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Theoretical analysis of conjugated heat transfer with a single domain formulation and integral transforms ☆

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ABSTRACT

The present work advances an analytical approach for conjugated conduction-convection heat transfer problems, by proposing a single domain formulation for modeling both the fluid stream and the channel wall regions. Making use of coefficients represented as space variable functions with abrupt transitions occurring at the fluid-wall interface, the mathematical model is fed with the information concerning the transition of the two domains, unifying the model into a single domain formulation with space variable coefficients. The Generalized Integral Transform Technique (GITT) is then employed in the hybrid numerical-analytical solution of the resulting convection-diffusion problem with variable coefficients, and critically compared for two alternative solution paths. A test problem is chosen that offers an exact solution for validation purposes, based on the extended Graetz problem including transversal conduction across the channel walls. The excellent agreement between approximate and exact solutions demonstrates the feasibility of the approach in handling more involved conjugated problems.

1. Introduction

The miniaturization of mechanical equipment such as heat exchangers is a subject of major interest in recent years, in light of the increasing demand on high performance devices and processes with high thermal efficiency, on sensors with rapid and accurate response, and on the ever growing need for heat dissipation in electronic devices, which tend to be conceived in even smaller dimensions and with more powerful data processing capacity.

For conception and design optimization, it is of crucial importance to employ reliable mathematical models and solution methodologies capable of describing the physical phenomena that take place in such micro-systems. Nevertheless, recent contributions have shown significant discrepancies between experimental results and macro-scale correlations and simulations [1,2] which may be the result of neglecting terms that are usually not important at the macro-scale, but whose effects may have significant importance in micro-scale heat transfer. In order to achieve simulated results with better agreement against experimental data, a lot of effort is being dispended in the proposition of models and solution methodologies to deal with fluid flow and heat transfer in microchannels, such as the consideration of slip flow in opposition to the classical no-slip condition, the inclusion of terms related to the viscous dissipation and axial diffusion which are often neglected in macro-scale problems, besides the investigation of corrugated walls effects in heat transfer enhancement [3–11]. Recently, Nunes et al. [12], motivated by the theoretical conclusions reached in [13], presented some experimental and theoretical results showing the importance of taking into account the heat conduction within metallic microchannel walls, leading to a conjugated problem which solution yields results in much better agreement with the available experimental data. The theoretical approach then employed was an extension of the work of [14], based on the Generalized Integral Transform Technique (GITT), a hybrid numerical-analytical technique for the solution of convection-diffusion problems [15–20], accounting for the longitudinal heat conduction along the asymmetric walls.

The present work is thus aimed at progressing into the analysis of conjugated heat transfer in channels, developing and validating a methodology for the approximate treatment of the conjugated problem reformulated into a single domain model. Thus, inspired by the well succeeded approach developed in reference [21] for the solution of heat conduction problems in heterogeneous media, we propose in this paper the reformulation of conjugated problems as a single region model that accounts for the heat transfer at both the fluid flow and the channel wall regions. By making use of coefficients represented as space variable functions, with abrupt transitions occurring at the fluid-wall interface, the mathematical model is fed with the information concerning the two original domains of the problem.

For the solution of the proposed mathematical model we again make use of the Generalized Integral Transform Technique (GITT). This approach is based on extending the classical integral transform

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Nomenclature

u	fluid velocity;
T	temperature field;
y	transversal coordinate;
z	longitudinal coordinate;
w	heat capacity;
k	thermal conductivity;
y_w	distance from the channel centerline to the external face of the channel wall;
y_i	distance from the channel centerline to the internal face of the channel wall;
Y	dimensionless transversal coordinate;
Z	dimensionless longitudinal coordinate;
U	dimensionless fully developed velocity profile;
θ	dimensionless temperature field;
K	dimensionless thermal conductivity;
Pe	Péclét number;
Nu	Nusselt number;
N	Truncation order of the temperature field expansions;
M	Truncation order of the auxiliary problem, Eqs. (14a,b)

Greek letters

ψ	eigenfunction of the constant coefficients eigenvalue problem;
ξ	eigenfunction of the space variable coefficients eigenvalue problem;
Ω	eigenfunction of the auxiliary problem corresponding to the variable coefficients eigenvalue problem;
ϕ	eigenfunction of the exact solution;

μ, β, λ, γ eigenvalues corresponding to ψ, ξ, Ω and ϕ, respectively;

Subscripts & Superscripts

i, n	order of eigenquantities
–	integral transform
~	normalized eigenfunction
*	filtered temperature field
s	quantity corresponding to the solid region (channel walls)
f	quantity corresponding to the fluid flow region
in	quantity corresponding to the entrance of the channel ($Z=0$)
w	quantity corresponding to the external face of the channel wall

method [22], making it sufficiently flexible to handle problems that are not *a priori* transformable, such as in the case of problems with arbitrarily space-dependent and nonlinear coefficients in either the equation or the boundary conditions. In order to validate the approximate solution herein obtained, a test problem was chosen based on an extended Graetz problem with transversal conduction across the wall, which results in an exact solution for the conjugated problem achieved with the Classical Integral Transform Technique (CITT), here used as a benchmark result. The developed approach is critically compared for two alternative solution paths. First, we propose the simplest possible auxiliary problem with constant coefficients to demonstrate this most flexible solution path via GITT. Then, the auxiliary problem is formulated by directly applying separation of variables to the homogeneous version of the original problem so that all the information concerning the transition of the two domains is represented by the space variable coefficients of the eigenvalue problem, which in this case is tackled with the GITT itself.

2. Problem formulation and solution methodology

The problem chosen to illustrate the proposed methodology involves a laminar incompressible internal flow of a Newtonian fluid between parallel plates, in steady-state and undergoing convective heat transfer due to a prescribed temperature, T_w, at the external face of the channel wall. The channel wall is considered to participate on the heat transfer problem through transversal heat conduction only. The fluid flows with a known fully developed velocity profile $u_f(y)$, and with an inlet temperature T_{in}. Fig. 1 depicts a schematic representation of this problem.

2.1. Single domain formulation

We assume that the flow is dynamically developed and thermally developing. Then, the formulation of the conjugated problem as a single region model that accounts for the heat transfer phenomena at both the fluid flow and the channel solid wall is achieved by making use of coefficients represented as space variable functions where the abrupt transitions occur at the fluid–solid wall interface. The conjugated problem is then given by the following single domain formulation with space variable coefficients:

$$u(y)w_f\frac{\partial T(y,z)}{\partial z} = k(y)\frac{\partial^2 T}{\partial z^2} + \frac{\partial}{\partial y}\left(k(y)\frac{\partial T}{\partial y}\right), 0<y<y_w, z>0 \tag{1a}$$

$$T(y, z=0) = T_{in} \tag{1b}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0,\ T(y=y_w, z) = T_w \tag{1c, d}$$

where the following space variable functions have been used:

$$u(y) = \begin{cases} u_f(y), & \text{if} 0<y<y_i \\ 0, & \text{if} y_i<y<y_w \end{cases} \tag{1e}$$

$$k(y) = \begin{cases} k_f, & \text{if } 0<y<y_i \\ k_s, & \text{if } y_i<y<y_w \end{cases} \tag{1f}$$

where w_f is the heat capacity of the fluid, k_s is the thermal conductivity of the channel wall, k_f is the thermal conductivity of the fluid, and $u_f(y)$is the known parabolic velocity profile of the fully developed flow. Defining the following dimensionless groups:

$$Z = \frac{z/y_w}{\text{RePr}} = \frac{z}{y_w\text{Pe}};\ Y = \frac{y}{y_w};\ U = \frac{u}{4u_{av}};\ \theta = \frac{T-T_{in}}{T_w-T_{in}};\ K = \frac{k}{k_f}$$
$$\text{Re} = \frac{u_{av}4y_w}{\nu};\ \text{Pr} = \frac{\nu}{\alpha}; \text{Pe} = \text{RePr} = \frac{u_{av}4y_w}{\alpha};\ \alpha = \frac{k_f}{w_f} \tag{2}$$

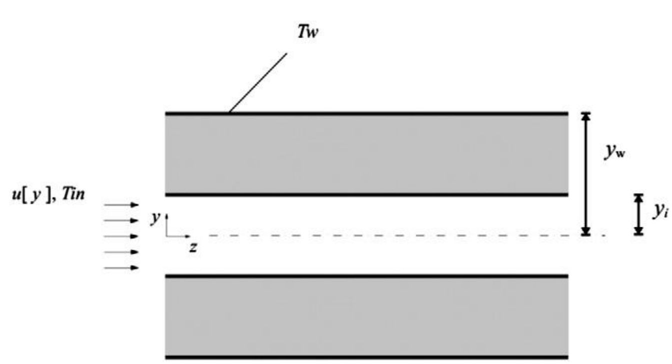

Fig. 1. Schematic representation of the conjugated heat transfer problem.

we obtain the dimensionless formulation of this problem

$$U(Y)\frac{\partial\theta(Y,Z)}{\partial Z}=\frac{K(Y)}{\mathrm{Pe}^2}\frac{\partial^2\theta}{\partial Z^2}+\frac{\partial}{\partial Y}\left(K(Y)\frac{\partial\theta}{\partial Y}\right),\ 0<Y<1,\ Z>0 \tag{3a}$$

$$\theta(Y,Z=0)=0 \tag{3b}$$

$$\left.\frac{\partial\theta}{\partial Y}\right|_{Y=0}=0,\quad \theta(Y=1,Z)=1 \tag{3c, d}$$

where

$$U(Y)=\begin{cases}U_f(Y), & \text{if } 0<Y<Y_i=y_i/y_w\\ 0, & \text{if } Y_i<Y<1\end{cases} \tag{3e}$$

$$K(Y)=\begin{cases}1, & \text{if } 0<Y<Y_i=y_i/y_w\\ k_s/k_f, & \text{if } Y_i<Y<1\end{cases} \tag{3f}$$

To improve the computational performance of the formal solutions to be derived below, it is recommended to reduce the importance of the boundary source terms, so as to enhance the eigenfunction expansions convergence behavior [18]. One possible approach for achieving this goal is the proposition of analytical filtering solutions, and in this work the proposed filter is just the temperature at the external wall, as represented in the following expression:

$$\theta(Y,Z)=1+\theta^*(Y,Z) \tag{4}$$

The filtered problem is thus rewritten by substituting Eq. (4) into Eqs. (3a)–(3f).

For validation purposes, in order to obtain a problem that still offers a fairly simple exact solution, we neglect the axial conduction term in Eq. (3a), which is a feasible assumption for this problem when dealing with high Péclét numbers and walls with a low thermal conductivity, such as micro-channel walls made of polymeric materials. The following equations provide the filtered problem formulation when the axial conduction term is neglected:

$$U(Y)\frac{\partial\theta^*(Y,Z)}{\partial Z}=\frac{\partial}{\partial Y}\left(K(Y)\frac{\partial\theta^*}{\partial Y}\right),\ 0<Y<1,\ Z>0 \tag{5a}$$

$$\theta^*(Y,Z=0)=\theta^*_{Z=0}=-1 \tag{5b}$$

$$\left.\frac{\partial\theta^*}{\partial Y}\right|_{Y=0}=0,\theta^*(Y=1,Z)=0 \tag{5c, d}$$

Two alternatives for handling problem (5) above via integral transforms are now more closely considered. The first methodology involves the consideration of an auxiliary eigenvalue problem with constant coefficients, thus without transferring to the eigenfunction expansion basis the information on the space variation of the original problem coefficients. Although this could be the simplest and most flexible solution path when employing the GITT, it is not necessarily the most effective from the computational point of view, as we shall examine within the results and discussion section. The second methodology is based on considering the auxiliary eigenvalue problem with all the information on the space variable coefficients, as obtained from applying separation of variables to the homogeneous version of the originally posed problem. In this case, the GITT itself must eventually be employed in the solution of the proposed eigenvalue problem, but may also result in marked simplification of the following steps in the solution methodology, with an overall gain in convergence rates and computational efficiency, as we shall later on discuss.

2.2. Solution with constant coefficients eigenvalue problem

Following the GITT formalism, the transform/inverse pair for solving problem (5) with a constant coefficients eigenvalue problem is defined as follows:

$$\text{transform}: \bar{\theta}^*_n(Z)=\int_0^1\tilde{\psi}_n(Y)\theta^*(Y,Z)dY \tag{6a}$$

$$\text{inverse}: \theta^*(Y,Z)=\sum_{n=1}^{\infty}\tilde{\psi}_n(Y)\bar{\theta}^*_n(Z) \tag{6b}$$

where

$$\tilde{\psi}_n(Y)=\frac{\psi_n(Y)}{\sqrt{N_n}},\text{normalized eigenfunctions} \tag{6c}$$

$$N_n=\int_0^1\psi_n^2(Y)dY,\text{normalization integrals} \tag{6d}$$

where the eigenfunctions $\psi_n(Y)$ come from the eigenvalue problem solution, which is here first chosen as the simplest possible auxiliary problem, to demonstrate this flexible solution path which is very attractive for implementation in automatic solvers such as the recently developed UNIT code (UNified Integral Transforms) [23]:

$$\frac{d^2\psi_n(Y)}{dY^2}+\mu_n{}^2\psi_n(Y)=0 \tag{7a}$$

$$\left.\frac{d\psi_n}{dY}\right|_{Y=0}=0,\quad \psi_n(Y=1)=0 \tag{7b, c}$$

Operating Eq. (5a) on with $\int_0^1\tilde{\psi}_n(Y)(\cdot)dY$, one obtains:

$$\begin{aligned}\int_0^1U(Y)\frac{\partial\theta^*}{\partial Z}\tilde{\psi}_n(Y)dY&=\int_0^1\tilde{\psi}_n(Y)\frac{\partial}{\partial Y}\left(K(Y)\frac{\partial\theta^*}{\partial Y}\right)dY=\\&=\int_0^1\frac{\partial}{\partial Y}\left(\tilde{\psi}_n(Y)K(Y)\frac{\partial\theta^*}{\partial Y}\right)dY-\int_0^1\frac{d\tilde{\psi}_n(Y)}{dY}K(Y)\frac{\partial\theta^*}{\partial Y}dY\end{aligned} \tag{8a}$$

which can be rewritten as:

$$\int_0^1U(Y)\frac{\partial\theta^*}{\partial Z}\tilde{\psi}_n(Y)dY=\tilde{\psi}_n(Y)K(Y)\frac{\partial\theta^*}{\partial Y}\Big)_0^1-\int_0^1\frac{d\tilde{\psi}_n(Y)}{dY}K(Y)\frac{\partial\theta^*}{\partial Y}dY \tag{8b}$$

where

$$\tilde{\psi}_n(Y)K(Y)\frac{\partial\theta^*}{\partial Y})_0^1=0 \tag{8c}$$

Thus:

$$\int_0^1U(Y)\frac{\partial\theta^*}{\partial Z}\tilde{\psi}_n(Y)dY=-\int_0^{y_w}\frac{d\tilde{\psi}_n(Y)}{dY}k(Y)\frac{\partial\theta^*}{\partial Y}dY \tag{8d}$$

Using the inverse formula (6b) into Eq. (8d), one obtains:

$$\int_0^1U(Y)\left[\sum_{m=1}^{\infty}\frac{d\bar{\theta}^*_m}{dZ}\tilde{\psi}_m(Y)\right]\tilde{\psi}_n(Y)dY=-\int_0^1\frac{d\tilde{\psi}_n(Y)}{dY}K(Y)\left[\sum_{m=1}^{\infty}\bar{\theta}^*_m\frac{d\tilde{\psi}_m(Y)}{dY}\right]dY \tag{8e}$$

which can be rewritten as:

$$\sum_{m=1}^{\infty}\frac{d\bar{\theta}^*_m}{dZ}\int_0^1U(Y)\tilde{\psi}_n(Y)\tilde{\psi}_m(Y)dY=-\sum_{m=1}^{\infty}\bar{\theta}^*_m\int_0^1K(Y)\frac{d\tilde{\psi}_n(Y)}{dY}\frac{d\tilde{\psi}_m(Y)}{dY}dY \tag{8f}$$

and concisely given in matrix form by:

$$\mathbf{A}\frac{d\bar{\theta}^*}{dZ} = -\mathbf{B}\bar{\theta}^* \tag{9a}$$

$$\text{where}: A_{nm} = \int_0^1 U(Y)\tilde{\psi}_n(Y)\tilde{\psi}_m(Y)dY, \qquad B_{nm} = \int_0^1 K(Y)\frac{d\tilde{\psi}_n(Y)}{dY}\frac{d\tilde{\psi}_m(Y)}{dY}dY \tag{9b, c}$$

The ordinary differential equations (ODE) system (9) can be analytically solved to provide results for the transformed temperatures, upon truncation to a sufficiently large finite order N, in terms of the matrix exponential function

$$\bar{\theta}^*(Z) = \exp\left(-\mathbf{A}^{-1}\mathbf{B}Z\right)\bar{\theta}^*(0) \tag{10a}$$

where $\bar{\theta}^*(0)$ are the transformed initial conditions, given by:

$$\bar{\theta}_n^*(0) = \int_0^1 \tilde{\psi}_n(Y)\theta_{Z=0}^* dY \tag{10b}$$

Once the transformed potentials $\bar{\theta}_n^*(Z)$, with $n=1,2,\ldots,N$, have been computed, the inversion formula can be recalled to yield the temperature field $\theta^*(Y, Z)$ representation at any desired position Y and Z. The original dimensionless temperature field $\theta(Y, Z)$ can then be approximately obtained in analytical form by:

$$\theta(Y,Z) = 1 + \theta^*(Y,Z) = 1 + \sum_{n=1}^{N}\bar{\theta}_n^*(Z)\tilde{\psi}_n(Y) \tag{11}$$

2.3. Solution with Variable Coefficients Eigenvalue Problem

In many applications, especially when dealing with inverse or optimization problems, it becomes crucial to adopt a solution methodology that is both accurate and computationally fast, so as to allow for intensive iterative analyses. In this context, it is desirable that the eigenvalue problem be chosen in order to contain as much information as possible about the original problem. The following eigenvalue problem has been formulated by directly applying separation of variables to problem (5) so that all the information concerning the transition of the two domains is represented within the eigenvalue problem, by means of the space variable coefficients $K(Y)$ and $U(Y)$. Thus,

$$\frac{d}{dY}\left(K(Y)\frac{d\zeta_i(Y)}{dY}\right) + \beta_i^2\zeta_i(Y)U(Y) = 0 \tag{12a}$$

$$\left.\frac{d\zeta_i}{dY}\right|_{Y=0} = 0, \quad \zeta_i(Y=1) = 0 \tag{12b, c}$$

Problem (12) does not allow for an explicit analytic solution, but the GITT itself can be used in order to provide a hybrid numerical-analytical solution. The GITT is here employed in the solution of this eigenvalue problem via the proposition of a simpler auxiliary eigenvalue problem, and expanding the unknown eigenfunctions in terms of the chosen basis. The chosen auxiliary problem is given by:

$$\frac{d^2\Omega_n(Y)}{dY^2} + \lambda_n^2\Omega_n(Y) = 0 \tag{13a}$$

$$\left.\frac{d\Omega_n(Y)}{dY}\right|_{Y=0} = 0, \Omega_n(Y=1) = 0 \tag{13b, c}$$

The proposed expansion of the original eigenfunction is then given by:

$$\zeta_i(Y) = \sum_{n=0}^{\infty}\tilde{\Omega}_n(Y)\bar{\zeta}_{i,n}, \quad \text{inverse} \tag{14a}$$

$$\bar{\zeta}_{i,n} = \int_0^1 \zeta_i(Y)\tilde{\Omega}_n(Y)dY, \quad \text{transform} \tag{14b}$$

The integral transformation of the eigenvalue problem with space variable coefficients is then performed by operating on Eq. (5a) with $\int_0^1 \tilde{\Omega}_n(Y)(\cdot)dY$, to yield the following algebraic problem in matrix form [21]:

$$(\mathbf{A}-\nu\mathbf{B})\bar{\boldsymbol{\zeta}} = 0, \text{with } \nu_i = \beta_i^2 \tag{15a}$$

$$\bar{\boldsymbol{\zeta}} = \left\{\bar{\zeta}_{n,m}\right\}; \mathbf{B} = \left\{B_{n,m}\right\}, B_{n,m} = \int_0^1 U(Y)\tilde{\Omega}_n(Y)\tilde{\Omega}_m(Y)dY \tag{15b – d}$$

$$\mathbf{A} = \left\{A_{n,m}\right\}; A_{n,m} = \int_0^1 \tilde{\Omega}_m(Y)\frac{d}{dY}\left(K(Y)\frac{d\tilde{\Omega}_n(Y)}{dY}\right)dY = -\int_0^1 K(Y)\frac{d\tilde{\Omega}_m(Y)}{dY}\frac{d\tilde{\Omega}_n(Y)}{dY}dY \tag{15e, f}$$

The algebraic problem (15a) can be numerically solved to provide results for the eigenvalues and eigenvectors, upon truncation to a sufficiently large finite order M, and then combined by the inverse formula (9a) to provide the desired original eigenfunctions.

Once the solution of the eigenvalue problem (12) is made available, the original problem (5) becomes completely transformable and the final solution is then obtainable by separation of variables, and given by:

$$\theta(Y,Z) = 1 + \theta^*(Y,Z) = 1 + \sum_{i=1}^{N}\bar{\theta}_{Z=0,i}^*\exp\left(-\beta_i^2 Z\right)\tilde{\xi}_i(Y), \qquad \bar{\theta}_{Z=0,i}^* = \int_0^1 U(Y)\tilde{\xi}_i(Y)\theta_{Z=0}^* dY \tag{16}$$

2.4. Exact Solution

Problem (5) has been here selected for illustration of the methodology since a straightforward exact solution can be readily obtained. For this purpose, the heat transfer problem is then modeled as a conduction problem for the solid wall, coupled at the interface $Y=Y_i$ with the internal convective problem for the fluid, as represented by the following equations:

Solid heat conduction equation:

$$0 = \frac{\partial^2\theta_s(Y,Z)}{\partial Y^2}, Y_i<Y<1, \ Z>0 \tag{17a}$$

$$\theta_s(Y=1,Z) = 1 \tag{17b}$$

Fluid heat convection equation:

$$U_f(Y)\frac{\partial\theta_f(Y,Z)}{\partial Z} = \frac{\partial^2\theta_f}{\partial Y^2}, 0<Y<Y_i, Z>0 \tag{17c}$$

$$\theta_f(Y,Z=0) = 0 \tag{17d}$$

$$\left.\frac{\partial\theta_f}{\partial Y}\right|_{Y=0} = 0 \tag{17e}$$

Interface conditions:

$$\left.\frac{\partial\theta_f}{\partial Y}\right|_{Y=Y_i}=\frac{k_s}{k_f}\left.\frac{\partial\theta_s}{\partial Y}\right|_{Y=Y_i}\ \text{and}\ \theta_f(Y_i,Z)=\theta_s(Y_i,Z) \tag{17f, g}$$

For the exact solution of the proposed problem, Eqs. (17), we first consider Eq. (17a) and the boundary conditions given by Eqs. (17b) and (17g), which readily yield the following expression for the solid wall temperature distribution:

$$\theta_s(Y,Z)=1-\frac{1-\theta_f(Y_i,Z)}{1-Y_i}+\frac{1-\theta_f(Y_i,Z)}{1-Y_i}Y \tag{18}$$

and the boundary condition given by Eq. (17f) can then be rewritten as:

$$\left.\frac{\partial\theta_f}{\partial Y}\right|_{Y=Y_i}+\frac{k_s}{k_f}\frac{1}{1-Y_i}\theta_f(Y_i,Z)=\frac{k_s}{k_f}\frac{1}{1-Y_i} \tag{19}$$

Thus, the problem for the fluid flow region becomes a Graetz type problem with third kind boundary condition:

$$U_f(Y)\frac{\partial\theta_f(Y,Z)}{\partial Z}=\frac{\partial^2\theta_f}{\partial Y^2},\ 0<Y<Y_i,\ Z>0 \tag{20a}$$

$$\theta_f(Y,Z=0)=0 \tag{20b}$$

$$\left.\frac{\partial\theta_f}{\partial Y}\right|_{Y=0}=0,\qquad \left.\frac{\partial\theta_f}{\partial Y}\right|_{Y=Y_i}+\frac{k_s/k_f}{1-Y_i}\theta_f(Y_i,Z)=\frac{k_s/k_f}{1-Y_i} \tag{20c, d}$$

Problem (20) has an exact analytical solution readily obtainable by the Classical Integral Transform Technique [16,22] and then the channel wall region temperature distribution, $\theta_s(Y, Z)$, can be directly obtained from Eq. (18). This exact solution for the wall and fluid flow regions will later on be used as a benchmark solution for the validation of the conjugated problem approximate solutions described in the previous sections. The exact solution for the fluid flow region is obtained from the solution of the following eigenvalue problem, formulated by directly applying separation of variables to the homogeneous version of problem (20):

$$\frac{d^2\phi(Y)}{dY^2}+U_f(Y)\gamma^2\phi(Y)=0 \tag{21a}$$

$$\left.\frac{d\phi}{dY}\right|_{Y=0}=0,\ \left.\frac{d\phi}{dY}\right|_{Y=Y_i}+\frac{k_s/k_f}{1-Y_i}\phi(Y_i)=0 \tag{21b, c}$$

which allows for an analytical solution in terms of hypergeometric functions that can be readily obtained using the routine DSolve of the *Mathematica* platform [6].

2.5. Nusselt Number Calculation

The main interest in convective heat transfer analysis is often to determine the local Nusselt number, $Nu(Z)$, to be evaluated here from both the approximate and the exact solutions. The following expressions for the local Nusselt number and for the bulk temperature are then employed:

$$Nu(Z)=-\frac{1}{\theta_{av}(Z)}\left.\frac{\partial\theta(Y,Z)}{\partial Y}\right|_{Y=Y_i},\quad \theta_{av}(Z)=\frac{\int_0^{Y_i}U_f(Y)\theta(Y,Z)dY}{\int_0^1 U(Y)dY} \tag{22a, b}$$

In order to avoid the direct evaluation of the derivative $\partial\theta/\partial Y|_{Y=Y_i}$ when using the approximate solutions, the following alternative integral balance formula [18] has been used:

$$\int_0^{Y_i}U(Y)\frac{\partial\theta}{\partial Z}dY=\int_0^{Y_i}\frac{\partial}{\partial Y}\left(\frac{\partial\theta}{\partial Y}\right)dY,\ \text{or}\ \left.\frac{\partial\theta}{\partial Y}\right|_{Y=Yi}=\int_0^{Y_i}U(Y)\frac{\partial\theta}{\partial Z}dY \tag{24a, b}$$

3. Results and Discussion

Fig. 2a,b below illustrate the behavior of the space variable coefficients that are feeding the single region model in Eq. (3a), $U(Y)$ and $K(Y)$, as space variable functions where the region from $Y=0$ to $Y=Y_i=0.5$ corresponds to the fluid flow domain and the region from $Y=Y_i=0.5$ to $Y=1$ corresponds to the channel wall. The dimensionless thermal conductivity has been calculated motivated by an application with a microchannel made of polyester resin ($k_s=0.16$ W/m°C) with water as the working fluid ($k_f=0.64$ W/m°C), so that $k_s/k_f=0.25$ [24].

The conjugated problem presented in this work has been solved using the approximate single region approach described in Sections 2.1–2.3 and compared to the exact solution of Section 2.4. Fig. 3a shows the transversal temperature profiles for a few different longitudinal positions along the flow, $Z=0.01$, 0.05, 0.1, 0.2, 0.3, 0.5, 0.75, 1.0 and 1.5, for the fluid and the channel wall regions obtained with the solution with the constant coefficients eigenvalue problem described in Section 2.2, with N = 20 terms as the truncation order. In these results it can be observed the excellent agreement between the approximate and exact solutions, which are essentially coincident to the graph scale. In Fig. 3b it can be seen the temperature evolution at the centerline of the channel ($Y=0$) for $Z=0$ up to 1.6, again using the single domain approach with constant coefficients eigenvalue problem, and it also shows the Graetz problem solution with first kind boundary condition, which is a simplification of this problem when the wall thermal resistance is neglected. In this test case, it can be concluded that the thermal resistance of the polymeric wall noticeably delays the increase of the fluid temperature along the flow.

Tables 1a, 1b illustrate the convergence behavior of the temperature profile for the single domain approach with constant coefficients eigenvalue problem (Section 2.2), respectively at $Z=0.01$ and $Z=0.05$, for different positions in the fluid flow region. The results are apparently fully converged to at least three digits for N = 50 in all selected positions. The exact solution results are fully converged to all five digits shown, which are achieved to within only five terms in the eigenfunction expansion. One may observe that in all selected positions the error of the approximate solution with respect to the exact one was smaller than 1.26%, which occurred at the interface position for $Z=0.01$.

The approximate solution for the single domain approach with variable coefficients eigenvalue problem, here proposed in Section 2.3, is also critically examined. First, Table 2 illustrates the excellent convergence behavior of the first 10 eigenvalues associated with the original problem, Eqs. (12), and Fig. 4 depicts the convergence behavior of the 10th eigenfunction, for different truncation orders in Eq. (14a), M = 10, 12, 14, and 16, where it can be noticed that with only 16 terms the 10th eigenfunction is fully converged to the graph scale.

The comparison of both approximate solutions with the exact solution at $Z=0.01$ is shown in Table 3, in which the single domain approach with variable coefficients eigenvalue problem has been

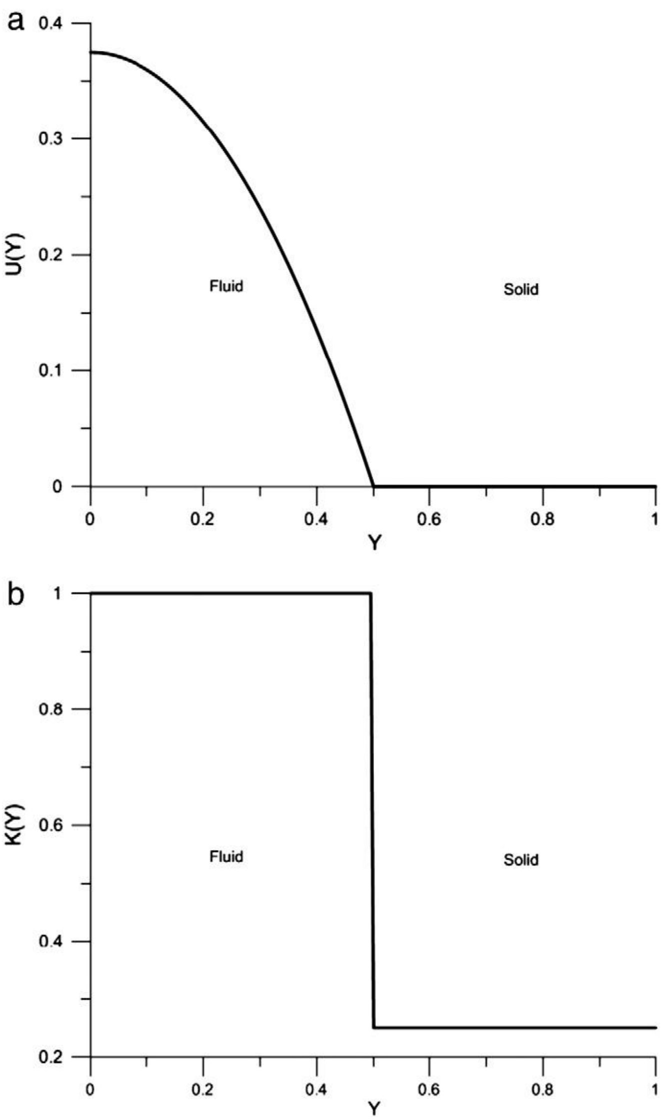

Fig. 2. a Representation of the space variable coefficients as space variable functions with the abrupt transition occurring at the interface fluid–solid wall: $U(Y)$.b Representation of the space variable coefficients as space variable functions with the abrupt transition occurring at the interface fluid–solid wall: $K(Y)$.

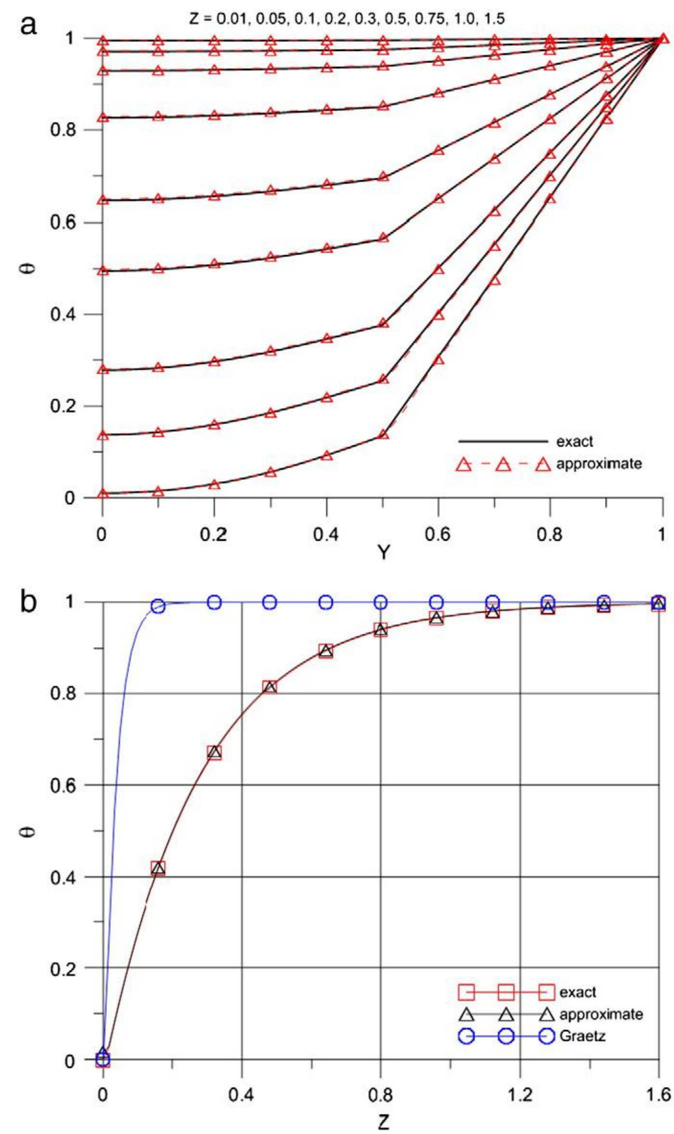

Fig. 3. a Temperature profiles calculated using the single domain approach with constant coefficients eigenvalue problem in comparison with the exact solution.b Comparison of the temperature evolution along the centerline of the channel (Y = 0) for Z = 0 up to 1.6, using the single domain approach with constant coefficients eigenvalue problem in comparison with the exact solution and Graetz problem solution with first kind boundary condition.

obtained with M = 50 terms in the eigenvalue problem solution and N = 5 terms in the temperature expansion, achieving full convergence to the five digits shown. It can be noticed that the relative error dropped significantly with the variable coefficients approach, which is computationally faster and more adequate for inverse and optimization problems. Fig. 5 finally depicts both the approximate and the exact computations of the local Nusselt number, where an excellent agreement can also be observed. It is also clear that the thermal development in the case under consideration, with the establishment of a fully developed asymptotic Nusselt number, is noticeably delayed with respect to the classical Graetz problem with prescribed wall temperature (case without wall conjugation). It is also evident that marked differences between the estimates of Nusselt number, with and without wall conjugation, can be achieved.

4. Conclusions

A single domain approach has been developed and validated for the approximate analytical treatment of conjugated heat transfer problems, here illustrated for laminar thermally developing channel flow, modeling the heat transfer phenomena at both the fluid flow and the channel wall regions. By making use of coefficients represented as space variable functions, with abrupt transitions occurring at the fluid-channel wall interface, the mathematical model is fed with the information concerning both domains of the problem. The chosen test problem has been tackled with the Generalized Integral Transform Technique (GITT), with both a straightforward constant coefficients eigenvalue problem and a more elaborate space variable

Table 1a
Convergence behavior of the temperature profile for the single domain approach with constant coefficients eigenvalue problem, at $Z = 0.01$, for the fluid flow region.

order	$Y = 0$	$Y = 0.25$	$Y = 0.5$
N = 10	0.010441	0.041455	0.14716
N = 20	0.010389	0.042408	0.14068
N = 30	0.010507	0.042456	0.13856
N = 40	0.010482	0.042288	0.13754
N = 50	0.010500	0.042329	0.13704
Exact solution	0.010413	0.042193	0.13534
Relative error	0.84%	0.32%	1.26%

Table 1b
Convergence behavior of the temperature profile for the single domain approach with constant coefficients eigenvalue problem, at $Z=0.05$, for the fluid flow region.

order	$Y=0$	$Y=0.25$	$Y=0.5$
N = 10	0.13810	0.17176	0.26606
N = 20	0.13806	0.17258	0.26048
N = 30	0.13816	0.17262	0.25865
N = 40	0.13807	0.17247	0.25778
N = 50	0.13815	0.17251	0.25735
Exact solution	0.13764	0.17195	0.25547
Relative error	0.37%	0.33%	0.74%

Table 2
Convergence behavior of the first ten eigenvalues in problem (12).

Eigenvalue β_i	N = 30	N = 60	N = 90	N = 120
1	1.89403	1.89112	1.89014	1.88965
2	14.3682	14.3671	14.3667	14.3665
3	27.3581	27.3574	27.3571	27.3570
4	40.3901	40.3896	40.3894	40.3893
5	53.4355	53.4352	53.4350	53.4349
6	66.4872	66.4869	66.4868	66.4867
7	79.5423	79.5421	79.5420	79.5419
8	92.5995	92.5994	92.5993	92.5992
9	105.658	105.658	105.658	105.658
10	118.718	118.718	118.718	118.718

coefficients case, in which the auxiliary problem itself is handled by GITT. An excellent agreement between the approximate and exact solutions was obtained, demonstrating the feasibility of the general approach herein proposed. It was also observed that the solution path that accounts for the space variable behavior within the eigenfunction expansion basis, provides better convergence rates and more efficient computational performance, in comparison to the solution with an eigenvalue problem with constant coefficients, and should in principle be preferred when dealing with more involved conjugated problems.

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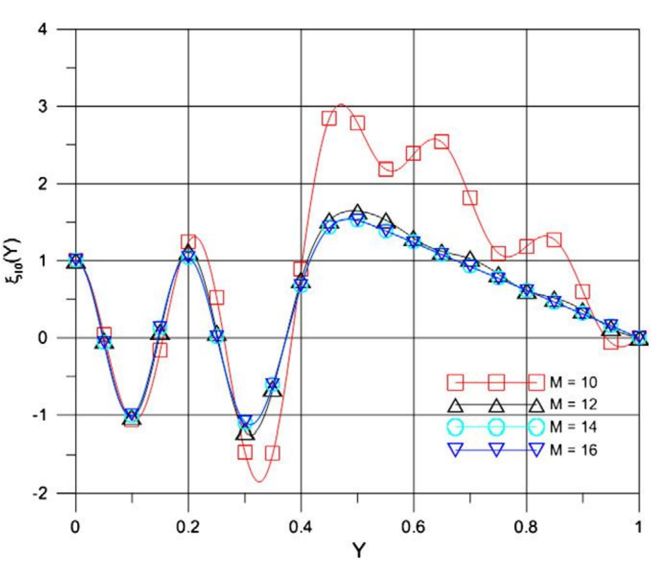

Fig. 4. Convergence behavior of the 10th eigenfunction of problem (12), for different truncation orders in Eq. (14a), M = 10, 12, 14, and 16.

Table 3
Comparison between the approximate solutions (Sections 2.2 and 2.3) and the exact solution for the temperature at $Z=0.01$.

Y	Approx. sol. (Section 2.2)	Approx. sol. (Section 2.3)	Exact sol.	Relative error (Section 2.2)	Relative error (Section 2.3)
0.00	0.010501	0.010422	0.010413	0.84%	0.086%
0.10	0.015238	0.015246	0.015230	0.053%	0.11%
0.15	0.021530	0.021465	0.021430	0.42%	0.12%
0.20	0.030564	0.030435	0.030396	0.55%	0.13%
0.25	0.042329	0.042249	0.042192	0.32%	0.13%
0.30	0.056922	0.056854	0.056776	0.25%	0.14%
0.35	0.074221	0.074001	0.073900	0.43%	0.14%
0.40	0.093561	0.093249	0.093122	0.47%	0.14%
0.45	0.11408	0.11399	0.11384	0.21%	0.13%
0.50	0.13705	0.13605	0.13534	1.26%	0.53%

(Section 2.2) - single region approach with constant coefficients eigenvalue problem.
(Section 2.3) - single region approach with variable coefficients eigenvalue problem.

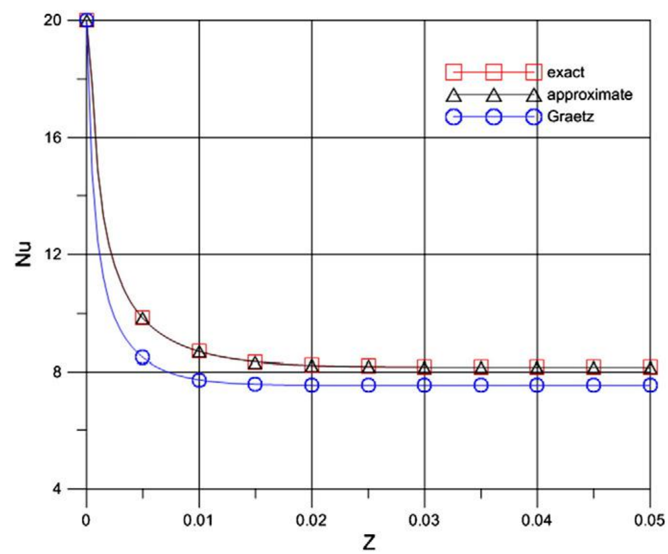

Fig. 5. Nusselt number calculated from the approximate (Sections 2.2 and 2.3) and the exact solutions, compared to the classical Graetz problem with prescribed wall temperature.

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