots, u_n, t) dt + h_i(x_i(t_f), t_f), \quad i = 1, \ldots, n \] (1.17)

or, in the case of infinite horizon, the objective has one term: the profit from time 0 until \( \infty \).

\[ J_i = \int_{t_0}^{\infty} g_i(x_1, \ldots, x_n, u_1, \ldots, u_n, t) dt, \quad i = 1, \ldots, n \] (1.18)

The firm \( i \) attempts to maximize \( J_i \) subject to the dynamics

\[ \dot{S}_i = f_i(x_1, \ldots, x_n, u_1, \ldots, u_n, t), \quad x_i(0) = x_{i0}, \quad i = 1, \ldots, n \] (1.19)

where the controls \( u_i \) are advertising rates, the state variables \( x_i \) are market shares or sales.

Assuming that competitors cannot collude and that each of them has complete information on the best strategies of its opponent, or that it can infer them, Nash equilibria are sought. Informally, a Nash equilibrium is a set of strategies where one player cannot improve its outcome unilaterally. Nash equilibria can be found using Hamilton-Jacobi-Bellman-Isaacs equations, obtained from the objective (1.17) and the constraints (1.19).

The Hamiltonian for the differential game just described is [28]:

\[ H_i = g_i + \sum_{j=1}^{n} \lambda_{ij} \dot{x}_j, \quad i = 1, \ldots, n \] (1.20)
where $\lambda_{ij}$ are called costate variables. The necessary conditions for a Nash equilibrium are given by
\[
\frac{\partial H_i}{\partial u_i} = 0, \quad i = 1, \ldots, n
\] (1.21)
and for a given a finite horizon $T$:
\[
\dot{\lambda}_{ij} = -\frac{\partial H_i}{\partial x_j} - \sum_{k \neq i} \frac{\partial H_i}{\partial u_k} \frac{\partial u_k^*}{\partial x_j}, \quad i, j = 1, \ldots, n
\]
\[
\lambda_{ii}(t_f) = \frac{\partial H_i}{\partial x_i} \bigg|_{t=t_f}, \quad i, j = 1, \ldots, n
\] (1.22)
\[
\lambda_{ij}(t_f) = 0, \quad i, j = 1, \ldots, n
\]
where $u_k^*$ are Nash equilibrium strategies. If there is an infinite horizon, the constraints on the costates $\lambda_{ii}(t_f), \lambda_{ij}(t_f)$ are replaced by an assumption that the problem approaches a steady state.

There are two types of Nash equilibria: open-loop and closed-loop. Open-loop equilibrium strategies depend only on time $t$. Closed-loop strategies depend on states $S_i$ as well as time $t$. A closed-loop equilibrium that does not depend on initial conditions is called a feedback strategy. Finding closed loop strategies is difficult because the sum of partial derivatives in (1.22) is nonzero in general and the system of differential equations are not solvable in many cases. The closed-loop strategies can be determined only in special cases.

Remark: Even though many of the early models were descriptive, they were conducive to the use of differential game theory in order to derive the associated optimal dynamic advertising policies. In fact, most of the economics and management literature is focused on designing management strategies that optimize some pre-determined performance indices, using tools from differential game theory, to investigate Nash equilibria that represent some possible equilibrium market shares [38–40], and the recent surveys [29], [30], [31]. Several studies also emphasize the appearance of chaos in duopolistic or oligopolistic models (see, e.g., [41], [42] and references therein).

1.2.5 Duopoly differential games with dynamics derived from Vidale-Wolfe or Lanchester models

Table 1.1 lists duopoly differential games that use Vidale-Wolfe or Lanchester dynamics. We can make the following observations from the entries in the table:

- Note that the feedback strategies found in most of the cases use the approach suggested by Case in 1979 [2]. Only Bass et al. [8] and Krishnamoorthy et al. [10] do not cite Case, however they also use the value function approach with infinite horizon and make assumptions about the form of functions. Krishnamoorthy et al. uses
a value function $V$ that is linear $(V(S_i, S_j) = k_i(T - S_i - S_j))$ where $i, j = 1, 2$ $i \neq j$ and two types of demand function, one linear and the other isoelastic. Bass et al. uses a value function $V$ that is linear $(V_i(S_1, S_2) = \alpha_i + \rho_i S_i + \gamma_i S_{3-i} \text{ where } i = 1, 2$ and $\alpha_i, \rho_i, \gamma_i \in \mathbb{R}$).

- In most of the cases the open-loop solution is computed numerically.
- The last entry in Table 1.1 corresponding to Jorgensen et al. [11], is not directly related to Lanchester or Vidale-Wolfe. They bring attention to state-redundant differential games which have the property that the open-loop solution and the closed-loop solution are the same. The study of differential games with this property began with the work of Leitmann-Schmittendorf.

1.2.6 Characteristics of differential game solutions

- The usability of a solution of a differential game depends on its type of equilibrium strategy, i.e., closed-loop or open-loop strategy.
  
  - A closed-loop strategy that is also a feedback strategy is preferred so that the competitors can adjust their strategies as the state changes. A closed-loop strategy must satisfy (1.19), (1.21) and (1.22), i.e. a system of partial differential equations where the state equations have initial conditions and the costate equations have final conditions.
  
  - Open-loop strategies are viable only if competitors decide their strategy at the beginning of the interaction and do not make any adjustment thereafter. In the case of open-loop strategies, they must satisfy (1.19), (1.21) and (1.22). However, in this case the summation in (1.22) is zero because the strategy depends only on time not on the states and open-loop equilibria can be computed. Also, if the time horizon is infinite the terminal time horizon conditions should be replaced by steady state assumptions.

- It is assumed that each competitor has full knowledge of the competitive interaction and the profit structure of the other competitors [14, p. 7, chap. 2].

- Solving differential games with either open-loop or closed-loop Nash equilibrium strategies is difficult [9]. In 1995, Erickson [28] stated that without some development in the area of partial differential equations closed-loop solutions for oligopolies will continue being difficult to obtain. In 2004, Jarrar et al. [7] stated that in many cases researchers simplify the HJB equations to make them analytically tractable and find a solution.
<table>
<thead>
<tr>
<th>Published by</th>
<th>Dynamics</th>
<th>Type of solution</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deal 1979 section E.1</td>
<td>V-W</td>
<td>OLNE numerical. Finite horizon.</td>
<td></td>
</tr>
<tr>
<td>Case 1979 section E.2</td>
<td>Lanchester</td>
<td>CLNE FS analytical. Infinite horizon</td>
<td>Zero discount factor</td>
</tr>
<tr>
<td>Fruchter et al. 1997 section E.6</td>
<td>Lanchester</td>
<td>CLNE numerical, OLNE numerical. Infinite horizon</td>
<td>Solves numerically an IBVP (initial value boundary problem) instead of a TBVP (two boundary). Nonzero discount factor. Time dependent CLNE proportional to OLNE.</td>
</tr>
<tr>
<td>Wang et al. 2007 section E.10</td>
<td>Lanchester.</td>
<td>CLNE numerical. OLNE numerical. Infinite horizon</td>
<td>Empirical study. Square root of advertising in the dynamics is not always suitable for the problem.</td>
</tr>
<tr>
<td>Krishnamoorthy et al. 2010 section E.11</td>
<td>V-W. Price and advertising</td>
<td>CLNE FS analytical. Infinite horizon</td>
<td>Value function is linear on the state. Two forms of demand function.</td>
</tr>
<tr>
<td>Jorgensen et al. 2010 section E.12</td>
<td>State-Redundant games.</td>
<td>Analytical</td>
<td>For state-redundant games and some linear-state games the OLNE solution is the same as the CLNE FS solution. For instance, Leitmann-Schmittendorf.</td>
</tr>
</tbody>
</table>

Table 1.1: Duopoly differential games and their solutions. CLNE= closed-loop Nash equilibrium, OLNE= open-loop Nash equilibrium, FS= Feedback strategy, i.e. subgame perfect (not depending on initial conditions)
- The use of numerical algorithms for solving competitive advertising is widespread, only special cases can be solved analytically. See Table 1.1.

- Even though the closed-loop solution and in particular the feedback strategy solution, which depends only on the state, is more appealing to decision makers of the firms, in many cases it is the open-loop solution that is calculated because it is more tractable [11].

- As a remark, note that when facing the tradeoff between a closed-loop solution and the tractability of the open-loop solutions, researchers have looked for problem structures where both solutions are the same, i.e. the open-loop solution is a closed-loop solution [11]. These are very special rather crafted cases.
Chapter 2

Proposed approach to study advertising duopoly dynamics

In this chapter, we will put together the observations regarding the types of models and approaches available and propose an approach that allows some flexibility in modeling different types of strategic behavior by the competing firms and, in order to do so, examines the consequences of adopting such strategies, rather than attempting to compute them as optimizers of some performance criterion.

2.1 Outline of the proposed approach

Our perspective is to investigate market share dynamics of some nonprice duopoly models subject to a competitive advertising policy. Once the possible market share outcomes have been characterized, they are also evaluated with respect to a performance index. The primary concern is not with maximizing a performance index, rather the objective for firm $F_i$ is to take action to try to prevent its current market share falling below its target market share. This approach can be regarded as being complementary to the differential games approach, which starts out by optimizing a performance index, through the calculation of an optimal advertising effort (which is often difficult to calculate and implement).

As we recall from the previous section, the problems with finding the optimal control for a differential game are as follows:

- It is generally not analytically computable.

- In most cases it is not in feedback form for a finite horizon.

- The approaches taken so far are to make simplifications or assumptions with regard to the dynamics in order to obtain an analytical solution or a numerical solution.

- In general both firms need complete information (state and parameters) of the system. However, information of a competitor may not be readily available.
We propose an alternative approach in which:

- Firm $F_i$ takes action to try to reach at least its target market share.
- A performance index is used to evaluate the advantage of different advertising controls, but not to design them, as in the differential game approach.
- A simple and naturally implementable policy is used. It is based on switching control that relies only on the observed state of the system, but in which each firm does not use private or privileged information of its competitor in order to design its policy.

In this context, we will make the following assumptions in order to model the behavior of firms:

A1. Both firms have complete access to current market share information (i.e., to states $x_1, x_2$).

A2. The total market capacity is normalized to unity (applicable when the state variable is the market share of the firm).

A3. The market environment is purely competitive, i.e. there is no collusion or market division between the firms.

A4. A firm changes its advertising expenditure only when it perceives that it has a smaller market share than its target market share.

A5. The change in the previous assumption occurs in a predetermined manner, by switching on an additional positive advertising effort directed at regaining lost market share (and possibly surpassing the competitor's market share).

Given these assumptions, our focus is on analyzing the resulting dynamics and identifying initial market shares which achieve the following goal:

G The goal of a firm is to reach or surpass its target market share. The firm with an initial market share that is smaller than its target applies an additional advertising effort to reach its target. Once a firm surpasses its target market share, it stops making its additional advertising expenditure.

Evidently, if the goal of surpassing a target share is unachievable, then it is of interest to know if the long term outcome for the firm is its target market share.
2.2 Proposed unified model of advertising dynamics

This section first describes a general model which contains, as special cases, the models of interest previously discussed. We propose a unified model (UM) that can generate different models with appropriate parameter choices:

\[
\begin{align*}
\dot{x}_1 &= u_{11}(x_1, x_2)(1 - x_1 - x_2) - v_{12}(x_1, x_2) + v_{21}(x_1, x_2) \\
\dot{x}_2 &= u_{22}(x_1, x_2)(1 - x_1 - x_2) - w_{21}(x_1, x_2) + w_{12}(x_1, x_2)
\end{align*}
\]  

(2.1)  

(2.2)

where \( u_{11}(x_1, x_2), u_{22}(x_1, x_2), v_{12}(x_1, x_2), v_{21}(x_1, x_2), w_{12}, w_{21}(x_1, x_2) \) are the terms to be defined, and the states \( x_1, x_2 \) are the market shares of firms \( F_1, F_2 \) unless otherwise specified.

The UM model generalizes the model in [34]. Some models generated from (2.1)-(2.2) are listed below:

- The Vidale-Wolfe model without decay terms. The values for the terms in (2.1)-(2.2) are:

\[
\begin{align*}
  u_{11} &= k_1, & v_{12} = v_{21} = 0 \\
  u_{22} &= k_2, & w_{21} = w_{12} = 0
\end{align*}
\]  

(2.3)  

(2.4)

- The Lanchester model. The values for the terms in (2.1)-(2.2) are:

\[
\begin{align*}
  u_{11} &= 0, & v_{12} = v_{21} = k_1 x_2 \\
  u_{22} &= 0, & w_{21} = w_{12} = k_2 x_1
\end{align*}
\]  

(2.5)  

(2.6)

- The Lotka-Volterra model. The values for the terms in (2.1)-(2.2) are:

\[
\begin{align*}
  u_{11} &= b_1 x_1, & v_{12} = ((a_{11} - b_1)x_1 + (a_{12} - b_1)x_1)x_1, & v_{21} = 0 \\
  u_{22} &= b_2 x_2, & w_{21} = ((a_{22} - b_2)x_2 + (a_{21} - b_2)x_1)x_2, & w_{12} = 0
\end{align*}
\]  

(2.7)  

(2.8)

2.3 Associated Performance Index

Often, profit is also a criterion of interest. Deal [11] proposed a performance index which uses market share and profit (see equation (1.16)), and we use the following simplified version which corresponds to the index used by Sethi in [40, p. 196, chap. 7] without the discount factor, since a finite horizon problem is being considered:

\[
J_i(r, x_i(t), u_i(t), t) = \int_{t_0}^{t_f} [r x_i(t) - u_i(t)] dt
\]  

(2.9)
where $i$ refers to firm $i$, $r$ is the maximum sales revenue factor of the market, $t_0$ the initial time of the planning horizon and $t_f$ the final time of the planning horizon. We use $J_i$ as an abbreviation for $J_i(r, x_i(t), u_i(t), t)$. Note that a firm makes profits when its index $J(x_i)$ is positive, whereas it makes a loss when its index $J(x_i)$ is negative. The term $rx_i(t)$ represents the income of firm $F_i$, so that the factor $r$ converts market share $x_i$ to a money unit, i.e. thousand of dollars. The factor $r$ can be estimated as the product of the maximum number of items that can be sold in the market multiplied by the average price of the product.

### 2.4 Motivation for using state dependent control

Consider the following system which is the simplified Vidale-Wolfe model for a monopoly firm $F$ without the decay term.

$$\dot{S}(t) = u(t)(M - S(t))$$

(2.10)

where $S$ is the sales, $M$ is total sales the market can support, $u(t)$ is the advertising effort or control. Suppose that $F$ can sell at most an amount $M^* < M$; this could happen because it has a limited production given its current factory capability. Since the term $(M - S(t))$ on the right hand side of (2.10) is always positive for all possible $S(t)$, it follows that, in order for $S(t)$ to increase, we should have $u(t) > 0$ whenever $S(t) \in [0, M^*]$. When $S(t)$ exceeds $M^*$, it is reasonable to suppose that spends zero advertising effort, i.e., $u(t) = 0$ for $S(t) \geq M^*$. One particular choice of continuous control that achieves this is $k(M^* - S(t))$ for $S(t) \in [0, M^*]$ and $u(t) = 0$ for $S(t) \geq M^*$. Another, possibly more realistic strategy recognizes that, even when a given maximum sales level has been achieved, advertising effort should merely be reduced and not extinguished, in order to avoid losing the sales level it has achieved, due to ongoing externalities. The two strategies just described are depicted in Figures [2.1](a) and (b), with the former being continuous and the latter discontinuous.

Additional motivation for decision makers to use switching control comes from the concept of performance dashboards from [43, p. 4-5, chap. 1]. A performance dashboard communicates strategic objectives and enables business people to measure, monitor, and manage the key activities and processes needed to achieve their goals. To achieve all this, a performance dashboard provides three main sets of functionality:

- Monitor critical business processes and activities using metrics that trigger alerts when performance falls below predefined targets.

- Analyze the root cause of problems by exploring relevant and timely information from multiple perspectives at various levels of detail.
Figure 2.1: Control $u(t)$ applied to achieve the sales goal of at least $M^*$. The continuous case is shown in (a) the control $u(t) = k(M^* - S(t)) > 0$ when current sales $S(t) < M^*$ and $u(t) = 0$ when $M^* < S(t)$. The discontinuous case is shown in (b) where the control is a switching control $u(t) = M_{left}$ when $S(t) < M^*$ and $u(t) = M_{right}$ when $M^* < S(t)$.

- Manage people and processes to improve decisions, optimize performance, and steer the organization in the right direction.

Performance dashboards provide benefits such as actionable information delivered in a timely fashion that lets users take action to fix a problem [43] p. 15, chap. 1].

The goal of the dashboard is to present organized data to the decision maker in an easy-to-understand format. In addition to providing the data of the current metric, the dashboard provides values of other metrics, at comparable times, that would help the decision maker in the understanding of the current metric. A dashboard is more than simple reporting, a dashboard is also interactive. [44] p. 402, chap. 10]. Figure 2.2 shows a speedometer which is one way of displaying part of a dashboard.

Figure 2.2: A speedometer for indicator Sales in millions of $ using a standard coloring for the status of a metric: red=In trouble (from 0 to 22), yellow=Attention needed (from 22 to 44), and green=Normal (from 44 to 70).

Following the dashboard usage in business management, one can see that an switching control strategy can be regarded as a corrective action when, for instance, the state of the metric market share or sales changes its status from green (normal) to yellow (attention needed).
2.5 Objectives of the firms

The goal of firm $F_i$ is to have, at the saturated market, at least its target market share $target_i$. The target of a firm defines the switching line. This line goes from the origin to the target market share of the firm on the saturated market share line. This switching line specifies the change of controls of the firm: When a firm perceives that its market share is below its target share, that firm turns on an extra control. Depending on the value of $target_1 + target_2$ the possibilities are:

1. The sum $target_1 + target_2 = 1$. In this case, both firms have the same switching line. For instance, if both firms want to have at least with 0.5 of the market share in the long run then the switching line is the equal share line $x_2 - x_1 = 0$.

2. The sum $target_1 + target_2 > 1$. In this case, there are two switching lines, one for each firm. Even though in a duopoly the market shares of the two firms comprise the total market, it is highly probable, given the competition between the firms, that each firm wants a bigger market share than the other firm. For instance, suppose that firm $F_1$ desires 0.6 of the market share and $F_2$ desires 0.7 of the final market share. The switching line for $F_1$ is $x_2 - \frac{0.4}{0.6}x_1 = 0$ and $F_1$ achieves its goal when $x_1 > \frac{0.6}{0.4}x_2$. The switching line for $F_2$ is $x_2 - \frac{0.7}{0.3}x_1 = 0$ and $F_2$ achieves its goal of having at least 0.7 of the market share when $x_2 > \frac{0.7}{0.3}x_1$.

3. The sum $target_1 + target_2 < 1$. In this case, there are two switching lines, one for each firm. Even though this case seems to have no conflict, depending on the initial conditions and the parameters of the system, a firm might or might not reach its target market share.

2.6 Classes of Control Functions used by each firm

The control laws to be used are simple laws. A policy $u = kx$ which is state dependent with gain parameter $k > 0$, is said to be a constant rate control, and if $u = k > 0$ (not dependent on state), then it is called a constant effort control. In general, the controls we will use are constant effort, constant rate or a sum of them.

Specifically we propose the use of switching control or variable structure control [45]. Each firm uses the state $x$ to test if its target market share is currently fulfilled. For example, the firm $F_i$ could use a constant effort control while its goal is achieved, but uses a constant effort plus a constant rate control when its goal is not fulfilled. In other words, $F_i$ applies its additional constant rate control to try to fulfill its goal. For instance, if the goal of each firm is to have at least 0.5 of the total market at steady state, the
switching control would be

\[ u_i(x_1, x_2) = \begin{cases} 
  k_i x_i + c_i & \text{if } x_i < x_j \\
  k_i x_i & \text{if } x_i > x_j
\end{cases} \quad i = 1, 2; \ j \neq i \quad (2.11) \]

Note that, in order to define a dynamical system subject to control (2.11) on the discontinuity line \( x_2 - x_1 = 0 \), we will use the Filippov rule (see Appendix A, [46]).

### 2.7 Objective of this thesis

In addition to proposing modifications of advertising models, this thesis focuses on answering the following question, that follows naturally from the assumptions made above:

- Is it possible to identify initial market shares and advertising effort parameters for which the firm with an initial market share below its target will (respectively, will not) be able to attain it?

Also, we want to define a reasonable performance index that will allow a classification of advertising strategies.
Chapter 3

Dynamics of a modification to the Vidale-Wolfe-Deal model under feedback control

3.1 Introduction

We chose to work with the Vidale-Wolfe model [22] because it directly relates market share with advertising and it was tested against real world data. Deal [11] was the first to use the Vidale-Wolfe model for the duopoly case. In addition it has been an inspiration for many models [29].

Before presenting the model Vidale-Wolfe-Deal with switching control (VWDsC), we give some notation related to second order dynamical systems with discontinuous right-hand sides. For general second order models (i.e. having a state vector \( \mathbf{x} = [x_1(t), x_2(t)]^T \in \mathbb{R}^2 \)), an ODE with one discontinuity boundary \( \Sigma \) and lower and upper vector fields \( \mathbf{f}^L = [f_1^L(x), f_2^L(x)]^T \) and \( \mathbf{f}^U = [f_1^U(x), f_2^U(x)]^T \) on either side of \( \Sigma \) is written as:

\[
\dot{x} = \begin{cases} 
\mathbf{f}^L(x), & x \in L \\
\mathbf{f}^U(x), & x \in U 
\end{cases} 
\] (3.1)

where the regions \( L, U \) are separated by \( \Sigma = \{x : H(x) = 0\} \). \( L \) corresponds to the lower region where \( H(x) < 0 \) and \( U \) to the upper region where \( H(x) > 0 \).

Next, we proceed to the basic control policy which is used in the model VWDsC:

\[
u_i(x_1, x_2) = \begin{cases} 
k_i x_i + c_i & \text{if } x_i < x_j \\
k_i x_i & \text{if } x_i > x_j 
\end{cases}
\] (3.2)

where \( i = 1, 2 \) and \( j \neq i \). Policy (3.2) can be interpreted as follows: if firm \( F_i \) is leading the market (i.e., \( x_i > x_j \)), it uses a constant effort advertising policy \( (k_i, x_i) \); however at the moment at which the competing firm \( F_j \) becomes the leader (exceeds the market share of
firm $F_i$), firm $F_i$ switches to an additional (constant rate) effort ($c_i$) in its attempt to avoid falling behind its competitor.

In order to proceed, some terminology is introduced. The policy (3.2) is one that switches from constant effort to constant effort plus constant rate and vice-versa, whenever the equal market share line $x_1 = x_2$ is crossed. Since the switching policy is discontinuous on this boundary, the equal share line is also referred to as discontinuity boundary and denoted by the symbol $\Sigma$.

Using this previous notation and the policies (3.2) as $u_{ii}$ and $u_{ij} = w_{ij} = 0$ with $i, j = 1, 2$ in model UM (2.1 - 2.2), the equations that define the model VWDsC are as follows:

The upper field $f^U$, which defines the dynamics in the region $U$, is given by:

$$
\begin{align*}
\dot{x}_1 &= (k_1 x_1 + c_1)(1 - x_1 - x_2) \\
\dot{x}_2 &= k_2 x_2(1 - x_1 - x_2)
\end{align*}
$$

(3.3) (3.4)

The lower field $f^L$, which defines the dynamics in the region $L$, is given by:

$$
\begin{align*}
\dot{x}_1 &= k_1 x_1 (1 - x_1 - x_2) \\
\dot{x}_2 &= (k_2 x_2 + c_2)(1 - x_1 - x_2)
\end{align*}
$$

(3.5) (3.6)

On $\Sigma$, the dynamics is defined following the Filippov rule which is based, roughly speaking, on a convex combination of the lower and upper fields (see Appendix A [46, p. 50-52, chap. 2] for mathematical details).

Remark: The policy described above belongs to the class of switching policies, in which the switching is based on the observations of the states (market shares). In fact, many other choices of control laws are possible for models derived from the Vidale-Wolfe model and represent different hypotheses about the behavior of competing firms (see, for example, [14, 29, 42, 47, 48]).

Figure 3.1 identifies the regions and the relevant lines for model (3.3)-(3.6). The discontinuity boundary $\Sigma$ is the line $x_2 = x_1$, also written as $H(x) = x_2 - x_1 = 0$. The line $1 - x_1 - x_2 = 0$ corresponds to a saturated market, and is denoted as Sat. The segment $Sat^U$ (resp. $Sat^L$) corresponds to firm $F_2$ (resp. $F_1$) ending with a bigger saturated market share. The point $E$ is the equal share saturation point, i.e. where both firms have half of the total saturated market. The triangle $T$, defined as $\{x \in \mathbb{R}_+^2 : x_1 + x_2 \leq 1\}$, is the set of all feasible states. The region $U$ is the set $\{x \in \mathbb{R}_+^2 : x_1 + x_2 \leq 1 \land x_2 > x_1\}$ where the dynamics (3.3)-(3.4) corresponding to $f^U$ is active. The region $L$ is the set $\{x \in \mathbb{R}_+^2 : x_1 + x_2 \leq 1 \land x_1 > x_2\}$ where the dynamics (3.5)-(3.6) corresponding to $f^L$ is active.

Given the definition of the switching controls, it is clear that, if a trajectory attains
the equal market share line $\Sigma$ at a point below the equal saturated market share $E$ (see Figure 3.1), then both firms switch on their extra-advertising efforts. Depending on the relative magnitude of their efforts, a resultant motion, called sliding, may occur for some segment of $\Sigma$. The endpoint of this sliding motion, marked by a dot in Figure 3.1, is $S_{\text{end}} = (s_{\text{end}}, s_{\text{end}})$. For brevity, we will use only one coordinate when referring to points on the equal share line $\Sigma$, i.e. we will use $s_{\text{end}}$ when referring to the sliding end point.

A separatrix is a trajectory which separates obvious distinct regions in the phase plane [49, p. 111, chap. 3]. Recall that the goal of each firm is to obtain at least 50% of the saturated market share. For each firm $F_i$, there is a trajectory $'Y_i$ such that it separates the phase plane into two regions: starting in one of the regions it achieves its goal in the long term, while starting in the other it does not. In model (3.3)-(3.6), we call separatrix the trajectory $'Y_i$ which separates regions (of initial conditions) which correspond to different outcomes (e.g. Firm $F_1$ has a larger long term market share than Firm $F_2$, or vice-versa).

Observe that separatrices are associated to a vector field as well as to a firm and depending on the parameters of the system, a given firm can have a separatrix derived from $f^U$ or from $f^L$, hence we use the notation $'Y_i^U$ to denote the separatrix for firm $F_i$, $i = 1, 2$ obtained using the dynamics of the field $f^Y$, $Y = U, L$. Figure 3.1) shows the case where there is one separatrix in region $L$, other cases will be shown in section 3.2.3.

![Figure 3.1: The upper region $U$, the lower region $L$, the switching line $\Sigma = \{x : H(x) = x_2 - x_1 = 0\}$, the segment $Sat$ defined by $1 - x_1 - x_2 = 0$. The segment $Sat^U$ (resp. $Sat^L$) is the portion of $Sat$ that lies in region $U$ (resp. $L$). $T$ is the triangle defined by the points $(1, 0), (0, 1), (0, 0)$. The point marked $E$ is the equal share saturation point, i.e. where both firms have half of the total market. In this figure the sliding end is $s_{\text{end}} < 0.5$ and there is one separatrix $'Y_2^U = 'Y_1^U$](image)

The set of initial shares ($T$) is partitioned into regions by separatrices. In the cases where there is only one separatrix, one of the firms does not fulfill its goal of at least 50% of the market share even when it begins with a bigger initial market share. For instance, if the separatrix is $'Y_2^U = 'Y_1^U$ (see Fig. 3.2(c) and (d)) and the initial market share $x_{20} > x_{10}$ is below $'Y_2^U$, then firm $F_2$ ends with less than half when market saturation is reached.

Given this kind of behavior, the evolution of the market shares after the equal market share is reached until market saturation is attained has to be described.

A trajectory that departs from an initial condition in $U$ or $L$ (i.e. not on $\Sigma$) and attains $\Sigma$ is said to have reached $\Sigma$, and the period in which this occurs is known as the reaching
phase. Once the trajectory from $U$ (resp. $L$) reaches $\Sigma$ there are the following possibilities (formal mathematical definitions of sliding and crossing are given in Appendix A):

1. The trajectory crosses from $U$ into $L$ (resp. from $L$ into $U$)

2. The trajectory moves along $\Sigma$, which is known as sliding, and then crosses from $U$ into $L$ (resp. from $L$ into $U$) finally ending on $Sat^L$ (resp. $Sat^U$).

3. The trajectory slides on $\Sigma$ up to a certain point, and then returns to $U$ (resp. to $L$) finally ending on $Sat^U$ (resp. $Sat^L$).

4. The trajectory slides on $\Sigma$ until it attains the saturated equal market share point $E$.

For brevity we will refer case 1 as reaching and crossing ($RC$), case 2 as reaching, sliding and crossing ($RSC$), case 3 as reaching, sliding and returning ($RSRet$) and case 4 as ($RS$). This cases are shown in Figure 3.2.

### 3.2 Main result for model VWDsC

The main result for model VWDsC is a complete description of market share dynamics. Theorem 3.2.1 characterizes the market share dynamics having initial conditions $x_2(0) > x_1(0)$, i.e. trajectories $\tau$ beginning in $U$, which attain with the equal share line $\Sigma$. The case for $\tau$ beginning in $L$, i.e. $x_1(0) > x_2(0)$ is analogous and it is omitted for brevity. Other possibilities, completing the description of the market share dynamics, are given in remarks 1 and 2, which follows Theorem 3.2.1.

**Theorem 3.2.1 (Characterization of market share dynamics for model (3.3)-(3.6) with controls (3.2)).** Suppose that $\tau$ is a trajectory that starts in region $U$, i.e. firm $F_1$ starts out with a smaller market share $x_1(0) < x_2(0)$ and that the parameters $k_1, c_1, k_2, c_2$ are known, then the trajectories reaching the equal market share line $\Sigma$ can have one of the following behaviors:

1. If the sliding end point $s_{end} = 0.5$ then any trajectory $\tau$ with initial conditions in region $U$ and below separatrix $\varphi_1^U$ reaches the equal share line and slides until the saturated equal share point $E$ on the saturated market line (see Figure 3.2a)

2. If the sliding end point $s_{end} < 0.5$ and $k_2 > k_1$, then the separatrix is $\varphi_2^L = \varphi_1^U$ and some trajectories beginning in region $U$ reach the equal share line $\Sigma$, move on it until the point $s_{end}$ and return to $U$ finally ending on the saturated market $Sat$ with firm $F_2$ having more than 50% of the market share. (see Figure 3.2b)

3. If the sliding end point $s_{end} < 0.5$ and $k_1 > k_2$, then the separatrix is $\varphi_2^U = \varphi_1^U$ and the trajectories beginning in region $U$ and below $\varphi_2^U$ reach the equal share line $\Sigma$. 

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(a) If the trajectories reach the equal market share on the sliding segment $\Sigma_s$, they stay on the equal market share until the point $s_{end}$ and then cross into region $L$ finally ending on the saturated market $Sat$ with firm $F_1$ having more than 50% of the market share. (see Figure 3.2c)

(b) If the trajectories reach the equal market share line on the crossing segment $\Sigma_c$, they cross into region $L$ finally ending on the saturated market $Sat$ with firm $F_1$ having more than 50% of the market share. (see Figure 3.2d)

Proof. Proof of Theorem 3.2.1 given in Appendix C Demonstrations of properties of model VWD_sc.

Figure 3.2: Possible outcomes for firm $F_2$, which has been chosen, without loss of generality, to have a bigger initial market share. The sliding end point $s_{end}$ is marked by a dot. (a) $RS$: With initial conditions that lie below $\varphi^U_2$, $F_2$ loses its initial advantage, but ends with equal market share at saturation, (b) $RS_{Ret}$: For some trajectories firm $F_2$’s initial advantage reduces to an equal share, but at saturation $F_2$ has more than half of the market, (c) $RSC$: Firm $F_2$ loses its initial advantage, has a period of equal share and at saturation line ends with less than half of the market, (d) $RC$: Firm $F_2$ loses its initial advantage and ends, at saturation line, with less than half of the market, without experiencing a period of equal share.

Remarks:

1. If firm $F_2$’s separatrix is $\varphi^U_2$, then trajectories beginning above $\varphi^U_2$ do not attain the equal share line $\Sigma$ and they lead to firm $F_2$ ending with more than 50% of the market share (see Figure 3.2a, 3.2c and 3.2d).

2. If firm $F_2$’s separatrix is $\varphi^L_2$ (see Figure 3.2b), then any trajectory beginning in $U$, whether it attains $\Sigma$ or not, leads to firm $F_2$ ending with more than 50% of the market share. This last case applies also to trajectories beginning in $L$ and above $\varphi^L_2$.

3. When the firms spend a period of time on the equal share line, i.e. sliding occurs, both firms turn on their extra effort which leads to a larger advertising cost.

4. There are trajectories which cannot occur, when parameters have a given relation: When $k_2 > k_1$ (resp. $k_1 > k_2$) it is not possible that firm $F_1$ (respectively $F_2$) ends
with a larger saturated market share. See Propositions C.0.5 and C.0.6 in Appendix C.

All the proofs are detailed in Appendix C. Demonstrations of properties of model V WD s C.

3.2.1 The state variables (market shares) \( x_1, x_2 \) are nondecreasing along the trajectories

For all trajectories originating in the interior of the triangle \( T \), equations (3.3)-(3.6) imply that the state derivatives \( \dot{x}_1, \dot{x}_2 \) are always nonnegative, this implies that they are non-decreasing. The variables \( x_1, x_2 \) always increase, except at \( Sat \) where \( x_1, x_2 \) remain unchanged, see section 3.2.2 next.

3.2.2 Equilibria in \( U \), their location and stability properties

We observe that the set of points \( \{(a, 1-a) : a \in [0, 0.5]\} \) is an equilibrium set for the dynamics in \( U \) defined by \( f^U \). This is because the term \( (1-x_1-x_2) \) is a factor in the expression defining \( f^U \), implying that the vector field \( f^U \) vanishes on the set \( \{(a, 1-a) : a \in [0, 0.5]\} \) defined by the upper half of the line \( Sat \).

If the market is unsaturated (i.e., \( 1-x_1-x_2 > 0 \)), the only equilibrium point for (3.3), (3.4) is \( x_1 = -c_1/k_1, x_2 = 0 \). Under the assumptions that \( k_i, c_i \) are positive for \( i = 1, 2 \), this means that this equilibrium point lies on the negative segment of the \( x_1 \) axis and is outside the first quadrant. In other words, it is a virtual equilibrium, in the sense that it is outside of the feasible region \( T \). Also, it is an unstable node.

Behavior of points on \( Sat \)

The Jacobian matrix \( J_1(x_1, x_2) \) for (3.3), (3.4) is given by:

\[
\begin{bmatrix}
-2k_1x_1 + k_1 - c_1 - k_1x_2 & -c_1 - k_1x_1 \\
-k_2x_2 & -2k_2x_2 - k_2x_1 + k_2
\end{bmatrix}
\] (3.7)

Substituting the coordinates of a point \( (a, 1-a) \) on \( Sat \) in (3.7) we have \( J_1(a, 1-a) \)

\[
\begin{bmatrix}
-k_1a - c_1 & -k_1a - c_1 \\
-k_2(1-a) & -k_2(1-a)
\end{bmatrix}
= -u \mathbb{I}^T,
\] (3.8)

where \( u = (k_1a + c_1, k_2(1-a)) \in \mathbb{R}^2 \) and \( \mathbb{I} = (1, 1) \in \mathbb{R}^2 \). Abbreviating \( J_1(a, 1-a) \) by \( J_1 \), the equations \( J_1u = -(u \mathbb{I}^T)u \) and \( J_1w = 0w \), for \( w \in \mathbb{I}^\perp (\mathbb{I}^\perp = \{v : v \mathbb{I}^T = 0\}) \) show that \(-\mathbb{I}^T u \) is a negative eigenvalue associated with the direction of the eigenvector \( u \), and 0 (zero) is an eigenvalue associated with an eigenvector in \( \mathbb{I}^\perp \) (the eigenvector \( w \) can be chosen as \( (1, -1) \)).
Since for \( a \in [0, 0.5) \) and any choices of \( k_1, c_1, k_2 \) (all positive), the eigenvector \( \mathbf{u} \) points towards \( Sat \) and furthermore the eigenvector \((1, -1)\) is parallel to \( Sat \), we can conclude that trajectories in the interior of \( T \) will approach \( Sat \). Once a point on \( Sat \) is attained, the trajectory stops evolving \((\dot{x}_1 = \dot{x}_2 = 0)\). In this sense, \( Sat \) is a continuum of points that attract trajectories originating in \( T \), and behave like equilibria. The points on \( Sat \) are attractive but a perturbation that changes the state from point \( P_1 \) on \( Sat \) to another point \( P_2 \) on \( Sat \) will stay on \( P_2 \).

### 3.2.3 Types of separatrices

Separatrices associated with the regions \( U \) and \( L \) are defined as follows: \( \varphi^U \) (resp. \( \varphi^L \)) is a trajectory confined to \( U \) (resp. \( L \)) starting at an initial condition on the \( x_2 \) (resp. \( x_1 \)) axis and ending at point \( E \). At least one separatrix will exist because a separatrix is a special trajectory that reaches \( Sat \) at the equal market share saturated market \( E \) and one of the firms or both could reach the equal share saturated market \( E \).

Figure 3.3a shows the case in which two separatrices, denoted \( \varphi^U_1 \) and \( \varphi^L_2 \) occur (recall that the subscript indicates the firm to which the separatrix applies). When both separatrices exist, the region in between them (shown in white in Figure 3.3a) is a region where the firms tie, i.e. each firm ends up with half the saturated market share. There are parameter values for which only one separatrix exists \( \varphi^U_1 = \varphi^U_2 \) (resp. \( \varphi^L_1 = \varphi^L_2 \)) and it applies to both firms. This is shown in Figure 3.3b (resp. Figure 3.3c). Firm \( F_2 \) (resp. \( F_1 \)) achieves its goal above (resp. below) its separatrix.

![Figure 3.3: Types of separatrices](image)

If only one separatrix exists then it applies to both firms (see Figure 3.3b and 3.3c). If two separatrices exist, the separatrix for firm \( F_1 \) is \( \varphi^U_1 \) and the separatrix for firm \( F_2 \) is \( \varphi^L_2 \) (see Figure 3.3a). The mathematical details needed to calculate the separatrix of a firm are given in Appendix B.
3.2.4 Reaching, sliding and crossing behaviors with regard to the equal share line $\Sigma$

If only one separatrix $\varphi_2^U = \varphi_1^U$ (resp. $\varphi_2^L = \varphi_1^L$) exists, then trajectories beginning in $U$ (resp. in $L$) and below the separatrix $\varphi_2^U$ (resp. above $\varphi_1^L$) lead to firm $F_2$ (resp. $F_1$) not attaining its goal of at least 50% of the saturated market share. These trajectories, on reaching the equal share line $\Sigma$ could slide on $\Sigma$ and then cross it into $L$, or could directly cross $\Sigma$ into $L$. If there are two separatrices $\varphi_2^L$ and $\varphi_1^U$, trajectories beginning in the region in between them reach $\Sigma$ and then slide on it until the saturated equal market share point $E$. Thus, it remains to find additional conditions on the policy parameters $(k_i, c_i)$ that result in these different types of possible behavior. Notice that it is necessary to define the segment $\Sigma_s \subset \Sigma$ on which sliding occurs. Similarly, it is necessary to define a segment $\Sigma_c \subset \Sigma$ on which crossing occurs.

The details about reaching, sliding and crossing the equal share line $\Sigma$ are given in Appendix C Demonstration of properties of model VWDSc.

3.3 Designing advertising policies: a pictorial version of the main results for model VWDSc

To facilitate the use of the results given in section 3.2, this section provides a pictorial version of the main results, identifying regions of the phase plane (i.e. initial conditions) that lead to different outcomes for the long term market shares of the competing firms. In order to do this, we assume, without loss of generality, that firm $F_1$ starts out with a lower initial market share (i.e. $x_1(0)$ is in $U$).

Figure 3.4 shows regions of all initial conditions in $U$ for which firm $F_1$ can attain or surpass the market share of firm $F_2$. In Figure 3.4a firm $F_1$ will attain 50% of the saturated market share while in Figure 3.4b firm $F_1$ will end with more than 50% of the saturated market share. If the separatrix is $\varphi_2^L = \varphi_1^L$, then there are no initial conditions in $U$ for which $F_1$ attains at least 50% of the saturated market. For this reason the case $\varphi_2^L = \varphi_1^L$ is not included in Figure 3.4.

![Figure 3.4](image.png)

Figure 3.4: The gray regions of $U$ contain the initial conditions for which: (a) $F_1$ attains 50% of the saturated market, (b) $F_1$ surpasses the saturated market share of $F_2$. 

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Figure 3.5 shows a gray region for which $F_1$ attains equal market share in the long term. In Figure 3.5a, only initial conditions on the separatrix, shown in gray, lead to both firms ending with 50% of the saturated market share. If the separatrix is $\varphi^U_2 = \varphi^U_1$, then there are no initial conditions in $U$ for which $F_1$ attains 50% of the saturated market. For this reason the case $\varphi^U_2 = \varphi^U_1$ is not included in Figure 3.5.

Figure 3.5: The gray regions of $U$ contain the initial conditions for which both firms end with an equal saturated market share in the long term. In (b) these initial conditions lie on the separatrix $\varphi^U_2 = \varphi^U_1$ and the region collapses to a single curve (the separatrix).

Figure 3.6 shows gray regions of initial conditions for which $F_1$ can never attain the saturated market share of its competitor, firm $F_2$.

Figure 3.6: The gray regions of $U$ contain the initial conditions in $U$ for which the firm $F_1$ end with a saturated market share smaller than the saturated market share of $F_2$.

In summary, supposing that firm $F_1$ has a lower initial market share, the strategy determined by Theorem 3.2.1 is to use a constant rate $k_1 > k_2$. With this choice, the separatrix $\varphi^U_1$ exists and if the initial market shares are below it, then firm $F_1$ is guaranteed 50% of the saturated market share (see Figure 3.4a) or even more than 50% of the market share (see Figure 3.4b). In this case, firm $F_1$ should also calculate its performance index to evaluate if its investment plans are better aligned with having 50% of the saturated market share or having more than 50% of it (the calculation of a performance index will be discussed in the next section). If the initial conditions are above the separatrix $\varphi^U_1$, then firm $F_1$ will end with less than 50% of the saturated market share, not attaining its goal. Again, the question of whether to enter the market or not, should be decided using a performance index.
3.4 Simulation of different scenarios

Next, we present simulations with different parameters displayed on the phase plane. Without loss of generality, the scenarios are commented on from the viewpoint of firm $F_2$. Recall that the goal of each firm is to end with at least 50% of the saturated market share. We assume that $F_2$ is already in the market so that its initial market share satisfies $x_2(0) > x_1(0)$. Also, suppose that the parameters $k_2, c_2, k_1, c_1$ are known. The entering firm $F_1$, i.e. $F_1$ enters the market, also wants to get at least half of the saturated market. The simulations were done using the software Berkeley Madonna [51].

Notice that in all scenarios firm $F_2$ achieves its goal of having at least half of the market for any trajectory $\tau$ is above its separatrix, which can be $\varphi^U_2$ or $\varphi^L_2$. Firm $F_1$ achieves its goal for any trajectory $\tau$ below its separatrix, which can be $\varphi^U_1$ or $\varphi^L_1$. If there are two separatrices, the region in between $\varphi^U_1$ and $\varphi^L_2$ is a draw region where each firm ends with half of the saturated market.

In the first scenario, shown in Figure 3.7, we investigate if there is a trajectory such that firm $F_1$ reaches the equal market share line and finally ends with more than half of the saturated market share when $k_1 < k_2$, i.e. when $F_1$ makes an insufficient effort. The answer is no. If $k_1 < k_2$ there is no trajectory that reaches the equal market share line, remains on it and finally ends with more than half of the saturated market share (Proposition C.0.5) neither a trajectory that reaches the equal market share line on the crossing segment and finally ends with more than half of the saturated market share (Proposition C.0.6). However, if $k_2 > k_1$ and $c_1/(k_2 - k_1) < 0.5$, there are trajectories $\tau$ that reach the equal market share $\Sigma$, remains on it and then return to $U$, so that finally $F_2$ ends with more than half of the saturated market. This corresponds to Theorem 3.2.1 2 with $s_{end} < 0.5$ and is highlighted in gray color in Figure 3.7.
Figure 3.7: Insufficient effort by initially disadvantaged firm $F_1$. Phase plane of model (3.3)-(3.6) with parameters $k_1 = 1, c_1 = 0.3, k_2 = 2, c_2 = 0.3$. The highlighted gray trajectory $\tau$ starts in $U$ and is such that firm $F_1$ reaches the equal market share, remains on it but finally firm $F_2$ ends with more than half of the saturated market share. This corresponds to Theorem 3.2.1 2.

In the next scenario, shown in Figure 3.8 if firm $F_1$ increases its the constant effort $c_1 > c_2$ but keeps its constant rate $k_1 < k_2$, does firm $F_1$ ends with more than half of the saturated market share? If the sliding end point $s_{end} = 0.5$ there are two separatrices $\varphi_1^U$ and $\varphi_2^L$. The trajectories with initial conditions below $\varphi_1^U$ and above $\varphi_2^L$ reach the equal market share line, remains on it and finally reach the saturated equal market share point $E = 0.5$, so that finally firms $F_2$ and $F_1$ end with half of the saturated market share. For instance, when the parameters are $k_1 = 1, c_1 = 2, k_2 = 2, c_2 = 0.3$, trajectories that reach $\Sigma$ slide until 0.5 as shown in Figure 3.8. These cases correspond to Theorem 3.2.1 1 where firm $F_2$ retains half of the saturated market share.
If $s_{\text{end}} = 0.5$, there are separatrices $\varphi_1^U$ and $\varphi_2^L$. All trajectories $\tau$ with initial conditions below $\varphi_1^U$ and above $\varphi_2^L$ end at the saturated equal market share $E$, i.e. firm $F_2$ finally ends with half of the saturated market share. The firm $F_1$ does a bigger constant effort $c_1 > c_2$ but a smaller constant rate $k_1 < k_2$ when compared to the previous scenario. The phase plane of model (3.3)-(3.6) corresponds to parameters $k_1 = 1, c_1 = 2, k_2 = 2, c_2 = 0.3$. The highlighted gray trajectory $\tau$ is such that it reaches the equal market share $\Sigma$ and remains on it until the saturated equal market share point $E$. This corresponds to Theorem 3.2.1.

In the last scenario, shown in Figure 3.9, if firm $F_1$ increases its constant rate $k_1 > k_2$, does $F_1$ end with more than half of the saturated market share? If the sliding end coordinate $s_{\text{end}} < 0.5$, the answer is yes, firm $F_1$ can end with more than half of the saturated market share. Figure 3.9 shows trajectories sliding on $\Sigma$ and then crossing into region $L$ (this corresponds to Theorem 3.2.1 3a) and trajectories directly crossing into region $L$ (this corresponds to Theorem 3.2.1 3b). Note that if firm $F_2$ had used a constant effort $c_2 = 0.8$ it would have attained half of the market share (this corresponds to Theorem 3.2.1 1). With a constant effort $c_2 = 0.8$, all trajectories that attain the equal market share line $\Sigma$ stay on it until the saturated equal market share point $E$. 

Figure 3.8: If $s_{\text{end}} = 0.5$, there are separatrices $\varphi_1^U$ and $\varphi_2^L$. All trajectories $\tau$ with initial conditions below $\varphi_1^U$ and above $\varphi_2^L$ end at the saturated equal market share $E$, i.e. firm $F_2$ finally ends with half of the saturated market share. The firm $F_1$ does a bigger constant effort $c_1 > c_2$ but a smaller constant rate $k_1 < k_2$ when compared to the previous scenario. The phase plane of model (3.3)-(3.6) corresponds to parameters $k_1 = 1, c_1 = 2, k_2 = 2, c_2 = 0.3$. The highlighted gray trajectory $\tau$ is such that it reaches the equal market share $\Sigma$ and remains on it until the saturated equal market share point $E$. This corresponds to Theorem 3.2.1.1.
Figure 3.9: Firm $F_2$ loses its initial advantage. The phase plane of model (3.3)-(3.6) corresponds to parameters $k_1 = 3.1$, $c_1 = 1.8$, $k_2 = 2$, $c_2 = 0.3$. All trajectories below the separatrix $\varphi'^{L}_2 = \varphi'^{L}_1$ that reach the equal market share line on the sliding segment $\Sigma_s$, remain on it and then they cross into region $L$, with firm $F_1$ ending with more than half of the saturated market share (see Theorem 3.2.1 3a). The trajectories that reach the equal market share line on the crossing segment $\Sigma_c$ directly cross into region $L$ (see Theorem 3.2.1 3b). The highlighted gray trajectory is such that it reaches the equal market share line $\Sigma$, remains on it for some time and then it crosses into region $L$ where firm $F_1$ ends up with more than half of saturated the market share.

3.5 Evaluating outcomes of the proposed policies using a performance index in model VWDsC

Market share is one viewpoint of the interaction between the two firms. We use the index (2.9) with the control functions given in (3.2) to measure the performance of each firm. A firm makes profits when its index $J(x_i)$ is positive, whereas it makes a loss when its index $J(x_i)$ is negative. The simulations were done using the software Berkeley Madonna [51].

Figure 3.10 shows a situation in which there is only one separatrix $\varphi'^{L}_2 = \varphi'^{L}_1$ which means that firm $F_2$ has a larger region (above the separatrix) of initial conditions for which it has a long term market share greater or equal to 50%. In figure 3.10 the sales revenue coefficient $r = 2$ is low enough that despite the long term market share advantage of firm $F_2$, its performance index is always negative for initial conditions on the axis. The light (resp. dark) gray region demarcates sets of initial conditions for which the performance index of firm $F_1$ is positive (resp. negative). This means that for the chosen (low) sales revenue coefficient, firm $F_2$ makes a loss despite its larger market share, while firm $F_1$ operates with a profit despite its lower market share. This suggests that the level
of advertising expenditure of firm $F_2$ is excessive.

Figure 3.11 shows that with a bigger sales revenue coefficient $r = 4$, there are three final outcomes of the indexes: only firm $F_2$ has a positive index, both firms have positive indexes or only firm $F_1$ has a positive index. Besides, note that most of the trajectories correspond to both firms making profit even though in several of those trajectories a firm does not fulfill its target market share at saturated market.

Thus, the value of the sales revenue coefficient is decisive for assessing profitability with respect to market share. In fact this analysis should be carried out to test any desired choice of parameters for different sales revenue coefficients in order to design the appropriate advertising policies of type (3.2).

Figure 3.10: $F_2$ spends more on advertising, i.e. it makes bigger efforts $k_2$, $c_2$ when compared to $F_1$. With a sales revenue coefficient $r = 2$, both firms have negative index values in the dark gray region and Firm $F_2$ has negative index values and firm $F_1$ has positive index values in the light gray region.

Figure 3.11: With a bigger sales revenue coefficient is $r = 4$ there are trajectories where both firms make a profit, i.e. their indexes are positive, even where their corresponding long term market share is less than 50%.

3.6 Remarks on model VWDsC

- The entering firm $F_1$ needs its constant effort $k_1 > k_2$ to obtain more than half of the long term market share.

- The entering firm $F_1$ needs its constant effort $k_1 = k_2$ to obtain half of the long term market share.

- Not using the discount rate, which brings profit future values to present values is a mathematical simplification. However, in the simulations the system reaches its steady state in a small number of time units so that it can be considered a finite time horizon, so that it is unnecessary to consider all profits in terms of present value.
The performance index identifies which trajectories make a profit, thus the index and the saturated market share values can be used to make advertising expenditure decisions. For example, if the revenue factor of the market is low, then high spending on advertising could lead to big market share but low profit.
Chapter 4

Modification of Vidale-Wolfe-Deal model with Predatory Advertising

4.1 Introduction

We study a modification of the Vidale-Wolfe-Deal model that has switched predation, this model will be called VWDsP. We assume that both firms want to attain at least half of the market share. In this model each firm uses the same control $u_{ii}$ in regions $U$ and $L$:

\begin{align*}
  u_{11} &= k_1 x_1 + c_1 \quad (4.1) \\
  u_{22} &= k_2 x_2 + c_2 \quad (4.2)
\end{align*}

The switching control is on the predatory advertising. A firm $F_i$ turns on its predatory advertising when it is losing i.e., when $x_i < x_j$. The predatory advertising by firm $F_1$ is:

\begin{equation}
  w_{21} = \begin{cases} 
  c_{21} x_2 & \text{if } x_1 < x_2 \\
  0 & \text{if } x_1 > x_2
  \end{cases} \quad (4.3)
\end{equation}

and the predatory advertising by firm $F_2$ is:

\begin{equation}
  v_{12} = \begin{cases} 
  c_{12} x_1 & \text{if } x_2 < x_1 \\
  0 & \text{if } x_2 > x_1
  \end{cases} \quad (4.4)
\end{equation}

We replace (4.1), (4.1), (4.3) and (4.4) in model (2.1) to get the VWDsP model: The upper field $f^U$, which defines the dynamics in the region above $\Sigma$, is

\begin{align*}
  \dot{x}_1 &= (k_1 x_1 + c_1)(1 - x_1 - x_2) \quad (4.5) \\
  \dot{x}_2 &= (k_2 x_2 + c_2)(1 - x_1 - x_2) - c_{21} x_2 \quad (4.6)
\end{align*}

when $x_2 > x_1$
The lower field $f^L$, which defines the dynamics in the region below $\Sigma$, is

\[
\begin{align*}
\dot{x}_1 &= (k_1 x_1 + c_1) x_1 (1 - x_1 - x_2) - c_{12} x_1 \\
\dot{x}_2 &= (k_2 x_2 + c_2) (1 - x_1 - x_2)
\end{align*}
\] (4.7)

when $x_1 > x_2$

On $\Sigma$, the dynamics is defined following the Filippov rule (see Appendix A, [46, p. 50-52, chap. 2]).

### 4.2 Equilibrium points of model VWDsP

The saturated market line $Sat(1 - x_1 - x_2 = 0)$ is not stable anymore. In region $U$, $\dot{x}_1$ is zero on $Sat$ but $\dot{x}_2$ is not. In region $L$, $\dot{x}_1$ is not zero on $Sat$ but $\dot{x}_2$ is. Let a trajectory $\tau$ begin in region $U$ and satisfy $1 - x_{10} - x_{20} > 0$, with this setting the dynamics of the system is nondecreasing for market share $x_1$ so that $\tau$ tries to reach $1 - x_1 - x_2$. However, market share $x_2$ does not allow $\tau$ to reach the saturated market line $1 - x_1 - x_2$ because of the term $-c_{21} x_2$.

In order to calculate the equilibrium points we will work with the field $f^U$ (the calculation of the equilibria of $f^L$ is analogous). The system (4.5)-(4.6) has three equilibria:

\[
P_1 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}
\] (4.9)

The eigenvalues of the Jacobian at equilibrium point $P_1$ are

\[
\begin{align*}
\lambda_1(P_1) &= \frac{-c_1 - c_2 - c_{21} - k_1 - \sqrt{(c_1 + c_2 + c_{21} + k_1)^2 - 4c_{21}(k_1 + c_1)}}{2} \\
\lambda_2(P_1) &= \frac{-c_1 - c_2 - c_{21} - k_1 + \sqrt{(c_1 + c_2 + c_{21} + k_1)^2 - 4c_{21}(k_1 + c_1)}}{2}
\end{align*}
\] (4.10) (4.11)

\[
P_2 = \begin{bmatrix} \frac{-c_1}{k_1 k_2} \\ \frac{c_1 k_2 - c_2 k_1 - c_{21} k_1 + k_1 k_2 + \sqrt{(c_1 k_2 - c_2 k_1 - c_{21} k_1 + k_1 k_2)^2 + 4c_2 c_{21} k_1^2}}{2k_1 k_2} \end{bmatrix}
\] (4.12)

and the eigenvalues of the Jacobian at equilibrium point $P_2$ are

\[
\begin{align*}
\lambda_1(P_2) &= \frac{c_1 k_2 + c_2 k_1 + c_{21} k_1 + k_1 k_2 - \sqrt{(c_2 k_1 + c_1 k_2 - c_{21} k_1 + k_1 k_2)^2 + 4c_2 c_{21} k_1^2}}{2k_2} \\
\lambda_2(P_2) &= -\frac{\sqrt{(c_2 k_1 + c_1 k_2 - c_{21} k_1 + k_1 k_2)^2 + 4c_2 c_{21} k_1^2}}{k_1}
\end{align*}
\] (4.13) (4.14)
and the eigenvalues of the Jacobian at equilibrium point $P_3$ are

$$
\lambda_1(P_3) = \frac{c_1 k_2 + c_2 k_1 + c_{21} k_1 + k_1 k_2 + \sqrt{(c_2 k_1 + c_1 k_2 - c_{21} k_1 + k_1 k_2)^2 + 4 c_2 c_{21} k_1^2}}{2 k_2}
$$

(4.16)

$$
\lambda_2(P_3) = \frac{\sqrt{(c_2 k_1 + c_1 k_2 - c_{21} k_1 + k_1 k_2)^2 + 4 c_2 c_{21} k_1^2}}{k_1}
$$

(4.17)

Propositions 4.2.1, 4.2.2 and 4.2.3 below refer to trajectories beginning in $U$ with their dynamics defined by (4.5)-(4.6) which are attracted by the virtual stable node $P_1$ (which is virtual because it lies in region $L$ but corresponds to the field $f^U$)

Proposition 4.2.1 (Equilibrium point $P_1$ is stable). For all positive parameter values, the equilibrium point $P_1$ of system (4.5)-(4.6) is stable and virtual.

Proof. The equilibrium $P_1 = [1, 0]^T$ corresponds to the dynamics defined by $f^U$. However, it lies in region $L$ (because $x_1 > x_2$) where the field $f^L$ defines the dynamics, so that $P_1$ is virtual.

If $(c_1 + c_2 + c_{21} + k_1)^2 - 4 c_{21} (k_1 + c_1) > 0$ then we can write

$$
\sqrt{(c_1 + c_2 + c_{21} + k_1)^2 - 4 c_{21} (k_1 + c_1)} = (c_1 + c_2 + c_{21} + k_1)^2 - 4 c_{21} (k_1 + c_1) - \alpha > 0
$$

(4.18)

where $\alpha$ is a real positive number. Using (4.18) in the eigenvalues equations (4.10) and (4.11), we obtain

$$
\lambda_1(P_1) = -2 (c_1 + c_2 + c_{21} + k_1) + \alpha < 0
$$

(4.19)

$$
\lambda_2(P_1) = -\alpha
$$

(4.20)

because both eigenvalues are real negative numbers, the equilibrium point $P_1$ is a stable node in this case.

If $(c_1 + c_2 + c_{21} + k_1)^2 - 4 c_{21} (k_1 + c_1) < 0$ then the eigenvalues are a pair of conjugate imaginary numbers with negative real part and the equilibrium point $P_1$ is a stable focus.

Then the equilibrium point $P_1$ of system (4.5)-(4.6) is virtual and stable.

Proposition 4.2.2 (Condition for equilibrium point $P_2$ to be a saddle.). If $2k_1 (c_{21} - \sqrt{c_2 c_{21}}) > 0$ then the equilibrium point $P_2$ of system (4.5)-(4.6) is a saddle.

Proof. The eigenvalue $\lambda_2(P_2)$, given in (4.14), is negative because it is the negative of the square root of the sum of two squared numbers. In order to analyze the eigenvalue
\( \lambda_1(P_2) \), given in (4.13), we use the triangle inequality to obtain
\[
\sqrt{(c_2k_1 + c_1k_2 - c_2k_1 + k_1k_2)^2 + 4c_2c_2k_1^2} < (c_2k_1 + c_1k_2 - c_2k_1 + k_1k_2) + 2k_1 \sqrt{c_2c_2}
\] (4.21)

substituting (4.21) into (4.13) leads to
\[
\lambda_1(P_2) > 2k_1(c_2 - \sqrt{c_2c_2})
\] (4.22)
because the LHS of (4.22) is positive by hypothesis, \( \lambda_1(P_2) > 0 \). Because \( \lambda_1(P_2) > 0 \) and \( \lambda_2(P_2) < 0 \), the equilibrium point \( P_2 \) is a saddle.

Remark: If \( 2k_1(c_2 - \sqrt{c_2c_2}) < 0 \), the eigenvalue \( \lambda_1(P_2) \) could still be positive. It is the condition \( \lambda_1(P_2) > 0 \) and \( \lambda_2(P_2) < 0 \) that makes the equilibrium point \( P_2 \) a saddle.

Proposition 4.2.3 (Equilibrium point \( P_3 \) is an unstable node). For all positive parameters Equilibrium point \( P_3 \) of system (4.5)-(4.6) is unstable.

Proof. Because all the system parameters are positive numbers and the expressions of the eigenvalues corresponding to point \( P_3 \) (4.16) and (4.17), the eigenvalues \( \lambda_1(P_3) > 0 \) and \( \lambda_2(P_3) > 0 \). This implies that the equilibrium point \( P_3 \) is an unstable node.

4.3 Dynamics of trajectories reaching the equal market share line \( \Sigma \) in model VWDS

Determining the crossing \( \Sigma_c \) and sliding \( \Sigma_s \) sets: To describe trajectories that reach the equal market share line \( \Sigma \), we need to determine the crossing set \( \Sigma_c \) and the sliding set \( \Sigma_s \). To calculate these sets, we use the crossing condition (4.23). (See Appendix A).

\[
\langle \nabla H(w), f^U(w) \rangle \langle \nabla H(w), f^L(w) \rangle > 0
\] (4.23)

where \( w \in \Sigma \), i.e. \( w = (w, w) \) for model (4.5)-(4.6).

An interpretation for condition (4.23) is that each inner product projects the corresponding field onto \( \nabla H(w) \), which is the normal to \( \Sigma \). The convention is that a positive (resp. negative) sign indicates that the projection points into region \( U \) (respectively \( L \)). Then, one way for obtaining the crossing set \( \Sigma_c \) is to calculate where \( \langle \nabla H(w), f^U(w) \rangle \) and \( \langle \nabla H(w), f^L(w) \rangle \) have the same sign. The advantage of analyzing each product separately is that it simplifies the algebra.

If the system parameters are known, using the point \( w = (w, w) \) as argument for the
inner products in (4.23) defines the scalar functions

\[ g_U(w) := \langle \nabla H(w), f'^U(w) \rangle \]  
(4.24)

\[ g_L(w) := \langle \nabla H(w), f'^L(w) \rangle \]  
(4.25)

In the case of model (4.5)-(4.8), the equal market share line \( \Sigma \) is described by \( H(x) = x_2 - x_1 = 0 \) and its normal is \( \nabla H(x) = [-1 \ 1]^T \), with this information the corresponding scalar functions are:

\[ g_U(w) = (2k_1 - 2k_2)w^2 + (2c_1 - 2c_2 - c_{21} - k_1 + k_2)w - (c_1 - c_2) \]  
(4.26)

\[ g_L(w) = (2k_1 - 2k_2)w^2 + (2c_1 - 2c_2 + c_{12} - k_1 + k_2)w - (c_1 - c_2) \]  
(4.27)

Since the right hand sides of (4.26)-(4.27) are parabolas in \( w \), \( \text{sign}(g_U(w)) \) and \( \text{sign}(g_L(w)) \) can be determined by calculus. Once the signs are calculated, the sets can be combined to obtain the crossing set \( \Sigma_c \), which is comprised of the points where the values of parabolas have the same sign, and sliding set \( \Sigma_s \), which is the complement of \( \Sigma_c \), i.e. the points where the values of parabolas \( w \) are zero or have different signs. The zeros of (4.26) are:

\[ v_1 = \frac{-\beta + \sqrt{\beta^2 + 8(k_1 - k_2)(c_1 - c_2)}}{4(k_1 - k_2)} \]  
(4.28)

\[ v_2 = \frac{-\beta - \sqrt{\beta^2 + 8(k_1 - k_2)(c_1 - c_2)}}{4(k_1 - k_2)} \]  
(4.29)

where \( \beta = (2c_1 - 2c_2 - c_{21} - k_1 + k_2) \) and the zeros of (4.27) are:

\[ w_1 = \frac{-\gamma + \sqrt{\gamma^2 + 8(k_1 - k_2)(c_1 - c_2)}}{4(k_1 - k_2)} \]  
(4.30)

\[ w_2 = \frac{-\gamma - \sqrt{\gamma^2 + 8(k_1 - k_2)(c_1 - c_2)}}{4(k_1 - k_2)} \]  
(4.31)

where \( \gamma = (2c_1 - 2c_2 + c_{12} - k_1 + k_2) \)

The second derivatives of (4.26) and (4.27) are:

\[ g_U(w)'' = 4(k_1 - k_2) \]  
(4.32)

\[ g_L(w)'' = 4(k_1 - k_2) \]  
(4.33)

We describe the process of determining the crossing set \( \Sigma_c \) and the sliding set \( \Sigma_s \) with the following algorithm:

**Algorithm for calculating the crossing \( \Sigma_c \) and the sliding set \( \Sigma_s \)**

1. For each field \( F = U, L \) determine the sign of \( g_F(w) \). The signs are calculated
using the fact that (4.26) and (4.27) are parabolas on the variable \( w \) and thus their second derivatives are constants.

- Let the \( w_1, w_2 \) be the zeros of \( g_F(w) \)
- If \( \text{sign}(g_F(w)') < 0 \), then \( g_F(w) \) has a maximum
  - If \( w_1, w_2 \) are real numbers, \( \text{sign}(g_F(w)) \) determined as follows:
    \[
    \text{sign}(g_F(w)) = \begin{cases} 
    1 & \text{if } w \text{ is between its zeros } w_1, w_2 \\
    0 & \text{if } w \text{ is any of its zeros } w_1, w_2 \\
    -1 & \text{for any other value of } w 
    \end{cases} \tag{4.34}
    \]
  - If \( w_1, w_2 \) are conjugate imaginary numbers, then \( \text{sign}(g_F(w)) = -1 \).
- If \( \text{sign}(g_F(w)') > 0 \), then \( g_F(w) \) has a minimum
  - If \( w_1, w_2 \) are real numbers, \( \text{sign}(g_F(w)) \) is determined as follows:
    \[
    \text{sign}(g_F(w)) = \begin{cases} 
    -1 & \text{if } w \text{ is between } w_1, w_2 \\
    0 & \text{if } w = w_1 \text{ or } w = w_2 \\
    1 & \text{for any other value of } w 
    \end{cases} \tag{4.35}
    \]
  - If \( w_1, w_2 \) are conjugate imaginary numbers, then \( \text{sign}(g_F(w)) = 1 \).

2. Combine the signs of the fields obtained in step 1 to calculate the sets

- the crossing set is
  \[
  \Sigma_c = \{ w | \text{sign}(g_U(w)) \text{sign}(g_L(w)) > 0 \} \tag{4.36}
  \]
- the sliding set is the complement of \( \Sigma_c \), i.e. its elements are all \( w \) for which the projections do not have the same sign.

### 4.4 Pseudoequilibrium points of model VWDsP

A pseudoequilibrium point is an equilibrium that may come into existence because of the use of switching control (See Appendix A). The pseudoequilibrium candidates are obtained using the fact that at a pseudoequilibrium the fields must be aligned and have opposite directions. Thus for the system (4.5)-(4.8) it must hold that

\[
\frac{(k_1x_1 + c_1)(1 - x_1 - x_2)}{(k_1x_1 + c_1)(1 - x_1 - x_2) - c_{12}x_1} = \frac{(k_2x_2 + c_2)(1 - x_1 - x_2) - c_{21}x_2}{(k_2x_2 + c_2)(1 - x_1 - x_2)} < 0 \tag{4.37}
\]
considering that the pseudoequilibrium is on \( \Sigma \) where \( x_1 = x_2 = w^* \) and after some algebraic operations, (4.37) can be rewritten as

\[
1 - \frac{c_{12}w^*}{(k_1 w^* + c_1)(1-2w^*)} = 1 - \frac{c_{21}w^*}{(k_2 w^* + c_2)(1-2w^*)} < 0 \tag{4.38}
\]

The pseudoequilibrium candidates for model (4.5), (4.8), which were obtained using the proportion in (4.38), are points on \( \Sigma \) having coordinates \( w^* \):

\[
w_1^* = -\frac{(A - B) - \sqrt{(A - B)^2 + 4B(2c_1c_{21} + 2c_{12}c_2)}}{4B} \tag{4.39}
\]
\[
w_2^* = -\frac{(A - B) + \sqrt{(A - B)^2 + 4B(2c_1c_{21} + 2c_{12}c_2)}}{4B} \tag{4.40}
\]
\[
w_3^* = 0 \tag{4.41}
\]

where \( A = 2c_1c_{21} + 2c_{12}c_2 + c_{12}c_{21} \) and \( B = c_{21}k_1 + c_{12}k_2 \).

Notice that for \( w_3^* \) the proportion is positive, so that it cannot be a pseudoequilibrium.

**Proposition 4.4.1 (Feasibility interval for a pseudoequilibrium point).** If \( w^* \) is a pseudoequilibrium of model (4.5)-(4.8) then the following inequalities are satisfied

(i) A pseudoequilibrium can occur in the interval

\[ 0 < w^* < 1/2 \]

(ii) The relation of the predation effort with the other control efforts at a pseudoequilibrium is

\[
c_{21}w^* > (k_2 w^* + c_2)(1 - 2w^*)
\]
\[
c_{12}w^* > (k_1 w^* + c_1)(1 - 2w^*)
\]

**Proof.** Given that the parameters \( k_1, c_1, k_2, c_2, c_{12}, c_{21} \) are positive, the condition \( (1 - 2w^*) > 0 \) must hold to fulfill (4.38). This leads to \( w^* < 1/2 \). Since we are only interested in positive values for the states \( x_1, x_2 \), item (i) is proved and the interval where a pseudoequilibrium can occur is

\[ 0 < w < 1/2 \tag{4.42} \]

Also, for the proportion in (4.38) to be negative, the following conditions must hold

\[
c_{21}w^* > (k_2 w^* + c_2)(1 - 2w^*)
\]
\[
c_{12}w^* > (k_1 w^* + c_1)(1 - 2w^*)
\]

and item (ii) is proved.
Proposition 4.4.2 (Signs of pseudoequilibrium candidates). The pseudoequilibrium candidates $w_1^*$ given by (4.39) and $w_2^*$ given by (4.40) have the following properties:

(i) $w_1^*$ is a negative number.

(ii) $w_2^*$ is a positive number.

Proof. By multiplying $w_1^*w_2^*$ we get

$$w_1^*w_2^* = -4B(2c_1c_{21} + 2c_{12}c_2)$$

which is a negative number because the parameters of model (4.5)-(4.8) are all positive numbers. Then, one factor has to be positive and the other negative.

Because the square root is a nondecreasing function we can write

$$\sqrt{(A - B)^2 + 4B(2c_1c_{21} + 2c_{12}c_2)} = (A - B) + \alpha$$

where $\alpha > 0$. Substituting (4.44) into (4.40)

$$w_2^* = \alpha$$

Because $\alpha > 0$, $w_2^*$ is a positive number and (ii) is proved.

Since $w_1^*w_2^* < 0$ by (4.43) and $w_2^* > 0$, $w_1^*$ is a negative number and (i) is proved.

Given that the states have to be positive, the only possible pseudoequilibrium for model (4.5), (4.8) is $w_2^* > 0$.

Proposition 4.4.3 (Pseudoequilibrium feasibility satisfied by $w_2^*$). If the pseudoequilibrium candidates $w_1^*$ and $w_2^*$, resp. defined in (4.39) and (4.40), satisfy

$$w_1^* + w_2^* > 0$$

then $w_2^* < \frac{1}{2}$.

Proof. Adding $w_1^* + w_2^*$ we get

$$w_1^* + w_2^* = \frac{1}{2} - \frac{A}{2B}$$

Using the hypothesis $w_1^* + w_2^* > 0$ in the previous we obtain

$$\frac{A}{B} < 1$$

Rewriting $w_2^*$ in (4.40) we obtain

$$w_2^* = -(A - B) + \sqrt{(A - B)^2 + 4B(A - c21c12)}$$

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Since the terms $A, B$ and the parameters are nonnegative we can write

$$w_2^* < \frac{-(A - B) + \sqrt{(A - B)^2 + 4BA}}{4B} \quad (4.50)$$

Applying the triangular inequality to the RHS of the previous inequality and simplifying terms we obtain

$$w_2^* < \frac{-(A - B) + \sqrt{(A - B)^2 + 4BA}}{4B} < \frac{1}{2} \frac{\sqrt{A}}{B} \quad (4.51)$$

Using (4.48) in (4.51)

$$w_2^* < \frac{-(A - B) + \sqrt{(A - B)^2 + 4BA}}{4B} < \frac{1}{2} \frac{\sqrt{A}}{B} < \frac{1}{2} \quad (4.52)$$

so that $w_2^* < \frac{1}{2}$ and $w_2^*$ satisfies Proposition 4.4.1.

\[\square\]

**Lemma 4.4.4 (Pseudoequilibrium of Model VWDsP).** Trajectories of the system (4.5)-(4.8) end at a pseudoequilibrium point.

### 4.5 Simulations of model VWDsP

We have done several simulations of the model (4.5)-(4.8) and the trajectories ended at a pseudoequilibrium point on the equal market share line $\Sigma$. Depending on the parameters, a trajectory that reaches $\Sigma$ may slide on it, cross it or have both behaviors at different time values.

Table 4.1 shows different parameter values and the corresponding pseudoequilibrium. Notice how increasing parameter $k_1$ (lines 1-5) and parameter $c_1$ (lines 6-9) increases the market share value attained at the pseudoequilibrium. However, increasing the predation $c_{21}$ done by firm $F_1$ on firm $F_2$ (lines 10-13) decreases noticeably the market share value attained at the pseudoequilibrium.

We also ran simulations with the parameters $k_2, c_2, c_{21}, k_1, c_1, c_{12}$ taking a permutation of the values 0.25, 1.25, 2.5, 3.75, 5 and all ended at a pseudoequilibrium point on the equal share line $\Sigma$ below (.5, .5) and above (0, 0) so that the trajectories of model (4.5)-(4.8) end at a pseudoequilibrium on the equal market share line independent of the efforts done by the firms.

Next, we present the qualitative analysis of two scenarios: in the first, both firms $F_1$ and $F_2$ do the same predation effort. In the second scenario, firm $F_1$ does a predation effort that is bigger than the effort of firm $F_2$.

The first scenario of model (4.5)-(4.8) has the parameters $k_1 = 1.5, c_1 = 0.7, c_{21} = 0.15, k_2 = 2, c_2 = 1, c_{12} = 0.15$. Firm $F_1$ and $F_2$ apply the same predation effort.
Table 4.1: One parameter variation in model VWDsP. Increasing $k_1$ or $c_1$ moderately increases the market share value at pseudoequilibrium. On the other hand, increasing the predation effort $c_{21}$ decreases noticeably the market share value at pseudoequilibrium.

The corresponding phase plane is shown in Figure 4.1 where one can see all trajectories go to a pseudoequilibrium on the equal market share line. The equilibrium points of the fields are shown in Tables 4.2 and 4.3, note that field $f^L$ has a virtual stable equilibrium in region $U$, respectively field $f^U$ has a virtual stable equilibrium in region $L$.

Dynamics of a trajectory on the equal market share line $\Sigma$

- Crossing and sliding sets are determined using the algorithm 4.3. The projections for determining the sets are $\langle \nabla H(w), f^U(w) \rangle = -w^2 - 0.25w + 0.3$, that has the zeros $w_1 = -0.68681$ and $w_2 = 0.43681$, and $\langle \nabla H(w), f^L(w) \rangle = -w^2 + 0.05w + 0.3$, that has the zeros $w_1 = -0.52329$ and $w_2 = 0.57329$. Because we are interested only $w$ values that are nonnegative and below saturation ($w = 0.5$), the crossing set is $\Sigma_c = [0, 0.43681]$ (where trajectories cross into region $U$) and the sliding set is $\Sigma_s = [0.43681, 0.5]$. Figure 4.3 shows the the plots of the signs of the projections and their products which are used by the algorithm for determining the segments on the equal share line.

- One of the pseudoequilibrium ratios corresponding to pseudoequilibrium condition for scenario 1 is $1 - \frac{1.0(1.05w^3 + 0.0075w^2 - 0.255w)}{12.0w^2 + 0.2w^3 - 5.8w - 0.05w + 0.7}$. Scenario 1 has a pseudoequilibrium $w^*_2 = 0.48925$ because its field component ratio is negative with numerator value of 0.03084 and denominator value of $-0.04255$.
In the second scenario of model (4.5)-(4.8) firm $F_1$ does a bigger predation effort than the predation effort of firm $F_2$, i.e. $c_{21} \approx 7.33c_{12}$. This scenario exemplifies the situation where large predation effort (by firm $F_1$) or small predation effort (by firm $F_2$) leads to a pseudoequilibrium on the equal market share line. The parameters in this scenario are: $k_1 = 1.5$, $c_1 = 0.7$, $c_{21} = 1.1$, $k_2 = 2$, $c_2 = 1$, $c_{12} = 0.15$. Figure 4.4 shows the...
phase plane and the equilibrium points of the fields are shown in Tables 4.4 and 4.5, note that field $f^L$ has a virtual stable equilibrium in region $U$ and field $f^U$ has a virtual stable equilibrium in region $L$.

Dynamics of a trajectory on the equal market share line $\Sigma$

- Crossing and sliding sets are determined using the algorithm 4.3. The projections for determining the sets are $\langle \nabla H(w), f^U(w) \rangle = -1.0w^2 - 1.2w + 0.3$, that has the zeros $w_1 = -1.4124$ and $w_2 = 0.2124$, and $\langle \nabla H(w), f^L(w) \rangle = -1.0w^2 + 0.05w + 0.3$, that has the zeros $w_1 = -0.52329$ and $w_2 = 0.57329$. Because we are interested only in $w$ values which are nonnegative and below saturation ($w = 0.5$) the crossing set is $\Sigma_c = [0, 0.2124]$ (where trajectories cross into region $U$) and the sliding set is $\Sigma_s = [0.2124, 0.5]$. Figure 4.6 shows the the plots of the signs of the projections and their products which are used by the algorithm for determining the segments on the equal share line.

- One of the pseudoequilibrium ratios corresponding to pseudoequilibrium condition for scenario 2 is $-\frac{1.0}{12w^4 + 0.2w^3 - 5.8w^2 - 0.05w + 0.7}$. Scenario 2 has a pseudoequilibrium $w^2 = 47869$ because its field component ratio is negative with numerator value of 0.06043 and denominator value of $-0.01137$.

![Figure 4.4: Model VWDS\textregistered, scenario 2: Firm $F_1$ doing big spending on advertising when compared to firm $F_2$ ($c_{21} \approx 7.33c_{12}$).](image1)

![Figure 4.5: Model VWDS\textregistered, scenario 2: Time plots of the market shares $x_1(t)$, $x_2(t)$ with initial conditions $x_1(0) = 0.04$ and $x_2(0) = 0.05$.](image2)
4.6 Evaluating outcomes of the proposed policies using a performance index for model VWDsP

We use the index (2.9). Note that a firm makes a profit when its index $J_i$ is positive, whereas it makes a loss when its index is negative.

The market share $x_i(t)$ evolution is given by the system (4.5), (4.8) and the control functions $u_i(t)$ used for the index are listed below. The control function for each firm in the upper field $f^U$ are

$$u_1 = (k_1 x_1 + c_1) \quad (4.53)$$
$$u_2 = (k_2 x_2 + c_2) - c_{21} \quad (4.54)$$

when $x_2 > x_1$
The control function for each firm in the lower field $f^L$ are

$$u_1 = (k_1 x_1 + c_1) - c_{12} \quad (4.55)$$

$$u_2 = (k_2 x_2 + c_2) \quad (4.56)$$

when $x_1 > x_2$

On $\Sigma$, both firms turn on their extra effort

$$u_1 = (k_1 x_1 + c_1) - c_{12} \quad (4.57)$$

$$u_2 = (k_2 x_2 + c_2) - c_{21} \quad (4.58)$$

when $x_1 = x_2$

The scenarios were simulated using the software Matlab. The first scenario of model (4.5)-(4.8) has the parameters $k_1 = 1.5, c_1 = 0.7, c_{21} = 0.15, k_2 = 2, c_2 = 1, c_{12} = 0.15$ and $r = 4.5$ where both firms do the same predation effort, i.e. $c_{21} = c_{12}$. In this setting both firms end at a pseudoequilibrium point below 0.5 on the equal market share line $\Sigma$.

Even if both firms end on the equal market share line, a firm needs to identify on which trajectories it makes a profit. Figure 4.7 shows the trajectories in the phase plane according to their index value at final time, these indexes were calculated using a revenue factor $r = 4.5$. Only firm $F_2$ makes a profit on the solid line trajectories, both firms make a profit on the dashed line trajectories and only firm $F_1$ makes a profit on the dashed-dotted line trajectories, firm $F_1$ profits in more trajectories because it is doing smaller efforts ($k_1 < k_2$ and $c_1 < c_2$). Figure 4.8 shows $J_2 V s. J_1$ to help identify compromise solutions, i.e. conditions that end in positive $J_i$ and those values cannot improve anymore.

![Figure 4.7: Model VWDsP, scenario 1: Trajectories plotted according to their index values at final time.](image1)

![Figure 4.8: Model VWDsP, scenario 1: $J_1(x_1(t), x_2(t), t)$ vs. $J_2(x_1(t), x_2(t), t)$](image2)
Figures 4.9 and 4.10 show the index values when the market share of the entering firm \((x_i < x_j)\) is fixed and the market share of the other firm is increasing. From these figures one can see that the firm that begins with a smaller market share has in most of the cases a smaller and decreasing performance index value.

Figures 4.9: Model VWDsP, scenario 1: Value of indices when the initial value \(x_1(0) = 0.04\) is fixed.

Figures 4.10: Model VWDsP, scenario 1: Value of indices when the initial value \(x_2(0) = 0.05\) is fixed.

The second scenario of model VWDsP has firm \(F_1\) with a bigger predation effort \(c_{21} = 1.1\).

A firm needs to identify on which trajectories it makes a profit regardless of its final market share, which in this case is on the equal market share. Figure 4.11 shows the trajectories in the phase plane according to their index value at final time, these indexes were calculated using a revenue factor \(r = 4.5\). Only firm \(F_2\) makes a profit on the solid line trajectories, both firms make a loss on the dotted line trajectories and only firm \(F_1\) makes a profit on the dashed-dotted line trajectories, firm \(F_1\) profits less than in the first scenario because now it is spending far more in predatory advertising. Figure 4.12 shows \(J_2\) vs. \(J_1\) to help identify compromise solutions, i.e. conditions that end in positive \(J_i\) and those values cannot improve anymore.
Figures 4.11 and 4.12 show the index values when the market share of the entering firm \((x_i < x_j)\) is fixed and the market share of the other firm is increasing. From these figures one can see that the firm that begins with a smaller market share has in most of the cases a smaller and decreasing performance index value.

4.7 Remarks on model VWDsP

- The saturated market is not reached, i.e. the long run equal market share is below the saturated equal market share.

- As the predation effort \(u_{ij}\) increases, the steady state market share decreases, see
Table 4.1

- Even though it may seem reasonable for a firm $F_i$ to turn on a predatory advertising spending $u_{ji}$ when its target market is below its goal ($x_i < x_j$), from the analysis of model VWDsP we see that this leads to both firms reaching an equal market share in the long run independent of the efforts they might apply, and this may not be reasonable since it would mean that a firm spending very little on advertising would end with the same market share as a firm spending big amounts. Further analysis by the decision maker would be needed to decide whether this could happen: for example, in some markets with unsatisfied buyers, frequently changing from one firm to the other.

4.8 Model Vidale-Wolfe-Deal model with 2 switching lines (VWD2s)

A model which is more realistic in terms of target market shares is the model VWD2s. In this model, the target market shares of the firms are different, firm $F_1$ desires a long term market share $target_1$, and $F_2$ desires a long term market share $target_2$ which satisfy $target_1 + target_2 > 1$. In this setting, each $F_i$ has its own switching line.

The model VWD2s is as follows:

1. $F_2$ fulfills its target in the region $U$ which is above $\Sigma_{UM}$, which is defined by the line $x_2 - \frac{target_2}{1-target_2}x_1 = 0$. The upper field $f^U$ operating in $U$ is $\Sigma_{UM}$, is

   \[
   \begin{align*}
   \dot{x}_1 &= (k_1x_1 + c_1)(1 - x_1 - x_2) \\
   \dot{x}_2 &= k_2x_2(1 - x_1 - x_2)
   \end{align*}
   \tag{4.59}
   \]

   when $x_2 > \frac{target_2}{1-target_2}x_1$

2. The region $M$ is where neither firm obtains its target market share. It is below $\Sigma_{UM}$ and above $\Sigma_{ML}$, which is defined by the line $x_2 - \frac{1-target_1}{target_1}x_1 = 0$. The middle field $f^M$ operating in $M$ is

   \[
   \begin{align*}
   \dot{x}_1 &= (k_1x_1 + c_1)(1 - x_1 - x_2) \\
   \dot{x}_2 &= (k_2x_2 + c_2)(1 - x_1 - x_2)
   \end{align*}
   \tag{4.61}
   \]

   when $x_2 < \frac{target_2}{1-target_2}x_1$ and $x_2 > \frac{1-target_1}{target_1}x_1$

3. The region $L$ is where firm $F_1$ obtains its target market share. It is below $\Sigma_{ML}$,
which is the line \( x_2 - \frac{1 - \text{target}_1}{\text{target}_1} x_1 = 0 \). The lower field \( f^L \) operating in \( L \) is

\[
\begin{align*}
\dot{x}_1 &= k_1 x_1 (1 - x_1 - x_2) \\
\dot{x}_2 &= (k_2 x_2 + c_2) (1 - x_1 - x_2)
\end{align*}
\]

(4.63)

(4.64)

when \( x_2 < \frac{1 - \text{target}_1}{\text{target}_1} x_1 \)

4. On the switching lines \( \Sigma_{UM} \) and \( \Sigma_{ML} \) the dynamics is defined following the Filippov rule which is based, roughly speaking, on a convex combination of the lower and upper fields with respect to the corresponding switching line (see Appendix A [46, p. 50-52, chap. 2] for mathematical details).

Figure 4.15 identifies the regions and the relevant lines for model VWD2s which is defined by (4.59)-(4.64). It uses \( \text{target}_2 = 0.8 \) and \( \text{target}_1 = 0.7 \). \( \Sigma_{UM} \) is the line \( x_2 - \frac{0.8}{0.2} x_1 = 0 \) and \( \Sigma_{ML} \) is the line \( x_2 - \frac{3}{7} x_1 = 0 \). The line \( 1 - x_1 - x_2 = 0 \) is the full market line which corresponds to the saturated market, which will be called \( \text{Sat} \). The interior of the triangle \( \mathcal{T} \), defined as \((1, 0), (0, 1), (0, 0)\) is the set of all feasible states. The region \( U \) is the set \( \{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1 \wedge x_2 > 4x_1 \wedge x_2 > \frac{3}{7} x_1\} \) where the dynamics corresponding to \( f^U \) is acting. The region \( M \) is the set \( \{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1 \wedge x_2 < 4x_1 \wedge x_2 > \frac{3}{7} x_1\} \) where the dynamics corresponding to \( f^M \) is acting. The region \( L \) is the set \( \{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1 \wedge x_2 < 4x_1 \wedge x_2 < \frac{3}{7} x_1\} \) where the dynamics corresponding to \( f^L \) is acting.

Figure 4.15: The regions: upper \( U \), middle \( M \) and lower \( L \). The switching lines \( \Sigma_{UM} = \{x : H(x) = x_2 - 4x_1 = 0\} \), \( \Sigma_{ML} = \{x : H(x) = x_2 - \frac{3}{7} x_1 = 0\} \). The segment \( \text{Sat} \) segment which is defined by \( 1 - x_1 - x_2 = 0 \). \( \mathcal{T} \) is the interior of the triangle defined by the points \((1, 0), (0, 1), (0, 0)\)

4.9 Simulations of model VWD2s

The scenarios were simulated using the software Matlab. The first scenario of model (4.59)-(4.64) has the parameters \( k_1 = 1.5, c_1 = 0.7, c_{21} = 0.15, k_2 = 2, c_2 = 1, c_{12} = 0.15 \) and \( r = 4.5 \) where the control efforts of the firms are not very different and the outcome in terms of final index values is divided evenly between the firms.
In the first scenario, shown in Figures 4.16 and 4.17, the firms do similar control efforts and the long run market share values are evenly distributed on the saturated market share. In the switching lines $\Sigma_{UM}$ and $\Sigma_{ML}$ crossing is the behavior observed.

In the second scenario, shown in Figures 4.18 and 4.19, firm $F_1$ does bigger control efforts than firm $F_2$ and most of the trajectories go down where firm $F_1$ has a bigger long run market share. In the switching lines crossing and sliding are the behaviors observed.

Figure 4.16: Model VWD2s, scenario 1: Firm $F_1$ and $F_2$ spend similar amounts in advertising.

Figure 4.17: Model VWD2s, scenario 1: Time plots of the market shares $x_1(t)$, $x_2(t)$ with initial conditions $x_1(0) = 0.04$ and $x_2(0) = 0.05$.

Figure 4.18: Model VWD2s, scenario 2: Firm $F_1$ spends more than $F_2$ in advertising.

Figure 4.19: Model VWD2s, scenario 2: Time plots of the market shares $x_1(t)$, $x_2(t)$ with initial conditions $x_1(0) = 0.04$ and $x_2(0) = 0.05$. 

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4.10 Evaluating outcomes of the proposed policies using a performance index for model VWD2s

We use the index (2.9). Note that a firm makes a profit when its index $J_i$ is positive, whereas it makes a loss when its index is negative.

In the first scenario, shown in Figures 4.20-4.22, the control efforts of the firms are not very different and the outcome in terms of final index values is divided evenly between the firms. Figure 4.20 shows the trajectories in the phase plane according to their index value at final time, these indexes were calculated using a revenue factor $r = 3.5$. Only firm $F_2$ makes a profit on the solid line trajectories, both firms make a profit on the dashed line trajectories and only firm $F_1$ makes a profit on the dashed-dotted line trajectories, firm $F_1$ profits in more trajectories because it is doing smaller efforts ($k_1 < k_2$ and $c_1 < c_2$).

![Graph showing trajectories and index values](image)

Figure 4.20: Model VWD2s, scenario 1: Trajectories plotted according to their index values at final time.

Figures 4.21 and 4.22 show the index values when the market share of the entering firm ($x_i < x_j$) is fixed and the market share of the other firm is increasing. From these figures one can see that the firm that begins with a smaller market share has in most of the cases a smaller and decreasing performance index value.
In the second scenario, shown in Figures 4.23-4.25, firm $F_1$ does a bigger control effort than firm $F_2$. In this setting more trajectories go down, i.e. where $F_1$ end with a bigger market share in the long run. Figure 4.23 shows the trajectories in the phase plane according to their index value at final time, these indexes were calculated using a revenue factor $r = 3.5$. Only firm $F_2$ makes a profit on the solid line trajectories, and both firms make a loss on the dotted line trajectories. Firm $F_1$ does not make a profit because its big spending on advertising does not generate profit for the market revenue factor $r = 3.5$.

![Figure 4.21: Model VWD2s, scenario 1: Value of indices when the initial value $x_1(0) = 0.04$ is fixed.](image1)

![Figure 4.22: Model VWD2s, scenario 1: Value of indices when the initial value $x_2(0) = 0.05$ is fixed.](image2)

Figures 4.24 and 4.25 show the index values when the market share of the entering firm ($x_i < x_j$) is fixed and the market share of the other firm is increasing. From these figures one can see that the firm that begins with a smaller market share has in most of the
cases a smaller and decreasing performance index value.

Figure 4.24: Model VWD2s, scenario 2: Value of indices when the initial value \( x_1(0) = 0.04 \) is fixed.

Figure 4.25: Model VWD2s, scenario 2: Value of indices when the initial value \( x_2(0) = 0.05 \) is fixed.

4.11 Remarks on model VWD2s

- Model VWD2s extends model VWDsC and the analysis done for model VWDsC applies to model VWD2s on each discontinuity line, for firm \( F_2 \) (resp. \( F_1 \)) the line of interest is \( \Sigma_{UM} \) (resp. \( \Sigma_{ML} \)).
  - The saturated market share line is a set of attractive points.
  - The state variables are nondecreasing, they increase while they are below the saturated market share \( Sat \) and remain unchanged once they reach \( Sat \).
  - Trajectories that reach the switching line \( \Sigma_{UM} \) (or \( \Sigma_{ML} \)) have the same behavior as in model VWDsC, the difference is the slope of the switching lines.
  - The model has no pseudoequilibrium point (neither on \( \Sigma_{UM} \) nor on \( \Sigma_{ML} \)).
  - Each firm has a separatrix, in the case of firm \( F_2 \) (resp. firm \( F_1 \)) a separatrix is calculated using its target saturated market share point \( (1 - target_2, target_2) \) (resp. \( (target_1, 1 - target_1) \)) which is where \( \Sigma_{UM} \) (resp. \( \Sigma_{ML} \)) and \( Sat \) intersect. A separatrix for firm \( F_2 \) (resp. \( F_1 \)) may be \( \varphi^U \) or \( \varphi^M \) (resp. \( \varphi^M \) or \( \varphi^L \)), i.e. obtained with fields \( f^U \) or \( f^M \) (resp. \( f^M \) or \( f^L \)).
  - Firm \( F_2 \) (resp. \( F_1 \)) fulfills its target market share above (resp. below) its separatrix.

- The difference is that a separatrix may span more than one region: for firm \( F_2 \) (resp. \( F_1 \)) it may span region \( U \) or regions \( M \) and \( L \) (resp. regions \( M \) and \( U \) or region \( L \)).
Chapter 5

Duopoly model based on Lotka-Volterra dynamics with switched predation (LVsP)

5.1 Introduction

Wang et al. [35] proposed a duopoly model with Lotka-Volterra competing species dynamics where the state variable is sales:

\[
\begin{align*}
\dot{x}_1 &= x_1(b_1 - a_{11}x_1) - a_{12}x_1x_2 \\
\dot{x}_2 &= x_2(b_2 - a_{22}x_2) - a_{21}x_1x_2
\end{align*}
\]  \hspace{1cm} (5.1)

in this model, all the coefficients \(a_{ij}, b_i\) are assumed nonnegative and it is stated that \(b_i = f(\text{advertising})\).

Wang et al. [35] showed that model (5.1,5.2) fulfills the desired properties of advertising function [25], but there is no exact hint on how to obtain the coefficient \(b_i\) as a function of advertising. Because of this we consider this model theoretical. However, recent papers use Lotka-Volterra models and this indicates that they should be explored. In 2016, Marasco et al. [52] used a nonautonomous Lotka-Volterra system to model market share dynamics that depend on utility functions of the competing firms. Because of the special functional forms they impose, Marasco et al. can obtain an analytical solution to the system. In 2014, Hung et al. [36] modeled the saturated market dynamics using a Lotka-Volterra model.

In analogy with the biological models, the value \(b_i/a_{ii}\) could be thought of as a carrying capacity (i.e., market saturation level) for firm \(i\) and the so-called “interaction” term \(a_{ij}x_ix_j\) as a term describing predatory advertising. Note that if the state variable to be used is market share, the Lotka-Volterra model does not result in invariance of to-
tal market share (always equal to one), ensuring instead that each market share variable is below or equal to a given upper bound (via the carrying capacity of each firm). One way for model (5.1, 5.2) to enforce the invariance of total saturated market share is setting $b_1 = a_{11} = a_{12} = b_2 = a_{22} = a_{21} = 1$, but this reduces the model (5.1, 5.2) to Vidale-Wolfe-Deal’s model where $u_i = x_i$.

The fact that the Lotka-Volterra model cannot enforce the saturated market restriction led to the choice, in this chapter, of the number of clients as the state variable instead of market share. In this case the predatory term $a_{ij}x_ix_j$, i.e. advertising, removes clients from one firm. In the context of a duopoly for an essential good, it is also reasonable to argue that clients removed from one firm go to the other firm and, motivated by this observation, we propose the following duopoly model with Lotka-Volterra type dynamics, together with controlled predatory terms:

$$\dot{x}_1 = x_1(b_1 - a_{11}x_1) - u_{12}x_1x_2 + u_{21}x_1x_2 \quad (5.3)$$
$$\dot{x}_2 = x_2(b_2 - a_{22}x_2) + u_{12}x_1x_2 - u_{21}x_1x_2 \quad (5.4)$$

where $b_i, a_{ii}$ defines the saturation level for firm $i$, $u_{ij}$ is a control coefficient determining predatory advertising by firm $j$ on firm $i$ (i.e., firm $j$ capturing clients of firm $i$). In what follows, it will be postulated that the advertising effort or expenditure by a firm modifies the values of the control coefficients $u_{ij}$ depending on its perception of its client base.

The policy $u_{ij}$ is a predatory advertising effort made by firm $F_j$ on firm $F_i$ and it is as follows:

$$u_{ij} = \begin{cases} 
  c_{ij} & \text{if } x_j < x_i, \\
  d_{ij} & \text{if } x_j > x_i
\end{cases} \quad (5.5)$$

where $i, j = 1, 2$, $i \neq j$ and the policy $c_{ij} > d_{ij}$, i.e. firm $F_j$ applies a bigger predatory advertising constant effort on firm $F_i$ clients when $x_j < x_i$, this is when $F_j$ has a smaller number of clients.

After using the policies (5.5) and some algebra in model (5.3, 5.4) its fields are obtained:

The upper field $f^U$, which defines the dynamics in region $U$, is given by

$$\dot{x}_1 = x_1(b_1 - a_{11}x_1) + (-d_{12} + c_{21})x_2 \quad (5.6)$$
$$\dot{x}_2 = x_2(b_2 - (-d_{12} + c_{21})x_1 - a_{22}x_2) \quad (5.7)$$

The lower field $f^L$, which defines the dynamics in region $L$, is given by

$$\dot{x}_1 = x_1(b_1 - a_{11}x_1 + (-c_{12} + d_{21})x_2) \quad (5.8)$$
$$\dot{x}_2 = x_2(b_2 - (-c_{12} + d_{21})x_1 - a_{22}x_2) \quad (5.9)$$
On $\Sigma$, the dynamics is defined following the Filippov rule (see Appendix [46] p. 50-52, chap. 2). Note that all coefficients $b_i, a_{ij}, c_{ij}, d_{ij} (i, j = 1, 2, i \neq j)$ are real nonnegative numbers.

## 5.2 Equilibria of model LVsP

We analyze the equilibrium points in field $f^L$ (the analysis of field $f^U$ is analogous). The equilibria are the zeros of the RHS (right hand side) of system (5.6), (5.7). In order to find if an equilibrium point is stable or not, the jacobian of the system is used:

$$
\begin{bmatrix}
-2a_{11}x_1 + b_1 + (c_{21} - d_{12})x_2 & (c_{21} - d_{12})x_1 \\
(c_{21} - d_{12})x_2 & b_2 - (c_{21} - d_{12})x_1 - 2a_{22}x_2
\end{bmatrix}
$$

(5.10)

The four equilibria points of the model (5.6), (5.7) are:

1. Equilibrium point $P_1 = (0, 0)$ and its corresponding jacobian matrix $J(P_1) = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$ with the eigenvalues readily on the diagonal, so that $P_1$ is an unstable node because $b_i > 0, i = 1, 2$.

2. Equilibrium point $P_2 = (0, b_2/a_{22})$ and its corresponding jacobian matrix $J(P_2) = \begin{bmatrix} b_1 & a_{22} \\ b_2 & -a_{22} \end{bmatrix}$ with the eigenvalues on the diagonal.

3. Equilibrium point $P_3 = (b_1/a_{11}, 0)$ and its corresponding jacobian matrix $J(P_3) = \begin{bmatrix} -b_1 & a_{11} \\ 0 & b_2 \end{bmatrix}$ with the eigenvalues on the diagonal.

4. Equilibrium point $P_4 = \left( \frac{a_{22}b_1 + b_2c_{21} - b_2d_{12}}{a_{11}a_{22} + c_{21} - 2c_{21}d_{12} + d_{12}^2}, \frac{a_{11}b_2 - b_1c_{21} + b_1d_{12}}{a_{11}a_{22} + c_{21} - 2c_{21}d_{12} + d_{12}^2} \right)$ and its corresponding jacobian matrix is

$$
\begin{bmatrix}
-2a_{11}(a_{22}b_1 + b_2b_1) & + \frac{\beta(a_{11}b_2 - b_1b_1)}{a_{11}a_{22} + \beta^2} + b_1 \\
-\frac{\beta(a_{11}b_2 - b_1b_1)}{a_{11}a_{22} + \beta^2} & -2a_{22}(a_{11}b_2 - b_1b_1) + \frac{\beta(a_{22}b_1 + b_2b_1)}{a_{11}a_{22} + \beta^2} + b_2
\end{bmatrix}
$$

where $\beta = c_{21} - d_{12}$.

The type of the equilibrium points $P_2, P_3$ and $P_4$ depends on parameter values, for instance using the parameters $a_{11} = 1, a_{22} = 1, b_1 = 10.9, b_2 = 6.8, c_{12} = 7.1, c_{21} = 5.5, d_{12} = 6.3, d_{21} = 4.6$ the equilibrium points $P_2, P_3$ and $P_4$ are respectively saddle, saddle and stable focus.

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5.3 Dynamics of trajectories reaching the equal number of clients line $\Sigma$ in model LVsP

Determining the crossing $\Sigma_c$ and sliding $\Sigma_s$ sets: To describe a trajectory that reaches the equal number of clients $\Sigma$, we need to determine if that trajectory reaches the equal number of clients $\Sigma$ in the crossing set $\Sigma_c$ or in the sliding set $\Sigma_s$. For this we will use the Algorithm 4.3.

In the case of model (5.6)-(5.8), the equal number of clients line $\Sigma$ is described by $H(x) = x_2 - x_1 = 0$ and its normal is $\nabla H(x) = [-1 \ 1]^T$, and the following scalar functions are parabolas, then we can use Algorithm 4.3.

$$g_U(w):= \langle \nabla H(w), F_U(w) \rangle = (a_{11} - a_{22} - 2c_{21} + 2d_{12})w^2 + (b_2 - b_1)w$$

$$g_L(w):= \langle \nabla H(w), F_L(w) \rangle = (a_{11} - a_{22} + 2c_{12} - 2d_{21})w^2 + (b_2 - b_1)w$$

where $w = (w, w)$ which is a point on the equal number of clients line $\Sigma$.

Algorithm 4.3 calculates $\text{sign}(g_U(w))$ and $\text{sign}(g_L(w))$ using their respective zeros and second derivatives. Once the signs of the values of the points in each parabola are calculated, the sets can be combined to obtain $\Sigma_c$, the crossing set that is comprised of the points where the values of parabolas have the same sign, and $\Sigma_s$, the sliding set that is comprised of the points where the values of the parabolas have opposite sign. To apply Algorithm 4.3 the zeros needed are:

$$w_U^1 = 0 \quad \text{Zeros of } g_U(w)$$
$$w_U^2 = \frac{b_1 - b_2}{a_{11} - a_{22} - 2c_{21} + 2d_{12}}$$

$$w_L^1 = 0 \quad \text{Zeros of } g_L(w)$$
$$w_L^2 = \frac{b_1 - b_2}{a_{11} - a_{22} + 2c_{12} - 2d_{21}}$$

The needed second derivatives are:

$$g_U(w)'' = 2a_{11} - 2a_{22} - 4c_{21} + 4d_{12}$$

$$g_L(w)'' = 2a_{11} - 2a_{22} + 4c_{12} - 4d_{21}$$

Next we enunciate a proposition which for the given case exemplifies the usage of the fact that $g_U(w)$ and $g_L(w)$ are parabolas. (For the the calculations Algorithm 4.3 is used and it takes into account other cases)

**Proposition 5.3.1** (Calculating the crossing set $\Sigma_c$ and sliding set $\Sigma_s$). Suppose that

(i) $b_1 < b_2$

(ii) $g_U(w)'' < 0$

(iii) $g_L(w)'' > 0$
then for nonnegative $w$ values

- the crossing set is $\Sigma_c = \{w|w \in (0, w_2^U)\}$
- the sliding set is $\Sigma_s = \{w|w \in [w_2^U, \infty)\}$

Proof. First we find the sign of $w_2^U$. Comparing (5.13) and (5.15) we find that the denominator of $w_2^U$ is $2g_U(w)''$ then by assumption (ii) the denominator of $w_2^U$ is negative. The numerator of $w_2^U$ is negative by assumption (i). Because of this we have

$$w_2^U > 0 \quad (5.17)$$

The function $g_U(w)$ is a parabola on variable $w$ by (5.11) and by assumption (ii) its second derivative is negative, then it has a maximum. From this and (5.17) we have

$$\text{sign}(g_U(w)) = \begin{cases} 
-1 & \text{if } w \in (-\infty, 0) \\
1 & \text{if } w \in (0, w_2^U) \\
-1 & \text{if } w \in (w_2^U, \infty)
\end{cases} \quad (5.18)$$

Now we find the sign of $w_2^L$. Comparing (5.14) and (5.16) we find that the denominator of $w_2^L$ is $2g_L(w)''$ then by assumption (ii) the denominator of $w_2^L$ is positive. The numerator of $w_2^L$ is negative by assumption (i). Because of this we have

$$w_2^L < 0 \quad (5.19)$$

The function $g_L(w)$ (5.12) is a parabola and by assumption (iii) its second derivative is positive, then it has a minimum. From this and (5.19) we have

$$\text{sign}(g_L(w)) = \begin{cases} 
1 & \text{if } w \in (-\infty, w_2^L) \\
-1 & \text{if } w \in (w_2^L, 0) \\
1 & \text{if } w \in (0, \infty)
\end{cases} \quad (5.20)$$

By the crossing condition (A.5) (in Appendix A) and the definitions (5.11)-(5.12), the crossing set $\Sigma_c$ is comprised of the values $w$ where $\text{sign}(g_U(w)) = \text{sign}(g_L(w)) \neq 0$ and the sliding set $\Sigma_s$ is its complement. Combining this and the results in (5.18) and (5.20), and considering we are interested in the nonnegative values of $w$, the sets for crossing and sliding on the equal number of clients line $\Sigma$ are

$$\Sigma_c = \{w|w \in (0, w_2^U)\} \quad (5.21)$$
$$\Sigma_s = \{w|w \in [w_2^U, \infty)\} \quad (5.22)$$

\[\Box\]
Determining pseudoequilibrium points: A pseudoequilibrium point is an equilibrium that may come into existence because of the use of switching control (See Appendix A). The pseudoequilibrium candidates are obtained using the fact that at a pseudoequilibrium the fields must be aligned and have opposite directions. The pseudoequilibrium candidates for model (5.6), (5.9) are:

\[
w_1^* = 0 \\
w_2^* = \frac{b_1 + b_2}{a_{11} + a_{22}}
\] (5.23) (5.24)

Note that

- the pseudoequilibrium candidates obtained are independent of the predation coefficients \(c_{ij}, d_{ij}\) where \(i \neq j\) and \(i, j = 1, 2\).

- the candidate \(w_1^* = 0\) is not stable because the point \((0, 0)\) is unstable for both fields \(f_U^\) and \(f_L^\).

- the candidate \(w_2^*\) can be a pseudoequilibrium only if \(w_2^* \in \Sigma_s\).

5.4 Simulations of different scenarios of model LVsP

The first scenario of model (5.6)-(5.9) has the parameters \(b_1 = 6.8, a_{11} = 1, c_{21} = 3.6, d_{21} = 1.6, b_2 = 10.9, a_{22} = 1, c_{12} = 5.5, d_{12} = 4.6\). In this case, the efforts \(c_{21}, d_{21}\) are small when compared to those of firm \(F_2\). The corresponding phase plane is shown in Figure 5.1 where one can see that firm \(F_2\) ends with all the clients and firm \(F_1\) is extincted from the market. The equilibrium points of the fields are shown in Tables 5.1 and 5.2, note that both fields have stable equilibrium points in region \(U\). The stable equilibrium of field \(f_L^\) is virtual but it still attracts trajectories beginning in \(L\) to \(U\). It is virtual because it corresponds to the dynamics of field \(f_L^\) but lies in region \(U\) where the dynamics is given by field \(f_U^\).
Figure 5.1: Model LVsP, scenario 1: Firm F₁ doing little spending on advertising when compared to firm F₂.

Table 5.1: Model LVsP, scenario 1: Equilibrium points of field $f^L$ with parameters $b_1=6.8$, $a_{11}=1$, $d_{21}=1.6$, $b_2=10.9$, $a_{22}=1$, $c_{12}=3.6$, $d_{12}=1.6$

<table>
<thead>
<tr>
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<th>x</th>
<th>Eigs($J(x)$)</th>
<th>Type</th>
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</thead>
<tbody>
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<td>Stable node</td>
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<td>−9.1323</td>
<td>Saddle</td>
<td></td>
</tr>
<tr>
<td>2.30845</td>
<td>9.0268</td>
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</tr>
</tbody>
</table>

Table 5.2: Model LVsP, scenario 1: Equilibrium points of field $f^U$ with parameters $b_1=6.8$, $a_{11}=1$, $d_{21}=1.6$, $b_2=10.9$, $a_{22}=1$, $c_{12}=3.6$, $d_{12}=1.6$

<table>
<thead>
<tr>
<th>$f^U$</th>
<th>x</th>
<th>Eigs($J(x)$)</th>
<th>Type</th>
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<td>[10.9]</td>
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<td>Unstable node</td>
</tr>
<tr>
<td>0</td>
<td>10.9</td>
<td>[−10.9]</td>
<td>Stable node</td>
</tr>
<tr>
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<td>[17.7]</td>
<td>Saddle</td>
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<td>−10.317</td>
<td>Saddle</td>
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</tr>
<tr>
<td>8.85</td>
<td>3.517</td>
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</tr>
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</table>

Dynamics of a trajectory on the equal number of clients line $Σ$

- Crossing and sliding sets are determined using the algorithm 4.3 for their calculation. The projections for determining the sets are $\langle \nabla H(w), f^L(w) \rangle = 2.0 w^2 + 4.1 w$, that has the zeros $w_1 = 0.0$ and $w_2 = −2.05$, and $\langle \nabla H(w), f^U(w) \rangle = 7.8 w^2 + 4.1 w$, that has the zeros $w_1 = 0.0$ and $w_2 = −0.5256$. Figure 5.2 shows the plots of the signs of the projections and their products to help visualize the procedure. Note that we are interested only in nonnegative $w$ values, then the crossing set is $Σ_c = (0, \infty)$, and all the trajectories that reach $Σ$ cross into $U$.

- There is no pseudoequilibrium since candidate $w_2^* = 8.85$ belongs to the crossing set $Σ_c$.  

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The second scenario of model (5.6)-(5.9) has the parameters $b_1 = 6.8, a_{11} = 1, c_{21} = 4.4, d_{21} = 3, b_2 = 10.9, a_{22} = 1, c_{12} = 5.5, d_{12} = 4.6$. In this case firm $F_1$ does more effort than in the previous scenario, but it is not enough for beating the number of clients of firm $F_2$. Firm $F_2$ ends with bigger number of clients but now firm $F_1$ is not extincted from the market. The corresponding phase plane is shown in Figure 5.3. The equilibrium points of the fields are shown in Tables 5.3 and 5.4, note that both fields have stable equilibrium points in region $U$. In the case of field $f^L$ this equilibrium is virtual but still attracts trajectories beginning in $L$ to $U$.

- Crossing and sliding sets are determined using the algorithm 4.3 for their calculation. The projections for determining the sets are $\langle \nabla H(w), f^U(w) \rangle = 0.4 w^2 + 4.1 w$, that has the zeros $w_1 = -10.25$ and $w_2 = 0.0$, and $\langle \nabla H(w), f^L(w) \rangle = 5.0 w^2 + 4.1 w$, that has the zeros $w_1 = -0.82$ and $w_2 = 0.0$. Figure 5.4 shows the plots of the signs of the projections and their products. Note that we are
interested only in nonnegative \( w \) values, then the crossing set is \( \Sigma_c = (0, \infty) \), and all the trajectories that reach \( \Sigma \) cross into \( U \).

- There is no pseudoequilibrium since candidate \( w^*_2 = 8.85 \) belongs to the crossing set \( \Sigma_c \).

**Table 5.3:** Model LVsP, scenario 2: Equilibrium points of field \( f^L \) with parameters \( b_1 = 6.8, a_{11} = 1, d_{21} = 3, b_2 = 10.9, a_{22} = 1, c_{12} = 5.5 \)

<table>
<thead>
<tr>
<th>( f^L )</th>
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<th>( \text{Eigs}(J(x)) )</th>
<th>Type</th>
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<tbody>
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<td>0</td>
<td>10.9</td>
<td>Unstable node</td>
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<tr>
<td></td>
<td>0</td>
<td>6.8</td>
<td></td>
</tr>
<tr>
<td>6.8</td>
<td>0</td>
<td>27.9</td>
<td>Saddle</td>
</tr>
<tr>
<td>10.9</td>
<td>-20.45</td>
<td>-10.9</td>
<td>Stable node</td>
</tr>
<tr>
<td>-2.82069</td>
<td>-9.3998</td>
<td>8.3722</td>
<td>Saddle</td>
</tr>
</tbody>
</table>

**Table 5.4:** Model LVsP, scenario 2: Equilibrium points of field \( f^U \) with parameters \( b_1 = 6.8, a_{11} = 1, c_{21} = 4.4, b_2 = 10.9, a_{22} = 1, d_{12} = 4.6 \)

<table>
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<th>( f^U )</th>
<th>( x )</th>
<th>( \text{Eigs}(J(x)) )</th>
<th>Type</th>
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</thead>
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<tr>
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<td>10.9</td>
<td>Unstable node</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>6.8</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>10.9</td>
<td>-10.9</td>
<td>Saddle</td>
</tr>
<tr>
<td>6.8</td>
<td>0</td>
<td>12.26</td>
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</tr>
<tr>
<td>-10.25</td>
<td>-11.4913</td>
<td>-4.7395</td>
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<tr>
<td>11.7885</td>
<td>11.4913</td>
<td>-4.7395</td>
<td>Stable node</td>
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</tbody>
</table>

The third scenario of model (5.6)-(5.9) has the parameters \( b_1 = 6.8, a_{11} = 1, c_{21} = 4.9, d_{21} = 4.5, b_2 = 10.9, a_{22} = 1, c_{12} = 5.5, d_{12} = 4.6 \). In this scenario, firm \( F_1 \) again increases its efforts and now manages to end with the same number of clients of firm \( F_2 \). Some trajectories starting in region \( L \) crosses into region \( U \) but after some time return to the equal number of clients line \( \Sigma \) to end at a pseudoequilibrium point. The corresponding phase plane is shown in Figures 5.5 and 5.5. The only difference between these phase planes is that the step size in Figure 5.5 is smaller and this results in the reduction of the chattering observed in the sliding trajectories. Figures 5.7 and 5.8 that shows the time plot \( x_i(t) \) of a trajectory that has crossing and sliding behaviors, again the trajectories are the same what is different between this figures is the setpsize, which is smaller in the figure without noticeable chattering. The equilibrium points of the fields are shown in

![Figure 5.4: Model LVsP, scenario 2: Signs of the inner products used in the calculation of conditions for crossing and sliding.](image)
Tables 5.5 and 5.6 note that in this case both fields have stable virtual equilibrium points: the virtual equilibrium of \( f^L \) is \( x = [0 \ 10.9]^T \) and the virtual equilibrium of field \( f^U \) is \( x = [9.23853 \ 8.12844]^T \). The trajectories beginning in region \( L \) (resp. \( U \)) are attracted to region \( U \) (resp. \( L \)) by the virtual equilibrium of field \( f^L \) (resp. \( f^U \)).

![Figure 5.5: Model LVsP, scenario 3: Firm \( F_1 \) increases again its effort and this introduces a pseudoequilibrium. Firms \( F_1 \) and \( F_2 \) coexist with an equal number of clients. The phase plane shows trajectories crossing and then returning to \( \Sigma \)](image)

![Figure 5.6: Model LVsP, scenario 3: Same simulation as in Figure 5.5, but here the stepsize = 0.0002 is smaller and this reduces most of the chattering on the equal number of clients line \( \Sigma \).](image)

<table>
<thead>
<tr>
<th>( f^L )</th>
<th>( x )</th>
<th>Eigs(( J(x) ))</th>
<th>Type</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>10.9</td>
<td>6.8</td>
<td>Unstable node</td>
</tr>
<tr>
<td>6.8</td>
<td>0</td>
<td>17.7</td>
<td>Saddle</td>
</tr>
<tr>
<td>0</td>
<td>10.9</td>
<td>-10.9</td>
<td>Stable node</td>
</tr>
<tr>
<td>-2.05</td>
<td>-10.317</td>
<td>3.517</td>
<td>Saddle</td>
</tr>
</tbody>
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<table>
<thead>
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<th>Eigs(( J(x) ))</th>
<th>Type</th>
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</thead>
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<td>6.8</td>
<td>Unstable node</td>
</tr>
<tr>
<td>0</td>
<td>10.9</td>
<td>-10.9</td>
<td>Saddle</td>
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<td>0</td>
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<td>Saddle</td>
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<td>9.23853</td>
<td>-8.6835</td>
<td>-2.5398i</td>
<td>Stable focus</td>
</tr>
<tr>
<td>8.12844</td>
<td>-8.6835</td>
<td>2.5398i</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.5: Model LVsP, scenario 3: Equilibrium points of field \( f^L \) with parameters \( b_1=6.8 \), \( a_{11}=1 \), \( d_{21}=4.5 \), \( b_2=10.9 \), \( a_{22}=1 \), \( c_{12}=5.5 \)

Table 5.6: Model LVsP, scenario 3: Equilibrium points of field \( f^U \) with parameters \( b_1=6.8 \), \( a_{11}=1 \), \( c_{21}=4.9 \), \( b_2=10.9 \), \( a_{22}=1 \), \( d_{12}=4.6 \)
Crossing and sliding sets are determined using the algorithm 4.3 for their calculation. The projections for determining the sets are $\langle \nabla H(w), f^U(w) \rangle = 4.1 \, w - 0.6 \, w^2$, that has the zeros $w_1 = 0.0$ and $w_2 = 6.83$, and $\langle \nabla H(w), f^L(w) \rangle = 2.0 \, w^2 + 4.1 \, w$, that has the zeros $w_2 = 0.0$ and $w_2 = -2.05$. Figure 5.9 shows the the plots of the signs of the projections and their products. Note that we are interested only in nonnegative $w$ values, then the crossing set is $\Sigma_c$ interval $(0, 6.8333)$ and the sliding set is $\Sigma_s = [6.8333, \infty)$. Trajectories that reach $\Sigma_c$ cross into $U$ and trajectories that reach $\Sigma_s$ slide until a pseudoequilibrium point.

There is a pseudoequilibrium point $w^*_2 = 8.85$ that belongs to the sliding set $\Sigma_s$.

The fourth scenario of model (5.6)-(5.9) has the parameters $b_1 = 6.8$, $a_{11} = 1$, $c_{21} = 5.5$, $d_{21} = 3$, $b_2 = 10.9$, $a_{22} = 1$, $c_{12} = 5.5$, $d_{12} = 4.6$. The corresponding phase plane is shown in Figure 5.10 and 5.11. The only difference between these phase
planes is that the step size in Figure 5.11 is smaller and this results in the reduction of the chattering observed in the sliding trajectories. When compared to the third scenario, firm \( F_1 \) increases its effort \( c_{21} = 5 > 4.9 \) but decreases \( d_{21} = 3 < 4.5 \), with these efforts firm \( F_1 \) manages to end with the same number of clients of firm \( F_2 \). But now the observed trajectories only slides on \( \Sigma \) until they reach a pseudoequilibrium point. The equilibrium points of the fields are shown in Tables 5.7 and 5.8, note that in this case both fields have stable virtual equilibrium points: the virtual equilibrium of field \( f^L \) is \( x = [0 \ 10.9]^T \) and the virtual equilibrium of field \( f^U \) is \( x = [9.1768 \ 2.64088]^T \). So that the trajectories beginning in region \( L \) (resp. \( U \)) are attracted to region \( U \) (resp. \( L \)).

![Figure 5.10: Model LVsP, scenario 4: Firm \( F_1 \) increases \( c_{21} \) and diminishes \( d_{21} \). In the long term, firms \( F_1 \) and \( F_2 \) coexist with an equal number of clients. The phase plane shows that trajectories slides to a pseudoequilibrium.](image1)

![Figure 5.11: Model LVsP, scenario 4: Same simulation as in Figure 5.10 but here the stepsize = 0.0002 is smaller and this reduces most of the chattering on the equal number of clients line \( \Sigma \).](image2)

<table>
<thead>
<tr>
<th>( f^L )</th>
<th>( \mathbf{x} )</th>
<th>Eigs(J(( \mathbf{x} )))</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0]</td>
<td>[0]</td>
<td>[10.9 6.8]</td>
<td>Unstable node</td>
</tr>
<tr>
<td>6.8</td>
<td>0</td>
<td>27.9</td>
<td>Saddle</td>
</tr>
<tr>
<td>0</td>
<td>6.8</td>
<td>-20.45</td>
<td>Saddle</td>
</tr>
<tr>
<td>-2.82069</td>
<td>3.84828</td>
<td>-9.3998</td>
<td>Saddle</td>
</tr>
</tbody>
</table>

Table 5.7: Model LVsP, scenario 4: Equilibrium points of field \( f^L \) with parameters \( b_1 = 6.8, a_{11} = 1, d_{21} = 3, b_2 = 10.9, a_{22} = 1, c_{12} = 5.5 \)

<table>
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<th>Eigs(J(( \mathbf{x} )))</th>
<th>Type</th>
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</thead>
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<td>0</td>
<td>-6.8</td>
<td>Saddle</td>
</tr>
<tr>
<td>9.1768</td>
<td>2.64088</td>
<td>-5.9088 + 2.9918i</td>
<td>Stable focus</td>
</tr>
</tbody>
</table>

Table 5.8: Model LVsP, scenario 4: Equilibrium points of field \( f^U \) with parameters \( b_1 = 6.8, a_{11} = 1, c_{21} = 5.5, b_2 = 10.9, a_{22} = 1, d_{12} = 4.6 \)

- Crossing and sliding sets are determined using the algorithm 4.3 for their calculation. The projections for determining the sets are \( \langle \nabla H(w), f^U(w) \rangle = 4.1 w - 1.8 w^2 \), that has the zeros \( w_1 = 0.0 \) and \( w_2 = 2.278 \), and \( \langle \nabla H(w), f^L(w) \rangle = \)
$5.0 \ w^2 + 4.1 \ w$, that has the zeros $w_1 = -0.82$ and $w_2 = 0.0$. Figure 5.12 shows the plots of the signs of the projections and their products. Note that we are interested only in nonnegative $w$ values, then the crossing set is $\Sigma_c = (0, 2.2778)$ and the sliding set is $\Sigma_s = [2.2778, \infty)$. Trajectories that reach $\Sigma_c$ cross into U and trajectories that reach $\Sigma_s$ slide until a pseudoequilibrium point.

- There is a pseudoequilibrium point $w_2^* = 8.85$ that belongs to the sliding set $\Sigma_s$.

Finally, the fifth scenario of model (5.6)-(5.9) has the parameters $b_1 = 6.8$, $a_{11} = 1$, $c_{21} = 7.1$, $d_{21} = 6.3$, $b_2 = 10.9$, $a_{22} = 1$, $c_{12} = 5.5$, $d_{12} = 4.6$. When compared to the fourth scenario, firm $F_1$ increases its efforts $c_{21} = 7.1 > 5$ and $d_{21} = 6.3 > 3$, with these efforts firm $F_1$ ends with a bigger number of clients than firm $F_2$. The corresponding phase plane is shown in Figure 5.13 where the observed trajectories cross into $L$. The equilibrium points of the fields are shown in Tables 5.9 and 5.10, note that in this case both fields have stable equilibrium points in region $L$. In the case of field $f^U$, it is a stable virtual equilibrium and the trajectories beginning in region $U$ are attracted to region $L$. 

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**Figure 5.12:** Model LVsP, scenario 4: Signs of the inner products used in the calculation of conditions for crossing and sliding.
Table 5.9: Model LVsP, scenario 5: Equilibrium points of field $f^L$ with parameters $b_1=6.8$, $a_{11}=1$, $d_{21} =6.3$, $b_2 =10.9$, $a_{22} =1$, $c_{12} =5.5$

<table>
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<tr>
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<td>[15.52]</td>
<td>[-10.9]</td>
<td>Saddle</td>
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Table 5.10: Model LVsP, scenario 5: Equilibrium points of field $f^U$ with parameters $b_1=6.8$, $a_{11} =1$, $c_{21} =7.1$, $b_2 =10.9$, $a_{22} =1$, $d_{12} =4.6$

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<td>Saddle</td>
<td></td>
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</table>

- Crossing and sliding sets are determined using the algorithm 4.3 for their calculation. The projections for determining the sets are $\langle \nabla H(w), f^U(w) \rangle = 4.1w - 5.0w^2$, that has the zeros $w_1 = 0.0$ and $w_2 = 0.82$, and $\langle \nabla H(w), f^L(w) \rangle = 4.1w - 1.6w^2$, that has the zeros $w_1 = 0.0$ and $w_2 = 2.5625$. Figure 5.14 shows the plots of the signs of the projections and their products. Note that we are interested only in nonnegative $w$ values, then the crossing set is $\Sigma_c = (0, 0.82) \cup (2.5625, \infty)$ and the sliding set is $\Sigma_s = [0.82, 2.5625]$. Trajectories that reach $\Sigma_c$ in the interval $(0, 0.82)$ cross into $U$ and trajectories that reach $\Sigma_c$ in the interval $(2.5625, \infty)$ cross into $L$.

- There is no pseudoequilibrium point because the $w_2^* = 8.85$ belongs to the crossing set $\Sigma_c$.
5.5 Evaluating outcomes of the used policies using a performance index for model LVsP

We use the index (2.9). Note that a firm makes a profit when its index $J_i$ is positive, whereas it makes a loss when its index is negative. The number of clients $x_i(t)$ evolution is given by the system (5.6), (5.7) and the controls $u_i(t)$ are given in (5.5).

The scenarios were simulated using the software Matlab. In the first scenario, the controls $c_{21} = 3.6$ and $d_{12} = 1.6$ actioned by firm $F_1$ are small when compared to the controls $c_{21} = 5.5$ and $d_{21} = 4.6$ of firm $F_2$. In this setting firm $F_2$ ends with more clients than firm $F_1$, actually firm $F_1$ ends without clients. Figure 5.15 shows the trajectories in the phase plane according to their index value at final time, these indexes were calculated using a revenue factor $r = 1$ and firm $F_2$ is the only firm that makes a profit. There is little variation of the index values as can be seen in Figures 5.17 and 5.18. The small increases of the index values are more noticeable in Figure 5.16 where the indexes are plotted against each other.
In the next scenario, the controls $c_{21} = 5.5$ and $d_{12} = 3$ actioned by firm $F_1$ are similar in magnitude when compared to the controls $c_{21} = 5.5$ and $d_{21} = 4.6$ of firm $F_2$. Now, both firms end with the same number of clients. Figure 5.19 shows the trajectories according to their indexes at final time, these indexes were calculated using a revenue factor $r = 1$ and both firms make a profit. The firm that begins with a smaller number of clients makes a smaller profit: (a) In Figure 5.21 the initial conditions are $x_1(0) = 0.5$ and $x_2(0) \in [0.03, 12]$, the index $J_2$ grows as $x_2(0)$ grows and it is bigger than $J_1$ when $x_2(0) > x_1(0)$. (b) In Figure 5.22 the initial conditions are $x_2(0) = 0.03$ and $x_1(0) \in [0, 10]$, the index $J_1$ increases as $x_1(0)$ increases and it is bigger than $J_2$ when $x_1(0) > x_2(0)$.
In the last scenario, the controls \( c_{21} = 7.1 \) and \( d_{12} = 6.3 \) actioned by firm \( F_1 \) are bigger in magnitude when compared to the controls \( c_{21} = 5.5 \) and \( d_{21} = 4.6 \) of firm \( F_2 \). In this situation firms \( F_1 \) and \( F_2 \) coexist in the long term, with firm \( F_1 \) having more clients than firm \( F_2 \). Figure 5.23 shows the the trajectories in the phase plane according to their index value at final time, these indexes were calculated using a revenue factor \( r = 1 \). Firm \( F_1 \) is the only firm that makes profits, even though firms coexist in the long term. In Figure 5.25 the index \( J_2 \) increases as \( x_2 \) increases but the growth is small and the index remains negative. In Figure 5.26 the index \( J_1 \) increases as \( x_1 \) increases. When the index values are plotted against each other, i.e. \( J_2 \) vs. \( J_1 \), as in Figure 5.24 the indexes do not seem to create a front although their values have rather small variations (see Figures 5.25 and 5.26).
5.6 Remarks on model LVsP

- The entering firm, i.e. the firm with a smaller initial market share, can introduce a pseudoequilibrium on the equal number of clients line $\Sigma$ using an appropriate control value. Using a control value larger than this, the entering firm can achieve a larger number of clients than its competitor.

- The firm decides which advertising control efforts to apply using its performance index in a finite horizon, i.e. the firm evaluates the costs and benefits it obtains with different advertising efforts.
• If the state variable is market share, the Lokta-Volterra model does not result in invariance of total market share (always equal to one), ensuring instead that each market share variable is below or equal to a given upper bound. This fact led to the choice, in this chapter, of the number of clients as the state variable instead of market share.
Chapter 6

Conclusions and Future Work

This thesis focuses on the analysis of qualitative behavior of the dynamics of models, making use of simple policies, whereas the emphasis in the existing literature has been mainly on the optimization of quadratic (or related) performance indices, as well as on the differential game-theoretic aspects of such models.

The relatively simple choices of the switching behavior, variable structure control theory made possible a complete qualitative analysis, for any choice of initial condition, and for several different strategies and scenarios. This is to be contrasted with a specific trajectory produced by optimal control approaches. In fact, this possibility of complete qualitative analysis is sometimes referred to as a flight simulator mode [53] and should, in our opinion, be considered a strength of the proposed approach.

The contributions of this thesis are as follows:

- Proposal of the models VWD2s (Vidale-Wolfe-Deal with two switching lines) and LVsP (Lotka-Volterra with switched predation and conservation).

- The qualitative analysis of the proposed models, under switching control based on measurement of each firm’s market share. The proposed switching control, unlike those resulting from optimal control/differential game approaches, is simple and thus implementable.

- Use of a performance index to identify trajectories that make a profit at saturated market in conjunction with the qualitative analysis of the market share dynamics to aid in the choice of advertising policies.

Some possible future works are listed:

- Developing a user friendly interface for the pieces of software elaborated for this thesis, in order to simulate the dynamics of a duopoly subject to switching policies.

- Discussing whether firms can switch between distinct models, because it has been assumed that both firms follow the same model. However, in line with the switching
behavior that the firms are allowed to have, this leaves open the analysis of what happens if the disadvantaged firm decides to change its strategy (e.g., switch to a different one) and a game theoretic analysis would be appropriate in this case.

- Using delays in the estimates of market shares (which may introduce chaos or limit cycles).
- Using a combined logistic function to enforce the saturated market restriction and relating it to a Lotka-Volterra model.
- Using the current value of the associated performance index in the switching decision.
Bibliography


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Appendices
Appendix A

Filippov solutions on the discontinuity boundary

Since the model used in this paper presents discontinuities in the vector fields on the equal share line, it is necessary to define the vector field on the discontinuity boundary, so that trajectories that reach it can be continued in an unambiguous manner. We will use the concept of a Filippov solution [46]. We recapitulate Filippov theory for the second order models (state vector $x = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$) very briefly below, following the very clear presentation in [54]. The ODE with one discontinuity boundary $\Sigma$ and vector fields $f^L = [f^L_1(x), f^L_2(x)]^T$ and $f^U = [f^U_1(x), f^U_2(x)]^T$ on either side of $\Sigma$ is written as:

$$\dot{x} = \begin{cases} f^L(x), & x \in L \\ f^U(x), & x \in U \end{cases}$$

(A.1)

where the regions $L, U$ are separated by $\Sigma$. $\Sigma$ is defined by $H(x) = 0$. $L$ corresponds to the region where $H(x) < 0$ and $U$ to the region where $H(x) > 0$.

According to Filippov [46], solutions of (A.1) can be constructed by concatenating standard solutions in $L, U$ and sliding solutions on $\Sigma$, which are obtained using the Filippov convex rule. The crossing set $\Sigma_c \subset \Sigma$ is defined as the set of all points $x \in \Sigma$, at which the two vectors $f^L(x), f^U(x)$ have nonzero components of the same sign, transversal to $\Sigma (\langle \nabla H(x), f^U(x) \rangle \langle \nabla H(x), f^L(x) \rangle > 0)$. By definition, at such a point $x$, a trajectory of (A.1) reaching $x$ from $L$ (resp. $U$) crosses $\Sigma$ and is concatenated with a trajectory entering $U$ (resp. $L$) from $x$. The sliding set $\Sigma_s$ is defined as the complement to $\Sigma_c$ in $\Sigma$. The sliding set may contain singular sliding points, i.e. points with zero as component on the normal to $\Sigma$. Either both vectors $f^L(x)$ and $f^U(x)$ are tangent to $\Sigma$, or one of them vanishes while the other is tangent to $\Sigma$, or both vanish at a sliding singular point.

The Filippov rule associates the following convex combination $\Phi(x)$ of the two vectors $f^L(x), f^U(x)$ to each nonsingular sliding point $x \in \Sigma_s$, where $\langle \cdot, \cdot \rangle$ denotes the
standard inner product in \( \mathbb{R}^2 \).

\[
\Phi(x) = \lambda f^L(x) + (1 - \lambda) f^U(x)
\]  \hspace{1cm} (A.2)

\[
\lambda = \frac{\langle \nabla H(x), f^U(x) \rangle}{\langle \nabla H(x), f^U(x) - f^L(x) \rangle}
\]  \hspace{1cm} (A.3)

where \( \lambda \) was obtained using the condition that \( \Phi(x) \) lies on the tangent line to \( \Sigma \) at \( x \).

The equation

\[
\dot{x} = \Phi(x), \quad x \in \Sigma_s
\]  \hspace{1cm} (A.4)

is a scalar differential equation on \( \Sigma_s \), which is smooth on one-dimensional sliding subintervals of \( \Sigma_s \). Solutions of this equation are called \textit{sliding solutions}.

Equilibria of \( \text{(A.4)} \), for which the vectors \( f^L(x), f^U(x) \) are anticollinear (\( \angle(f^U, f^L) = 180^\circ \)) and transversal to \( \Sigma_s \) in opposite directions (\( \langle \nabla H(x), f^U(x) \rangle \langle \nabla H(x), f^L(x) \rangle < 0 \)), are called \textit{pseudo-equilibria} of \( \text{(A.1)} \) and also referred to as quasi-equilibria by Filippov [46]. Thus pseudo-equilibria lie in the interior of sliding segments. A non-singular sliding point \( x \in \Sigma_s \) for which \( f^L = 0, \lambda = 1 \) or \( f^U = 0, \lambda = 0 \) implies \( \Phi = 0 \) and is called a \textit{boundary equilibrium point}. A tangent point \( T \) is a point at which the vectors \( f^L, f^U \) are non-zero, but one of them is tangent to \( \Sigma \). A sliding segment terminates either at a boundary equilibrium, or at a tangent point.

Phase portraits of Filippov systems can have multiple sliding segments and be rather complex. Figure A.1 (a) is a rather simple phase portrait to exemplify previous definitions. In this example \( \Sigma \) is a line. The depicted area for the given fields has a sliding subsegment \( \Sigma_s \), and two crossing subsegments \( \Sigma_c \). Points \( A \) and \( C \) are boundary equilibrium points. Figure A.1 (b) shows the corresponding fields \( f^U, f^L \) at \( V, W, B, Z \). Here \( V, Z \) are crossing points, \( W, B \) are sliding points. Notice that \( B \) is a pseudoequilibrium point.

\[
\begin{align*}
&U \\
&L \longrightarrow \Sigma_s \longrightarrow \Sigma_c \longrightarrow \Sigma \\
&A \quad B \quad C
\end{align*}
\]

\[
\begin{align*}
&U \\
&L \longrightarrow \Sigma_s \longrightarrow \Sigma_c \longrightarrow \Sigma \\
&V \quad W \quad B \quad C \quad Z
\end{align*}
\]

Figure A.1: (a) Sample phase portrait showing segments \( \Sigma_s \) and \( \Sigma_c \). (b) \( V, Z \) crossing points, \( W, B \) sliding points. \( B \) is a pseudoequilibrium point, i.e. \( \angle(f^U, f^L) = 180^\circ \)

The conditions for deciding if a trajectory will cross or slide at point \( x \in \Sigma \) or to determine if \( x \) is a pseudo-equilibrium are:
• Crossing \((x \in \Sigma_c)\):

\[
\langle f^L(x), \nabla H(x) \rangle \langle f^U(x), \nabla H(x) \rangle \geq 0 \quad \text{(A.5)}
\]

• Sliding \((x \in \Sigma_s)\):

\[
\langle f^L(x), \nabla H(x) \rangle \langle f^U(x), \nabla H(x) \rangle \leq 0 \quad \text{(A.6)}
\]

• Pseudo-equilibrium : \((x \in \Sigma_s)\) and \(\exists w \in \mathbb{R}^+\) such that

\[
f^L(x) = -wf^U(x) \quad \text{(A.7)}
\]

The anticollinearity between \(f^U(x)\) and \(f^L(x)\) is expressed by (A.7). Condition (A.7) is simpler to apply, resulting in equations of lower degree in the components of \(x\), than the equivalent condition \(\frac{\langle f^L(x), f^U(x) \rangle}{\|f^L(x)\|\|f^U(x)\|} = -1\).
Appendix B

Algorithm for obtaining separatrices of model VWDsC

We use the dynamics (3.3), (3.4) (resp. (3.5), (3.6)) to calculate $\varphi^U$ (resp. $\varphi^L$) in the phase plane $x_1 - x_2$ from the corresponding expression for $\frac{dx_2}{dx_1}$. Furthermore, we use $\frac{dx_2}{dx_1}|_E$ as an abbreviation for the limiting expression for $\frac{dx_2}{dx_1}$ as it approaches the saturated equal market share $E$ from region $U$ (resp. from region $L$).

Thus, in order to derive a condition that determines if a trajectory starting in $U$ will reach the equal share $E$, we obtain $\frac{dx_2}{dx_1}$ from (3.3), (3.4):

$$\frac{dx_2}{dx_1} = \frac{k_2 x_2}{k_1 x_1 + c_1} \quad (B.1)$$

The solution of (B.1) using the initial conditions $(x_{20}, x_{10})$ is

$$x_2 = \frac{x_{20}}{(k_1 x_{10} + c_1) k_2/k_1 (k_1 x_1 + c_1)^{k_2/k_1}}. \quad (B.2)$$

Renaming $x_{10}$ as $X_1$ and $x_{20}$ as $X_2$ and setting $(x_1, x_2) = (0.5, 0.5)$ in (B.2) yields the locus of all initial conditions $(X_1, X_2)$ that end at $(0.5, 0.5)$, i.e. the separatrix $\varphi_U$

$$X_2 = 0.5 \frac{(k_1 X_1 + c_1)^{k_2/k_1}}{(k_1 0.5 + c_1)^{k_2/k_1}} \quad (B.3)$$

Analogously, trajectories in $L$ satisfy:

$$\frac{dx_2}{dx_1} = \frac{k_2 x_2 + c_2}{k_1 x_1} \quad (B.4)$$

The solution of (B.4) using the initial conditions $(x_{20}, x_{10})$ is

$$x_2 = \frac{k_2 x_{20} + c_2}{k_2 (x_{10})^{k_2/k_1} x_1^{k_2/k_1} - \frac{c_2}{k_2}} \quad (B.5)$$
and, as before setting \( x_{10} = X_1, x_{20} = X_2 \) and \( (x_1, x_2) = (0.5, 0.5) \) yields the equation for the separatrix \( \varphi_L \):

\[
X_2 = \frac{1}{k_2} \left( \frac{1}{2} + \frac{c_2}{k_2} \right) k_2 (2X_1)^{k_2/k_1} - c_2
\]  

(B.6)

Remark: The geometrical idea used to decide whether a separatrix exists or not is shown in Figure B.1. The trajectory defining a separatrix \( \varphi_U \) (resp. \( \varphi_L \)) calculated from (B.3) (resp. (B.6)) must lie in its defining region \( U \) (resp. \( L \)), which means that its slope at \( E \), calculated from (B.1) (resp. (B.4)) must be less than (resp. greater than) unity, so that it comes from region \( U \) and points into the region \( L \) (resp. comes from \( L \) and points into \( U \)) in the neighborhood of \( E \).

![Figure B.1: The fields \( f^U \) or \( f^L \) approaching the equal market share point \( E \) from the left can be as shown. In the case of the dashed line, the slope is less than unity and the field indicates that the trajectory comes from \( U \). In the case of the continuous line, the slope is greater than unity and the field indicates that the trajectory comes from \( L \).](image)

We describe the process of determining the existence of separatrices as the following algorithm:

**Algorithm (Existence of separatrices)**

1. Calculate the slope \( s_{f^U} = \frac{dx_2}{dx_1} \big|_E \) using (B.1) (resp. the slope \( s_{f^L} = \frac{dx_2}{dx_1} \big|_E \) using (B.4)).

2. If \( s_{f^U} > 1 \) (resp. \( s_{f^L} < 1 \)) then the separatrix \( \varphi_U \) (resp. \( \varphi_L \)) does not exist. In this case, the curve (B.3) comes from \( L \), i.e. it partially lies in \( L \) (resp. (B.6) comes from \( U \), i.e. it partially lies in \( U \)) , but it was derived from the equations of \( f^U \) (resp. \( f^L \)).

3. For region \( U \): If \( s_{f^U} < 1 \)
   
   (a) Set \( X_1 = 0 \) in (B.3) and solve for \( X_2 \). This determines the point at which (B.3) intersects the \( x_2 \) axis.

   (b) If \( X_2 \in [0, 1] \), then
Use $x_{10} = X_1$ and $x_{20} = X_2$ as the initial conditions in (B.2). The resulting equation defines the separatrix $\varphi_U$.

(c) If $X_2 \notin [0, 1]$ or $\not \exists x_2$, then the separatrix $\varphi^U$ does not exist.

4. For region $L$: If $s_{f_L} > 1$

(a) Set $X_2 = 0$ in (B.6) and solve for $X_1$. This determines the point at which (B.6) intersects the $x_1$ axis.

(b) If $X_1 \in [0, 1]$, then

Use $x_{10} = X_1$ and $x_{20} = X_2$ as the initial conditions in (B.5). The resulting equation defines the separatrix $\varphi^L$.

(c) If $X_1 \notin [0, 1]$ or $\not \exists x_1$, then the separatrix $\varphi^L$ does not exist.

5. Once $\varphi_U$ and $\varphi_L$ are determined, they are associated with each firm:

(a) If only one separatrix exists, then both firms are associated with it. If only $\varphi_U$ (resp. $\varphi_L$) exists, then the separatrix for firm $F_2$ is $\varphi^U_2$ (resp. $\varphi^L_2$) and the separatrix for firm $F_1$ is $\varphi^U_1$ (resp. $\varphi^L_1$)

(b) If both separatrices exist, then the separatrix for firm $F_2$ is $\varphi^L_2$ and the separatrix for firm $F_1$ is $\varphi^U_1$

Remark: All trajectories with initial conditions above (resp. below) separatrix of firm $F_2$ (resp. $F_1$) lead to firm $F_2$ (resp. $F_1$) attaining its goal of gaining at least 50% of the saturated market share and if both separatrices exist, then all trajectories originating from initial conditions in the region between them will converge to the saturated equal market share point $E$. Since a separatrix is just a trajectory in the phase plane that ends at the saturated market line $Sat$, it follows that trajectories below (resp. above) a separatrix always remain below it (resp. above it). This is because if the trajectories were to intersect at a point that is not an equilibrium there would be more than one value for the derivative and the uniqueness of the derivative would be violated.
Appendix C

Demonstrations of properties of model VWDsC

Theorem 3.2.1 (Characterization of market share dynamics for model (3.3)-(3.6) with controls (3.2)). Suppose that $\tau$ is a trajectory that starts in region $U$, i.e. firm $F_1$ starts out with a smaller market share $x_1(0) < x_2(0)$ and that the parameters $k_1, c_1, k_2, c_2$ are known. then the trajectories reaching the equal market share line $\Sigma$ can have one of the following behaviors:

1. If the sliding end point $s_{end} = 0.5$ then any trajectory $\tau$ with initial conditions in region $U$ and below separatrix $\varphi^U_1$ reaches the equal share line and slides until the saturated equal share point $E$ on the saturated market line (see Figure 3.2a).

2. If the sliding end point $s_{end} < 0.5$ and $k_2 > k_1$, then the separatrix is $\varphi^L_2 = \varphi^L_1$ and some trajectories beginning in region $U$ reach the equal share line $\Sigma$, move on it until the point $s_{end}$ and return to $U$ finally ending on the saturated market Sat with firm $F_2$ having more than 50% of the market share (see Figure 3.2b).

3. If the sliding end point $s_{end} < 0.5$ and $k_1 > k_2$, then the separatrix is $\varphi^U_2 = \varphi^U_1$ and the trajectories beginning in region $U$ and below $\varphi^U_2$ reach the equal share line $\Sigma$,

   (a) If the trajectories reach the equal market share on the sliding segment $\Sigma_s$, they stay on the equal market share until the point $s_{end}$ and then cross into region $L$ finally ending on the saturated market Sat with firm $F_1$ having more than 50% of the market share. (see Figure 3.2c)

   (b) If the trajectories reach the equal market share line on the crossing segment $\Sigma_c$, they cross into region $L$ finally ending on the saturated market Sat with firm $F_1$ having more than 50% of the market share. (see Figure 3.2d)

Proof. 1. If sliding end point $s_{end} = 0.5$ then any trajectory $\tau$ with initial conditions in region $U$ and below separatrix $\varphi^U_1$ reaches the equal share line and slides until
the saturated equal share point $E$ on the saturated market line.

When $s_{\text{end}} = 0.5$, there are two separatrices. To see this, suppose there is only one separatrix, without loss of generality suppose $\varphi^U$ does not exist $\varphi^L$ exists. A $s_{\text{end}}$, the field $f^U$ points from $L$ into $U$ since $\varphi^U$ does not exist and the field $f^L$ also points from $L$ into $U$ since $\varphi^L$ exists (see the remark in Appendix B. Algorithm for obtaining separatrices). Then, this implies that $s_{\text{end}}$ is a crossing point which is a contradiction with $s_{\text{end}}$ being a sliding point. Then, two separatrices exist and the separatrix corresponding to firm $F_1$ is $\varphi^L_1$. Next, determine $X_2$ from (B.3) by setting $X_1 = x_1(0)$, if $x_2(0) < X_2$ the trajectory $\tau_{x_1(0),x_2(0)}$ is below $\varphi^U_1$. Because trajectories in the phase plane can only cross at singular points, the trajectory $\tau_{x_1(0),x_2(0)}$ runs below $\varphi^U_1$ and reaches $\Sigma_s$. By Proposition C.0.7 there is no pseudo-equilibrium point, so that $\tau_{x_1(0),x_2(0)}$ slides on $\Sigma$ until the saturated equal market share point $E$ which is an attractive point.

2. If sliding end point $s_{\text{end}} < 0.5$ and $k_2 > k_1$, then the separatrix is $\varphi^L_2 = \varphi^U_1$ and some trajectories beginning in region $U$ reach the equal share line $\Sigma$, move on it until the point $s_{\text{end}}$ and return to $U$ finally ending on the saturated market Sat with firm $F_2$ having more than 50% of the market share. When $s_{\text{end}} < 0.5$ and $k_2 > k_1$, the separatrix is $\varphi^L_2 = \varphi^U_1$: Suppose that separatrix $\varphi^U$ exists, this implies there is a trajectory $\tau$ that reaches $\Sigma_c$ at a crossing point. But this is a contradiction with Proposition C.0.6, so that the only separatrix is $\varphi^L_2 = \varphi^U_1$. Next, determine if $\tau_{x_1(0),x_2(0)}$, obtained intersects with $\Sigma$. If it intersects $\Sigma$ it does it on a sliding point (intersection on a crossing point is impossible by Proposition C.0.6). After reaching $\Sigma$, $\tau_{x_1(0),x_2(0)}$ slides on it and at sliding end it returns to $U$ (since crossing into $L$ is impossible by Proposition C.0.5). By Proposition C.0.4 and Proposition C.0.6 once crossing occurs there is no reaching $\Sigma$ anymore and $\tau_{x_1(0),x_2(0)}$ will end in Sat$^U$ with Firm $F_2$ having more than 50% of the market.

3. If sliding end point $s_{\text{end}} < 0.5$ and $k_1 > k_2$, then the separatrix is $\varphi^L_2 = \varphi^U_1$ and the trajectories beginning in region $U$ and below $\varphi^L_2$ reach the equal share line $\Sigma$. Suppose the separatrix is $\varphi_L$, then at the saturated market share $f^L$ points from $L$ into $U$. But this contradicts Proposition C.0.6 which when applied to the situation $k_1 > k_2$ states that there is no trajectory starting in $L$ and reaching the crossing segment $\Sigma_c$. Then the separatrix has to be $\varphi^U_1 = \varphi^L_2$.

(a) If the trajectories reach the equal market share on the sliding segment $\Sigma_s$, they stay on the equal market share until the point $s_{\text{end}}$ and then cross into region $L$ finally ending on the saturated market Sat with firm $F_1$ having more than 50% of the market share.

In this case the trajectory $\tau_{x_1(0),x_2(0)}$ reaches the sliding segment $\Sigma_s$ and slides
on it until \( s_{\text{end}} \) (by Proposition C.0.7 there is no pseudo-equilibrium point). When it reaches \( s_{\text{end}} \), trajectory \( \tau_{x_1(0),x_2(0)} \) crosses into \( L \). To see this, suppose it returns to \( U \), so that the final saturated market would be in \( \text{Sat}^U \) above the saturated equal market share \( E \) which is the intersection of \( \varphi^U_1 \) with \( \text{Sat} \), then \( \tau_{x_1(0),x_2(0)} \) would intersect \( \varphi^U_1 \) at a point below \( \text{Sat} \) which is not singular point of the system which is not possible. Then, trajectory \( \tau_{x_1(0),x_2(0)} \) has to end in \( \text{Sat}^L \) with Firm \( F_1 \) having more than 50% of the market.

(b) If the trajectories reach the equal market share line on the crossing segment \( \Sigma_c \), they cross into region \( L \) finally ending on the saturated market \( \text{Sat} \) with firm \( F_1 \) having more than 50% of the market share.

In this case the trajectory \( \tau_{x_1(0),x_2(0)} \) reaches the crossing segment \( \Sigma_c \). At it, \( \tau_{x_1(0),x_2(0)} \) could cross into \( L \) or return to \( U \). Suppose that it returns to \( U \), this leads to the contradiction of \( \tau_{x_1(0),x_2(0)} \) intersecting with \( \varphi^U_1 = \varphi^U_2 \) at a nonsingular point as in \( (a) \). Then, \( \tau_{x_1(0),x_2(0)} \) has to cross into \( L \) and end in \( \text{Sat}^L \) with Firm \( F_1 \) having more than 50% of the market.

\[ \square \]

**Proposition C.0.1 (Characterization of the sliding segment \( \Sigma_s \)).** The sliding segment \( \Sigma_s \) for the model (3.3)-(3.6) is one of the following:

(i) If \( k_2 > k_1 \), \( \Sigma_s \) is the segment from 0 to \( s_{\text{end}} = \min\{0.5, \frac{c_1}{k_2-k_1}\} \).

(ii) If \( k_1 > k_2 \), \( \Sigma_s \) is the segment from 0 to \( s_{\text{end}} = \min\{0.5, \frac{c_2}{k_1-k_2}\} \).

(iii) If \( k_2 = k_1 \), \( \Sigma_s \) is the segment from 0 to \( s_{\text{end}} = 0.5 \).

**Proof.** We analyze sliding of the model (3.3)-(3.6) using the tangents to a trajectory evaluated in the neighborhood of \( x \in \Sigma \). This leads to simpler algebraic operations than the standard sliding condition (A.6) from Appendix A. Filippov solutions on the discontinuity boundary. Evaluating that the sliding condition is satisfied using the tangents is equivalent to using the intersection of signs of the projections of the fields onto the normal to \( \Sigma \), (which is another simplification to solving directly (A.6) and will be used in later chapters). Still, we consider that the use of tangents on the equal share line \( \Sigma \) is straightforward for model (3.3)-(3.6).

Since the coefficients \( k_1, k_2, c_1, c_2 \) as well as the variables \( x_1, x_2 \) are positive, the value of the tangent is positive. The tangents \( \tan_{f^U}(x_1, x_2) \) and \( \tan_{f^L}(x_1, x_2) \) where the vector field \( f^L \) points into \( U \) and simultaneously the vector field \( f^U \) points into \( L \), thus satisfy

\[
\tan_{f^U}(x_1, x_2) = \frac{k_2 x_2}{k_1 x_1 + c_1} < 1
\]

\[
\tan_{f^L}(x_1, x_2) = \frac{k_2 x_2 + c_2}{k_1 x_1} > 1
\]
for a point \((x_1, x_2) = (a, a)\) on \(\Sigma\) this leads to

\[-c_2 < a(k_2 - k_1) < c_1\]  \hspace{1cm} (C.1)

First, considering the case \(k_2 > k_1\) in (C.1) yields the sliding condition

\[-\frac{c_2}{k_2 - k_1} < a < \frac{c_1}{k_2 - k_1}\]  \hspace{1cm} (C.2)

From (C.2) and since \(a \in \Sigma\) and \(a \in \mathcal{T}\), then \(a\) belongs to the segment from 0 to \(s_{\text{end}} = \min\{0.5, \frac{c_1}{k_2 - k_1}\}\) so that (i) is proved.

Considering the case \(k_2 < k_1\) in (C.1) yields the sliding condition

\[-\frac{c_1}{k_1 - k_2} < a < \frac{c_2}{k_1 - k_2}\]  \hspace{1cm} (C.3)

From (C.3) and since \(a \in \Sigma\) and \(a \in \mathcal{T}\), then \(a\) belongs to the segment from 0 to \(s_{\text{end}} = \min\{0.5, \frac{c_1}{k_1 - k_2}\}\) so that (ii) is proved.

Third, considering \(k_1 = k_2\) in (C.1) yields

\[-c_2 < c_1\]  \hspace{1cm} (C.4)

In this case, all \(x \in \Sigma\) that are not pseudo-equilibrium points are sliding points (see Appendix A Filippov solutions on the discontinuity boundary) because (C.4) is always satisfied. From (C.4) and since \(a \in \Sigma\) and \(a \in \mathcal{T}\), then \(a\) belongs to the segment from 0 to 0.5

\[\square\]

**Proposition C.0.2 (Configuration of vector fields \(f^L, f^U\) on the segment \(\Sigma_s\)).** *The configuration of the vector fields approaching the sliding segment \(\Sigma_s\) for the model (3.3)-(3.6) is \(f^U\) pointing into \(L\) and \(f^L\) pointing into \(U\)*

**Proof.** Proposition C.0.2 is true because Proposition C.0.1 was derived from that configuration of fields, which is shown in Figure C.1a and the other configuration where \(f^U\) is pointing into \(U\) and \(f^L\) is pointing into \(L\) is not possible (Figure C.1b) as shown below.

![Figure C.1: Configurations of fields \(f^U\) and \(f^L\) for a point belonging to \(\Sigma_s\); The configuration in (a) is the only possible configuration. The configuration of fields in (b) is impossible.](image-url)

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For this configuration the tangents $\tan_{f^U}(x_1, x_2)$ and $\tan_{f^L}(x_1, x_2)$ where the vector field $f^L$ points into $L$ and simultaneously the vector field $f^U$ points into $U$, thus satisfy

$$\tan_{f^U}(x_1, x_2) = \frac{k_2x_2}{k_1x_1 + c_1} > 1$$
$$\tan_{f^L}(x_1, x_2) = \frac{k_2x_2 + c_2}{k_1x_1} < 1$$

for a point $(x_1, x_2) = (a, a)$ on $\Sigma$ this yields

$$c_1 < a(k_2 - k_1) < -c_2 \quad (C.5)$$

from (C.5) and considering that the model parameters are positive real numbers $a$ has to be smaller than a negative real number and bigger than a positive real number which is a contradiction, so this configuration of the fields is not possible. In the case $k_2 > k_1$, $a > \frac{c_1}{k_2 - k_1}$ and $a < -\frac{c_2}{k_2 - k_1}$. In the case $k_1 > k_2$, $a < -\frac{c_1}{k_1 - k_2}$ and $a > \frac{c_2}{k_1 - k_2}$.

**Proposition C.0.3 (Characterization of the crossing segment $\Sigma_c$).** The crossing segment $\Sigma_c$ for the model (3.3)-(3.6) is one of the following:

(i) If $k_2 > k_1$ and $\frac{c_1}{k_2 - k_1} < 0.5$, $\Sigma_c$ is the segment from $\frac{c_1}{k_2 - k_1}$ to 0.5.

(ii) If $k_1 > k_2$ and $\frac{c_2}{k_1 - k_2} < 0.5$, $\Sigma_c$ is the segment from $\frac{c_2}{k_1 - k_2}$ to 0.5.

(iii) If $c_2 = c_1 = 0$ or $k_2 > k_1$ and $c_1 = 0$ or $k_1 > k_2$ and $c_2 = 0$, $\Sigma_c$ is the segment from 0 to 0.5.

**Proof.** Since by definition $\Sigma_c \cup \Sigma_s = \Sigma$ and a point $x$ on $\Sigma$ belongs either to $\Sigma_s$ or $\Sigma_c$, we can compute $\Sigma_c = \Sigma - \Sigma_s$.

From proposition [C.0.1](1), if $k_2 > k_1$, $\Sigma_s$ is the segment from 0 to $s_{end}$ where $s_{end} = \min\{0.5, \frac{c_1}{k_2 - k_1}\}$. Thus, in order to have a crossing segment in $T$, we must have $\frac{c_1}{k_2 - k_1} < 0.5$. In this case $\Sigma_c$ is the segment from $\frac{c_1}{k_2 - k_1}$ to 0.5 and (i) is proved.

From proposition [C.0.1](2), if $k_1 > k_2$, $\Sigma_s$ is the segment from 0 to $s_{end}$ where $s_{end} = \min\{0.5, \frac{c_2}{k_1 - k_2}\}$. Thus, in order to have a crossing segment in $T$, we must have $\frac{c_2}{k_1 - k_2} < 0.5$. In this case $\Sigma_c$ is the segment from $\frac{c_2}{k_1 - k_2}$ to 0.5 and (ii) is proved.

Let $c_1 = c_2 = 0$ in (C.1), this leads to $0 < a < 0$, so that $\Sigma_s = 0$ and $\Sigma_c$ is the segment from 00 to 0.5. If $k_2 > k_1$ and $c_1 = 0$, then $s_{end} = 0$ and $\Sigma_c$ is the segment from 0 to 0.5. If $k_1 > k_2$ and $c_2 = 0$, then $s_{end} = 0$ and $\Sigma_c$ is the segment from 0 to 0.5. This proves (iii).

**Proposition C.0.4 (The sliding segment $\Sigma_s$ is followed by the crossing segment $\Sigma_c$).**

If the sliding segment $\Sigma_s$ and the crossing segment $\Sigma_c$ both exist, $\Sigma_s$ is followed by $\Sigma_c$.

Then, once crossing occurs there is no possibility of regaining market share for the firm that lost part of it.
Proof. Since both the sliding segment $\Sigma_s$ and the crossing segment $\Sigma_c$ exist, from propositions C.0.1 and C.0.3 we see that in $T$, only $\Sigma_s$ followed by $\Sigma_c$ can occur. 

Figure C.2: Suppose that $k_2 > k_1$, $s_{end} < 0.5$ and firm $F_1$ has an initial lower market share than $F_2$, then: (a) There is no trajectory $\tau$ that reaches the equal market share line on the sliding segment $\Sigma_s$, remains on it for some time and then crosses into $L$, with $F_1$ ending with a bigger market share than $F_2$ (b) There is no trajectory $\tau$ that reaches the equal market share line on the crossing segment $\Sigma_c$ and crosses into $L$, with $F_1$ ending with a bigger saturated market share than $F_2$.

Proposition C.0.5. If $k_2 > k_1$ and the sliding end point $s_{end} < 0.5$, then there is no trajectory $\tau$ that has initial conditions in $U$ and reaches $\Sigma$ at a point $(v, v)$ below the sliding end point $s_{end}$ and then crosses into $L$. (See Figure C.2(a))

Proof. To see this, we obtain an expression for $\tan \varphi_{UL}(c_1/(k_2 - k_1), c_1/(k_2 - k_1))$

$$\tan \varphi_{UL}(c_1/(k_2 - k_1), c_1/(k_2 - k_1)) = \frac{k_2}{k_1} + \frac{(k_2 - k_1)c_2}{c_1k_1}$$

(C.6)

Since $k_2 > k_1$ and all parameters $k_1, c_1, k_2, c_2$ are positive (C.6) is always greater than 1, i.e. the condition for crossing into $L$ is never met when $k_2 > k_1$. 

Proposition C.0.6. If $k_2 > k_1$ and the sliding end point $s_{end} < 0.5$, then there is no trajectory $\tau$ that has initial conditions in $U$ and reaches the crossing segment $\Sigma_c$. (See Figure C.2(b))

Proof. Consider a disk with center at $(x_1, x_2) \in \Sigma$ and radius $\delta$. Let $\tau$ be a trajectory that passes by $(x_1, x_2 = x_1 + \delta)$ in $U$. If the trajectory $\tau$ were to hit $\Sigma$ in the crossing segment $(x_1, x_2 = x_1 < 0.5)$, the tangent at $(x_1, x_2 = x_1 + \delta)$ has to be less than 1 as $\delta \to 0$ for crossing from $U$ into $L$

$$\lim_{\delta \to 0^+} \tan \varphi_U(x_1, x_1 + \delta) = \frac{k_2x_1}{k_1x_1 + c_1} < 1$$

(C.7)

Let the LHS of (C.7) be rewritten as

$$\frac{1}{k_1/k_2 + c_1/(k_2x_1)}$$

(C.8)
Since the crossing needs to be in $\Sigma_c$, we have $x_1 > c_1/(k_2 - k_1)$, so that $c_1 < x_1(k_2 - k_1)$. Using this in (C.8)

$$\frac{1}{k_1/k_2 + c_1/(k_2 x_1)} > \frac{1}{k_1/k_2 + (x_1(k_2 - k_1)/(k_2 x_1))} = 1$$  \hspace{1cm} (C.9)

But (C.9) conflicts with (C.7), so that it is not possible that a trajectory $\tau$ beginning in $U$ and satisfying $k_2 > k_1$ to intersect the equal market share on the crossing segment $\Sigma_c$.

Similarly, for a trajectory $\tau$ starting in region $L$ crossing directly into region $U$ can only occur when $k_2 > k_1$.

The next proposition shows that the model (3.3)-(3.6) does not have pseudoequilibrium points. Pseudo-equilibrium points are the stationary points of the differential equation (A.4), see Appendix A.

Proposition C.0.7 (Nonexistence of pseudo-equilibria). The model (3.3)-(3.6) does not have any pseudo-equilibrium points.

Proof. Substituting $x_1 = x_2 = a$ in (3.3)-(3.6), the pseudo-equilibrium condition (A.7) for their existence is

$$\frac{k_1 a + c_1}{k_1 a} = \frac{k_2 a}{k_2 a + c_2} < 0$$

which cannot be satisfied because the parameter values are positive numbers, and point $(a, a)$ belongs to the first quadrant. Hence, the model (3.3), (3.6) does not present any pseudo-equilibrium point. \hfill $\square$
Appendix D

Numerical considerations on simulating Odes with discontinuous RHS (right-hand sides)

In the first simulations, the package by Piirainen and Kuznetzov [55] was used for simulating Odes with discontinuous RHS. However, we had problems with some simulations showing wrong trajectories, trajectories which did not stop at the corresponding equilibrium point. For instance, the simulation of model VWDsC in Figure D.1 shows trajectories continuing upwards on the saturated market line $Sat$ and then stopping in the second quadrant where state variable $x_1$ is negative. The problem was reported but we received no answer. As of 2016, the package is not available for download anymore.

![Figure D.1: Simulation ran with Piirainen, Kuznetzov Matlab software. Trajectories should stop on the saturated line $Sat$, however in the simulation they continue moving upwards.](image)

After that, we used the application Berkeley Madonna [51] for running the simulations. However, we found wrong trajectories model VWDsC like the one shown in Figure
From the experience accumulated doing simulations with Berkeley Madonna application we inferred that Berkeley Madonna considers the function $if$ -used to implement Odes with discontinuous RHS- a continuous function and in some cases you get a reasonable behavior on the sliding segments and in other it disagrees with the theory of Odes with discontinuous RHS. Also, for more complex dynamics like the Lotka-Volterra derive model defining the dynamics with the function $if$ becomes very complex.

Figure D.2: Trajectory on the phase plane showing more than one crossing. This situation does not comply with Filippov’s theory for Odes with discontinuous RHS.

Diminishing the time step in the simulations ran with Berkeley Madonna reduced the chattering in many cases. However, this was not the case for some simulations of the model LVsP presented in Chapter 5 which still had a good amount of chattering.

Because of the previous situations, we implemented a simple discretization which supports the discretization methods Forward-Euler or RK4 (Runge Kutta 4) and uses the Filippov Rule for the field value on the sliding segments. The result are obviously better in smoothing the chattering on the sliding set $\Sigma$ because the field defined by the Filippov rule on the sliding segment is tangent to the discontinuity line. However, since we are working with finite precision arithmetic when doing simulations on computers, any point which lies inside an $\epsilon$-tube around the switching line and satisfies the sliding condition (A.6) is assigned the field (A.2) corresponding to the Filippov rule for sliding dynamics. See Figure D.3.

Figure D.3: A magnified section of the discontinuity line $\Sigma$. For a point that lies in the $\epsilon$-tube around $\Sigma$ and satisfies the sliding condition (A.6), the point is assigned the field (A.2) defined by the Filippov rule for the numerical discretization.
Consider an autonomous ODE system given in vector form:

\[ \dot{x} = f(x) \]  

(D.1)

For instance if \( x \in \mathbb{R}^2 \), \( x = [x_1, x_2]^T \) and \( f(x) = [f_1(x_1, x_2), f_2(x_1, x_2)]^T \)

The *Forward Euler* discretization [56, p. 257, chap. 5] calculates the next time value for \( x \) using the first order Taylor approximation:

\[ x(t + h) = x(t) + hf(x(t)) \]  

(D.2)

The fourth term *Runge-Kutta (RK4)* [56, p. 277-278, chap. 5] is more complex and requires more operations. It calculates the next time value for \( x \) using the following steps:

1. calculate \( k_1 = hf(x(t)) \)
2. calculate \( k_2 = hf(x(t) + 1/2k_1) \)
3. calculate \( k_3 = hf(x(t) + 1/2k_2) \)
4. calculate \( k_4 = hf(x(t) + k_3) \)
5. calculate \( \)

\[ x(t + h) = x(t) + 1/6(k_1 + 2k_2 + 2k_3 + k_4) \]  

(D.3)

A basic \( \epsilon \)-tube algorithm for calculating \( x \) for the next time step

1. Determine the field \( f \) to be used for calculating \( x(t + h) \)
   - if \( (x(t) \in \epsilon \text{-tube}) \) and \( x(t) \in \Sigma_x \), then the field to be used is \( f(x(t)) = \Phi(x(t)) \) with \( \Phi(x) \) defined in (A.2).
   - if \( x_2(t) > x_1(t) + \epsilon \), then the field to be used is \( f(x(t)) = f^U(x(t)) \)
   - if \( x_1(t) > x_2(t) - \epsilon \), then the field to be used is \( f(x(t)) = f^L(x(t)) \)

2. Calculate \( x(t + h) \) using the field \( f \) in the selected discretization method, i.e. in (D.2) for Forward Euler or (D.3) for RK4.

We show phase planes from simulations done with Berkeley Madonna (Figure [D.4]) and with the basic \( \epsilon \)-tube algorithm (Figure [D.5]). Obviously there is a trade-off between the size of \( \epsilon \)-tube and the discretization step size. One would like to use a thin \( \epsilon \)-tube but one also needs that the simulation solution falls inside the \( \epsilon \)-tube and this implies using a small step size. The thinner the tube, the smaller the step size.
Figure D.4: Phase plane plotted with simulation data calculated using Berkeley Madonna.

Figure D.5: Phase plane plotted with simulation data calculated using the basic $\epsilon$-tube algorithm.

Figure D.6 shows that there are small errors that can be seen when doing a zoom near the sliding end point $s_{end}$ when in the phase plane of Figure D.5, this is because of finite precision arithmetic and the use of the $\epsilon$-tube. The zoom also shows that taking into account the numerical errors, the sliding segment is effectively $s_{end}$. Figure D.7 corresponds to the same trajectory shown in Figure D.2.

Figure D.6: Zoom of the phase plane nearby the sliding end point $s_{end}$

Figure D.7: Trajectory calculated using the basic $\epsilon$-tube algorithm.

Remark: We assume that for a point that lies in the $\epsilon$-tube the firm, whose target market share is represented by $\Sigma$, has its extra effort turned on when calculating its associated performance index $J$. The justification is that the firm wants more market share than its target market share even though they accept at least their target market share.
Appendix E

Review of duopoly models

E.1 K. R. Deal, 1979 [1]

E.1.1 Model derived from Vidale-Wolfe.

E.1.2 Formulation

The objectives $J_1, J_2$ of firms $F_1, F_2$ are

$$\max_{u_1} J_1 = w_1 \frac{S_1(t_f)}{S_1(t_f) + S_2(t_f)} + \int_{t_0}^{t_f} (c_1 S_1(t) - u_1^2(t))dt$$ (E.1)$$

$$\max_{u_2} J_2 = w_2 \frac{S_2(t_f)}{S_1(t_f) + S_2(t_f)} + \int_{t_0}^{t_f} (c_2 S_2(t) - u_2^2(t))dt$$ (E.2)

and the system dynamics is

$$\dot{S}_1(t) = \beta_1 u_1(t) \frac{[M - S_1(t) - S_2(t)]}{M} - \delta_1 S_1(t)$$ (E.3)

$$\dot{S}_2(t) = \beta_2 u_2(t) \frac{[M - S_1(t) - S_2(t)]}{M} - \delta_2 S_2(t)$$ (E.4)

where $x_i(t)$ is the sales of firm $F_i$, $u_i(t)$ is the advertising expenditure of $F_i$, $\delta_i$ is the sales decay parameter, $\beta_i$ is the sales response parameter, $M$ is the total sales potential. In the objective function, $w_i$ is the weight that is given to the final market proportion obtained, $c_i$ is the net revenue coefficient, $u_i^2$ is the advertising cost and $c_i x_i(t)$ is the revenue for $F_i$.

Assumption: the size of the potential market is fixed at $M$ and this size is not changed by advertising.
E.1.3 Solution

Solves the open-loop Nash equilibrium using a numerical method.

E.2 J. Case, 1979 [2]

E.2.1 Model derived from

Lanchester. Its game formulation/solution is later used by others like Erickson, Chintagunta.

E.2.2 Formulation

Firm 1
\[
\max_{u_1} \int_0^\infty e^{-r_1 t} (g_1 x(t) - \frac{c_1}{2} u_1^2(t)) dt \quad x_1(t) = x(t) \quad (E.5)
\]

Firm 2
\[
\max_{u_2} \int_0^\infty e^{-r_2 t} (g_2 (1 - x(t)) - \frac{c_2}{2} u_2^2(t)) dt \quad x_2(t) = 1 - x(t) \quad (E.6)
\]

subject to
\[
\dot{x}(t) = u_1(t)(1 - x(t)) - u_2(t)x(t) \quad (E.7)
\]

where

\(i = 1, 2\) number of the firm

\(t\) time

\(x_1(t) = x(t)\) market share of firm 1 at time \(t\)

\(x_2(t) = 1 - x(t)\) market share of firm 2 at time \(t\)

\(u_i(t)\) advertising of firm \(i\) at time \(t\) (control variable)

\(r \geq 0\) discount factor

\(g_i > 0\) constant unit margin of firm \(i\)

\(c_i > 0\) constant cost parameter of firm \(i\)

\(x_0 \in [0, 1]\) initial market share of firm 1 at \(t = 0\)
E.2.3 Solution

The infinite horizon planning has an analytical closed-loop solution.

Case (1979) derives conditions which a pair of feedback strategies \((u_1(x), u_2(x))\) has to satisfy, if it constitutes a Nash equilibrium for the above game. Moreover, he calculates explicit expressions for \(u_i(x)\), \(i = 1, 2\), in the case of no discounting (i.e., \(r = 0\)).

E.3 G. Sorger, 1989 [2]

E.3.1 Model derived from

Uses a model derived from Lanchester. It is a modification of the Case Game.

E.3.2 Formulation

\[
\max_{u_i} = \int_0^T e^{-rt}(g_i x_i(t) - \frac{c_i}{2} u_i^2(t))dt \quad i = 1, 2, \quad x_1(t) = x(t), x_2(t) = 1 - x(t) \tag{E.8}
\]

subject to

\[
\dot{x}(t) = u_1(t)^2(1 - x(t)) - u_2(t)^2 x(t) \tag{E.9}
\]

where \(i = 1, 2\) is the firm index, \(F_i\) is the firm \(i\), \(x_1(t) = x(t)\) and \(x_2(t) = 1 - x(t)\) are the market shares of \(F_1\) and \(F_2\) at time \(t\), \(u_i(t)\) is the advertising rate of \(F_i\), \(r_i \geq 0\) is the constant discount factor of \(F_i\), \(g_i\) is the constant margin of \(F_i\), \(c_i > 0\) is the constant cost of \(F_i\) Total market potential is constant. Sorger uses the square root of the the market share in the dynamics equation to obtain a market share with a concave response so that it has diminishing returns, in contrast to Little in 1979 who used an exponent on the advertising effort to obtain a concave or an S-shaped response.

E.3.3 Solution

Finite time horizon, Open-Loop, Closed-loop) derives differential equation on the costate. Sorger does a qualitative analysis on that equation.

Infinite time horizon has analytical solution for Open-loop and Closed-loop.


E.4.1 Model derived from

Lanchester.

Uses econometric estimation of response and differential games. Estimates parameters \(k_1\) and \(k_2\) using a regression.
E.4.2 Formulation

\[ \dot{x} = \beta_1 \sqrt{u_1(1 - x)} - \beta_2 \sqrt{u_2} x \]  
(E.10)

\( x_1 = x \) is the market share of firm 1, \( x_2 = 1 - x \) is the market share of firm 2.

Discrete equation

\[ x_{t+1} - x_t = \beta_1 \sqrt{u_{1,t+1}(1 - x_t)} - \beta_2 \sqrt{u_{2,t+1}} x_t \]  
(E.11)

So that players need to know the previous period market share \( M \). The problem to be solved is

\[ \max_{u_i} \pi_i = \int_0^\infty e^{-rt}(g_ix_iS_{inst} - u_i)dt \]  
(E.12)

subject to

\[ \dot{x}_i = \beta_i \sqrt{u_i(1 - x_i)} - \beta_j \sqrt{u_j} x_i \quad i, j = 1, 2 \quad i \neq j \]  
(E.13)

where \( x_i \) is the market share of firm \( i \) and \( x_2 = 1 - x_1 \), \( u_i \) are the advertising controls. \( r \) is the discount factor, \( g_i \) is the net contribution as a fraction of dollar sales for firm \( i \), \( g_i \) is constant and independent of sales. \( S_{inst} \) is the instantaneous total sales in the market in dollars at time \( t \). \( r \) is the same for both firms. Parameters \( \beta_i \) have to be empirically estimated. Firms cannot collude, so that a Nash equilibrium is appropriate.

E.4.2.1 Open-loop

To obtain the OLNE (open-loop Nash equilibrium), maximize the Hamiltonian

\[ \max_{u_i} H_u = (g_ix_iS_{inst} - u_i) + \lambda_i(\beta_i \sqrt{u_i(1 - x_i)} - \beta_j \sqrt{u_j} x_i) \quad i, j = 1, 2 \quad i \neq j \]  
(E.14)

where \( \lambda_i \) are the costates.

The necessary conditions for the OLNE are

\[ \frac{\partial H_i}{\partial u_i} = 0, \quad \dot{x}_i = \frac{\partial H_i}{\partial \lambda_i}, \quad \dot{\lambda}_i = r\lambda_i - \frac{\partial H_i}{x_i} \quad i = 1, 2 \]  
(E.15)

After simplifications, the above system of equations is transformed in

\[ u_i = \left[ \frac{\lambda_i \beta_i (1 - x_i)}{2} \right]^2 \]  
(E.16)

\[ M_i = \beta_i \sqrt{u_i(1 - x_i)} - \beta_j \sqrt{u_j} x_i \quad i, j = 1, 2 \quad i \neq j \]  
(E.17)

\[ \dot{\lambda}_i = g_iS_{inst} + \lambda_i(r + \beta_i \sqrt{u_i} + \beta_j \sqrt{u_j}) \quad i, j = 1, 2 \quad i \neq j \]  
(E.18)

The system of equations given by (E.16), (E.17) and (E.18) does not have a closed form solution. Chintagunta and Vilcassim solved it numerically using an algorithm pro-
posed by Deal in 1979.

**E.4.2.2 Closed-loop**

They apply the Two-Player Game of Case (1979). The assumption made is that the discount factor \( r \) is strictly positive and is close to zero in magnitude.

The advertising controls obtained are

\[
 u_1 = \frac{1}{3} [2TR^2 - S + 2(T^2R^4 - STR^2)] \tag{E.19}
\]

\[
 u_2 = \frac{1}{3} \left[ \frac{2S}{R^2} - T + 2 \left( T^2 + \frac{S^2}{R^4} - \frac{ST}{R^2} \right) \right] \tag{E.20}
\]

where \( R = \frac{k_2}{k_1} \frac{x}{1-x}, S = g_1 S_{inst} x \) and \( T = g_2 S_{inst} (1-x) \).

*Parameters* \( \beta_1, \beta_2 \) in (E.10) were estimated using ordinary least squares and the historical datasets for Coke and Pepsi advertising expenditures.

**E.4.3 Solution**

- Closed-loop, analytical solution for infinite time horizon. Solution as in Case (1979).
- Open-loop, numerical solution.
- Solutions obtained are also analyzed using empirical data.

**E.5 G. Erickson, 1992 [4]**

**E.5.1 Model derived from**

Lanchester.

**E.5.2 Formulation**

Each firm seek to maximize its discounted profit

\[
 \max \int_0^\infty e^{-rt} h_1(x, u_1, u_2) \tag{E.21}
\]

dynamics is given by

\[
 \dot{x} = f(x, u_1, u_2), \quad x(0) \text{ given} \tag{E.22}
\]

also \( 0 \leq x \leq 1 \) and \( u_i > 0 \)
Firms cannot cooperate, so that it is reasonable that they search for Nash equilibria. A Nash equilibrium is a pair of strategies, one for each competitor, which has the property that no competitor would like unilaterally to change its strategy (Moorthy 1985). In a Nash equilibrium, each strategy is a competitor’s best strategy, given the strategies of its rival, where “best” means maximizing the profit integral.

There are two kinds of Nash equilibria that can be pursued: open-loop, where advertising is a function of time, $u_i = u_i(t, x(0))$, in which $x(0)$ is the starting value for market share, and closed-loop, where advertising is a function of time and the current state of the system $u_i = u_i(t, x, x(O))$. Unfortunately, open-loop and closed-loop equilibria are generally different. The most frequently used approach in differential games has been to develop open-loop equilibria, primarily because they are easier to compute (Case 1979).

Open-loop equilibria are by definition time consistent, in that if at some intermediate point the competitors are asked to reconsider their strategies they would refuse to change them (Fershtman 1987a). Open-loop strategies are not subgame perfect, they change if initial conditions change. To be subgame perfect, an equilibrium must not depend upon initial conditions. Specifically, strategies $u_i(t, x)$ that depend upon current values of state variables as well as time and that do not depend upon initial conditions are termed feedback strategies (Fershtman 1 987b).

In general, closed-loop equilibria have been difficult to obtain, since they tend to involve partial differential equations (Starr and Ho 1969; Fershtman 1987a).

E.5.2.1 Case approach to Lanchester game

An approach by Case (1979), however, offers hope for a class of problems, those involving a single state variable, for which only ordinary, and not partial, differential equations are required. In particular, Case’s approach can be applied to competitive situations involving duopolistic competition for market share.

Case (1979, pp. 210-215) considers what he terms perfect equilibria, which are time invariant (stationary) functions of state variables. This definition of ”perfect” equilibria differs from what is elsewhere defined as ”perfect” (e.g., Friedman 1986). CASE, JAMES H., Economics and the Competitive Process, New York University Press, New York, 1979.

Procedure by Case

- Define the Hamiltonians

$$H_i = h_i(x, u_1, u_2) + \lambda_i f(x, u_1, u_2) \quad (E.23)$$

where $\lambda_i$ is the costate and each firm maximizes its $H_i$

- Now determine $u_i(x, \lambda_1, \lambda_2)$ and $A_2(x, \lambda_1, \lambda_2)$ that form a Nash equilibrium for the auxiliary game
\[ \max H_i, \quad i = 1, 2 \quad (E.24) \]

Define the Hamilton-Jacobi equations

\[
h_i(x, \dot{u}_1(x, V'_1(x), V'_2(x)), \dot{u}_2(x, V'_1(x), V'_2(x))) \\
+ V'_i f(x, V'_1(x), V'_2(x)), \dot{u}_2(x, V'_1(x), V'_2(x))) \\
= rV_i(x) + c_i \quad i = 1, 2 \quad (E.25)
\]

where the \( c_i \) are arbitrary real numbers and \( r \) is the discount factor, the \( V_i(M) \) are the value functions for firm \( i \), they are the discounted profit for firm \( i \) on an optimal advertising path. The costate and value functions are related by \( \lambda_i = V'_i(x) \), for different starting levels \( x \). If the system of ordinary differential equations defined by (E.25) can be solved for \( V_1(x) \) and \( V_2(x) \), a perfect equilibrium is derived:

\[ u_i(M) = \dot{u}_i(x, V'_1(x), V'_2(x)) \quad (E.26) \]

An equilibrium (E.26), forms an optimal strategy for each firm \( i \) and for any initial value \( x \). Because \( c_i \) are arbitrary, there are infinite number of perfect advertising equilibria.

### E.5.3 Lanchester differential game

The objective for each firm

\[ \max_{u_i} \int_0^\infty e^{-rt}(g_i x_i - u_i)dt, \quad i = 1, 2 \quad (E.27) \]

subject to

\[ \dot{x}_i = \beta_i u_i^{\alpha_i} (1 - x_i) - \beta_j u_j^{\alpha_j} x_i, \quad i, j = 1, 2 \quad i \neq j \quad (E.28) \]

where \( u_i \) is the advertising effort of firm \( i \), \( r \) is the discount factor, \( g_i \) is the gross profit rate of firm \( i \), \( x_i \) is the market share of firm \( i \) and \( x_1 = x \), \( x_2 = 1 - x \), \( \beta_i \) is a constant for firm \( i \).

### E.5.4 Closed-loop solution

Analytical solutions can be derived only for \( r = 0 \). The following relations result and they define \( u_i \) implicitly in term of \( x \).

\[ g_i x_i + \frac{1 - \alpha_j}{\alpha_i} u_i - \frac{\beta_{3-i}}{\alpha_i \beta_i} u_i^{1-\alpha_i} u_{3-i}^{\alpha_{3-i} x_i^{1-\alpha_{3-i}}}, \quad i = 1, 2 \quad (E.29) \]
Explicit forms for $u_i$ when $\alpha_i = 0.5$, $i = 1, 2$ in (E.29) are given below

$$u_i(x) = \frac{2C^2_i E_{3-i} - E_i + 2\sqrt{E^2_i - \frac{C^2_i E_i E_{3-i} + C^2_i E^2_{3-i}}{3}}} i = 1, 2$$ (E.30)

where

$$C_i = \frac{x_i}{x_{3-i}}, \quad D_i = \frac{\beta_i}{\beta_{3-i}}, \quad E_i = g_i x - c_i, \quad i = 1, 2$$ (E.31)

Note that $x$ and $u_i$ are simultaneous in different equations and also nonlinear.

### E.5.4.1 Open-loop solution

The following differential equations are derived from the necessary conditions of the open-loop strategies

$$\dot{u}_i = 2u_i \left( r + \frac{\beta_{3-i}\sqrt{u_{3-i}}}{x_{3-i}} - \frac{g_i\beta_i x_{3-i}}{2u_i} \right), \quad i = 1, 2$$ (E.32)

After estimating parameters, the system (E.32) is solved numerically.

### E.5.4.2 Conclusions

For two cases: Soda drinks (Coke, Pepsi) and Beer drinks (Miller, Anheuser-Busch) they estimate parameters for the model and conclude that closed-loop perfect equilibrium fits better the historical datasets.

### E.5.5 Solution

- Closed-loop, analytical solution for infinite time horizon. Solution as in Case(1979).
- Open-loop, numerical solution.
- Solutions obtained are also analyzed using empirical data.


#### E.6.1 Model derived from

Lanchester.
E.6.2 Formulation

Two firms compete using a system dynamics described using Lanchester model they determine the optimal advertising strategy for maximum discounted profits. New mathematical approach. Discounted profit for each firm

\[
\max_{u_k} J_k(u_1, u_2) = \int_0^\infty \left[ g_k x_k(t) - c_k u_k^2(t) \right] e^{-rt} dt, \quad k = 1, 2 \tag{E.33}
\]

subject to

\[
\dot{x}(t) = \beta_1 u_1(t)(1 - x(t)) - \beta_2 u_2(t)x(t) \tag{E.34}
\]

where \( x_1 + x_2 = 1 \) and \( x_k \) is the fraction of the market that belongs to firm \( k \). The constant \( g_k \) is the gross profit rate of firm \( k \), \( r \) is the discount rate and \( c_k \) is the effectiveness of advertising buying power, usually \( c_1 = c_2 = 1 \). The variables \( u_1, u_2 \) are the controls based on advertising, in particular \( u_k, \ k = 1, 2 \) is the square root of the advertising expenditures of firm \( k \). When the current state is also the initial state, the computed optimal controls become a feedback strategy which depends on \((x, t)\). The effectiveness of the combat of firm \( k \) is measured by \( \beta_k u_k(t) \).

E.6.3 Solution

Numerical solution. They transformed the problem into an initial value problem. Closed-loop solution depends on time and state.

E.6.3.1 Open-loop

The first order necessary conditions using the variational approach as in Bryson and Ho 1975 on the differential game are

\[
\dot{\lambda}_k(t) = r \lambda_k(t) + (\beta_1 u_1(t) + \beta_2 u_2(t))\lambda_k(t) + (-1)^k g_k, \quad k = 1, 2 \quad \lim_{t \to \infty} \lambda_k(t)e^{-rt} = 0 \tag{E.35}
\]

\[
\dot{u}_k(t) = \frac{(-1)^{k+1}}{2c_k} c_k^{-1} \beta_k \lambda_k(t)(1 - x_k(t)), \quad k = 1, 2 \tag{E.36}
\]

Substituting (E.36) in (E.35) and (E.34) they obtain the following two-point boundary value problem TPBVP value obtained

\[
\dot{x} = \frac{1}{2} \left[ c_1^{-1} \beta_1^2 (1 - x)^2 \lambda_1 - c_2^{-1} \beta_2^2 (x)^2 \lambda_2 \right], \quad x(0) = x_0 \tag{E.37}
\]

\[
\dot{\lambda}_k = r \lambda_k + \frac{1}{2} \left[ c_1^{-1} \lambda_k \beta_1^2 (1 - x) - c_2^{-1} \beta_2^2 \lambda_2 x \right] - (-1)^{k+1} g_k, \quad \lim_{t \to \infty} \lambda_k(t)e^{-rt} = 0 \tag{E.38}
\]
where \( k = 1, 2 \). In their lemma 1 they let \( \lambda_k(t) = (-1)^{k+1}g_k e^{rt}\Phi(t), \ k = 1, 2 \), also \( \lambda_k \) satisfies (E.37) and (E.38). Using this assumptions \( \Phi(t) \) satisfies a backward differential equation

\[
\Phi'(t) = \frac{1}{2}[c_1^{-1}\beta_1^2 g_1 (1 - x^P) - c_2^{-1}\beta_2^2 g_2 x^P]e^{rt}\Phi^2(t) - e^{-rt} \lim_{t \to \infty} \Phi(t) = 0 \quad (E.39)
\]

Note the change in notation \( x \) was replaced by \( x^P \) to denote \( x \) during planning.

Let

\[
\Phi(t)e^{rt} = \psi(x^P) \quad (E.40)
\]

to get a new backward differential equation for \( \psi(x^P) \)

\[
\psi'(x^P)\psi(x^P)[c_1^{-1}\beta_1^2 g_1 (1 - x^P)^2 - c_2^{-1}\beta_2^2 g_2 (x^P)^2] - [c_1^{-1}\beta_1^2 g_1 (1 - x^P) - c_2^{-1}\beta_2^2 g_2 x^P]\psi(x^P) = 2r\psi(x^P), \quad \lim_{t \to \infty} \psi(x^P)e^{-rt} = 0 \quad (E.41)
\]

where

The TPBVP (E.37), (E.38) can be transformed in the following IVP (initial value problem)

\[
\dot{x}^P = \frac{1}{2}[c_1^{-1}\beta_1^2 (1 - x^P)^2 g_1 - c_2^{-1}\beta_2^2 (x^P)^2 g_2]e^{rt}\Phi(t), \quad x(0) = x_0 \quad (E.42)
\]

\[
\dot{\Phi}(t) = \frac{1}{2}[c_1^{-1}\beta_1^2 (1 - x^P)g_1 - c_2^{-1}\beta_2^2 x^P g_2]e^{rt}\Phi^2(t) - e^{-rt}, \quad \Phi(0) = \psi(x_0) \quad (E.43)
\]

where \( \psi(x_0) \) is obtained from the backward equation (E.41)

The control for the open-loop problem is

\[
u_{OL}^k = \frac{1}{2}c_k^{-1}\beta_k g_k \Phi(t)e^{rt}(1 - x_k^P) \quad k = 1, 2 \quad (E.44)
\]

where \( x_1^P = x^P \) and \( x_2^P = 1 - x^P \) and \( x^P \) satisfies (E.42) and (E.43).

**E.6.3.2 Closed-loop**

The closed loop is time variant and depends linearly on the actual market share. Time variant coefficients incorporate the discount factor. Its computation requires the solution of a backward differential equation and a set of two nonlinear differential equations for an initial value problem. Let the close-loop strategies be

\[
u_k = \frac{1}{2}c_k^{-1}\beta_k g_k \Phi(t)e^{rt}(1 - x_k), \quad k = 1, 2 \quad (E.45)
\]

The closed-loop advertising expenditures are proportional to the open-loop advertising expenditures and to the square of the competitor’s actual market share. From (E.44)
and (E.45), we get

\[ u_k^* = u_k^{OL} \frac{1 - x_k}{1 - x_k^0}, \quad k = 1, 2 \]  

(E.46)

**E.7 Q. Wang, Z. Wu, 2001 [6]**

**E.7.1 Model derived from**

Lanchester and Vidale-Wolfe.

**E.7.2 Formulation**

*Model*

\[
\max_{u_i \geq 0} J_i = \int_0^{t_f} (M p_i x_i - u_i^2) dt + g_i x_i(t_f)
\]  

(E.47)

subject to

\[
\frac{dx_i}{dt} = \beta_i u_i (1 - x_i) - \beta_j u_j x_i - \delta_i x_i, \quad i, j = 1, 2 \quad i \neq j \quad x_i(0) = x_{i0}
\]  

(E.48)

where \( x_i \) is the market share of firm \( i \), \( u_i \) is a variant of the advertising expenditure of firm \( i \), \( \beta_i \) are response constants to the advertising of firm \( i \), \( \delta_i \) is a decay constant, \( p_i \) is the unit price at which firm \( i \) sells, \( M \) is the total market and \( g_i \) is a valuation of the market share of firm \( i \) at time \( T \). The market share satisfies \( 0 \leq x_1 + x_2 \leq 1 \) and the total market is normalized to 1, *which is a subtle difference from other models and allows for market growth until saturation*. In other models, \( x_1 + x_2 = 1 \). The model does not use any discount factor.

**E.7.3 Solution**

Numerical algorithm for open-loop and closed-loop Nash equilibrium solutions.

The solutions they obtain are numerical, for OLNE and CLNE. i.e., they solve the differential equations for the necessary conditions numerically. Algorithm starts estimating an initial value for each auxiliary variable \( \lambda_i(0) \) and \( \phi_i(0) \) (for CLNE only). The necessary conditions of Nash optimality are then solved forward in time by using the fourth order Range-Kutta method. The values of auxiliary variables at terminal time \( T \), \( \lambda_i(T) \) and \( \phi_i(T) \), are with target values \( g_i \) and 0. If they are within predetermined ranges a solution is found. Otherwise, the differential equations are solved again with the initial values \( \lambda_i(0) \) adjusted by \( \delta_i \) and \( \phi_i(0) \) by \( \sigma_i \) (See expressions for \( \delta_i \) and \( \sigma_i \) in the paper) They say their model is better for the datasets tested.

E.8.1 Derived from
Lanchester.

E.8.2 Formulation

\[ \max_{u_1} \int_0^\infty e^{-rt}(g_1 x - u_1) dt \] (E.49)

\[ \max_{u_2} \int_0^\infty e^{-rt}(g_2(1 - x) - u_2) dt \] (E.50)

subject to

\[ \dot{x}(t) = \beta_1 \sqrt{u_1(t)(1 - x(t))} - \beta_2 \sqrt{u_2(t)x(t)}, \quad x(0) = x_0 \] (E.51)

where \( x_1(t) = x(t) \) is the market share of firm 1 at time \( t \) and \( x_2(t) = 1 - x(t) \) is the market share of firm 2 at time \( t \). \( \beta_i, \ i = 1, 2 \) is a constant that denotes the advertising effectiveness of firm \( i \), \( u_i(t) \) is the advertising budget of firm \( i \), \( g_i \) is the gross margin per point of market share for firm \( i \) and \( r \) is the discount rate.

From page 996: However, in an infinite-horizon setting, the assumption of a zero-discount rate lacks conceptual appeal.

E.8.3 Solution

They proposed a numerical algorithm for solving Markow perfect equilibrium for the Lanchester duopoly model where the discount factor can be nonzero.

Discussion

The standard sufficient condition to determine a stationary MPNE (Markov Perfect Nash Equilibrium) is to find bounded and continuously differentiable functions \( V_i(x) \), \( i = 1, 2 \) , satisfying, for all \( x, 0 \geq x \leq 1 \), the Hamilton-Jacobi-Bellman (HJB) equations

\[ rV_1(x) = \max_{a_1} \left[ g_1 x - a_1^2 + V'_1(x)(\beta_1 a_1(t)(1 - x(t)) - \beta_2 a_2(t)x(t)) \right] \] (E.52)

\[ rV_2(x) = \max_{a_2} \left[ g_2(1 - x) - a_2^2 + V'_2(x)(\beta_1 a_1(t)(1 - x(t)) - \beta_2 a_2(t)x(t)) \right] \] (E.53)

where \( a_i^2 = u_i \) the change is for computations purposes, \( V_i(x) \) is the value function of firm \( i \) and \( V'_i(x) \) its derivative. \( u_i^2 \) is used for computations.

The usual algorithm for obtaining MPNE strategies is:

- Assume interior solutions and take derivatives of (E.52), (E.53) with respect to \( a_1 \),
\(a_2\), respectively, and equate to zero

\[ a_1 = \frac{\beta_1}{2} (1 - x)V_1'(x), \quad a_2 = -\frac{\beta_2}{2} x V_2'(x) \]  \hspace{1cm} (E.54)

For \(M \in (0, 1)\) these equations can be rewritten as

\[ V_1'(M) = \frac{2a_1}{\beta_1(1 - x)}, \quad V_2'(x) = -\frac{2a_2}{\beta_2 x} \]  \hspace{1cm} (E.55)

- Substitute for \(u_i\) from (E.54) in (E.52), (E.53) to obtain two differential equations involving \(V_i(x), V'_i(x) i = 1, 2,\) and \(x\).

- Postulate a functional form for \(V_i(x), i = 1, 2\) —for example, a polynomial of a certain degree—and determine its coefficients by identification.

The main difficulty is that one cannot find such a functional form (at least nobody has found it yet); hence, two options are possible:

- The first is to obtain these value functions numerically.

- The second is to somehow simplify the HJB equations so that they become analytically tractable. The literature has adopted this last approach.

For instance, to solve the HJB equations and derive a stationary MPNE, Chintagunta and Vilcassim (1992) use an equivalent system defined in terms of the advertising variables \(a_1\) and \(a_2\) rather than the value functions. Indeed, they use (E.55)) to replace \(V_i(x)\) by their expressions in terms of \(a_i\) in the HJB equations. Assuming that the discount rate \(r = 0\), then the HJB differential equations system is reduced to a simple system of two algebraic equations with two unknown variables, \(a_1\) and \(a_2\).

Jarrar et al. use a system of differential equations on \(V_i\). They substitute \(a_i\) from (E.54) in (E.52), (E.53) to obtain

\[ rV_1(x) = g_1 x + \left( \frac{\beta_1}{2} (1 - x)V_1'(x) \right)^2 + \frac{\beta_2}{2} x^2 V_1'(x)V_2'(x) \]  \hspace{1cm} (E.56)

\[ rV_2(x) = g_2 (1 - x) + \left( \frac{\beta_2}{2} x V_2'(x) \right)^2 + \frac{\beta_1}{2} (1 - x)^2 V_1'(x)V_2'(x) \]  \hspace{1cm} (E.57)

after algebraic computations they obtain a system of \(V_i'\) on the RHS in terms of \(V_i\) on the LHS, which they solve numerically with ODE45 available in Matlab.

E.9.1 Model derived from
Lanchester. It includes brand and generic advertising.

E.9.2 Formulation
Change in primary demand $Q$ is given by
\[ \frac{dQ}{dt} = \dot{S}_1(t) + \dot{S}_2(t) = k_1a_1(t) + k_2a_2(t) \] (E.58)
where $S_i$ is the rate of change of the sales of $F_i$ (firm $i$), $a_i$ is the generic advertising of $F_i$, $k_i$ is the effectiveness of that advertising and $i = 1, 2$.

The generic advertising effect on the sales of firm $i$ is
\[ \dot{S}_{i,g}(t) = \theta_i(k_1a_1(t) + k_2a_2(t)) \] (E.59)

The brand advertising effect on the sales of firm $i$, which is a Lanchester model, is given by
\[ \dot{S}_{i,b}(t) = \beta_i u_i(t) \sqrt{M(t) - S_i(t)} - \beta_j u_j(t) \sqrt{S_i(t)} \quad i, j = 1, 2 \quad i \neq j \] (E.60)
where $u_i$ is the advertising of $F_i$, $\beta_i$ is the effectiveness of the advertising of $F_i$.

The total change of sales is \[ \dot{S}_i = \dot{S}_{i,g}(t) + \dot{S}_{i,b}(t), \] i.e., variation due to generic plus variation due to brand advertising.

\[ \dot{S}_i(t) = \beta_i u_i(t) \sqrt{M(t) - S_i(t)} - \beta_j u_j(t) \sqrt{S_i(t)} \]
\[ + \theta_i(k_1a_1(t) + k_2a_2(t)), \quad S_i(0) = S_{i0} \quad i, j = 1, 2 \quad i \neq j \] (E.61)

The controls of $F_i$ are its brand advertising $u_i(t)$ and its generic advertising $a_i(t)$. The discounted profit maximization problem is
\[ V_i(S_1, S_2) = \max_{u_i(t), a_i(t), p_i(t)} \int_0^\infty e^{-rt} \left( (1 - b_i p_i(t) + d_i p_{3-i}(t)) p_i(t) S_i(t) - C(u_i(t), a_i(t)) \right) dt, \] (E.62)
where $r_i$ is the discount rate, $p_i$ is the price charged, $b_i$ and $d_i$ are demand parameters, the factor $(1 - b_i p_i(t) + d_i p_{3-i}(t))$ of $p_i(t) S_i(t)$ is interpreted as the reduction in sales due to price competition and $C(u_i(t), a_i(t))$ is the total advertising expenditure of firm $F_i$. The
expenditure in advertising is quadratic

\[ C(u_i(t), a_i(t)) = \frac{c_i}{2}((a_i(t))^2 + (u_i(t))^2) \]  
(E.63)

### E.9.3 Solution

They obtain closed loop solutions. They obtain perfect Markov Nash equilibrium, i.e., they use an infinite time horizon. Work with symmetric and asymmetric firms. Symmetric is when the two firms have the same parameters, asymmetric otherwise. For firm \( i = 1, 2 \) the optimal decisions are:

- **Brand advertising**
  
  \[ u_i^*(t) = \frac{\beta_i}{c_i} (\rho_i - \gamma_i) \sqrt{S_{3-i}(t)} \]  
  (E.64)

- **Generic advertising**
  
  \[ a_i^*(t) = \frac{k_i}{c_i} (\theta_i \rho_i - \theta_{3-i} \gamma_i) \]  
  (E.65)

- **Price**
  
  \[ p_i^*(t) = \frac{d_i + 2b_{3-i}}{4b_1 b_2 - d_1 d_2} \]  
  (E.66)

- **Value function**
  
  \[ V_i(S_1, S_2) = \alpha_i + \rho_i S_i + \gamma_i S_{3-i} \]  
  (E.67)

where \( \alpha_1, \alpha_2, \rho_1, \rho_2, \gamma_1 \) and \( \gamma_2 \) solve the systems of six equations

\[ r_i \alpha_i - \frac{k_i^2}{2c_i} (\theta_i \rho_i + \rho_{3-i} \gamma_i)^2 \]

\[- \frac{k_{3-i}^2}{c_{3-i}} (\theta_1 \rho_1 + \theta_2 \gamma_1)(\theta_2 \rho_2 + \theta_1 \gamma_2) = 0 \quad i = 1, 2 \]  
(E.68)

\[ r_i \beta_i - m_i + \frac{\beta_{3-i}^2}{c_{3-i}} (\rho_i - \gamma_1)(\rho_2 - \gamma_2) = 0 \quad i = 1, 2 \]  
(E.69)

\[ r_i \gamma_i - \frac{\beta_i^2}{2c_i} (\rho_i + \gamma_i)^2 = 0 \quad i = 1, 2 \]  
(E.70)

Krishnamoorthy et al. conclude that the impact of generic brand advertising is limited and that brand advertising has more effects on the market shares in the long run.

E.10.1  Model derived from

Empirical study of Lanchester. Issues considered are: i) specification of market share response model ii) effect of inflation iii) performance of competitive strategies. It is shown a) square root in market share response equation is often inappropriate b) market share variations are more responsive to current advertising c) closed-loop Nash equilibrium strategies are better than open-loop strategies for maximizing profit d) general perfect equilibria

E.10.2  Model

\[
\max_{\lambda_i \geq 0} J_i = \int_0^{t_f} e^{-rt}(S_{inst}p_f x_i - u_i)dt + g_i x_i(t_f) \tag{E.71}
\]

subject to

\[
\frac{dx_i}{dt} = \beta_i(u_i(t))^{\alpha_i}(1 - x_i) - \beta_j(u_j(t))^{\alpha_j}x_i, \quad i, j = 1, 2 \quad i \neq j \tag{E.72}
\]

where \(S_{inst}\) is the instantaneous total sales in the market in dollars at time \(t\), \(r\) is the discount rate, \(x_i\) is the market share of firm \(i\), \(p_f\) is the unit profit margin of firm \(i\), \(g_i\) is the valuation of the ending market in present value, \(i = 1, 2\) and the planning interval is \([0, T]\), the advertising expenditure of firm \(i\) is \(u_i\). Perfect equilibria by Case uses an infinite planning horizon.

Little 1979, used \(0 < \alpha_i < 1\) to obtain a sales response that is concave from a Lanchester model.

Difference with 2001, the exponent \(\alpha_i, \alpha_j\) on the advertising controls. Also discount factor.

Remark:

The introduction of the power function generalizes the Lanchester model but makes more complex. To overcome this, later studies use the square root function \(\alpha_i = 0.5\) for mathematical tractability without empirical validation (Deal 1979, Sorger 1989, Chintagunta Vilcassim 1992)

CLNE solutions allow decision makers to react to current situations and are more appropriate for marketing problems. However, general CLNE solutions have been rarely discussed. Erickson 1992 and Chintagunta-Vilcassim 1992 use perfect equilibria closed-loop Nash equilibrium PECLNE solution developed by Case 1979. PECLNE is a special case of CLNE. PECLNE depends on states alone and they are obtained using an infinite time horizon. Empirical studies by Erickson 1992 and Chintagunta-Vilcassim 1992

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showed that they fit better the data analyzed than OLNE.

With the present state of knowledge, solving OLNE or CLNE is difficult. Numerical algorithms have been developed to obtain general OLNE or CLNE solutions to dynamic competitive advertising models (Deal 1979, Wang Wu 2001)

**E.10.3 Solution**

They solve open-loop Nash equilibrium (OLNE), closed-loop Nash equilibrium (CLNE), perfect equilibria closed-loop Nash equilibria (PECLNE) using the algorithm in Wang, Wu 2001, which is a numerical algorithm.

In their empirical studies they use nonzero discount factor and found that

- market share variations are more responsive to current advertising
- CLNE are better competitive advertising strategies for firms maximizing profits
- PECLNE equilibrium strategies are usually not good competitive advertising strategies for firms maximizing profits
- square roots of advertising in the system dynamic are not suitable for all industries

**E.11 A. Krishnamoorthy, A. Prasad, S. P. Sethi, 2010 [10]**

**E.11.1 Model derived from**

Vidale-Wolfe. The model combines price and advertising for a duopoly differential game.

**E.11.2 Formulation**

\[
\max_{u_1(t), p_1(t)} u_1(t) = \int_0^\infty e^{-r_1 t}((p_1 - m_1)\dot{s}_1(t) - \frac{c_1}{2} u_1^2(t)) dt \\
\max_{u_2(t), p_2(t)} u_2(t) = \int_0^\infty e^{-r_2 t}((p_2 - m_2)\dot{s}_2(t) - \frac{c_2}{2} u_2^2(t)) dt
\]

subject to

\[
\dot{s}_1(t) = \rho_1 u_1(t) \sqrt{M - S_1(t) - S_2(t)} D_1(p_1(t)) \\
\dot{s}_2(t) = \rho_2 u_2(t) \sqrt{M - S_1(t) - S_2(t)} D_2(p_2(t))
\]

- \(S_i(t)\) is the cumulative sales of firm \(i\) at time \(t\)
- \(M\) is the market potential
• $u_i(t)$ is the advertising effort of firm $i$ at time $t$

• $p_i(t)$ is the price of firm $i$ at time $t$

• $c_i$ is the coefficient of advertising cost of firm $i$

• $\beta_i$ is the effectiveness of advertising of firm $i$

• $m_i$ is the marginal cost of production of firm $i$

• $\alpha_i$ is the demand intercep of firm $i$ (linear demand)

• $\phi_i$ is the price sensitivity of firm $i$ (linear demand)

• $\eta_i$ is the price elasticity of firm $i$ (isoelastic demand)

• $r_i$ discount factor of firm $i$

• $V_i(s_i, s_j)$ is the value function for firm $i$ when its cumulative sales is $s_i$ and the cumulative sales of its competitor is $s_j$

• $D_i$ is the demand function for firm $i$

### E.11.3 Solution

Closed-loop solution using the assumption of a constant value for the derivative of the value function and two forms for the demand function: linear and isoelastic. The feedback strategies they find are:

$$p_i^*(s_i, s_j) = \frac{1}{2} \left( \frac{\alpha_i}{\phi_i} + m_i - \frac{\partial V_i}{\partial S_i} \right)$$  \hspace{1cm} (E.77)

$$u_i^*(s_i, s_j) = \frac{\beta_i}{4c_i\phi_i} \left( \alpha_i + \phi_i \left( \frac{\partial V_i}{\partial S_i} - m_i \right) \right)^2 \sqrt{M - S_i - S_j}$$  \hspace{1cm} (E.78)

• They assume a constant value for the derivative of the value function and two forms for the demand function: linear and isoelastic.

• The solution is analytical in terms of constants, which are calculated from a system of equations.

• They obtain an analytical solution of the infinite horizon formulation.

• The symmetric case, when all parameters are the same for both firms $c_i = c_j = c, r_i = r_j = r, q_i = q_j = q, m_i = m_j = m, \alpha_i = \alpha_j = \alpha, \text{ and } \phi_i = \phi_j = \phi$ is solved analytically.

• The case where firms can have different the parameter values are obtained numerically.
E.12 S. Jorgensen, G. Martín-Herrán, G. Zaccour, 2010

E.12.1 Leitmann-Schmitendorf model

The Leitmann-Schmitendorf model has a decay like Vidale-Wolfe and a cross term from Lanchester. Its system dynamics is

\[ \dot{x}_i(t) = u_i(t) - \frac{c_i}{2} u_i^2 - k_i u_j(t)x_i(t) - \delta_i x_i(t) \quad (E.79) \]

where \( x(t) \) is the sales rate at time \( t \), \( c_i, \delta_i, k_i \) are positive parameters. The effect of advertising is \( u_i(t) - \frac{c_i}{2} u_i^2 \); it has a decay like Vidale-Wolfe \( \delta_i x_i(t) \) and a Lanchester term \( k_i u_j(t) \). The objective functional is

\[ J_i(u_i) = \int_0^T (p_i x_i(t) - u_i(t)) dt \quad (E.80) \]

where \( p_i > 0 \) is the price per unit sold an \( u_i \) is the rate of advertising expenditure.

E.12.2 Tractability of calculating the equilibrium

Feedback Nash equilibrium allows players to take decisions throughout the game in contrast to open-loop equilibrium where the players choose their strategies at the beginning without changing it during the game. However, open-loop Nash equilibrium has been popular in applications because it is generally much easier to compute. Facing the trade-off between strategic appeal of feedback equilibrium and the tractability of open-loop equilibrium, researchers have looked for game structures where an open-loop equilibrium is subgame perfect, in an attempt to get the best of the two equilibrium types. The Leitmann-Schmitendorf differential game belongs to the class of State-Redundant differential games. In this class of games, the open-loop Nash equilibrium is subgame perfect.

E.12.3 Linear and state-redundant differential games

To define the notion of state redundancy, first consider an \( N \) player differential game played on the interval \([t_0, T]\). Let the control \( u_i(t) \in U_i \subseteq \mathbb{R}^m \) and the state \( x_i(t) \in X \subseteq \mathbb{R}^n \). The state equations describing the motion are

\[ \dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0 \quad (E.81) \]
where \( u(t) = (u_1(t), u_2(T), \ldots, u_N(t)) \). The payoff functional is
\[
J_i(u, t_0, x_0) = \int_{t_0}^{t_f} g_i(x(t), u(t), t) dt + S_i(x(t_f), t_f)
\]
(E.82)
where function \( g_i \) is player \( i \)'s instantaneous payoff and \( S_i \). The Hamiltonian of player \( i \) is
\[
H_i(x, u, \lambda_i, t) = g_i(x, u, t) + \lambda_i f(x, u, t)
\]
(E.83)
where \( \lambda_i(t) \) is a \( p \)-dimensional vector of costate variables. Candidates for Nash equilibrium have to satisfy the following necessary conditions
\[
\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0
\]
(E.84)
\[
\dot{\lambda}_i(t) = -\frac{\partial H_i(x(t), u(t), \lambda_i(t), t)}{\partial x}, \quad \lambda_i(t_f) = \frac{\partial S_i(x(t_f), t_f)}{\partial x(t_f)}
\]
(E.85)
\[
\frac{\partial H_i(x(t), u(t), \lambda_i(t), t)}{\partial u_i} = 0, \quad i = 1, \ldots, N
\]
(E.86)

**E.12.4 Linear state games**

Linear state games were introduced by Dockner et al. as the *state-separable*. The state-separability conditions are:
\[
\frac{\partial^2 H_i}{\partial u_i \partial x} = 0, \quad \text{for} \quad \frac{\partial H_i}{\partial u_i} = 0, \quad \frac{\partial^2 H_i}{\partial x^2} = 0, \quad \frac{\partial S_i}{\partial x^2} = 0
\]
(E.87)
for a weaker version of the condition \( \frac{\partial^2 H_i}{\partial u_i \partial x} = 0 \) see Dockner et al.

Cleamhaut and Wan introduced the trilinear game, which is a kind of linear state game, it is the first model where the open-loop Nash equilibrium is subgame perfect. A game is called trilinear when the Hamiltonians are linear in state and costate variables as well as functions of the control variables.

There are many applications of linear state games in economics and management: Breton et al., Jorgensen and Zaccour, Jorgensen et al., Martín-Herrán and Zaccour, Viscolani and Zaccour, Cellini and Lambertini.

Exponential games are also a subclass of linear state games. In this class the objective is
\[
\int_{t_0}^{t_f} g_i(u(t), t)e^{-\mu_i x(t)} dt
\]
(E.88)
where \( \mu_i \in \mathbb{R}^p \) are constants. The state equations are
\[
\dot{x}(t) = f(u(t), t), \quad x(t_0) = x_0
\]
(E.89)
Note that the state dynamics depend only on the controls and the state variables enter the objective functions in the exponent. Dockner et. al used the transformation \( y_i(t) = e^{-mu_i x(t)} \). Jorgensen let the control variables enter the into the dynamics and the objective in exponential form and uses the same procedure as Leitmann-Schmitendorf to show that the OLNE is a FNE which is subgame perfect. Yeung extends the game in Jorgensen to a setting where the Hamiltonians are not required to be linear in the state.

E.12.5 State-redundant games

Leitmann-Schmitendorf initiated the state-redundant differential games. A differential game is state-redundant if the following holds: If after the substitution for the solution of the costate, obtained from equation (E.85), in the Hamiltonian-maximization conditions (E.86), these hamiltonian equations are independent of the state variables and of their initial values, then the game is state-redundant. Note that state-redundant games are state-separable, but state-separable games are not necessarily state-redundant games. Leitmann-Schmitendorf models and Feichtinger model are examples of state-redundant games.