



Mixed thermal convection: fundamental issues and analysis of the planar case

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ABSTRACT

This paper aims to renew interest on mixed thermal convection research and to emphasize three issues that arise from the present analysis: (i) a clear definition of the reference temperature in the Boussinesq approximation; (ii) a practical delimitation of the three convective modes, which are the forced convection (FC), mixed convection (MC) and natural (or free) convection (NC); (iii) and, finally, a uniform description of the set FC/MC/NC in the similarity framework. The planar case, for which analytical solutions are available, allows a detailed illustration of the answers here advanced to the above issues.

Key words: mixed convection, Boussinesq approximation, similarity, buoyancy forces, entropy production, reverse flow.

INTRODUCTION

FUNDAMENTALS OF MIXED CONVECTION

In most of the works on convective heat transfer, (for instance Bejan 2004, Gebhart et al. 1988, Incropera and De Witt 1985, Kakaç and Yener 1995, Kaviany 1994, Padet 2010, Taine and Petit 1989), the mixed convection is shortly treated relative to the forced, natural, phase change convection modes, or sometimes simply omitted. This reality is related to the fact that mixed convection is a complex subject for which many elements are missing for a general and coherent approach.

This heat or mass transfer mechanism relies in fact on two types of mechanical forces: the pressure forces (generated by a pressure gradient $\overline{\text{grad}} p^*$) and the buoyancy forces (generated by a variable density in the gravitational field). The latter are due either to the fluid dilatibility as a function of temperature (thermal convection), or to a variation of species concentration in a mixture (mass convection).

In practical problems, the simultaneous consideration of the two forces is frequently avoided and two simpler convection modes are defined:

- *natural or free convection (NC)*, when buoyancy forces are dominant

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- forced convection (FC), when buoyancy forces are negligible.

Natural and forced convection are, thus, limiting cases for the more general convection model, usually called *mixed convection (MC)*.

For a dilatible, non-isothermal fluid flow, the governing equations expressing the steady-state conservation of mass, momentum, and energy are as follows, respectively:

$$\text{div } \rho \bar{V} = 0 \tag{1}$$

$$\overline{\text{grad}} \bar{V} \cdot \rho \bar{V} = \rho \bar{g} - \overline{\text{grad}} p + \mu (\Delta \bar{V} + \frac{1}{3} \overline{\text{grad}} \text{div} \bar{V}) \tag{2}$$

$$\rho C_p (\bar{V} \cdot \overline{\text{grad}} T) = \Phi + \lambda \Delta T \tag{3}$$

where $\overline{\text{grad}} \bar{V}$ is the tensor gradient of the velocity field (components $\partial V_i / \partial x_j$), and where the symbol Δ represents here the Laplacian operator (sum of second space derivatives).

In general, the problem is simplified by neglecting the dissipation function Φ and by adopting the so called *Boussinesq approximation*, which basically includes three steps:

- B₁: The density variations are assumed negligible, except in the gravity force term
- B₂: In this gravity term, the effect of pressure on the density is neglected, compared to that of temperature
- B₃: Density $\rho(T)$ is expressed as a linear function of the temperature in the vicinity of a reference value T_{ref} , as:

$$\rho = \rho_{ref} [1 - \beta(T - T_{ref})] \tag{4}$$

where ρ_{ref} and β are the density and the thermal expansion coefficient at T_{ref} .

By introducing the *driving pressure* p^* defined as:

$$\overline{\text{grad}} p^* = \overline{\text{grad}} p - \rho \bar{g} \tag{5}$$

equation (2) becomes, after division by ρ (the difference $\rho - \rho_{ref}$ being negligible according to B1):

$$\overline{\text{grad}} \bar{V} \cdot \bar{V} = - \beta(T - T_{ref}) \bar{g} - \frac{1}{\rho} \overline{\text{grad}} p^* + \nu \Delta \bar{V} \tag{6}$$

(a) (b) (c)

(notation of the source terms by (a), (b), (c) will be used later).

Equation (3) is thus reduced to:

$$\bar{V} \cdot \overline{\text{grad}} T = a \Delta T \tag{7}$$

THREE FUNDAMENTAL ISSUES

From the previous presentation, three questions are raised, for which some answers are suggested in this paper.

Practical characterization of FC and NC

At first, forced and natural convection have been defined as theoretical limiting cases of the actual convective transfer phenomenon, but the practical “frontiers” beyond which the limiting approaches may be used, must inevitably be adopted as arbitrary and conventional limits. The pure NC (zero pressure gradient) and the pure FC (zero buoyancy forces) are ideal cases, while dominant NC and dominant FC (simply called NC and FC) must be characterized by clear criteria based on the associated physics. Unfortunately, the existing literature provides fragmentary indications and these are poorly sustained by means of distinguishing among the three possible convective regimes.

Boussinesq approximation and the reference temperature

In second place, it can be noticed that the three terms (a), (b) and (c) in equation (6) are easy to compare due to the Boussinesq approximation, and especially after adopting the linearized form of $\rho(T)$, but a *clear definition of the reference temperature* T_{ref} is then needed. Different authors previously treated this question (Barletta and Zanchini 1999, Wang et al. 2003, Padet 2010, Timshenko 2010) but it seems that no answer of general applicability has yet been given.

Similarity: compatibility of descriptions in MC and NC

The third question concerns the nondimensional description of the three convection modes, using similarity criteria (or dimensionless numbers). The reference flow velocity may be the bulk velocity in internal flows or the freestream velocity in boundary layer flows. But these values might be zero or close to zero in natural convection currents and then a reference velocity is defined based on the thermal buoyancy. The consequence is that the description of the MC, which is based on the Richardson, Reynolds and Peclet numbers, has nothing to do with the description of the NC, which is based on the Grashof and Boussinesq numbers (Padet 2011).

In addition, another difficulty is related to the bulk temperature generally used to describe the heat transfer rates. It would be of interest then to reconsider the reference values and examine if a true continuity may be introduced for the transition MC-NC.

ANALYSIS

MIXED CONVECTION «FRONTIERS»

It is obvious that the governing equations (6) and (7) do not indicate any physical limit between the three convection modes. Then, the practical need to distinguish between FC, MC and NC is purely numerical and a question arises: what criterion should be used to adopt an approach like the following?

- the convection is forced if the buoyancy term (a) is less than x% of the viscous forces term (c)
- the convection is natural if the buoyancy term (a) is more than y% of the viscous forces term (c)

The values of x and y are obviously arbitrary and must be chosen such as to be accepted in practice. On the other hand, other comparison elements may be preferred, such as pressure drop, kinetic energy, wall friction, dissipation, or entropy production, as it will be shown later in this study.

In order to answer to these questions, the terms in the momentum conservation equation (6) must be compared using, not the local values, but the space-averaged ones, over the flow cross section. More precisely, an arithmetic average would not seem representative, as the local values can be positive or negative. A better comparison may be based on the quadratic averages $\overline{a^2}$, $\overline{b^2}$ and $\overline{c^2}$.

REFERENCE TEMPERATURE T_{ref} IN THE BOUSSINESQ APPROXIMATION

However, before we proceed, a clear definition of the reference temperature is essential, regardless of the approach undertaken for the purpose of this study, as the choice of T_{ref} can be of great influence on the computed values. It will be chosen here to base the definition on the density, rather than on the temperature itself.

The vectorial equation (6), when projected on the horizontal axis, loses the buoyancy term, while when projected on the upward vertical axis, this term becomes $\beta(T - T_{ref})g$.

The fluid mass in a volume V in the flow may be calculated as:

$$m = \int_V \rho \, dV = \int_V (\rho - \rho_{ref}) \, dV + \int_V \rho_{ref} \, dV$$

or, using the Boussinesq approximation:

$$m = - \int_V \rho_{ref} \beta (T - T_{ref}) \, dV + \rho_{ref} V \tag{8a}$$

where the last term is the mass of the same fluid at the reference temperature:

$$m_{ref} = \rho_{ref} V \tag{8b}$$

The resulting buoyancy force applied to volume V is then the difference in weight between the considered fluid and the fluid at T_{ref} .

$$(m - m_{ref}) \overline{g} = - \overline{g} \int_V \rho_{ref} \beta (T - T_{ref}) \, dV = - \rho_{ref} V \overline{\beta g} \left(\frac{1}{V} \int_V T \, dV - T_{ref} \right) \tag{8c}$$

where $\frac{1}{V} \int_V T dV = \bar{T}$ is the mean temperature in volume V .

These considerations are now applied to the two particular representative groups of systems.

Open systems

A simple open system is the flue gas channel of a fireplace, whose walls are at imposed constant temperatures. The ambient air is at temperature T_a . The buoyancy force is generated by the difference in weight between the ambient cold air and the inside hot gas, regardless of the type of flow (Archimedes' law). Thus, the reference values for the Boussinesq approximation are:

$$\rho_{ref} = \rho_a; T_{ref} = T_a \quad (8d)$$

Nothing is modified if the channel is bounded by two walls whose temperatures are different, and if the distance between the walls is increased so that to achieve a boundary layer flow at each of them. In any of these cases, the reference temperature remains the ambient temperature T_a .

Closed or infinite systems

In closed systems (like a differentially heated enclosure) or infinite systems (like a fully developed flow with no entrance or exit conditions, cf. below), the concept of ambient has no meaning and another reference value must be defined based on the entire system. Two commonly encountered systems are considered here.

(i) For a vertical rectangular enclosure (of volume $V = AH$, with depth $W = 1$), whose walls are at uniform temperatures $T_1 > T_2$, the convection is purely natural and the balance of the buoyancy forces is zero over V (eq. 8c), so that:

$$T_{ref} = \frac{1}{V} \int_V T dV - \bar{T} \cong \frac{T_1 + T_2}{2} \quad (9a)$$

This value is in agreement with Timshenko's conclusion, obtained in a different way (Timshenko 2010).

If the vertical axis is "x", the above equation can be expressed with a mean temperature over the horizontal cross sectional area:

$$T_{ref} = \frac{1}{H} \frac{1}{A} \int_H \int_A T dA dx \quad (9b)$$

(ii) The fully developed flow between two walls of uniform temperatures $T_1 > T_2$ is another common convection problem.

- in natural convection, this configuration corresponds to an enclosure of infinite height, with a temperature independent of x . Then:

$$T_{ref} = \frac{1}{A} \int_A T dA \quad (9c)$$

i.e. T_{ref} is the space-averaged temperature over the horizontal cross-section area of the flow.

The reference density ρ_{ref} results as the mean value in a slice of unitary height, such as:

$$m = m_{ref} = \rho_{ref} A \times 1 = \rho_{ref} A$$

- in mixed convection, nothing has to be changed in the previous reasoning, and definition (9c) is conserved.

The reference temperature, as defined by eqs. (9a,b,c), was sometimes empirically defined in the literature, and even estimated based on mathematical (Barletta and Zanchini 1999) or symmetry considerations (Padet 2010, ch.6). Here, its expression was derived based simply on physics and is equally valid for steady or dynamic regimes.

SIMILARITY AND CONVECTION

Similarity criteria in mixed convection

Since long time ago, the non-dimensionalization of governing equations became normal, inducing sometimes confusion between dimensionless numbers and similarity criteria. A similarity criterion is a number without dimension, but a dimensionless number is not necessarily a similarity criterion. For example, the Nusselt and Rayleigh numbers are not similarity criteria, although they are frequently treated as such (Padet 2011, ch.2).

By considering only the momentum conservation equation in convection problems, a rigorous similarity approach brings up the dimensionless expressions named as Richardson (*Ri*), Reynolds (*Re*), Euler (*Eu*), and Grashof (*Gr*) numbers (in equations (10) and (11) below, the symbol ⁺ denotes dimensionless values, the LHS terms representing momentum fluxes and the RHS terms being momentum sources, i.e. forces applied to the fluid).

- *In mixed convection*, the dimensionless equations and similarity criteria are:

$$\overline{\overline{\text{grad } V^+ \cdot V^+}} = Ri T^+ \overline{g^+} - Eu \overline{\text{grad } (p^*)^+} + \frac{1}{Re} \Delta V^+ \quad (10a)$$

- * *Ri*: relative to the buoyancy forces
- * *Eu*: relative to the pressure gradient
- * *1/Re*: relative to viscous forces

defined as:

$$Ri = \frac{g \beta \Delta T^0 L^0}{(V^0)^2} ; Eu = \frac{p^0}{\rho (V^0)^2} ; Re = \frac{V^0 L^0}{\nu} \quad (10b)$$

where superscript ⁰ means “reference value”; for instance, in internal flows, *L*⁰ is the hydraulic diameter, and *V*⁰ is the bulk velocity.

- *In natural convection*, the dimensionless equation and similarity criteria are:

$$\overline{\overline{\text{grad } V^+ \cdot V^+}} = T^+ \overline{g^+} + Eu_n \overline{\text{grad } (p^*)^+} + \frac{1}{Gr^{1/2}} \Delta V^+ \quad (11a)$$

- * *I*: relative to the buoyancy forces
- * *Eu_n*: relative to the pressure forces (id 10b)
- * *1/Gr^{1/2}*: relative to the viscous forces;

$$Gr = \frac{(V^0)^2 (L^0)^2}{\nu^2} \quad (11b)$$

where *V*⁰ = (gβΔ*T*⁰*L*⁰)^{1/2} in *Eu_n* and *Gr*.

- *In pure natural convection* (defined as the special case where $\overline{\text{grad } p^*} = 0$), eq. (11a) reduces to:

$$\overline{\overline{\text{grad } V^+ \cdot V^+}} = T^+ \overline{g^+} + \frac{1}{Gr^{1/2}} \Delta V^+ \quad (11c)$$

It can be noticed here that, as regard to similarity, a natural convection flow inside a channel could be considered as a case of mixed convection, since the bulk velocity inside this channel can be used as reference velocity V⁰.

Comparison of sources

The first question suggests a comparison of the momentum sources, namely the existing forces in the fluid flow. A similarity criterion *Γ* is defined as (Padet 2011, ch. 2):

$$\Gamma = \frac{\text{reference flux at the source}}{\text{reference flux transported by the fluid}} \quad (12a)$$

Then, for two sources (forces) denoted as 1 and 2, the denominator being the same:

$$\frac{\Gamma_1}{\Gamma_2} = \frac{\text{reference flux at source 1}}{\text{reference flux at source 2}} \quad (12b)$$

With regards to eq. (10), the order of magnitude of the ratio between the buoyancy forces and the viscosity forces is given by:

$$\frac{Ri}{1/Re} = Ri Re \quad (12c)$$

where the product *Ri Re* is named the *buoyancy ratio* (or *coefficient*). It is then expected to have this coefficient in the ratio $\overline{a^2}/\overline{c^2}$ of the terms in eq. (6).

In the same manner, the ratio between the buoyancy forces and the pressure forces $\overline{a^2}/\overline{b^2}$ will be expressed as a function of the product:

$$Ri/Eu \quad (12d)$$

Then, contrary to what is usually stated in the literature, the Richardson number alone is not representative for the Archimedes forces significance in the momentum balance, this role belonging to the combinations Ri Re or Ri/Eu. This confusion comes from the classical choice for the reference pressure in the dimensionless values, namely, a dynamic pressure: in this case, the Euler number is equal

to unity. Acceptable in forced convection, such a choice becomes illicit in mixed convection (as it will be shown for a particular case below).

Moreover, a second reason for such a confusion lies in the fact that another dimensionless number, also called Richardson number, is used for characterizing the stability of a stratified fluid, especially in micrometeorology (Padet 2013, Rubin and Atkinson 2001, Sutton 1953).

A similar reasoning applied to the energy conservation equation shows that the order of magnitude of the ratio between the viscous dissipation and the heat flux is given by the Brinkman number (Br), which is the product between the Eckert number (Ec) and the Prandtl number (Pr), as shown later in this paper.

Non-continuity of the description from MC to NC

For the third question, it is necessary to reconsider the definition of certain reference values used in the similarity criteria, in parallel with the questioning whether the Grashof number should be maintained; this number materializes the discontinuity between MC and NC, which can be observed in eqs. (10) and (11).

METHODS

FUNDAMENTALS

In order to clarify the questions raised above, it is useful to consider first sufficiently simple configurations that allow for analytical solutions. Thus, a *fully developed two-dimensional flow in the x-direction* is chosen, for which:

$$\left. \begin{aligned} \partial U / \partial x &= 0 \\ \partial T / \partial x &= 0 \\ \partial p^* / \partial x &= cte ; \partial p^* / \partial y = 0 \end{aligned} \right\} \quad (13)$$

These assumptions are valid for fluid flows between infinite walls 1 and 2, having uniform temperatures T_1 and T_2 . This type of system is often treated theoretically (Aung and Worku 1986, Barletta 1998, Barletta and Zanchini 1999, Boulama and Galanis 2004, Guyon et al. 1991, Bayazitoglu et al. 2007).

DEFINITION OF T_{ref} : A CONSEQUENCE

It is noticed from eqs. (8) and (9) that the balance of the buoyancy forces is zero over the fluid domain:

$$\int_A \rho_{ref} \beta (T - T_{ref}) dA = 0 \quad (14)$$

By using eqs. (13), the integration of eq. (6) over A becomes:

$$0 = -\frac{l}{\rho} \int_A \frac{dp^*}{dx} dA + \nu \int_A \Delta U dA \quad (15a)$$

If L is the perimeter of A , dl the elementary length of L , and τ_s the surface viscous shear stress, the balance of the forces applied to the fluid is the following:

$$A \frac{dp^*}{dx} = - \int_L \tau_s dl \quad (15b)$$

When the flow is confined between walls of different lengths, L_1 and L_2 , each with uniform shear stress, it can be written:

$$L_1 \tau_{s1} + L_2 \tau_{s2} = -A \frac{dp^*}{dx} \quad (15c)$$

It means that, the balance of the buoyancy forces being zero over the fluid domain, the pressure forces are balanced by the surface viscous forces, like in FC. At last, in pure NC, eq.(15c) is reduced to:

$$L_1 \tau_{s1} + L_2 \tau_{s2} = 0 \quad (15d)$$

A consequence of this property is the coexistence of an upward flow and a downward flow, due to the fact that the wall stresses are of opposite signs (Jeng et al. 1992).

EXAMPLE: planar geometry, steady regime

The system chosen for exemplification (Aung and Worku 1986, Barletta and Zanchini 1999, Guyon et al. 1991, Padet 2010), is a laminar non-isothermal flow between two parallel vertical plates, placed

at $y = 0$ and $y = e$ and having imposed uniform temperatures T_1 and T_2 , respectively (fig. 1). The fluid is isochor and the flow is globally ascendant (along x -coordinate) with a mean (bulk) velocity V_m . In addition, the driving pressure gradient dp^*/dx is constant (eq. 13).

TEMPERATURE AND VELOCITY FIELDS

The plates are assumed infinite, and the temperature and velocity fields are x -independent. So, the immediate result is that the y -velocity component, V_y , is zero everywhere. The conservation equations (6) and (7) become:

$$0 = g \beta(T - T_{ref}) - \frac{1}{\rho} \frac{dp^*}{dx} + \nu \frac{d^2 U}{dy^2} \quad (16a)$$

$$\frac{d^2 T}{dy^2} = 0 \quad (16b)$$

with boundary conditions:

$$\left. \begin{aligned} y = 0 & : T = T_1; U = 0 \\ y = e & : T = T_2; U = 0 \end{aligned} \right\} \quad (16c)$$

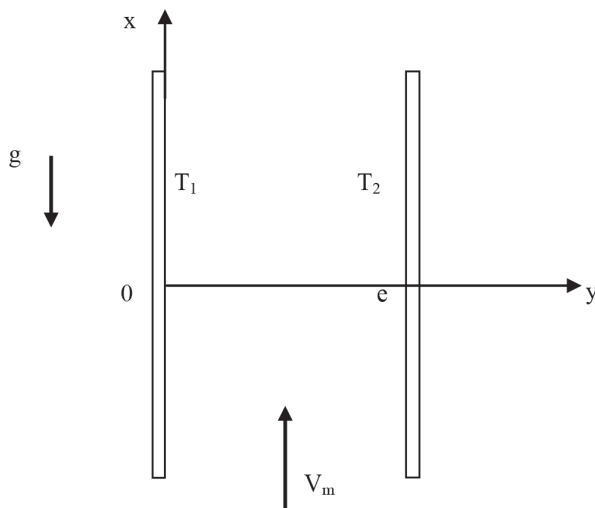


Figure 1 - Plane fully developed flow; $T_1 - T_2 = \Delta T > 0$.

Thus, the resulting temperature distribution is linear. By denoting $T_1 - T_2 = \Delta T$, the solution is as follows:

$$T = -\frac{\Delta T}{e} y + T_1 \quad (17)$$

According to eq. (9), here the reference temperature related to the Boussinesq approximation is:

$$T_{ref} = \frac{T_1 + T_2}{2} \quad (18)$$

Under these conditions, the velocity field derived from eq. (16a) is given by a third order polynomial:

$$U = \frac{g\beta}{6\nu e} \Delta T y^3 + \left(-\frac{g\beta}{4\nu} \Delta T + \frac{1}{2\mu} \frac{dp^*}{dx} \right) y^2 + \left(\frac{g\beta}{12\nu} \Delta T - \frac{1}{2\mu} \frac{dp^*}{dx} \right) ey \quad (19)$$

The Boussinesq approximation allows the separation in eq. (19) of the terms dp^*/dx with (characteristic to the pure FC) and the terms including $g\beta$ (characteristic to the pure NC). Thus, the velocity can be written as the sum of a FC component and a NC component:

$$U = U_f + U_n \quad (20a)$$

where:

$$U_f = \frac{1}{2\mu} \frac{dp^*}{dx} (y^2 - ey) \quad (20b)$$

$$U_n = \frac{g\beta \Delta T}{\nu} \left(\frac{y^3}{6e} - \frac{y^2}{4} + \frac{ey}{12} \right) \quad (20c)$$

The bulk velocity is computed based on eq. (19) as:

$$V_m = \frac{1}{e} \int_0^e U dy = -\frac{e^2}{12\mu} \frac{dp^*}{dx} \quad (21a)$$

and the volumetric flow rate is given by:

$$q_v = V_m e = -\frac{e^3}{12\mu} \frac{dp^*}{dx} \quad (21b)$$

The velocity field can also be expressed as a function of the bulk velocity V_m :

$$U = \frac{g\beta}{6\nu e} \Delta T y^3 - \left(\frac{g\beta}{4\nu} \Delta T + \frac{6V_m}{e^2} \right) y^2 + \left(\frac{g\beta}{12\nu} \Delta T + \frac{6V_m}{e^2} \right) ey \quad (22)$$

The velocity gradients at the wall surfaces can now be easily calculated:

- at the hot surface ($y = 0$):

$$\left(\frac{dU}{dy} \right)_{y=0} = \frac{g\beta}{12\nu} \Delta T e + 6 \frac{V_m}{e} \quad (23a)$$

- at the cold surface ($y = e$):

$$\left(\frac{dU}{dy} \right)_{y=e} = \frac{g\beta}{12\nu} \Delta T e - 6 \frac{V_m}{e} \quad (23b)$$

RESULTS

A first remark is that the thermal buoyancy does not affect the bulk velocity and the flow rate. This means that the complementary flow initiated at the hot surface by the buoyancy forces is exactly compensated by the flow deficit caused at the cold surface.

The observation of eq. (20b) allows for the following additional remarks:

- in pure FC, without buoyancy forces, the velocity profile U_f is the classical second order polynomial
- in pure NC, the velocity profile U_n is anti-symmetric and crosses zero at the channel centerline (fig. 2), showing a reverse flow at the cold surface, so that the bulk velocity is zero.

The critical case that indicates the transition to a reverse flow may be inferred from eq. (23b) by imposing a zero velocity gradient at the cold surface. The critical bulk velocity yields, thus, the expression:

$$V_{mc} = \frac{g\beta \Delta T e^2}{72\nu} \quad (24a)$$

It is also interesting to consider the mechanical frame of the flow. Then, the friction at the cold wall (23b) can be written as:

$$\tau_{se} = \mu \left(\frac{dU}{dy} \right)_{y=e} = \frac{\rho g\beta \Delta T}{12} e - 6 \frac{\mu V_m}{e} \quad (24b)$$

As for the velocity field, it can be observed that the total friction force is the sum of a force τ_{sn} due to the gravity field (NC) and of a force τ_{sf} related to the bulk velocity (FC):

$$\tau_{se} = \tau_{sn} - \tau_{sf} \quad (24c)$$

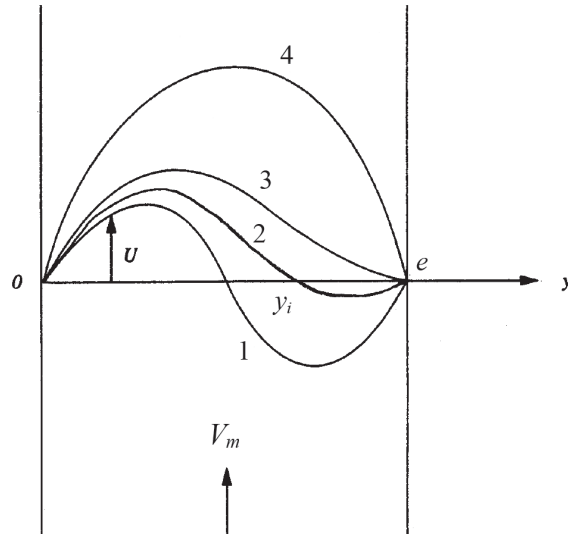


Figure 2 - Velocity fields – 1. Pure NC; 2. Reverse flow ($RiRe > 288$); 3. $RiRe = (RiRe)_c = 288$; 4. Pure FC.

Then a reverse flow appears when the buoyancy friction becomes dominant, the transition corresponding to $\tau_{se} = 0$.

Similarly, it can be written that the friction factor $C_f/2$ includes a buoyancy component and a forced convection component, as:

$$\frac{1}{2} C_{fe} = \frac{\tau_{se}}{\rho V_m^2} = \frac{1}{2} C_{fn} - \frac{1}{2} C_{ff} \quad (25a)$$

In the present case, the hydraulic diameter $D_h = 2e$. According to eq. (24b), eq. (24a) becomes:

$$\frac{1}{2} C_{fe} = \frac{1}{24} \frac{g\beta \Delta T D_h}{V_m^2} - 12 \frac{\mu}{\rho V_m D_h} \quad (25b)$$

This expression contains the Richardson and Reynolds numbers (eq. 10b, with $V^0 = V_m, L^0 = D_h$), and can be written as:

$$\frac{1}{2} C_{fe} = \frac{Ri}{24} - \frac{12}{Re} \quad (25c)$$

The value $C_{fe} = 0$ characterizes a limit between two kinds of flows; the buoyancy ratio has then a critical value $(RiRe)_c$ given by:

$$\frac{Ri}{24} - \frac{12}{Re} = 0 \Rightarrow (RiRe)_c = 288 \quad (25d)$$

where:

$$RiRe = \frac{g\beta\Delta T D_h^2}{V_m v} = \frac{g\beta\Delta T}{V_m v} 4e^2 \quad (25e)$$

It results that the existence of a reverse flow is governed solely by the thermal buoyancy ratio, and this coefficient has a critical value for the system considered here: $(RiRe)_c = 288$.

With different boundary conditions, (Desrayaud and Lauriat 2009) and (Kholai et al. 2010), have obtained numerical results indicating a single critical value a little bit less than 300. This result could then not be restricted only to the case considered in the present study.

A flash-back to the reference temperature seems now useful. As regard to the flow between two parallel walls, some authors choose for T_{ref} the value corresponding to the cold wall: $T_{ref} = T_2$. But, then, *the equilibrium condition expressed by eq. (15c) is no longer respected*. In particular, for pure NC (eq. 15d), the calculation leads to a non-zero volumetric rate (the upward flow rate is superior to the downward flow rate), the flow velocity symmetry being destroyed. It was exactly this symmetry condition, which is physically expected, that was used by Padet (Padet 2010) in reasoning the reference temperature defined by eq. (18). Therefore, the Boussinesq approximation must be used with precaution, especially since the use of nondimensional values aids in hiding this problem.

ENERGY DISSIPATION AND ENTROPY PRODUCTION

For a long period of time, the energy system optimization was based on energy criteria. However,

more recently, the interest moved towards exergy or entropy analysis. In particular, the type of analysis based on entropy production was born in association with irreversible thermodynamics (Glansdorf and Prigogine 1971). But this theory, conceptually very rich, did not offer to the engineering sciences the expected benefits, especially because of its complexity and lack of practical impact. Fortunately, it was re-launched by the works of (Bejan 1994, 1996) who used a simpler and more pragmatism approach for the optimization based on entropy production limitation. Most often, previous works are related to forced convection problems (Herpe 2007, Herpe et al. 2007, Mladin et al. 2006, Stanciu et al. 2005), and rarely to mixed convection problems (Ben Mansour et al. 2005, Zanchini 1998). The extension to mixed convection will be combined in this study with the research of usable criteria to distinguish between FC, MC and NC heat transfer.

It is assumed in what follows that the viscous dissipation influence on the temperature field is negligible, fact justified in the considered examples by Brinkman numbers [eq. (55c)] of order 10^{-7} . As a consequence, the temperature distribution will remain linear [eq. (17)], decoupled from the velocity field.

Viscous dissipation

The volumetric (local) dissipation source Φ''' is defined by:

$$\Phi'''(y) = \mu \left(\frac{dU}{dy} \right)^2 \quad [W/m^3] \quad (26)$$

The surface dissipation Φ'' results from integration over the flow width:

$$\Phi'' = \mu \int_0^e \left(\frac{dU}{dy} \right)^2 dy \quad [W/m^2] \quad (27a)$$

By using eq. (22) for the velocity distribution, the above expression becomes:

$$\Phi'' = 1.39 \times 10^{-3} \frac{(\rho g \beta \Delta T)^2 e^3}{\mu} + 12 \mu \frac{V_m^2}{e} \quad [W/m^2] \quad (27b)$$

indicating that dissipation takes place due to the buoyancy effects (term with ΔT), in addition to the dissipation that occurs in the isothermal flow.

The surface dissipation Φ'' can also be expressed in dimensionless form by adopting as a reference the surface dissipation for isothermal flow, Φ''^o :

$$\Phi''^o = 12 \mu \frac{V_m^2}{e} \quad [W/m^2] \quad (28)$$

Then, the dimensionless viscous dissipation Φ^+ is:

$$\Phi^+ = \frac{\Phi''}{\Phi''^o} = \frac{1.39 \times 10^{-3}}{12} \frac{(\rho g \beta \Delta T)^2 e^4}{\mu^2 V_m^2} + 1 \quad (29a)$$

It is interesting to note that eq. (29a) includes the thermal buoyancy ratio (25e), and Φ^+ may also be written as:

$$\Phi^+ = 7.23 \times 10^{-6} (Ri Re)^2 + 1 \quad (29b)$$

Entropy production

When considering the entropy production in mixed convection, a distinction must be made between the viscous source Φ'' (27a) that produces "viscous entropy" σ_v'' [W/m^2K], and the temperature gradient between the walls, that originates "thermal entropy", σ_{th}'' [W/m^2K].

The viscous entropy produced in a horizontal section of the flow (fig. 1) is calculated using the local dissipation source of eq. (26):

$$\sigma_v'' = \int_0^e \frac{\Phi'''(y)}{T(y)} dy \quad [W/m^2K] \quad (30a)$$

where T is the absolute temperature.

In first approximation, for mild temperature differences ($\Delta T/T < 10\%$), the production of viscous entropy may be expressed by reporting

the total dissipation of eq. (27b) to the reference temperature T_{ref} :

$$\sigma_v'' = \frac{\Phi''}{T_{ref}} = 1.39 \times 10^{-3} \frac{(\rho g \beta \Delta T)^2 e^3}{\mu T_{ref}} + 12 \frac{\mu V_m^2}{e T_{ref}} \quad (30b)$$

Once more, the Boussinesq approximation allows for distinguishing the terms related to either the buoyancy or to the fluid viscosity.

The local thermal entropy production σ_{th}'' [W/m^2K] is given by:

$$\sigma_{th}''(y) = \frac{\lambda}{T^2} \left(\frac{dT}{dy} \right)^2 \quad (31)$$

For the example considered here, the temperature gradient is constant and equal to $(\Delta T/e)$, leading to the total thermal entropy production in a horizontal section of the flow σ_{th}'' [W/m^2K]:

$$\sigma_{th}'' = \int_0^e \sigma_{th}'''(y) dy = \lambda \left(\frac{\Delta T}{e} \right)^2 \int_0^e \frac{1}{T^2} dy \quad (32a)$$

This expression is also evaluated in a first approximation, by assuming that:

$$\sigma_{th,approx}'' = \frac{\lambda}{e} \left(\frac{\Delta T}{T_{ref}} \right)^2 \quad (32b)$$

The total entropy production σ_{tot}'' [W/m^2K] results then as the sum:

$$\sigma_{tot,approx}'' = \sigma_v'' + \sigma_{th,approx}'' \quad (33a)$$

or, using eqs. (30) and (32b):

$$\sigma_{tot,approx}'' = 1.39 \times 10^{-3} \frac{(\rho g \beta \Delta T)^2 e^3}{\mu T_{ref}} + \frac{12 \mu V_m^2}{e T_{ref}} + \frac{\lambda}{e} \left(\frac{\Delta T}{T_{ref}} \right)^2 \quad (33b)$$

Figure 3 presents all the terms of eq. (33b) for a laminar water flow at a reference temperature of $20^\circ C$. It is shown that the total entropy is dominated by buoyancy at large values of separation distances e , and by heat transfer at small values of e .

Numerical calculations indicated that the approximate value in eq. (33b) of σ''_{tot} is very close to its exact value (the differences are inferior to 1% for ΔT less than 50 K). In addition, it is clear that the contribution of the second term, dependent on the mean velocity V_m , is negligible for all e values, under laminar flow conditions, as initially assumed.

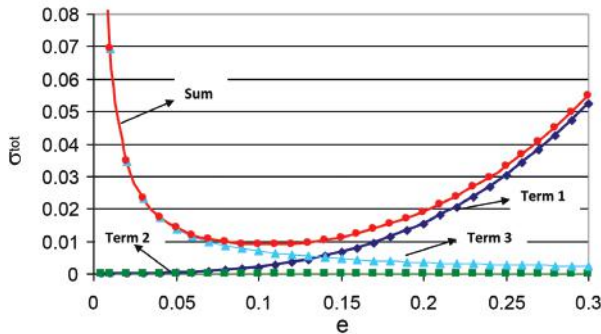


Figure 3 - Buoyancy entropy (1), isothermal entropy (2), thermal entropy (3), total (4). Water flow; $T_{ref} = 20^\circ C$; $\Delta T = 10^\circ C$; $V_m = 0.001 \text{ m/s}$ [eq.(33b)].

SELECTION CRITERIA FOR CF/CM/CN

Different criteria to distinguish between forced, mixed and natural convection (Padet 2010, Padet et al. 2004, 2005, 2009) are examined here. They are based either on the acting forces, eq. (16a), on the kinetic energy components (velocity field in eqs. (19) and (20b)), on the dissipation, eq. (27b), or on the entropy production, eq. (33b).

1st Criterion: comparison of the gravitational and viscous forces

By using eq.(16a), the terms (a) and (c) in eq. (6) become:

$$a = g \beta (T - T_{ref}); c = \nu \frac{d^2 U}{dy^2} \tag{34a}$$

or, as functions of y , with eqs.(17)-(19):

$$a = g \beta \Delta T \left(\frac{1}{2} - \frac{y}{e} \right) \tag{34b}$$

$$c = -g \beta \Delta T \left(\frac{1}{2} - \frac{y}{e} \right) - \frac{12 \nu V_m}{e^2} \tag{34c}$$

The squared averages over the range $[0, e]$ are, respectively:

$$\overline{a^2} = \frac{1}{12} g^2 \beta^2 \Delta T^2 \tag{35}$$

$$\overline{c^2} = \frac{1}{12} g^2 \beta^2 \Delta T^2 + 144 \frac{\nu^2 V_m^2}{e^4} \tag{36}$$

It is noted that, like it was concluded before, their ratio P^2 is expressed solely as a function of the thermal buoyancy coefficient $Ri Re$ defined by eq. (25e):

$$P^2 = \frac{\overline{a^2}}{\overline{c^2}} = \frac{(Ri Re)^2}{(Ri Re)^2 + 27648} \tag{37a}$$

or:

$$Ri Re = \frac{166.28 P}{\sqrt{1 - P^2}} \tag{37b}$$

The latter equation indicates that P is a number between 0 and 1, the limiting situations corresponding to:

$$\begin{aligned} P = 0 & \Rightarrow Ri Re = 0 & : & \text{FC} \\ P \rightarrow 1 & \Rightarrow Ri Re \rightarrow \infty & : & \text{NC} \end{aligned} \tag{37c}$$

In order to establish some practical limits, it can be admitted that the convection is mainly natural if the buoyancy forces are less than 5% relative to the viscous forces (when $P < 0.05$), and that the convection is mainly forced when they represent at least 95% ($P > 0.95$). Equation (37b) leads then to the following selection criteria:

$$\begin{aligned} P < 0.05 & \Rightarrow Ri Re < 8.3 & \text{for FC} \\ P > 0.95 & \Rightarrow Ri Re > 505.9 & \text{for NC} \end{aligned} \tag{38}$$

2nd Criterion: comparison of the gravitational and pressure forces

Referring again to eqs. (6) and (16a), terms (a) and (b) of eq. (6) are divided this time using also eq. (35) and noting from eq. (21a) that:

$$b = \frac{12 \nu}{e^2} V_m = cst \tag{39}$$

Then, the ratio Γ^2 of squared mean values is given by:

$$\Gamma^2 = \frac{\overline{a^2}}{b^2} = \frac{(Ri Re)^2}{27648} \tag{40a}$$

Again, this criterion is also a function of $RiRe$ only (as stated in eq. 12d). In fact, in this particular case where the pressure gradient may be expressed as a function of the bulk velocity, eq. (21b), the Euler number becomes (p^{*o} being a reference pressure, equal here to $(dp^* / dx)D_h$):

$$Eu = \frac{p^{*o}}{\rho V_m^2} = \frac{48}{Re} \tag{40b}$$

so that the ratio Ri/Eu is proportional to $RiRe$.

Equation (40a) may be re-written as:

$$Ri Re = 166.28 \Gamma \tag{40c}$$

showing that the equilibrium of gravitational and pressure forces ($\Gamma = 1$) is reached when $RiRe = 166.28$ and not when $Ri = 1$, as sometimes stated.

But differently to the ratio P in criterion 1, the ratio Γ varies from zero to infinity, as $a = 0$ in FC and $b = 0$ in NC. The distinguishing values are then adapted as follows: it is admitted that, in FC, the buoyancy forces represent maximum 5% of the pressure forces ($\Gamma < 0.05$), or in NC, the pressure forces are less than 5% of buoyancy forces ($b < 0.05\sqrt{\overline{a^2}}$, or $\Gamma > 20$). Equation (40a) provides the following bounds:

$$\begin{aligned} \Gamma < 0.05 &\Rightarrow Ri Re < 3325.6 \text{ for NC} \\ \Gamma > 20 &\Rightarrow Ri Re > 8.31 \text{ for FC} \end{aligned} \tag{41}$$

For forced convection, the boundary is about the same as that indicated by the first criterion. However, for natural convection, it is pushed at a much higher value for $RiRe$.

3rd Criterion: comparison of the friction constraints at the wall surfaces

Another selection criterion may be based on the comparison of the friction constraints at the surfaces, given by eqs. (23a) and (23b). These equations indicate that the velocity gradients are equal in magnitude and of opposite signs in FC, equal in magnitude and of same sign in NC (see fig. 2). Their relative difference may also be used for comparison between the two types of convection. Several trials indicated that a value of 10% for this difference would represent a good compromise in distinguishing FC and NC. The following mathematical expression is used for the new ratio K_f :

$$\frac{(dU / dy)_{y=0} \pm (dU / dy)_{y=e}}{(dU / dy)_{y=0}} = K_f \tag{42}$$

with a positive sign for FC and a negative one for NC.

By using the velocity profile, the ratio becomes again a function of $RiRe$ only:

- For dominant NC:

$$K_f = \frac{1}{\frac{Ri Re}{576} + 0.5} \tag{43a}$$

or

$$Ri Re = \frac{576}{K_f} - 288$$

If admitted $K_f < 0.1$:

$$Ri Re > 5472 \text{ for NC} \tag{43b}$$

- For dominant FC:

$$K_f = \frac{1}{\frac{144}{Ri Re} + 0.5} \tag{44a}$$

or

$$Ri Re = \frac{144}{\frac{1}{K_f} - 0.5}$$

If admitted $K_f < 0.1$:

$$RiRe < 15.2 \text{ for FC} \tag{44b}$$

Noteworthy, the values of $RiRe$ limiting FC/MC and MC/NC are of the same order of magnitude as in previous criteria.

4th Criterion: comparison based on kinetic energy

Another option to distinguish between different types of convection starts from the velocity distribution (20b), where the Boussinesq approximation allows the dissociation of the component U_n generated by buoyancy and the component U_f generated by the pressure gradient. Similarly, kinetic energy can be divided into K_{en} and K_{ef} defined as:

$$K_{en} = \frac{\rho}{2e} \int_0^e U_n^2 dy, K_{ef} = \frac{\rho}{2e} \int_0^e U_f^2 dy \tag{45}$$

The comparison of these two terms is in fact the comparison of the mean squared values of U_n and U_f . A new ratio is defined as K_e :

$$K_e = \frac{K_{en}}{K_{ef} + K_{en}} \tag{46}$$

which is equal to:

$$K_e = \frac{RiRe}{\sqrt{580608 + (RiRe)^2}}, \text{ or:} \tag{47}$$

$$RiRe = \frac{762 K_e}{\sqrt{1 - K_e^2}}$$

Like coefficient P , K_e is a parameter situated in the range $[0, 1]$. Even the expressions (39b) and (49) are similar.

By adopting the practical convention that, for natural convection, the gravitational kinetic energy represents at least 95% of the total kinetic energy ($K_e > 0.95$), while, for forced convection it represents less than 5% ($K_e < 0.05$), it results in:

$$\begin{aligned} RiRe > 2318 & \text{ for NC} \\ RiRe < 38.15 & \text{ for FC} \end{aligned} \tag{48}$$

5th Criterion: comparison based on the dissipation

The mechanical energy dissipation studied earlier may also be used to select between FC/MC/NC conditions. By looking at the equations (27) and (29), two ways of defining a selection criterion are apparent: compare the dissipation caused by the gravitational field (term with ΔT or $RiRe$) with the dissipation in an isothermal flow, or with the total dissipation.

a) First option leads to (see eq. (29)):

$$\begin{aligned} K_d &= \frac{7.23 \times 10^{-6} (RiRe)^2}{I} \\ &= 7.23 \times 10^{-6} (RiRe)^2 \end{aligned} \tag{49a}$$

This criterion may take values in the range $[0, +\infty]$. Then, it may be agreed that the convection is of forced type if the gravitational dissipation is less than 5% relative to the isothermal dissipation (i.e. $K_d < 0.05$), or vice-versa, that the convection is natural if the latter is less than 5% relative to the former (i.e. $K_d > 1/0.05 = 20$). And since equation (49a) may be rewritten as:

$$RiRe = 372 \sqrt{K_d} \tag{49b}$$

it is immediately obtained the following:

$$\begin{aligned} K_d > 20 & \Rightarrow RiRe < 1663 \text{ for NC} \\ K_d < 0.05 & \Rightarrow RiRe > 83 \text{ for FC} \end{aligned} \tag{50}$$

b) Second option leads to the criterion:

$$K_{dt} = \frac{7.23 \times 10^{-6} (RiRe)^2}{7.23 \times 10^{-6} (RiRe)^2 + I} \tag{51a}$$

which varies in the interval $[0, 1]$. It is then reasonable to admit that the convection is forced if the gravitational dissipation is less than 5% of the total (i.e. $K_{dt} < 0.05$), and natural when the same term represents more than 95% relative to the total dissipation (i.e. $K_{dt} > 0.95$). Equation (51a) may be expressed as:

$$RiRe = \left[\frac{I}{7.23 \times 10^{-6} \frac{K_{dt}}{1 - K_{dt}}} \right] \tag{51b}$$

which results in the following separation domains:

$$\begin{aligned} K_{dt} > 0.95 &\Rightarrow Ri Re < 1621 \quad \text{for } NC \\ K_{dt} < 0.05 &\Rightarrow Ri Re > 85 \quad \text{for } FC \end{aligned} \quad (52)$$

A comparison between the results associated with the first and the second options indicates that they are practically equivalent.

6th Criterion: comparison based on the entropy production

Another selection criterion occurs naturally as a continuity of the study presented sooner, where the entropy production is quantified.

a) A first possibility is to compare in eq. (33b) buoyancy entropy (first term) and isothermal flow entropy (second term). *The resulting ratio is exactly the criterion K_d already analyzed with regards to dissipation. Therefore, it is not necessary to define it again in relation to entropy production.*

b) A second approach means dividing the buoyancy entropy to the total entropy production:

$$K_S = \frac{1.39 \times 10^{-3} \frac{(\rho g \beta \Delta T)^2 e^3}{\mu T_{ref}}}{1.39 \times 10^{-3} \frac{(\rho g \beta \Delta T)^2 e^3}{\mu T_{ref}} + \frac{12 \mu V_m^2}{e T_{ref}} + \frac{\lambda}{e} \left(\frac{\Delta T}{T_{ref}} \right)^2}$$

or:

$$\frac{1}{K_S} = 1 + 8.63 \times 10^3 \frac{\mu^2 V_m^2}{(\rho g \beta \Delta T)^2 e^4} + \frac{1}{T_{ref}} \frac{\lambda \mu}{1.39 \times 10^{-3} (\rho g \beta)^2 e^4} \quad (53a)$$

This last expression may be equally rewritten in dimensionless form as:

$$\frac{1}{K_S} = 1 + \frac{1}{(Ri Re)^2} \left[1.38 \times 10^5 + 0.115 \times 10^5 \frac{\Delta T}{T_{ref}} \frac{1}{Br} \right] \quad (53b)$$

where:

$$Br = \mu V_m^2 / \lambda \Delta T = Pr Ec \quad (53c)$$

is the Brinkman number.

Approaching this criterion with benchmarks as $K_S = 0.05$ and $K_S = 0.95$, like it was done with K_{dt} , would lead to expressions that are hard to manipulate, because of the large number of dimensionless parameters, including in particular the ratio $\Delta T/T_{ref}$ (Bejan 1994). It seems then reasonable to push away the entropy related criteria when it comes to distinguishing among the three convection modes.

Additional remarks

A general remark over the proposed criteria is that the benchmark values of the product $RiRe$ have the same order of magnitude. Therefore, *a priori*, each of them seems acceptable.

On the other hand, it is noticed that all the analyzed criteria include the value $RiRe = 288$ indicating the occurrence of a reverse flow on the cold wall, as well as the value $RiRe = 166.28$ which characterizes the equilibrium between pressure and buoyancy forces. It can be seen that the condition for $RiRe = 288$ is achieved for $P = 0.866$ (1st criterion), $\Gamma = 1.73$ (2nd criterion) or for $K_e = 0.35$ (3rd Criterion), while the condition $RiRe = 166.28$ ($\Gamma = 1$, 2nd criterion) corresponds to $P = 0.707$ (1st criterion) or to $K_e = 0.213$ (3rd criterion).

The definition interval for the criteria P , Γ , etc may represent the base for a first selection. In fact, it is more comfortable to use values in the range $[0, 1]$ than in the range $[0, +\infty]$. This choice leads to the removal of the 2nd and 5th criteria (Γ and K_{dt} , respectively).

Among the left criteria, P (1st criterion) and K_f (3rd criterion) are based on the present forces, while K_e (4th criterion) and K_{dt} (5th criterion) are based on mechanical energy. In many applications, the latter are to be preferred. It results then that:

- with K_e : MC $\Leftrightarrow 38 < RiRe < 2318$
- with K_{dt} : MC $\Leftrightarrow 85 < RiRe < 1621$

The above values being just benchmarks, and given the fact that simple values are easy to remember and use, *it is proposed a compromise between the two criteria and admitted that, in*

practical applications, the mixed convection is characterized by the condition:

$$MC \Leftrightarrow 50 < RiRe < 2000 \quad (54)$$

Otherwise, unless marked precision is required, buoyancy may be neglected if the product $RiRe$ is inferior to 50 (FC is dominant), and pressure gradient is negligible if the product $RiRe$ is superior to 2000 (NC is dominant).

Numerical example

Consider, for instance, water flow at 30°C ($\beta \approx 4 \cdot 10^{-4} K^{-1}$; $\nu \approx 8.5 \cdot 10^{-7} m^2/s$), with $e = 2 \cdot 10^{-2} m$, $V_m = 3 \cdot 10^{-2} m/s$, $g = 9.81 m/s^2$. Then:

$$Re = 1412; Ri = 0.174 \Delta T; RiRe = 246 \Delta T$$

The corresponding temperature differences are, respectively:

$$RiRe = 50 : \Delta T = 0.2^\circ C$$

$$RiRe = 2000 : \Delta T = 8.13^\circ C$$

and the presence of a reverse flow ($RiRe > 288$) corresponds to:

$$\Delta T > 1.17^\circ C$$

CONTINUITY OF THE FC-MC-NC DESCRIPTION

Critics for two usual concepts and possible solutions

In the first part, it was underlined the problem raised by the dimensionless description of a flow when it comes close to a pure natural convection mode, as this brings up a discontinuity and imposes the replacement of Re and Ri by Gr . For the case presented here, the difficulty comes from the existence of a reverse flow for $RiRe > 288$, or of a recirculation zone in other cases. Then, the mean velocity, which is simply an arithmetic average, is no longer representative for the flow, or at least it only describes the global flow, ignoring the presence of two opposed fluxes. In particular, the mean flow is zero in pure natural convection.

This difficulty has effect on another concept, the bulk temperature, largely used as a reference to describe the convective heat transfer, and expressed by:

$$T_m = \frac{1}{V_m e} \int_0^e TU dy \quad (55)$$

such as $T_m \rightarrow \infty$ when $V_m \rightarrow 0$.

In fact, T_m is the temperature of an isothermal flow that would transport the same heat rate with the same mean velocity. As a result, *in pure NC, the transport of a finite energy with a zero velocity needs an infinite temperature*. This problem was brought up also by other authors, in particular by (Aung and Worku 1986).

Two approaches allow overcoming these difficulties. One approach, named bi-directional, separates the upward flow from the downward flow and considers, for example, only the upward flow. Another one may be based on the definition of a square mean flow velocity that would replace the mean velocity role in convection description. The two approaches are explored further.

Bi-directional approach

A way to re-establish the continuity in the dimensionless description of the convection is then to consider only the upward flow. This does not change anything if $RiRe < 288$. In the opposite situation, there is a location y_i where the flow changes direction (fig. 2), such as:

$$U(y_i) = 0 \quad (56a)$$

Equation (22) becomes now a simple second order equation with the roots:

$$y_i = e$$

$$y_i = \frac{e}{2} + 36 \frac{V_m \nu}{g \beta \Delta T e}$$

or:

$$y_i^+ = \frac{y_i}{e} = \frac{1}{2} + \frac{144}{Ri Re} \quad (56b)$$

For $RiRe > 288$, it is evident that $y_i < e$. Now, the following can be calculated:

- the upward flow rate:

$$q_{v\uparrow} = \int_0^{y_i} U(y) dy \tag{57a}$$

- the mean upward flow velocity:

$$V_{m\uparrow} = \frac{q_v}{y_i} \tag{57b}$$

The resulting flow rate is:

$$q_{v\uparrow} = \frac{g\beta\Delta T}{24\nu e} y_i^4 - \frac{g\beta\Delta T}{12\nu} y_i^3 - \frac{2V_m}{e^2} y_i^3 + \frac{g\beta\Delta T e}{24\nu} y_i^2 - \frac{3V_m}{e} y_i^2 \tag{58}$$

The Reynolds and Richardson numbers corresponding to the upward flow (having the hydraulic equivalent diameter $2y_i$) have now non-zero finite values in pure natural convection:

$$Re_{\uparrow} = \frac{2y_i V_{m\uparrow}}{\nu} = \frac{2q_{v\uparrow}}{\nu} \tag{59}$$

$$Ri_{\uparrow} = \frac{g\beta\Delta T 2y_i}{V_{m\uparrow}^2}$$

The Reynolds number is given now by:

$$Re_{\uparrow} = \frac{g\beta\Delta T}{12\nu^2 e} y_i^4 - \frac{g\beta\Delta T}{6\nu^2} y_i^3 - \frac{4V_m}{\nu e^2} y_i^3 + \frac{g\beta\Delta T e}{12\nu^2} y_i^2 + \frac{6V_m}{\nu e} y_i^2 \tag{60}$$

and, with $y_i^+ = y_i / e$:

$$Re_{\uparrow} = \frac{g\beta\Delta T e^3}{\nu^2} \left(\frac{(y_i^+)^4}{12} - \frac{(y_i^+)^3}{6} + \frac{(y_i^+)^2}{12} \right) \tag{61}$$

$$- \frac{2V_m e}{\nu} \left(2(y_i^+)^3 - 3(y_i^+)^2 \right)$$

(G₁)

(G₂)

The two terms G_1 and G_2 of eq. (61) are functions of $RiRe$ since y_i^+ is a function of this product. In addition, the factors that precede the parenthesis are nothing else, but $RiRe^2/8$ and Reynolds number Re , respectively:

$$Re_{\uparrow} = \frac{Ri Re^2}{8} G_1 (Ri Re) - Re G_2 (Ri Re) \tag{62}$$

It can be noticed that the Reynolds number of the upward flow is expressed only as a function of the Richardson and Reynolds numbers associated with the global flow, and there is no need to use the Grashof number. Of course, it is true that $Gr = Ri Re^2$, but there is no logical or physical reason to introduce Gr in the expression of eq. (62), given in addition the fact that Gr is not a similarity criterion in mixed convection: *Ri and Re are then sufficient to completely describe the whole convective domain.*

Two possible situations are now examined.

- **No reverse flow ($RiRe < 288$):**

$$y_i = e; \quad y_i^+ = 1; \quad G_1 = 0; \quad G_2 = -1; \tag{63}$$

resulting $Re_{\uparrow} \equiv Re$

The upward flow Reynolds number is identical to the classical one.

- **Pure NC:**

$$y_{in} = \frac{e}{2}; \quad y_{in}^+ = \frac{1}{2}; \quad G_{1n} = \frac{1}{192} \text{ and } G_2 = 0, \text{ resulting:}$$

$$Re_{n\uparrow} = \frac{Ri Re^2}{8 \times 192} = 6.51 \cdot 10^{-4} Ri Re^2 \tag{64a}$$

Equations (57b) and (58) allow, for the upward flow, the computation of velocity and flow rate:

$$V_{m,n\uparrow} = \frac{g\beta\Delta T e^2}{192\nu}; \quad q_{v,n\uparrow} = \frac{g\beta\Delta T e^3}{384\nu} \tag{64b}$$

And finally, the buoyancy ratio of the upward flow has the value, eq. (59):

$$(Ri Re)_{n\uparrow} = 192 \tag{64c}$$

The conclusions for the bidirectional approach are the following:

- *from FC to NC, the parameters of the upward flow keep finite values;*

- *a turning point still exists in the description for $RiRe = 288$;*

- the Grashof number is not needed, even in the pure NC.

It is noteworthy that analogue results are obtained when considering the friction factor.

Quadratic approach

Another way of avoiding the difficulties related to the definition of V_m is to consider square mean values. The velocity $\sqrt{\overline{U^2}}$ is then used, with:

$$\overline{U^2} = \frac{1}{e} \int_0^e U^2 dy \tag{65a}$$

It is noted then that the bulk velocity V_m is based on the mass flow rate, while the square mean velocity is based on the kinetic energy.

By using the local velocity decomposition (20), eq. (65a) becomes:

$$\overline{U^2} = \frac{1}{e} \int_0^e (U_n^2 + U_f^2 + 2 U_n U_f) dy \tag{65b}$$

The result can be summarized as:

$$\begin{aligned} \overline{U^2} = & \text{terms in } \left(\frac{\rho g \beta \Delta T e^2}{\mu} \right)^2 \\ & + \text{terms in } V_m^2 + \text{terms in } \frac{\rho g \beta \Delta T e^2 V_m}{\mu} \end{aligned} \tag{65c}$$

This allows the definition of a *quadratic Reynolds number* Re_Q :

$$Re_Q^2 = \frac{\overline{U^2} (2e)^2}{\nu^2} \tag{66a}$$

Which is of the form:

$$\begin{aligned} Re_Q^2 = & \text{terms in } Ri^2 Re^4 + \text{terms in } Re^2 \\ & + \text{terms in } Ri Re^3 \end{aligned} \tag{66b}$$

The turning point for $Ri Re = 288$ does not occur here and the Grashof number is still absent.

- In pure FC:

$Ri = 0$ and $U \equiv U_f$, resulting:

$$\sqrt{\overline{U_f^2}} = 1.095 V_m; \quad Re_{Qf} = 1.095 Re \tag{67}$$

The new quadratic Reynolds number does not coincide with the classical one. It is superior by about 10%.

- In pure NC:

In that case: $Re = 0$ and $U = U_n$, resulting:

$$Re_{Qn} = 1.44 \cdot 10^{-4} Ri Re^2 \neq 0 \tag{68}$$

The conclusions for the quadratic approach are:

- from FC to NC, the quadratic parameters keep finite values;
- there is a perfect continuity in describing FC-MC-NC;
- the Grashof number is not needed in the description.

Numerical example

Coming back to an isothermal flow, $Re = 1412$ corresponds to a flow rate of $q_v = 6.10^{-4} m^3/s$ (for a width $l = 1 m$).

In pure NC ($V_m = 0$), the same Reynolds number in the upward flow ($Re_{n\uparrow} = 1412$) can be obtained with a temperature gap of $\Delta T = 6.26^\circ C$, resulting in $Ri = 1.083$.

Then $V_{dn\uparrow} = 6.10^{-2} m/s$; $q_v = 6.10^{-4} m^3/s$: the velocity is two times the velocity of the isothermal flow, but as the cross section is divided by 2 ($e/2$ instead of e), the flow rate is not changed.

As for the quadratic approach, it leads to:

$$\sqrt{\overline{U_n^2}} = 6.6 \cdot 10^{-2} m/s; \quad Re_{Qn} = 311$$

It can be observed that the ratio "quadratic velocity" over $V_{dn\uparrow}$ is almost the same as in an isothermal flow (eq. 67).

The bulk temperature

The bulk temperature definition presented in eq. (55) may be adapted to both the bidirectional and quadratic approaches.

♣ The upward flow bulk temperature, as defined over the section $[0, y_i]$, becomes:

$$T_{m\uparrow} = \frac{I}{V_{m\uparrow} y_i} \int_0^{y_i} TU \, dy \quad (69)$$

If $RiRe < 288$ (no reverse flow, $y_i = e$, $V_{m\uparrow} \equiv V_m$), then $T_{m\uparrow} \equiv T_m$.

♦ The mean fluid temperature relative to the square mean velocity, is given by:

$$T_{mQ} = \frac{I}{(\sqrt{U^2})_e} \int_0^e TU \, dy \quad (70)$$

In particular, in forced convection, T_m is modified in the same proportion as Reynolds number, eq. (67):

$$T_{mQf} = \frac{I}{1.095} T_m \quad (71)$$

The analytical expressions of $T_{m\uparrow}$ and T_{mQ} are rather complex, but it is sufficient to indicate here that the mean fluid temperature has finite values in both situations. From the point of view of the raised question, they are thus equally satisfactory.

What approach to recommend?

A first general remark is that, in both approaches, a unifying formulation has been achieved for the entire convection domain and that this formulation includes only the Richardson and Reynolds numbers.

The summary of all advantages and disadvantages of each method follows.

♣ *The bidirectional approach*

- takes away the difficulties related to the definitions of the mean velocity V_m and of the mean fluid temperature T_m ;
- allows the distinction between uni- and bi-directional flow;
- does not imply any modification of usages for the uni-directional flow;
- is adaptable to boundary layer flows.

But

- it introduces a turning point between the uni- and bi-directional flow;

- it supposes the knowledge of y_i (where $U(y_i) = 0$).

♦ *The quadratic approach*

- also takes away the difficulties posed by V_m and T_m ;

- is also adaptable to the boundary layer flows.

But

- it is global, the uni- and bidirectional flows are no longer differentiated;

- it implies a significant modification of the current usages, V_m and Re being replaced by $\sqrt{U^2}$ and Re_Q respectively.

The above comparison seems to give some advantage to the bidirectional approach, which is more physical and more compatible with the usual formulations.

Expression of the heat transfer

The fully developed flow used as example in this study is not the most appropriate to study the thermal aspect of the problem, given the fact that the temperature field is purely conductive. In other configurations, selection between natural convection and forced convection has been based on Stanton (or Nusselt) number (Kakaç and Yener 1995, Taine and Petit 1989), which is not possible here: the flux density at the wall is given by $\varphi_p = \lambda \Delta T / e$ and, the hydraulic diameter being equal to $2e$, the Stanton number becomes:

$$St = \frac{\varphi_p}{\rho C_p V_m \Delta T} = \frac{2}{RePr}$$

Certainly, when using the bidirectional approach, it can be defined an upward flow Stanton number, like for the Reynolds number:

$$St_{\uparrow} = \frac{2}{Re_{\uparrow} Pr}$$

But in the present case, with a definition or another, the Stanton number has no property in the selection of the convective regime; and Nusselt number, a constant ($Nu = 2$), even less.

CONCLUSIONS

This article is aimed at underlining three difficulties encountered in convective heat transfer and to offer some answers.

The first difficulty represents a preamble: it concerns the expression of the reference temperature used in the Boussinesq approximation, which must not be arbitrarily chosen. In particular, in the case of a thermally and hydraulically fully developed fluid flow, a physical reasoning led to the conclusion that the reference temperature must be the average temperature field in a horizontal cross-section of the flow.

A second question, of practical nature, was raised in connection to the “borders” of the three convective domains (FC, MC and NC), and to the possible criteria to distinguish among them. Different criteria have been proposed based on existing forces, on mechanical energy, on dissipation and on entropy production. With the exception of the last one, all had in common the sole dependency on the buoyancy coefficient $Ri \cdot Re$ and all had comparable orders of magnitude. A synthesis of the results showed that a practical range for the mixed convection is from $Ri \cdot Re = 50$ to $Ri \cdot Re = 2000$. Below 50, buoyancy is negligible (dominant FC), and over 2000, the pressure gradient is negligible (dominant NC). Nevertheless, the presented values were obtained for a particular situation and must be used with precaution for other physical configurations.

The third problem, more fundamental, aims to unify the FC-MC-NC description in the similarity framework, in order to ensure the physical continuity of the dimensionless governing equations. Two approaches have been proposed, both based on the redefinition of the flow mean velocity: a bidirectional approach which distinguishes between the upward

flow and the downward flow when a reverse flow occurs, and another based on the square mean in the whole flow field. In both methods, the continuity of description is well done and the similarity criteria Ri and Re are sufficient to describe the totality of the convective domain, without recalling the Grashof number. However, the bidirectional approach is privileged by its surplus of physical transparence and closeness to the current usage.

NOMENCLATURE

$\overline{a^2}, \overline{b^2}, \overline{c^2}$	mean squares of buoyancy, pressure and viscous terms
e	distance between walls, m
Eu	Euler number (eq. 10b, 40b)
g	gravity acceleration, $m \cdot s^{-2}$
p^*	pressure, Pa
q_v	volumetric flow rate, $m^3 \cdot s^{-1}$
Ri	Richardson number (eq. 10b, 25c)
Re	Reynolds number
$Ri \cdot Re$	buoyancy ratio (eq. 25e)
T	temperature, $^{\circ}C$
T_1, T_2	hot wall / cold wall temperature, $^{\circ}C$
T	$= T_1 - T_2$
U	fluid velocity component in x direction, $m \cdot s^{-1}$
V_m	bulk (mean) velocity, $m \cdot s^{-1}$
x	vertical coordinate, m
y	horizontal coordinate, m
β	thermal expansion coefficient of fluid, K^{-1}
Φ'''	local viscous dissipation, $W \cdot m^{-3}$
Φ''	viscous dissipation on a plane section, $W \cdot m^{-2}$
μ	dynamic viscosity of fluid, $kg \cdot m^{-1} \cdot s^{-1}$
ν	kinematic viscosity of fluid, $m^2 \cdot s^{-1}$
ρ	fluid density, $kg \cdot m^{-3}$
σ''	entropy production on a plane section, $W \cdot m^{-2} \cdot K^{-1}$
$+$	dimensionless parameter

RESUMO

Esse artigo objetiva renovar o interesse na pesquisa em convecção térmica mista e enfatizar três temas que surgem da presente análise: (i) uma clara definição da temperatura de referência na aproximação de Boussinesq; (ii) uma delimitação prática dos três modos convectivos, que são a convecção forçada (CF), convecção mista (CM) e convecção natural (ou livre) (CN). (iii) e, finalmente, uma descrição uniforme do conjunto CF/CM/CN no contexto de similaridade. O caso plano, para o qual soluções analíticas estão disponíveis, permite uma ilustração detalhada das respostas aqui avançadas para os temas acima.

Palavras-chave: convecção mista, aproximação de Boussinesq, similaridade, forças de empuxo, produção de entropia, escoamento reverso.

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