# FOLIATIONS WITH MORSE SINGULARITIES

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# Resumo

Neste trabalho estudamos folheações suaves de codimensão um com singularidades de Morse sem conexões de sela em variedades fechadas. Nós estendemos os resultados de [2], [3], que são extensões do resultado clássico de Reeb em [15], [17] e o resultado de E Wagneur [44].

Em particular, estendemos o seguinte resultado de [2] que diz, um variedade fechado conexo e orientada três dimensional admitindo folheação de Morse ter mais singularidades centro de selas é difeomórfico de três esfera. Nós estendemos seu caso n-dimensional também que é em [3]. Nós também estender tipo teorema de Haefliger para $S^3$.

Em [2], [3] os resultados tem sido provado por meio da técnica de eliminar pares de centro-sela triviais de singularidades. Neste trabalho, provar os mesmos resultados em [2], [3] por acoplamento e eliminação de par de selas complementares.

# Abstract

In this work we study codimension one smooth foliations with Morse singularities without saddle connections on closed manifolds. We extend the results of [2], [3], which are extension of classical result of Reeb in [15], [17] and the result of E Wagneur [44].
In particular we extend the following result of [2] which says, a closed connected and oriented three dimensional manifold admitting Morse foliation having more center singularities than saddles is diffeomorphic to three spahere. We extend its n-dimensional case too which is in [3]. We also Extend Haefliger's type theorem for $S^3$.
In [2], [3] the results has been proved by using the technique of eliminating trivial center-saddle pairs of singularities. In this work we prove the same results in [2], [3] by coupling and eliminating of pair of complementary saddles.

# Contents

# Organização do texto

O texto encontra-se dividido em cinco capítulos. Em §1 definimos folheação e alguns temas relacionados com a folheação. Para motivação das singularidades de Morse também discutimos teorema de Haefliger. Em §2 definimos folheação de Morse e dar alguns de seus exemplos. Nós damos o conceito de dead branch, o que nós precisamos na prova de alguns teoremas em capitulo 3. Em §3 nos discutimos alguns teoremas usando o procedimento de eliminação de singularidades de centro-sela, a fim de substituir o uso de Poincaré-Bendixson. Em §4 discutimos um novo conceito de acoplamento e eliminação de duas singularidades de sela de índices complementares. Ao usar esse conceito damos estender os resultados do último capítulo. Em §5 estudamos folheações compactos com singularidades. Nós essencialmente investigar uma possível extensão do teorema reconhecimento esfera da Reeb [17] em diferentes contextos, em que admitem diferentes conjuntos singulares. Começamos com não-degenerados singularidades isoladas, mas também considerar alguns casos de um degenerado, mas conjunto singular regular, ou seja, um conjunto com a propriedade de que seus componentes conectados têm um sistema fundamental de compactos bairros "invariantes".

# Introduction

The interplay between the topology of a closed manifold and the combinatorics of the critical points of a real valued function of class $C^2$ defined on the manifold is a a well known fact of Morse theory ([11]). It is natural to expect a similar relationship for foliated manifolds. This became evident for the first time with the following result of G. Reeb ([17]), a consequence of his Stability Theorem ([1], [7], [15]):

**Theorem 1.** *Let $M$ be a closed oriented and connected manifold of dimension $m \geq 2$. Assume that $M$ admits a $C^1$ transversely oriented codimension one foliation $\mathcal{F}$ with a non empty set of singularities all of them centers. Then the singular set of $\mathcal{F}$ consists of two points and $M$ is homeomorphic to the $m$-sphere.*

Later on Eells and Kuiper classified the closed manifolds admitting a $C^3$ function with exactly three non-degenerated singular points ([42], [43]):

**Theorem 2.** *Let $M$ be a connected closed manifold (not necessarily orientable) of dimension $m$. Suppose $M$ admits a Morse function $f\colon M \to \mathbb{R}$ of class $C^3$ with exactly three singular points. Then:*

(i) $m \in \{2, 4, 8, 16\}$

(ii) *$M$ is topologically a compactification of $\mathbb{R}^m$ by an $\dfrac{m}{2}$-sphere*

(iii) *If $m = 2$ then $M$ is diffeomorphic to $\mathbb{R}P(2)$. For $m \geq 4$ $M$ is simply-connected and has the integral cohomology structure of the complex projective plane ($m = 4$), of the quaternionic projective plane ($m = 8$) on of the Cayley projective plane ($m = 16$).*

We will call these manifolds *Eells − Kuiper* manifolds. In both situations we have a closed manifold endowed with a foliation with Morse singularities where the number of centers is greater than the number of saddles. In [2] it has been proved that, in the case that the manifold is orientable of dimension three, this implies it is homeomorphic to the 3-sphere. In [3] it has been proved the n-dimensional case. We proceed to define the main notions we use.

E. Wagneur [44] in 1978 generalized the Reeb sphere theorem to Morse foliations with saddles. He showed that the number of centers cannot be too much as compared with the number of saddles, notably, $c \geq s+2$. So there are exactly two cases when $c > s$:
(1)$c = s+2$
(2)$c = s+1$

He obtained a description of the manifold admitting a foliation with singularities that satisfy (1).

Finally, in 2008, C. Camacho and B. Scardua considered the case (2), c=s+1. Interestingly, this is possible in a small number of low dimensions.

**Theorem.** Let $M^n$ be a compact connected manifold and F a Morse foliation on $M$. If $s = c+1$, then:
(1) $n = 2, 4, 8$ or 16
(2) $M^n$ is an Eells-Kuiper manifold.

A codimension one foliation with isolated singularities on a compact manifold $M$ is a pair $\mathcal{F} = (\mathcal{F}_0, \operatorname{sing} \mathcal{F})$ where $\operatorname{sing} \mathcal{F} \subset M$ is a discrete subset and $\mathcal{F}_0$ is a regular foliation of codimension one on the open manifold $M \backslash \operatorname{sing} \mathcal{F}$. We say that $\mathcal{F}$ is *of class* $C^k$ if $\mathcal{F}_0$ is of class $C^k$, $\operatorname{sing} \mathcal{F}$ is called the singular set of $\mathcal{F}$ and the *leaves* of $\mathcal{F}$ are the leaves of $\mathcal{F}_0$ on $M \backslash \operatorname{sing} \mathcal{F}$. A point $p \in \operatorname{sing} \mathcal{F}$ is a *Morse type* singularity if there is a function $f_p \colon U_p \subset M \to \mathbb{R}$ of class $C^2$ in a neighborhood of $p$ such that $\operatorname{sing} \mathcal{F} \cap U_p = \{p\}$, $f_p$ has a non-degenerate critical point at $p$ and the levels of $f_p$ are contained in leaves of $\mathcal{F}$. By the classical Morse Lemma ([**?**]) there are local coordinates $(y_1, \ldots, y_m)$ in a neighborhood $U_p$ of $p$ such that $y_j(p) = 0$, $\forall j \in \{1, \ldots, m\}$ and $f(y_1, \ldots, y_m) = f(p) - (y_1^2 + \cdots + y_{r(p)}^2) + y_{r(p)+1}^2 + \cdots + y_m^2$. The number $r(p)$ is called the Morse *index* of $p$. The singularity $p$ is a *center* if $r(p) \in \{0, m\}$ and it is a *saddle* otherwise. The leaves of $\mathcal{F}$ in a neighborhood of a center are diffeomorphic to the $(m-1)$-sphere. Given a saddle singular point $p \in \operatorname{sing} \mathcal{F}$ we have leaves of $\mathcal{F}\big|_{U_p}$ that accumulate on $p$, they are contained in the cone $\tau_p$: $y_1^2 + \cdots + y_{r(p)}^2 = y_{r(p)+1}^2 + \cdots + y_m^2 \neq 0$ and there are two possibilities:

either $r(p) = 1$ or $m-1$ and then $\tau_p$ is the union of *two* leaves of $\mathcal{F}\big|_{U_p}$ , or $r(p) \neq 1$ and $m-1$ and $\tau_p$ is a leaf of $\mathcal{F}\big|_{U_p}$. Any leaf of $\mathcal{F}\big|_{U_p}$ contained in $\tau_p$ is called a *local separatrix of* $\mathcal{F}$ at $p$, or a *cone leaf* at $p$ . Any leaf of $\mathcal{F}$ such that its restriction to $U_p$ contains a local separatrix of $\mathcal{F}$ at $p$ is called a *separatrix of* $\mathcal{F}$ *at* $p$ . A *saddle connection* for $\mathcal{F}$ is a leaf which contains local separatrices of two *different* saddle points. A *saddle self-connection* for $\mathcal{F}$ at $p$ is a leaf which contains two different local separatrices of $\mathcal{F}$ at $p$. A foliation $\mathcal{F}$ with Morse singularities *is transversely orientable* if there exists a vector field $X$ on $M$, possibly with singularities at $\operatorname{sing}\mathcal{F}$, such that $X$ is transverse to $\mathcal{F}$ outside $\operatorname{sing}\mathcal{F}$.

**Definition 1.** A *Morse foliation* $\mathcal{F}$ on a manifold $M$ is a transversely oriented codimension one foliation of class $C^2$ with singularities such that: (i) each singularity of $\mathcal{F}$ is of Morse type and (ii) there are no saddle connections.

Basic examples of Morse foliations are given by the levels of Morse functions $f\colon M \to \mathbb{R}$ of class $C^2$. Therefore any manifold of class $C^2$ supports a Morse foliation, i.e., the existence of a Morse foliation imposes no restriction on the topology of the manifold. Nevertheless, there are restrictions which come from the nature of the singularities of a Morse foliation $\mathcal{F}$ on $M$.

Our purpose is to study the effect of the presence of singularities of Morse type on the global topology of a codimension one foliation defined on a compact manifold of dimension $n \geq 2$. we introduce in §1. the notion of foliation with some examples for its motivation. We will also see an example illustrating the concept. In §2. we study codimension one smooth foliations with Morse singularities on closed manifold and some examples of Morse foliation. we present a method of elimination of singularities which form trivial pairing, via an isotopy of the foliation. In §3. we will study some results which define topology of manifolds admitting Morse foliations, which have been proved by using the technique of elimination of trivial center-saddle pairings. We prove generalizations of the Reeb and Milnor topological characterizations of the $n$-sphere. One of the fundamental theorems in codimension one foliation theory on compact manifolds with finite fundamental group is the existence of a leaf with nontrivial holonomy. The use of Reeb stability theorem in place of Poincare-Bendixon theorem paves the way of three dimensional version , for foliations with Morse singularities , of classical result of Haefliger. This is due to Haefliger and consists of two main steps. The first one consists in finding a closed transverse path to the foliation, thus inducing in a 2-disc a pull back foliation by lines with Morse singularities, transverse to the boundary, and the second, using Poincaré-Bendixson theorem, in finding a leaf with one-sided nontrivial holonomy. In §4. we shall introduce the concept of coupling and elimination of two saddles of complementary indices which are in stable connection. Using this method of elimination of complementary saddles , we shall extend the results of chapter 4. Along the same line of reasoning we present in §5. we study compact foliations with singularities. We Invetigate a possible extension of Reeb extension theorem [17] in a different context. We start with the case of non-generate singularity, we also consider degenerate singularity.

# Chapter 1

# The notion of foliation

## 1.1 Motivation

The notion of foliation has been originally conceived in a classical approach by C. Ehresmann and G. Reeb by the year of 1950 (see [24],[36]). This concept can be motivated by several different situations in Mathematics. In this part of the text we shall give some examples of such situations.

**Example 1** (Vector fields). *Let $X$ be a $C^1$-vector field on a manifold $M^m$ of dimension $m$. The Flow Box Theorem can be stated as follows:*

**Theorem 3** (Flow Box). *Given any point $p \in M$ such that $X(p) \neq 0$ (i.e., $p$ is not a singular point of $X$) there exists a local chart $\varphi\colon U \subset M \to \mathbb{R}^m$ of $M$ with $p \in U$ such that $\varphi(p) = 0$ and $\varphi_* X$ is the vector field $\varphi_* X = e_m$ on $\varphi(U) \subset \mathbb{R}^m$.*

Here we denote, as usual, by $\{e_1, \dots, e_m\}$ the canonical basis of the euclidian space $\mathbb{R}^m$. In simple words, $X$ corresponds, in a neighborhood of $p$ in $M$, to a vertical constant vector field on $\mathbb{R}^m$, after a suitable change of coordinates on $M$. As a consequence of the Flow Box Theorem one obtains the following:

**Theorem 4** (Local Flow Theorem). *Given any point $p \in M$ there exists a map $\varphi\colon X \times (-\varepsilon, \varepsilon) \to M$ defined in a product $U \times (-\varepsilon, \varepsilon)$ where $U$ is a neighborhood of $p$ in $M$ and $\varepsilon > 0$ a positive number, such that:*

*i)* $\varphi(q, 0) = q \ \forall q \in U$

*ii) For any $q \in U$ we have $\frac{\partial}{\partial t}\varphi(q,t) = X(\varphi(q,t))$.*

*So, $t \mapsto \varphi(q,t)$ is the only solution (trajectory) of $X$ starting from the point $q \in U$.*

*Also we have that $p \in M$ is a singular point for $X$ if, and only if, $\varphi(p,t) = p$, $\forall t \in (-\varepsilon, \varepsilon)$.*

Assume now that $X$ is *non-singular* in $M$. In this case, any point $p \in M$ has a neighborhood $U_p \in M$ where the trajectories of $X$ are arranged like curved lines.

Clearly, by uniqueness of such trajectories, in each intersection $U_p \cap U_q \neq \emptyset$ we have a gluing of such trajectories. This way the manifold $M$ is decomposed like a union of immersed curves $\gamma \subset M$ each one corresponding locally to a parameterized curve $t \mapsto \varphi(q,t)$ and, therefore, each $\gamma$ is a trajectory of $X$. However, $X$ may have no global parameterization $\varphi \colon M \times \mathbb{R} \to M$ for its solutions.

This shows that the trajectories of $X$ have sense as curves on $M$ but not necessarily as globally parameterized curves. In this case that is important is, actually, the *geometry* of the trajectories and the way they are embedded in $M$.

**Example 2** (Submersions)**.** Let $f \colon M^m \to \mathbb{R}$ be a $C^1$ submersion, that is, $df(p) \neq 0$, $\forall p \in M$. In this case the local form of submersion shows that for each $p \in M$ we may find an open neighborhood $U_p \ni p$ of $p$ in $M$ and a chart $\varphi_p \colon U_p \to \mathbb{R}^m$ such that $f \circ \varphi_p^{-1}(x_1, \dots, x_m) = x_m$, where $(x_1, \dots, x_m)$ are affine coordinates in $\mathbb{R}^m \supset \varphi_p(U_p)$. Thus each non-empty *level survface* $f^{-1}(c)$, $c \in \mathbb{R}$ is a (not necessarily connected) closed submanifold of codimension one of $M$. Moreover, as a consequence of the local form of submersions, these level surfaces are locally arranged as the fibers of the fibration

$$(x_1, \dots, x_m) \in \mathbb{R}^m \mapsto \mathbb{R} \ni x_m.$$

Again $M$ can be decomposed as a union of submanifolds with a geometrical sense.

## 1.2 Definition of foliation

Let $M$ be a differentiable manifold of class $C^r$, $r \geq 0$.

**Definition 2.** A *foliation of class* $C^s$ $0 \leq s \leq r$ and *dimension* $k$ on $M$ is given by an atlas $\{\varphi_j \colon U_j \to V_j\}_{j \in J}$ of class $C^s$ on $M$ such that for each intersection $U_i \cap U_j \neq \emptyset$ we have the change of coordinates satisfying the following *compatibility condition*:

$$\varphi_j \circ \varphi_i^{-1} \colon V_i \subset \mathbb{R}^n \to V_j \subset \mathbb{R}^n$$

preserving the fibration by horizontal planes $\mathbb{R}^k \times \{\ \cdot\ \}$ on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. Equivalently, if we consider coordinates $(x, y) \in \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$ then we have

$$\varphi_j \circ \varphi_i^{-1}(x, y) = (A_{ij}(x, y), B_{ij}(y)).$$

Such an atlas is called a *foliated atlas* for $\mathcal{F}$ on $M$, any chart $\varphi_j \colon U_j \to V_j$ s called a *foliated chart* for $\mathcal{F}$ and each $U_j$ is a *flow box* (or *distinguished*) *neighborhood* on $M$.

Given any foliated chart $\varphi \colon U \subset M \to V \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$ for a foliation $\mathcal{F}$ of dimension $k$ as above, we define the *plaques* of $\mathcal{F}$ on $U$ as the level surfaces $\varphi^{-1}((\mathbb{R}^k \times \{y\}) \cap V) \subset U$, that is, the inverse images of the horizontal planes $\mathbb{R}^k \times \{y\} \subset \mathbb{R}^n$ by $\varphi$.

The compatibility condition implies that if $U$ and $U'$ are two foliated neighborhood for $\mathcal{F}$ on $M$ with connected intersection, $U' \cap U \neq \emptyset$ then for each $p \in U' \cap U$ the plaques $P_p \ni p$ in $U$ and $P'_p \ni p$ in $U'$ satisfy

$$P_p \cap U \cap U' = P'_p \cap U \cap U'$$

In other words, the plaques of $U$ and $U'$ coincide in each connected component of $U \cap U'$.

**Definition 3.** *Given any point $p \in M$ the leaf of $\mathcal{F}$ through $\underline{p}$ is the union $L_p$ of all plaques of $\mathcal{F}$ that can joined to $\underline{p}$ by a path of plaques.*

Alternatively we may consider the following equivalence relation on $M$: two points $p, q \in M$ are equivalent say $p \sim q$ if and only if there exists a path $a \colon [0, 1] \xrightarrow{C^0} M$ with $a(0) = p$, $a(1) = q$ such that for each $t \in [0, 1]$ there exists a neighborhood $(t - \varepsilon, t + \varepsilon)$ such that $a([t - \varepsilon, t + \varepsilon] \cap [0, 1])$ is contained in a (*same*) plaque of $\mathcal{F}$. The *leaves* of $\mathcal{F}$ are the equivalence classes of this relation of $M$. In particular a leaf $L$ of a foliation $\mathcal{F}$ as above is a connected $k$-dimensional immersed submanifold of $M$. Also, for two leaves $L$ and $L'$ of $\mathcal{F}$ we have either $L \cap L' = \emptyset$ or $L = L'$.

**Example 3** (Reeb foliation)**.** Denote by $\overline{D}^2 = \{(x_1, x_2) \in \mathbb{R}^2;\ x_1^2+x_2^2 \leq 1\}$ the closed disc on $\mathbb{R}^2$. Let $f\colon \overline{D}^2 \times \mathbb{R} \to \mathbb{R}$ be given by $f(x_1, x_2, x_3) = \rho(x_1^2 + x_2^2) \cdot e^{x_3}$ where $\rho\colon \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}$ is a function such that $\rho(0) = 1$, $\rho(1) = 0$, $\rho'(t) < 0$, $\forall\, t > 0$.

The foliation $\mathcal{F}$ of $\mathbb{R}^3$ given by the level surfaces of $f$ is of codimension one and has the cylinder $C \simeq S^1 \times \mathbb{R}$ (given by $C = \{(x_1, x_2, x_3); x_1^2 + x_2^2 = 1\}$) as as leaf. Also the interior of $C$, given by $\{(x_1, x_2, x_3); x_1^2 + x_2^2 < 1\}$ is a union of leaves all diffeomorphic to $\mathbb{R}^2$, since they may be parameterized as $D^2 \mapsto \mathbb{R}^3$, $(x_1, x_2) \mapsto (x_1, x_2, \log(\frac{c}{\rho(r^2)}))$ where $c > 0$ and $r^2 = x_1^2 + x_2^2$. The leaves outside $C$ are diffeomorphic to cylinders.

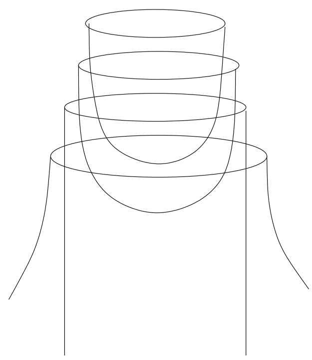

Figure 1.1:

For instance we tray take $\rho(r) := \exp\left(-\exp\left(\frac{1}{1-r^2}\right)\right)$. In this case the leaves of $\mathcal{F}$ outside $\overline{D}^2 \times \mathbb{R}$ are the cylinders $x_1^2 + x_2^2 = r^2$, $r > 1$ and leaves inside the solid cylinder parameterized by graphs of $x_3 = \exp\left(\frac{1}{1-r^2}\right) + \text{cte} \in \mathbb{R}$. Along $\overline{D}^2 \times [0, 1]$ we identify the points $(x_1, x_2, 0)$ and $(x_1, x_2, 1)$ obtaining this way a solid torus $\overline{D}^2 \times S^1$ still equipped with a foliation tangent to the boundary $\partial\overline{D}^2 \times S^1 \cong S^1 \times S^1$, having other leaves diffeomorphic to $\mathbb{R}^2$.

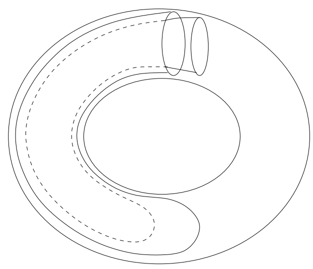

Figure 1.2:

This foliation on $\overline{D}^2 \times S^1$ is called (by the authors) *Reeb-foliation* of $\overline{D}^2 \times S^1$.

**Example 4** (integrable systems of differential forms)**.** Let $\omega_1, ..., \omega_r$ be differential 1-forms of class $C^r$ on a manifold $M$ and assume that they are linearly independent at each point $p \in M^n$. We may consider the distribution $\Delta$ of $n-r$ dimensional planes defined by $\Delta(p) \subset T_pM$ is

$$\Delta(p) = \{v \in T_pM, \omega_j(p) \cdot v = 0, j = 1, ..., r\}.$$

This distribution is called *integrable* if it is tangent to a $n-r$ dimensional foliation $\mathcal{F}$ on $M$. According to the Integrability Theorem [41] this occurs if and only if the system of 1-forms is *integrable* which means that we have $d\omega_j \wedge \omega_1 \wedge ... \wedge \omega_r = 0$ for all $j = 1, ..., r$. This occurs for instance if we have a closed 1-form $\omega$ with $\omega(p) \neq 0, \forall p \in M$. In this case we have a codimension one foliation $\mathcal{F}$ on $M$ which is defined by the Pffafian equation $\omega = 0$. The leaves of $\mathcal{F}$ are locally given by $f = cte$, where $f$ is a local primitive for $\omega$.

$G$ be a Lie group and denote by $\mathcal{G}$ the Lie algebra of $G$. The *Maurer-Cartan* form over $G$ is the unique 1-form $w\colon TG \to \mathcal{G}$ satisfying:

i) $w(X) = X, \forall X \in \mathcal{G}$

ii) $Lg^*w = w, \forall g \in G$; where $Lg\colon G \hookrightarrow G$ is the left-translation $x \in G \mapsto gx \in G$, $g \in G$ fixed.

The 1-form $w$ satisfies the *Maurer-Cartan formula* $dw + \frac{1}{2}[w, w] = 0$.

In fact, given $X, Y \in \mathcal{G}$ we have

$$dw(X, Y) = X.w(Y) - Y.w(X) - w([X, Y]) = -[X.Y].$$

But

$$[w, w](X, Y) = [w(X), w(Y)] - [w(Y), w(X)] = 2[X, Y]$$

because $X$ and $Y$ belong to $\mathcal{G}$ and $w(X) = X, \forall X \in \mathcal{G}$.

Thus we have $dw(X, Y) + \frac{1}{2}[w, w](X, Y) = 0, \forall X, Y \in \mathcal{G}$ which proves the Maurer-Cartan formula.

Let now $\{X_1, \ldots, X_n\}$ be a basis of $\mathcal{G}$. We have $[X_i, X_j] = \sum\limits_k c_{ij}^k X_k$ for some constants $c_{ij}^k \in \mathbb{C}$, skew-symmetric in $(i, j)$. The $c_{ij}^k$'s are the *structure constants* of $G$ in the basis $\{X_1, \ldots, X_n\}$.

Let now $\{w_1, \ldots, w_n\}$ be the dual basis to $\{X_1, \ldots, X_n\}$, with $w_j$ left-invariant. We have $dw_k = -\frac{1}{2} \sum_{i,j} c_{ij}^k w_i \wedge w_j$ and then it is easy to see that $w = \sum_k w_k X_k$ is the Maurer-Cartan form of $G$.

We recall the following theorem of Darboux and Lie:

**Theorem 5** ( [25] pag. 230)**.** *Let $\alpha$ be a differentiable 1-form on a manifold $M$ taking values on the Lie algebra $\mathcal{G}$ of $G$. Suppose $\alpha$ satisfies the Maurer-Cartan formula $d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$. Then $\alpha$ is locally the pull-back of the Maurer-Cartan form of $G$ by a differentiable map. Moreover the pull-back is globally defined if $M$ is simply-connected; and two such local maps coincide up to a left translation of $G$.*

**Corollary 1.** *Let $\alpha_1, \ldots, \alpha_n$ be linearly independent differentiable 1-forms on a manifold $M$. Suppose $d\alpha_k = -\frac{1}{2} \sum_{i,j} c_{ij}^k \alpha_i \wedge \alpha_j$ where the $c_{ij}^k$'s are the structure constants of a Lie group $G$ in the basis $\{X_1, \ldots, X_n\}$. Then, locally, there exist differentiable maps $\pi \colon U \subset M \to G$ such that $\alpha_j = \pi^* w_j$, $\forall\, j$ where $\{w_1, \ldots, w_n\}$ is the dual (left-invariant) basis of $\{X_1, \ldots, X_n\}$. Moreover if $M$ is simply-connected then we can take $U = M$ and if $\pi \colon U \to G$, $\overline{\pi} \colon \overline{U} \to G$ are two such maps with $U \cap \overline{U} \neq \phi$ and connected then we have $\overline{\pi} = Lg \circ \pi$ for some left-translation $Lg$ of $G$.*

This way we may construct foliated actions of Lie groups on manifolds by defining suitable integrable systems of 1-forms on the manifold. This gives rise to the notion of transversely homogeneous foliations which is a very important notion in the theory.

**Example 5** (Lie groups foliated actions)**.** Let $G$ be a Lie group and $M$ a differentiable manifold. A differentiable map $\varphi \colon G \times M \to M$ defines an *action* of $G$ in $M$ if:

(i) $\varphi(e, x) = x$, $\forall\, x \in M$

(ii) $\varphi(g_1 \circ g_2, x) = \varphi(g_1, \varphi(g_2, x))$, $\forall\, x \in M$, $\forall\, g_1, g_2 \in G$.

Here we denote by $e \in G$ the identity element (also called the *origin* of $G$). In other words $\varphi$ defines a group homomorphism

$$G \to \mathit{Diff}(M)$$

$$g \mapsto \varphi_g$$

*where the map $\varphi_g : M \to M$, $x \mapsto \varphi(g, x)$*

For instance a $(\mathbb{R},+)$-action On $M$ corresponds to a *flow* on $M$, i.e. to a complete vector field $X$ on $M$ given by $X(x) = \frac{\partial\varphi}{\partial t}(t,x)|_{t=0}$ as we have already seen.

An action $\varphi\colon G\times M \to M$ is*foliated* if all the *orbits* $\mathcal{O}_x = \{\varphi(g,x) \in M, g\in G\}$ have same dimension ($\forall\, x\in M$). Given any point $x\in M$ the *isotropy subgroup* of $x$ is defined by $G_x = \{g\in G;\ \varphi(g,x) = x\} < G$.

Since $G_x < G$ is a closed subgroup it is itself a Lie group (Cartan's Theorem) and also the quotient $G/G_x$ has the structure of a differentiable manifold. Actually we have $G_x = G_y\ \forall\, x,y$ belonging to a some orbit of $p$ and we may introduce the *isotropy subgroup of an orbit* as well.

Given any $x\in M$ we have a natural (diffeomorphism) identification $G/G_x \cong \mathcal{O}_x$ what given an immersed submanifold structure $\mathcal{O}_x \hookrightarrow M$.

An action $\varphi\colon G\times M\to M$ is *locally free* if the isotropy subgroups $G_x < G$ are discrete. In this case the action is foliated (assume $G$ connected). Finally, it is well-known that any foliated (and therefore any locally free) action of class $C^1$ is tangent to a foliation: There exists a foliation $\mathcal{F}$ on $M$ whose leaves are the orbits of $\varphi$ (use the Inverse Function Theorem).

For instance $G = \text{Aff}(\mathbb{R}) = \{(t\mapsto xt+y),\, x\in\mathbb{R}^*,\, y\in\mathbb{R}\} \cong \mathbb{R}^*\times\mathbb{R}$ is the *affine group* of $\mathbb{R}$, consisting of all affine maps $\mathbb{R}\to\mathbb{R}$, $t\mapsto xt+y$ of the line $\mathbb{R}$, is a Lie group that generates interesting actions.

**Example 6** (Fibre bundles)**.** A (differentiable) *fibre bundle* over a manifold $M$ is given by a differentiable map $\pi\colon E\to M$ from a manifold $E$, called *total space*, which is (the map) a submersion having the following *local triviality property*: for any $p\in M$ there exist a neighborhood $p\in U\subset M$ and a diffeomorphism $\varphi_U\colon \pi^{-1}(U)\subset E \underset{\sim}{\longrightarrow} U\times F$, where $F$ is fixed manifold called *typical fiber* of the bundle, such that the following diagram commutes

$$\pi^{-1}(U) \quad \overset{\varphi_U}{\longrightarrow} \quad U\times F \downarrow \pi \swarrow \pi_1 U$$

where $\pi_1\colon U\times F\to U$ is the first coordinate projection $\pi_1(x,f) = x$. In other words $\varphi_U$ is of the form $\varphi_U(\tilde{x}) = (\pi(\tilde{x}),\dots)$. Such a diffeomorphism $\varphi_U$ is called a *local trivialization* of the bundle and $U$ is a *distinguished neighborhood* of $p\in M$. Given $p\in M$ the *fiber over* $p$ is $\pi^{-1}(p)\subset E$ and by the local trivialization each fiber is an embedded submanifold diffeomorphic to $F$.

It is easy to see that the fibers of the bundle are the leaves of a foliation on $E$. Such a foliation is also called a *fibration*. This situation is quite usual as shows the following result:

**Theorem 6** (Ehresmann). *Let $f\colon M \to N$ be a $C^2$ submersion which is a proper map (i.e., $f^{-1}(K) \subset M$ is compact $\forall\, K \subset N$ compact). Then $f$ defines a fibre bundle over $N$.*

This is the case if $M$ is compact for instance. One very important result concerned with this framework is due to Tischler.

**Theorem 7** (Tischler). *A compact (connected) manifold $M$ fibers over the circle $S^1$ if, and only if, $M$ supports a closed non-singular 1-form.*

This is the case if $M$ admits a codimension one foliation $\mathcal{F}$ which is invariant by the flow of some non-singular transverse vector field $X$ on $M$.

**Example 7** (Holomorphic Foliations). A (real) manifold $M^{2n}$ is a *complex manifold* if it admits a differentiable atlas $\{\varphi_j\colon U_j \subset M \to \mathbb{R}^{2n}\}_{j\in J}$ whose corresponding changes of coordinates are holomorphic maps $\varphi_j \circ \varphi_i^{-1}\colon \varphi_i(U_i \cap U_j) \subset \mathbb{R}^{2n} \simeq \mathbb{C}^n \to \varphi_j(U_i \cap U_j) \supset \mathbb{R}^{2n} \simeq \mathbb{C}^n$.

Such an atlas is called *holomorphic*.

In this case all the basic concepts of differentiable manifolds (as tangent space, tangent bundle, etc...) can be introduced in this complex setting. This is the case of the concept of foliation:

**Definition 4.** *A holomorphic foliation $\mathcal{F}$ of (complex) dimension $k$ of a complex manifold $M$ is given by a holomorphic atlas $\{\varphi_j\colon U_j \subset M \to V_j \subset \mathbb{C}^n\}_{j\in J}$ with the compatibility property.*

*Given any intersection $U_i \cap U_j \neq \emptyset$ the change of coordinates $\varphi_j \circ \varphi_i^{-1}$ preserves the horizontal fibration on $\mathbb{C}^n \simeq \mathbb{C}^k \times \mathbb{C}^{n-k}$.*

Examples of such foliations are, like in the "real" case, given by non-singular holomorphic vector-fields, holomorphic submersions, holomorphic fibrations and holomorphic complex Lie group actions on complex manifolds.

**Remark 1.** (i) As in the “real” case, the study of holomorphic foliations may be very useful in the classification Theory of complex manifolds.

(ii) In a certain sense, the “holomorphic case” is more close to the “algebraic case” than the case of real foliations.

**Example 8** (Suspension of a foliation by a group of diffeomorphisms)**.** A well known way of constructing transversely homogeneous foliations on fibred spaces, having a prescribed holonomy group is the *suspension* of a foliation by a group of diffeomorphisms. This construction is briefly described below: Let $G$ be a group of $C^r$ diffeomorphisms of a differentiable manifold $N$. We can regard $G$ as the image of a representation $h\colon \pi_1(M) \to \text{Diff}^r(N)$ of the fundamental group of a complex (connected) manifold $M$. Considering the differentiable universal covering of $M$, $\pi\colon \widetilde{M} \to M$ we have a natural free action $\pi_1\colon \pi_1(M) \times \widetilde{M} \to \widetilde{M}$, i.e., $\pi_1(M) \subset \text{Diff}^r(\widetilde{M})$ in a natural way. Using this we define an action $H\colon \pi_1(M) \times \widetilde{M} \times N \to \widetilde{M} \times N$ in the natural way: $H = (\pi_1, h)$. The quotient manifold $\frac{\widetilde{M}\times N}{H} = M_h$ is called the *suspension manifold* of the representation $h$. The group $G$ appears as the *global holonomy* of a natural foliation $\mathcal{F}_h$ on $M_h$ (see [41]). We shall explain this construction in more details. Let $M$ and $N$ be differentiable manifolds of class $C^r$. Denote by $\text{Diff}^r(N)$ the group of $C^r$ diffeomorphisms of $N$. Given a representation of he fundamental group of $M$ in $\text{Diff}^r(N)$, say $h\colon \pi_1(M) \to \text{Diff}^r(N)$, we will construct a differentiable fiber bundle $M_h$, with base $M$, fiber $N$, and projection $P\colon M_h \to M$, and a $C^r$ foliation $\mathcal{F}_h$ on $M_h$, such that the leaves of $\mathcal{F}$ are transverse to the fibers of $P$ and if $L$ is a leaf of $\mathcal{F}$ then $P\,|_L\colon L \to M$ is a covering map. We will use the notation $G = h(\pi_1(M)) \subset Aut(N)$.

Let $\pi\colon \widetilde{M} \to M$ be the $C^r$ universal covering of $M$. A covering automorphism of $\widetilde{M}$ is a diffeomorphisms $f$ of $\widetilde{M}$ that satisfies $\pi \circ f = \pi$. If we consider the natural representation $g\colon \pi_1(M) \to Aut(\widetilde{M})$ (see [27]) then we know that:

(a)$g$ is injective. In particular $g(\pi_1(M))$ is isomorphic to $\pi_1(M)$.

(b)$g$ is properly discontinuous (see [27]).

We can therefore define an action $H\colon \pi_1(M) \times \widetilde{M} \times N \to \widetilde{M} \times N$ in a natural way:

If $\alpha \in \pi_1(M)$, $\tilde{m} \in \widetilde{M}$ e $n \in N$ then $H(\alpha, \tilde{m}, n) = (g(\alpha)(\tilde{m}), h(\alpha)(n))$.

Using (b) it is not difficult yo see that $H$ is properly discontinuous. Thus, the orbits of $H$ define an equivalence relation in $\widetilde{M} \times N$, whose corresponding quotient space is a differentiable manifold of class $C^r$.

**Definition 5.** The manifold $\frac{\widetilde{M}\times N}{H} = M_h$ is called the *suspension manifold* of the representation $h$.

Notice that $M_h$ is a $C^r$ fiber bundle with base $M$ and fiber $N$, whose projection $P\colon M_h \to M$ is defined by

$$P(o(\tilde{m}, n)) = \pi(\tilde{m})$$

where $o(\tilde{m}, n)$ denoted the orbit of $(\tilde{m}, n)$ by $H$.

Let us see how to construct the foliation $\mathcal{F}_h$. Consider the product foliation $\widetilde{\mathcal{F}}$ of $\widetilde{M} \times N$ whose leaves are of the form $\widetilde{M} \times \{n\}$, $n \in N$. It is not difficult to see that $\widetilde{\mathcal{F}}$ is $H$-invariant and therefore it induces a foliation of class $C^r$ and codimension $q = dim(N)$, $\mathcal{F}_h$ on $M_h$, whose leaves are of the form $P(\widetilde{L})$, where $\widetilde{L}$ is a leaf of $\widetilde{\mathcal{F}}$.

**Definition 6.** $\mathcal{F}_h$ is called the *suspension foliation* of $\mathcal{F}$ by $h$.

The most remarkable properties of this construction are summarized in the proposition below (see [25], [41]):

**Proposition 1.** *Let $\mathcal{F}_h$ be the suspension foliation of a representation $h\colon \pi_1(M) \to \mathrm{Diff}^r(N)$. Then:*

*(i) $\mathcal{F}_h$ is transverse to fibers of $P\colon M_h \to M$. Moreover, each fiber of $P$ cuts all the leaves of $\mathcal{F}_h$.*

*(ii)The leaves of $\mathcal{F}_h$ correspond to the orbits of $h$ in $N$ in a 1-to-1 correspondence.*

*(iii) [1] If $L$ is a leaf of $\mathcal{F}_h$ corresponding to the orbit of a point $p \in N$, then $P\mid_L\colon L \to M$ is a covering map (here $L$ is equipped with its natural intrinsic structure).*

*This implies that one fixed a point $p \in M$ and its fiber $N_p = P^{-1}(p)$, we obtain by lifting of paths in $\pi_1(M, p)$, to the leaves of $\mathcal{F}_h$, a group $G_p \subset \mathrm{Diff}^r(N_p)$, which is conjugate to $G$.*

[1]Due to (iii) we call $G$ the *global holonomy* of the suspension foliation $\mathcal{F}_h$.

*(iv) There exists a collection* $\{y_i\colon U_i \to N\}_{i\in I}$ *of submersions defined in open subsets* $U_i$ *of* $M_h$ *such that*

*(a)* $M_h = \bigcup_{i\in I} U_i$

*(b)* $\mathcal{F}_h\big|_{U_i}$ *is given by* $y_i\colon U_i \to N$.

*(c) if* $U_i \cap U_j \neq \phi$ *then* $y_i = f_{ij} \circ y_j$ *for some* $f_{ij} \in G$.

*(d) if* $L$ *is the leaf of* $\mathcal{F}_h$ *through the point* $q \in N_p$*, then the holonomy group of* $L$ *is conjugate to the subgroup of germs at* $q$ *of elements of the group* $G = h(\pi_1(M,p))$ *that fix the point* $q$.

## 1.3 Holonomy group of a leaf

The aim of this section is to introduce the notion of *holonomy group* of a leaf and some results related to this important notion. The concept of holonomy was first introduced by Ehresmann in [24], it is, a generalization of the concept of *first return map* of Poincaré, introduced by Poincaré, for the case of periodic orbits of real vector fields [37]. For instance, if $\gamma$ is a periodic orbit of a flow $\phi$ and $\Sigma$ is a transverse section that cuts $\gamma$ in a sole point $p \in \Sigma$, the holonomy of $\gamma$ relatively to $\Sigma$ is a diffeomorphism $f_\gamma\colon \Sigma^1 \to \Sigma$, where $\Sigma^1$ is a subsection of $\Sigma$, such that $p \in \Sigma^1$ is for any point $q \in \Sigma'$ the point where the positive orbit of $q$ by $\phi$ cuts $\Sigma$ at least once. We can define therefore $f$ by $f(q)$ = "first point where the positive orbit $\phi$ by $q$ cuts $\Sigma$ ". If $\Sigma^1$ is a section small enough contained in $\Sigma$, then $f$ will be a diffeomorphism over $f(\Sigma^1)$ with a fixed point at $p$. Usually, it is necessary to consider additional return of the orbits of the point of $\Sigma$, and this corresponds to consider the n-th. iterate of $f$, denoted by $f^{(n)}$, inductively defined by: $f^{(1)} = f$ e $f^{(n+1)} = f \circ f^{(n)}$. In general, however, for $n \geq 2$, $f^{(n)}$ may not be defined in all the points of $\Sigma^1$, so that we must take smaller domains $\Sigma^1 \supset \Sigma^2 \supset \ldots \supset \Sigma^n$. Due to this we shall consider the notion of *germ* we introduce below.

**Definition 7.** Let $X$ and $Y$ be topological spaces and $p \in X$. In the set of maps $f\colon V \to Y$, where $V$ is a neighborhood of $p$, we consider the following equivalence relation $\simeq$:

$$f \simeq g \iff \text{there is a neighborhood } W \text{ of } p \text{ such that } f\,|_W \equiv g\,|_W\,.$$

The equivalence class of $f$, denoted by $[f]_p$, is called the *germ of $f$ at $p$*.

The composition of two germs is well defined in terms of the composition of two representatives in a small common domain that contains the point $p$. The set of germs at $p \in X$ of local homeomorphisms at $p$ that leave $p$ fixed, will be denoted by $\mathrm{Hom}(X, p)$. When $X$ is a differentiable manifold of class $C^r$, we consider the set of $C^r$ local diffeomorphisms at $p$ leaving $p$ fixed, that will be denoted by $\mathrm{Diff}^r(X, p)$. The sets $\mathrm{Hom}(X, p)$ and $\mathrm{Diff}^r(X, p)$, as soon as they are defined, are groups for the operation of composition of functions.

The holonomy of a leaf $L$ of a foliation $\mathcal{F}$, of class $C^r$, on a manifold $M$, is a representation of the fundamental group of $L$ in the group of germs of diffeomorphisms $\mathrm{Diff}^r(\Sigma, p)$, of a transverse section $\Sigma$ to $\mathcal{F}$, that cuts $L$ at the point $p$, and the holonomy maps leave fixed the point $p$ (for more details we refer to [25],[41]).

Let $M$ be a differentiable manifold of dimension $n$, equipped with a foliation $\mathcal{F}$ of class $C^r$ and codimension $k$. Fixed a leaf $L$ of $\mathcal{F}$ and a continuous curve $\gamma\colon I \to L$ (called a *path* in $L$), where $I$ is the compact interval $[0, 1]$, we consider $\Sigma_0$ and $\Sigma_1$ transverse sections to $\mathcal{F}$ of dimension $k$, such that $p_0 = \gamma(0) \in \Sigma_0$ and $p_1 = \gamma(1) \in \Sigma_1$. We may use the distinguished charts for $\mathcal{F}$ in order to obtain such sections $\Sigma_0$ and $\Sigma_1$ with the property that for certain neighborhood $p_o \in U_0$ and $p_1 \in U_1$ the section $\Sigma_j$ cuts the plaque of $U_j$ exactly once.

Then, we consider a finite covering of the image $\gamma(I)$ by distinguished charts of $\mathcal{F}$, say $V_0, ..., V_m$, such that: (i) $V_0 = U_0$ e $V_m = U_1$. (ii) For each $j = 1, ..., m$, $V_{j-1} \cap V_j \neq \phi$. (iii) For each $j = 1, ..., m$, there exists a trivializing chart $U$ of $\mathcal{F}$ such that $V_{j-1} \cup V_j \subset U$. (iv) There exists a partition $\{0 = t_0 < t_1 < ... < t_m < t_{m+1} = 1\}$ of $I$ such that $\gamma[t_j, t_{j+1}] \subset V_j$ para $j = 0, ..., m$.

For each $j = 1, ..., m$ we consider $\Sigma_j^{'}$, a transverse section of $\mathcal{F}$ such that $\gamma(t_j) \in \Sigma_j^{'} \subset U_{j-1} \cap U_j$ e $\Sigma_j^{'}$ cuts each plaque of $U_{j-1}$ and each plaque of $U_j$ in at most one point. We also set $\Sigma_0^{'} = \Sigma_0$ and $\Sigma_{m+1}^{'} = \Sigma_1$. Using then (ii) and (iii), we conclude that if $q \in \Sigma_j^{'}$, then the plaque of $V_j$ that contains $q$, cuts $\Sigma_{j+1}^{'}$ in at most one point, and moreover if $q$ is in a small neighborhood $A_j$ of $\gamma(t_j)$ in $\Sigma_j^{'}$, then this plaque actually cuts $\Sigma_{j+1}^{'}$ in one point, say $f_j(q)$. Thus, we may define a map $f_j\colon A_j \to \Sigma_j^{'}$ such that $f_j(\gamma(t_j)) = \gamma(t_{j+1})$. Since the transverse sections we consider are $C^r$ submanifolds, the theorem of differentiable dependence of the solutions of ordinary differential equations, with respect to the initial conditions assures that the maps $f_j$ are $C^r$ maps. It follows that each map $f_j$ defines a $C^r$ diffeomorphism onto its image, because its inverse is defined in a natural analogous way. In general we cannot compose the germs $f_{j+1}$ and $f_j$, but we can compose their corresponding germs, for $f_j(\gamma(t_j)) = \gamma(t_{j+1})$. If we denote the germ of $f_j$ at $\gamma(t_j)$ by $[f_j]$, then we can consider the composed germ:

$$[f]_\gamma = [f_m] \circ ... \circ [f_0]$$

this is a germ of $C^r$ diffeomorphism at $p_0$, where, "*a priori´´*, $[f]_\gamma$ depends on the covering $V_0, ..., V_m$ and the intermediate sections. However, $[f]_\gamma$ does not depend on the auxiliary constructions:

**Lemma 1.** *The germ $[f]_\gamma$ depends only on $\gamma$, $\Sigma_0$ and $\Sigma_1$.*

**Definition 8.** The germ $[f]_\gamma$ is called the *holonomy of $\gamma$ with respect to the section $\Sigma_0$ and $\Sigma_1$*. In the case $\gamma$ is a closed curve in $L$, i.e., $p_0 = p_1$, and $\Sigma_0 = \Sigma_1$, $[f]_\gamma$ is an element of the group $\text{Diff}^r(\Sigma_0, p_0)$ and it is called *holonomy of $\gamma$ with respect to* $\Sigma_0$, or simply *holonomy of $\gamma$*.

Now we calculate the holonomy of a curve obtained by adjunction of two other curves. Let $\gamma,\ \delta\colon I \to L$ be two curves in $L$ such that $\gamma(0) = p_0$, $\gamma(1) = \delta(0) = p_1$ e $\delta(1) = p_2$. The adjunction (*adjunção*) of $\gamma$ and $\delta$ is the curve $\alpha\colon I \to L$ defined by:

$$\alpha(t) = \gamma(2t), \quad \text{for} \ \ t \in [0, 1/2] \ \ \text{and} \ \ \alpha(t) = \delta(2t-1), \quad \text{for} \ \ t \in [1/2, 1].$$

We denote it by $\alpha = \delta \star \gamma$. Using straightforward computation and the definitions we obtain:

**Lemma 2.** *Let $\gamma, \delta, p_0, p_1$ and $p_2$ be as above. Fixed transverse sections to $\mathcal{F}$, $\Sigma_0, \Sigma_1$ and $\Sigma_2$ through $p_0, p_1$ and $p_2$ respectively we have:*

$$[f]_{\gamma\star\delta} = [f]_\gamma \circ [f]_\delta.$$

*where the germs are the holonomy maps with respect to the sections $\Sigma_0, \Sigma_1$ and $\Sigma_2$.*

Thus we may introduce the following notion of "holonomy group" of a leaf of $\mathcal{F}$.

**Lemma 3.** *Let $M$,$\mathcal{F}$, $L$, $p_0, p_1 \in L$, $\Sigma_0$ and $\Sigma_1$ be as above. If the paths $\gamma, \delta \colon I \to L$ are such that $\gamma(0) = \delta(0) = p_0$, $\gamma(1) = \delta(1) = p_1$ and $\gamma$ and $\delta$ are homotopic in $L$ with fixed extremes, then $[f]_\gamma = [f]_\delta$.*

We recall that two paths $\gamma$ and $\delta$ as in the Lemma above are *homotopic with fixed extremes in $L$* if there exists a continuous map $H \colon I \times I \to L$ such that (i)$H(t, 0) = \gamma(t)$ and $H(t, 1) = \delta(t)$ $\forall t \in I$. (ii)$H(0, s) = p_0$ and $H(1, s) = p_1$ $\forall s \in I$.

We will use the notation $\gamma \sim \delta$. In the case $p_0 = p_1$ we know that $\sim$ is an equivalence relation [27]. The equivalence class (*homotopy class*) of a path $\gamma$ with extremes at $p_0$ is denoted by $[\gamma]$. The set of all these equivalence classes is the *fundamental group* or *first homotopy group* of $L$ with base point at $p_0$. This group is usually denoted by $\pi_1(L, p_0)$. The composition law for this group is defined from the adjunction of paths in a natural way and is denoted by $\star$. Therefore given two homotopy classes $[\gamma]$ and $[\delta]$ in $\pi_1(L, p_0)$, we may fix representatives $\gamma$ and $\delta$ for these classes and define $[\delta] \star [\gamma] = [\delta \star \gamma]$. Endowed with this composition the set $\pi_1(L, p_0)$ is a group whose identity element if the equivalence class of the constant path $e(t) \equiv p_0, t \in I$.

Thus we may define:

**Definition 9.** Let $M$ be a differentiable manifold equipped with a $C^r$ foliation $\mathcal{F}$ of codimension $k$, and let $L$ be a leaf of $\mathcal{F}$, $p \in L$ and $\Sigma$ a transverse section to $\mathcal{F}$ such that $p \in \Sigma$. The *holonomy representation of $L$ with respect to $p$ and $\Sigma$* is the map $H = H_{L,p,\Sigma} \colon \pi_1(L, p) \to \mathrm{Diff}^r(\Sigma, p)$, defined by:

$$H([\gamma]) = [f]_\gamma$$

where $\gamma$ is a representative of $[\gamma]$ and $[f]_\gamma$ is the germ of holonomy of $\gamma$ with respect to $\Sigma$. The map $H$ is well defined by the above lemmas. The *holonomy group of $L$ with respect to $p$ and $\Sigma$* is the image $H(\pi_1(L, p))$ that we will denote by $\mathrm{Hol}(L, p, \Sigma)$.

The following result is one of the main results in this chapter:

**Proposition 2.** (i) *The holonomy representation is a group homomorphism: if $a, b \in \pi_1(L, p)$, then $H(a \star b) = H(a) \circ H(b)$.*

(ii) *Let $L$ be a leaf of the foliation $\mathcal{F}$ of codimension $k$, $p_0, p_1 \in L$ and $\Sigma_0, \Sigma_1$ transverse sections to $\mathcal{F}$ that contain $p_0$ and $p_1$ respectively. Fix a curve $\alpha\colon I \to L$ such that $\alpha(0) = p_0$ and $\alpha(1) = p_1$. Let $[f]_\alpha$ be the germ at $p_0$ of the holonomy of $\alpha$, between the transverse sections $\Sigma_0$ and $\Sigma_1$. Then $[f]_\alpha$ conjugates* $\mathrm{Hol}(L, p_0, \Sigma_0)$ *and* $\mathrm{Hol}(L, p_1, \Sigma_1)$*, that is:*

$$\mathrm{Hol}(L, p_0, \Sigma_0) = ([f]_\alpha)^{-1} \circ \mathrm{Hol}(L, p_1, \Sigma_1) \circ [f]_\alpha$$

*In particular* $\mathrm{Hol}(L, p_0, \Sigma_0)$ *and* $\mathrm{Hol}(L, p_1, \Sigma_1)$ *are isomorphic.*

Since the transverse sections are diffeomorphic to open subsets of de $\mathbb{R}^k$, the following definition is natural:

**Definition 10.** Let $L$ be a leaf of a $C^r$ foliation $\mathcal{F}$ of codimension $k$. The *holonomy group* of $L$, denoted by $\mathrm{Hol}(L)$, is the collection of all the groups of germs at $q \in \mathbb{R}^k$, of diffeomorphisms of $\mathbb{R}^k$ that leave $q$ fixed and that are conjugate to $\mathrm{Hol}(L, p, \Sigma)$, where $p \in L$ and $\Sigma$ is a transverse section to $\mathcal{F}$ passing through $p$. We say that the holonomy group of $L$ is *conjugate* to a given group $G$, if $G \in \mathrm{Hol}(L)$.

Finally, we show how to calculate the holonomy in an analytical way. Let therefore $L$ be a leaf of a $C^r$ codimension $k$ foliation $\mathcal{F}$ on the manifold $M$. We recall the following facts from differential topology:

(i) each path $\gamma\colon I \to L$ is homotopic in $L$ with fixed extremes to a $C^r$ smooth curve (see [8], [29],[30]).

(ii) Given a Riemannian metric $g$ in and given an open subset $A \subset L$, with compact closure, there exists $r > 0$ such that for every $\epsilon > 0$ with $\epsilon < r$, we have a *normal tubular neighborhood* of class $C^r$, $\pi\colon V \to A$ and radius $\epsilon$ of $A$ (see [29],[30]).

Such a *normal tubular neighborhood* of class $C^r$and radius $\epsilon$ of a submanifold $A$ of $M$, is an open subset $V$ of $M$, $V \supset A$, and a submersion of class $C^r$,$\pi\colon V \to A$, with the following properties:

(a) $\pi(p) = p \;\; \forall p \in A$.

(b) for every $p \in A$, the fiber $F_p \doteq \pi^{-1}(p)$, is diffeomorphic to a disk in $\mathbb{R}^k$, that is normal to $A$ at $p$ and has radius $\epsilon$ with respect to the metric $g$.

These claims are consequence of the Tubular Neighborhood Theorem (see [8],[29],[30]) and the fact that of $A \subset L$ has compact closure, then $A$ is a $C^r$ submanifold of $M$ of codimension $k$. By the Tubular Neighborhood Theorem $\pi\colon V \to A$ is a fibration with fiber diffeomorphic to a disk in $\mathbb{R}^k$.

Thus, in order to calculate the holonomy of a path $\alpha$ in $L$ we may assume that $\alpha$ is regular. Fix a $C^r$ curve $\gamma\colon I \to L$ such that $\gamma(0) = p_0$ e $\gamma(1) = p_1$. Since $c = \gamma(I)$ is compact,we conclude that $c$ has a neighborhood $A$ in $L$ with compact closure. Let $\pi\colon V \to A$ be a normal tubular neighborhood of radius $\epsilon > 0$ of $A$, where $\epsilon$ is small enough so that the fibers $F_p, p \in A$, of $\pi$ are transverse to $\mathcal{F}$.

Assume that $\gamma$ is injective. In this case the set $\Lambda = \pi^{-1}(\gamma(I))$ is a $k+1$ dimensional submanifold of $M$, whose boundary is $\Sigma_0 \cup \Sigma_1$, where $\Sigma_0 = F_{p_0}$ and $\Sigma_1 = F_{p_1}$ (for $\pi$ is a submersion).

We may therefore define a $C^r$ vector field in $\Lambda$ such that:

(I) $\gamma$ is the trajectory of $p_0$ by $X$.

(II) the trajectories of $X$ are contained in the leaves of $\mathcal{F}$.

(III) if $q \in \Sigma_0$ is in a suitable neighborhood $U$ of $p_0$, then the trajectory cuts $\Sigma_1$ in a single point, say $f(q)$.

(IV) the germ of $f$ at $p_0$ is the holonomy of $\Sigma_0$ in $\Sigma_1$.

Given $q \in \Lambda$, we consider the linear map

$$T_q = D\pi(q)\,|_{T_q\mathcal{F}}\colon T_q\mathcal{F} \to T_{\gamma(t)}L = T_{\gamma(t)}\mathcal{F}$$

where $\gamma(t) = \pi(q)$. Since the fibers of $\pi$ são are transverse to $\mathcal{F}$, it is not difficult to see that $T_q$ is an isomorphism. Thus we put:

$$X(q) = T_q^{-1}(\gamma^{'}(t)).$$

Notice that for any $q \in \Lambda$ we have $X(q) \in T_q\mathcal{F}$. This implies (II). On the other hand, clearly we have $X(\gamma(t)) = \gamma^{'}(t)$, and this implies (I). Observe now that claim (III) is true for the orbit of $X$ through por $p_0$ (i.e., $\gamma$). Thus, the same holds for the orbits of the points close to $p_0$, so that (III) is true. Claim (IV) follows from (II).

In the case $\gamma$ is not injective we have mainly the same construction, except for the fact that $\Lambda$ is only an immersed submanifold, and the vector field $X$ can be multiply defined at a point $q \in \Lambda$ such that $\pi(q) = \gamma(t_1) = \gamma(t_2)$, where $t_1 \neq t_2$. In order to overcame this difficulty we may consider a partition $\{0 = t_0 < t_1 < ... < t_m = 1\}$ of $I$ such that for all $j = 1, ..., m$, the restriction $\gamma_j = \gamma \mid_{[t_{j-1},t_j]}$ is injective and then apply the same arguments above in order to obtain the holonomy maps between intermediate sections $F_{\gamma(t_{j-1})}$ and $F_{\gamma(t_j)}$. The holonomy map is then obtained by composing these intermediate holonomy maps.

The map $\pi = \pi_1\big|_\Lambda \colon \Lambda \to \gamma(I)$ is a fibration, whose fibers have dimension $k$ and are transverse to $\mathcal{F}$. Given a point $q \in \Sigma_0$, close to $p$, the orbit of $X$ that passes through $q$, is the *lifting of $\gamma$ by the fibers of $\pi$, to the leaf of $\mathcal{F}$ that passes through $q$*. We will call $\gamma_q$ the *lifting of $\gamma$ through the point $q$*.

## 1.4 The theorem of Haefliger

Let $\mathcal{F}$ be a foliation of codimension $q \geq 1$ on a manifold $M^n$. A submanifold $\Sigma \subset M$ is *transverse* to $\mathcal{F}$ if for any point $p \in \Sigma$ the leaf $L_p$ of $\mathcal{F}$ that contains the point $p \in \Sigma \cap L_p$ is transverse (as an immersed submanifold) to $\Sigma$ at the point $p$. This means, according to the usual definition of transversality, that we have $T_p(L_p) + T_p\Sigma = T_pM$.

Clearly we may have $\dim \Sigma \leq n$. If $\dim \Sigma = q$, i.e., $T_p(L_p) \oplus T_p\Sigma = T_pM$; then we say that $\Sigma$ is a *transverse section* of $\mathcal{F}$ on$M$.

The Flow Box Theorem state that any $C^1$ vector field $X$ on $M^n$ defines a dimension one foliation $\mathcal{F}_X$ outside the singular $\text{sing}(X)$ on $M$. Also, given any non singular point $p \in M \setminus \text{sing}(X)$ we have a neighborhood $p \in U_p \subset M \setminus \text{sing}(X)$ such that $\mathcal{F}_X|_{U_p}$ admits a (local) transverse section $\sum_p^{n-1}$ containing the point $p$. Using the definition of foliation one may prove:

**Proposition 3.** *Given $\mathcal{F}$ a codimension $q \geq 1$ foliation on $M^n$ and any point $p \in M$ there exists a smooth embedding $\varphi\colon D^q \to M$ of the $q$-dimensional disc $D^q = \{(x_1, \dots, x_q) \in \mathbb{R}^q$, $\sum_{j=1}^q x_j^2 < 1\}$, such that the image $\sum^q = \varphi(D^q) \subset M$ is a transverse section of $\mathcal{F}$.*

Thus, the existence of *local* transverse sections is a consequence of the definition of foliation. Nevertheless, the existence of *global transverse sections* is not so usual. By a *global transverse section* $\sum$ to a foliation $\mathcal{F}$ on $M$ we shall mean a *compact or closed* submanifold $\sum \subset M$ such that $\mathcal{F}$ is transverse to $\sum$ everywhere.

The following result of A. Haefliger shows an obstruction to the existence of global transverse sections to codimension one analytic submanifolds:

**Theorem 8.** *Let $\mathcal{F}$ be a codimension one $C^2$ foliation $\mathcal{F}$ on $M$ such that there exists an embedding $\gamma\colon S^1 \to M$ whose image $\Sigma = \gamma(S^1)$ is transverse to $\mathcal{F}$ and homotopic to a point on $M$. Then $\mathcal{F}$ is not analytic. Actually, there exist a leaf $L_0$ of $\mathcal{F}$ and a closed path $\alpha \subset \pi_1(L_0)$ whose corresponding holonomy map is conjugate to a germ $h \in$ Diff$((-\varepsilon, \varepsilon), 0)$ having the property that $h|_{(-\varepsilon,0]} = Id$ and $h|_{[0,+\varepsilon)} \neq Id$.*

Haefliger's Theorem above implies that a real analytic codimension one foliation $\mathcal{F}$ on a manifold $M$ having finite fundamental group $\#\pi_1(M) < \infty$, admits no closed transverse sections. This is proved by passing to the universal covering for instance.

**Rough Idea of the proof of Haefliger's Theorem:**

First one may assume that the embedding $\gamma\colon S^1 \to M$ is $C^\infty$ and small deformations of this embedding are still transverse to $\mathcal{F}$. Since $\gamma$ is homotopic to zero in $M$ we may assume that$\gamma$ bounds an immersed disc in $M$ say, there exists an embedding $\Gamma\colon \overline{D}^{n-1} \to M$ of the closed disc $\overline{D}^{n-1} = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1},\ \sum_{j=1}^{n-1} x_j^2 \leq 1\}$ such that $\gamma = \Gamma|_{\partial\overline{D}^{n-1}} = S^1$.

The fact that $\mathcal{F}$ is transverse to $\gamma$ and $D^2$ is simply-connected implies that we may assume that the pull-back $\Gamma^*\mathcal{F}$ is foliation defined in a neighborhood of $D^2 \subset \mathbb{R}^2$, transverse to $S^1$, of dimension one, induced by a $C^1$ vector field $X$ pointing inwards the disc $\overline{D}^2$ along the boundary $\partial\overline{D}^2 = S^1$.

By performing small perturbations of $\Gamma$ we may assume that the singularities of $X$, all inside $D^2$, are of Morse type so that they are either centers $d(x^2 + y^2) = 0$ or saddles $d(x^2 - y^2) = 0$.

Using the Poincaré-Hopf Theorem for $X$ (recall $X \pitchfork \partial D^2$) we conclude there exists some center singularity in $D^2$. Now Zorn's Lemma implies that we may find some maximal region $R \subset\subset D$ obtained as the work of periodic orbits begging around some center singularity in $D^2$.

Let $\alpha^* := \partial R \subset D^2$ then $\alpha^*$ is a periodic orbit of $X$ whose Poincaré map is, in one side of the transverse section, the identity and, in the other side of the transverse section, is not identity.

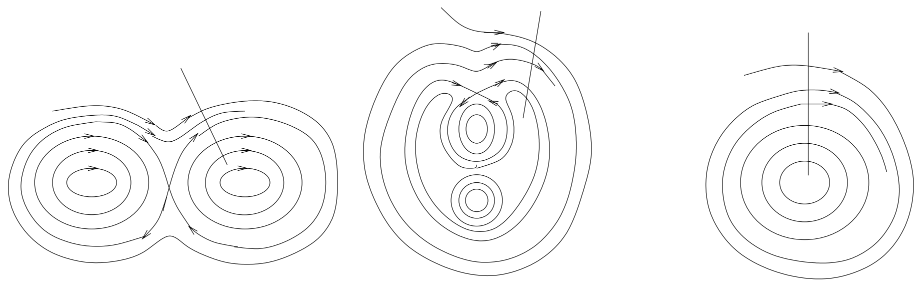

Figure 1.3:

The path $\alpha = \Gamma(\alpha^*)$ satisfies the statement. Q.E.D

An interesting, very important, consequence of the above results is the following:

**Corollary 2.** *Let $\mathcal{F}$ be a codimension one real analytic foliation on a simply-connected manifold $M$ (or, more generally, $|\pi_1(M)| < \infty$). then the leaves of $\mathcal{F}$ are closed, the leaf space $M/\mathcal{F} = \mathfrak{X}$ is a (may be non-Hausdorff) 1-manifold.*

The above corollary is easily proved owing to the following remark:

**Lemma 4.** *A codimension one foliation $\mathcal{F}$ on a manifold $M$, exhibiting some non-closed leaf necessarily exhibits some closed transverse section.*

**Proof:** Let $L_0$ be a non-closed leaf of $\mathcal{F}$. Given $p \in \overline{L}_0 \setminus L_0$ we choose a local Flow Box $p \in U_p$ and a small transverse section $\Sigma_p \ni p$ $\Sigma_p \subset U_p$.

We have $L_p \neq L_0$. Given two points $p_1, p_2 \in \Sigma_p \cap L_0$ belonging two distinct plaques of $\mathcal{F}|_{U_p}$ we choose a path $a \colon [0,1] \to L_0$ joining $p_1$ to $p_2$.

Now we choose a fibration transverse to $\mathcal{F}$ having basis along $a(I)$.

Now it is easy to cut $\Sigma_p$ and glue (emend) it with a piece of transverse section $\sigma$ to $\mathcal{F}$ contained in the fibers of the fibration above.

Then we may modify this construction in order to obtain a "smooth" transverse section to $\mathcal{F}$.

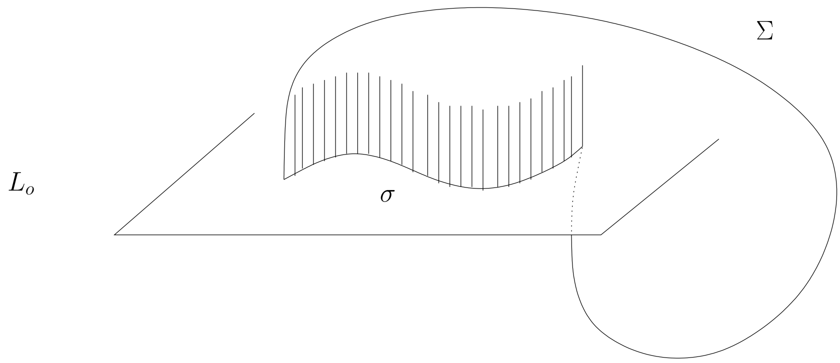


Figure 1.4:

Q.E.D.

**Remark 2.** One may ingredient in the proof of Haefliger's Theorem is the Poincaré-Bendixson theorem (that in the sketch above could be applied to $X$ in order to show that $\partial R$ is a limit cycle exhibiting therefore the specially behaved Poincaré map).

Haefliger's result has been extended by Plante and Thurston as follows:

**Theorem 9** (Plante-Thurston). *Let $M$ be a real compact manifold such that $\pi_1(M)$ has polynomial growth. If $M$ admits a real regular codimension one analytic transversely oriented foliation then $H^1(M, \mathbb{R}) \neq 0$.*

**Problem 1.** *Is there any kind of relation between the topology of the ambient manifold and the obstruction to the existence of non-singular codimension one foliations also in the complex case though no result like Haefliger's theorem is known yet?*

For instance we have the following question:

**Problem 2.** *Let $M$ be a compact complex surface. Is there any holomorphic regular foliation $\mathcal{F}$ by curves on $M$?*

# Chapter 2

# Morse foliations

## 2.1 Preliminaries

A codimension one $C^\infty$ foliation with isolated singularities on compact manifold $M$ is a pair $\mathcal{F} = (\mathcal{F}_o, \text{sing}\mathcal{F})$, where $\text{sing}\mathcal{F} \subset M$ is a discrete subset and $\mathcal{F}_o$ is a regular foliation of codimension one on the open manifold $M - \text{sing}\mathcal{F}$. We say that $\mathcal{F}$ is of class $C^k$ if $\mathcal{F}_o$ is of class $C^k$, $\text{sing}\mathcal{F}$ is called singular set of $\mathcal{F}$ and the leaves of $\mathcal{F}$ are the leaved of $\mathcal{F}_o$ on $M \setminus \text{sing}\mathcal{F}$.

A smooth real valued function $f : M \to \mathbb{R}$ is a Morse function if all critical points of $f$ are non-degenerate, i.e. Hessian matrix at all critical points are non-singular.

A point $p \in \text{sing}(F)$ is a Morse type singularity if there is a function $f_p : U_p \to \mathbb{R}$ of class $C^2$ in a neighbourhood of $p$ such that $\text{sing}(F) \cap U_p = \{p\}$, $f_p$ has non-degenerate critical point at $p$ and the levels of $f_p$ are contained in leaves of $\mathcal{F}$, i.e. $\mathcal{F}|_U$ is given by $df_p = 0$.

By classical Morse lemma [3] there is a system of coordinates $x = (x_1, x_2, ......x_n)$ on a neighbourhood $U_p$ of $p$ such that $x_i(p) = 0$, $\forall i \in \{1, 2, .....n\}$ and

$$f(x_1, x_2, ....., x_n) = f(p) - (x_1^2 + .....x_{r(p)}^2) + (x_{r(p)+1}^2 + .....x_n^2).$$

The number $r(p) = Ind(f, p)$ is called Morse index of $p$.

The singularity $p$ is a center singularity if $r(p) \in \{0, n\}$ and is called saddle singularity if $r(p) \in \{1, 2, .....n-1\}$. The leaves of $\mathcal{F}$ in a neighbourhood of a center singularity are diffeomorphic to $(n-1)$ sphere. Given a saddle singularity $p \in \text{sing}\mathcal{F}$ we have leaves of $\mathcal{F}|_{U_p}$ contained in the cone

$$\tau_p : x_1^2 + .......x_{r(p)}^2 = x_{r(p)+1}^2 + ..... + x_n^2 \neq 0,$$

and there are two possibilities for cone leaves:

either $\tau_p$ is union of two leaves of $\mathcal{F}|_{U_p}$ if $r(p) \in \{1, n-1\}$ , or $\tau_p$ is a leaf of $\mathcal{F}|_{U_p}$ if $r(p) \notin \{1, n-1\}$ i.e. $r(p)$ is other than 1 and $n-1$ .

Any leaf of $\mathcal{F}|_{U_p}$ contained in $\tau_p$ is called a local separatrix of $\mathcal{F}$ at $p$ , or a cone leaf at $p$. Any leaf of $\mathcal{F}$ such that its restriction to $U_p$ contains a local separatrix of $\mathcal{F}$ at $p$ is called a separatrix of $\mathcal{F}$ at $p$. Before defining the Morse foliation we should know what does saddle connection means. A saddle connection for $\mathcal{F}$ is a leaf which contains local separatrices of two different saddle singularities. A saddle self-connection for $\mathcal{F}$ at $p$ is a leaf which contains two different local separatrices of $\mathcal{F}$ at $p$.

## 2.2 Motivation for Morse foliations

**Definition 11.** *Let $\mathcal{F}$ be a codimension one $C^\infty$ foliation on a manifold $M$. A leaf $L$ of $\mathcal{F}$ has one-side holonomy if there is closed curve $c \subset L$ and $x_0 \in c$ whose holonomy map*

$$f : Dom(f) \subset \Sigma \to \Sigma$$

*on a transverse segment $\Sigma$ intersecting $c$ at $x_0$, $\Sigma \cap c = \{x_0\}$, satisfies the following conditions:*

*(i) $f =$ Identity in one of two connected componenets of $\Sigma - x_0$*

*(ii) $f \neq$ Identity in any neighbourhood of $x_0$ in $\Sigma$.*

Observe that a leaf with one-side holonomy can not be simply connected. But on the contrary every leaf which is not simply connected not necessarily has one-side holonomy, for example:

The torus fiber of a torus bundle over $S^1$ is not simply connected leaf without one-side holonomy.

### Haefliger theorem

**Theorem 10.** *Let $\mathcal{F}$ be a codimension one $C^2$ foliation on a manifold $M$. Suppose there exists a closed curve $\gamma : S^1 \to M$ with the following properties:*

*(i) $\gamma$ is transverse to $\mathcal{F}$, $\gamma \pitchfork \mathcal{F}$.*

*(ii) $\gamma$ is homotopic to a point in $M$ i.e. $\gamma$ is null homotopic.*

*Then there exists one-side holonomy leaf.*

**Proof.** Let $\mathcal{F}$ be a codimension one $C^2$ foliation on a manifold $M$. Let $\gamma : S^1 \to M$ be a null homotopic closed curve which is transverse to $\mathcal{F}$. So we can extend $\gamma : S^1 \to M$ to a map $A : D^2 \to M$. By Weierstrass approximation theorem, we can assume that $A$ is $C^\infty$. The map $A$ then approximated by a $C^\infty$ map $g : D^2 \to M$ transverse to $\mathcal{F}$ with $g^*(\mathcal{F}) = \mathcal{F}^*$, except at finite number of points $\{p_1, p_2, ......p_l\}$ where the tangency of $g$ with the leaves of $\mathcal{F}$ is nondegenerate i.e. center or saddle. Moreover $g$ satisfies the following properties:

(i) $g|_{\partial D^2}$ is transverse to $\mathcal{F}$ except at finite number of tangency points $\{p_1, p_2, ......, p_l\}$.

(ii) For every tangency point $p_i \in D^2$ of $g$ with $\mathcal{F}$ there exists a foliation box $U$ of $\mathcal{F}$ with $g(p_i) \in U$ and a distinguished map $\pi : U \to \mathbb{R}$ such that each $p_i$ is nondegenrate singularity of $\pi \circ g : g^{-1}(U) \to \mathbb{R}$ that is singularities of $\mathcal{F}^*$ are centers and saddles.

(iii) If $T = \{p_1, p_2, ......p_l\}$ is the set of tangency points of $g$ with $\mathcal{F}$, then $g(p_i)$ and $g(p_j)$ are contained in distinct leaves of $\mathcal{F}$ for every $i \neq j$. In particular the singular foliation $\mathcal{F}^* = g^*(\mathcal{F})$ has no distinct connected saddles; that is $\mathcal{F}^*$ has no saddle connections.

For saddle point $p_i$ of $\mathcal{F}^*$ we have four integral manifolds $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ of $\mathcal{F}^*|_V$ in a neighbourhood $V$ of $p_i$. These leaves accumulate on $p_i$ and are called local separatrices of $p_i$. If $\gamma$ is a leaf of $\mathcal{F}^*$ such that $\gamma \cap V$ contains a local separatrix of $p_i$ then we say that $\gamma$ is a separatrix of $p_i$, and if $\gamma \cap V$ contains two local separatrices of $p_i$ then we say $\gamma$ is a self connection of a saddle. When $\gamma$ is a separatrix of two distinct saddles, we say that $\gamma$ is a saddle connection and that the two saddles are connected.

If two distinct saddles $p_i$ and $p_j$ of $\mathcal{F}^*$ are connected then $g(p_i)$ and $g(p_j)$ are contained in the same leaf of $\mathcal{F}$. When this occur we can modify $g$ to obtain a map $\tilde{g} : D^2 \to M$ near $g$, such that $\tilde{g}(\tilde{p_i})$ and $\tilde{g}(\tilde{p_j})$ for $\tilde{p_i} \neq \tilde{p_j}$ saddles of $\tilde{g^*}(\mathcal{F})$ are in distinct leaves of $\mathcal{F}$. Hence we can suppose that $\mathcal{F}^*$ possibly having self connections but does not have distinct connected saddles.

Since $D^2$ is simply connected, there is a vector field $Y$ on $D^2$ with singularities $\{p_1, p_2, .......p_l\}$ whose regular orbits are leaves of $\mathcal{F}$. From this we obtain a foliation with singularities on $D^2$, $\mathcal{F}^* = g^*(\mathcal{F})$ transverse to the boundary and whose singular set $\{p_1, p_2, .......p_l\}$ is made up of centers and saddles. We say $\mathcal{F}^*$ is $C^r$ locally orientable since for every point $p \in D^2$ there exists a neighbourhood $U$ of $p$ and $C^r$ vector field $Y$ on $U$ such that;

$$Y(q) = 0, if q \in T$$

$$Y(q) \neq 0, if q \in U \setminus T$$

and $Y(q)$ is tangent to the leaf of $\mathcal{F}^*$ through $q$. It is clear that $\mathcal{F}^*$ is $C^2$ orientable close to the singularities. Far from the singularities we have that $\mathcal{F}^*$ is $C^2$ locally orientable by the tubular "flow box theorem". So we can say that $\mathcal{F}^*$ is orientable by the following proposition:

**Proposition 4.** *If $\mathcal{F}^*$ is a $C^r$ locally orientable foliation with singularities on $D^2$, then $\mathcal{F}^*$ is $C^r$ orientable.*

By applying Poincare-Bendixon theory to the vector field $Y$ we obtain a closed curve $\Gamma$, invariant under $Y$, and a transverse segment $\Sigma$ to $Y$ such that it is possible to define a first-return map $f$, in a neighbourhood of $x_0 = \Gamma \cap \Sigma$ in $\Sigma$, following the positive orbits of $Y$, which is identity on one of the components of $\Sigma - x_0$, but is not identity on any neighborhood of $x_0$ in $\Sigma$. The image of $\Gamma$ under $g$ defines a closed curve $g(\Gamma)$ in a leaf of $\mathcal{F}$ whose holonomy is conjugate to $f$.

**Corollary 3.** *Let $M$ be a compact manifold with finite fundamental group, and let $\mathcal{F}$ be a codimension one $C^2$ foliation on $M$. Then there exists one-side holonomy leaf.*

**Remark:** A codimension one foliation on a compact manifold with finite fundamental group is not real analytic.

**Example:** The Reeb foliation on $S^3$ is a codimension one foliation and finite fundamental group of $S^3$ is finite, so Reeb foliation has one-side holonomy leaves on $S^3$. By above remark Reeb foliation on $S^3$ is not real analytic.

## 2.3 Definition and Examples of Morse foliations

By knowing about the center and saddle singularities and also about saddle connections in the proof of Haefliger theorem, now we are in position to define Morse Foliations and give some examples.

**Definition 12.** *A Morse foliation $\mathcal{F}$ on a manifold $M$ is a $C^\infty$ singular codimension one transversely oriented foliation with isolated singularities such that:*

*(i) Each singularity p of $\mathcal{F}$ is of Morse type; (center or saddle) i.e. $p$ is non degenerate critical point of $f : U \to \mathbb{R}$ where $p \in U$ such that sing$(\mathcal{F}) \cap U = \{p\}$ and $\mathcal{F}|_U$ is given by $df = 0$.*

*(ii) Each singular leaf $L$ contains a unique singularity, i.e. there are no saddle connections.*

The foliations given by the levels of $C^\infty$ Morse functions $f : M \to \mathbb{R}$ are basic examples of Morse foliations. Therefore any $C^\infty$ smooth manifold supports a Morse foliation. There is no restriction on the topology of manifold for the existence of Morse foliation on $M$, but there are restrictions which come from the nature of singularities of Morse foliation $\mathcal{F}$ on $M$.

**Examples of Morse foliations**

**Definition 13.** *Let $M^n$ be a compact connected manifold of dimension $n \geq 3$. The foliation $\mathcal{F}$ on $M$ is a (singular) Seifert fibration of $M$ if its leaves are compact with finite fundamental group.*

**Example 9.** *Suppose we have Seifert fibration of $S^3$ i.e. singular foliation $\tilde{\mathcal{F}}$ of $S^3$ by centers and spheres $S^2$.*

*The inverse modification of foliation $\tilde{\mathcal{F}}$ by introducing the dead branches (we will discuss this in next section) we obtain foliation $\mathcal{F}$, which is a $C^\infty$ codimension one foliation with $c$ centers and $s$ saddles satisfy $c \geq s$ and has no saddle connections. The foliation $\mathcal{F}$ is $C^\infty$ Morse foliation on $S^3$.*

### Singular Reeb foliation

**Example 10.** *Singular Reeb foliation is an analogous of the Reeb foliation on the solid torus but exhibiting two Morse singularities in a center-saddle combination. We begin with a central sphere in $\mathbb{R}^3$ and introduce a center in the south pole and a saddle in the north pole as indicated in the figure below:*

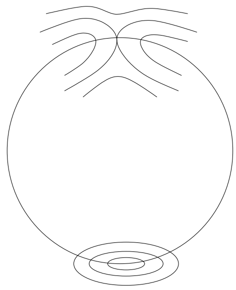

Figure 2.1:

*We obtain a foliation $\mathcal{F}$ in a singular solid torus as indicated below:*

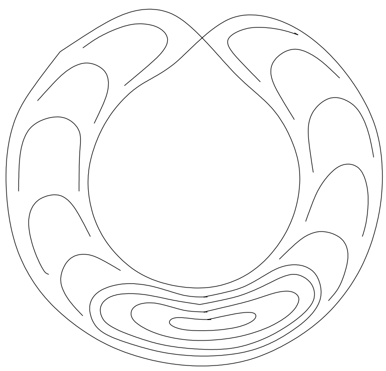

Figure 2.2:

*The leaves of this foliation $\mathcal{F}$ on solid torus:*

- *Inside leaves are diffeomorphic to $S^2$.*
- *The boundary leaf which is boundary of singular solid torus contains a self connection of pair of separatrices for saddle singularity.*

*By extension of the foliation $\mathcal{F}$ to the exterior of the singular solid torus, the leaves are diffeomorphic to the $S^1 \times S^1$ having trivial outside holonomy, as shown in figure:*

Figure 2.3:

*The singular solid torus, which contains the separatrices of the saddle, is diffeomorphic to a sphere which is made by pair of pinchings on sphere and joined these two points.*

### Foliation on Eells-Kuiper manifold

**Example 11.** *Let $M^n$ be a connected closed manifold. Suppose $M^n$ admits a Morse function $f : M \to \mathbb{R}$ of class $C^3$ with exactly three singular points. Then $M^n$ is a topologically a compactification of $\mathbb{R}^n$ by $\frac{n}{2}$- sphere for $n \in \{2, 4, 8, 16\}$, called Eells-Kuiper manifolds.*

*Levels of the Morse function $f : M \to \mathbb{R}$ on Eells-Kuiper manifolds define codimension one foliation with exactly three Morse singularities (two centers and one saddle).*

### Singular Seifert fibration

**Example 12.** *For foliation with Morse singularities on $M^n, n \geq 3$ we denote the set, $\mathcal{C}(\mathcal{F})$=Union of all centers and leaves diffeomorphic to $S^{n-1}$ in $M^n$.*

*Given center singularity $p \in sing(\mathcal{F})$ we denote $\mathcal{C}_p(\mathcal{F})$ as, Connected component of $\mathcal{C}(\mathcal{F})$ that contains $p$. Since $\mathcal{C}_p(\mathcal{F})$ is open in $M$ so we have $\mathcal{C}_p(\mathcal{F}) = M$ if and only if $\partial\mathcal{C}_p(\mathcal{F}) = \emptyset$. So singularities of $\mathcal{F}$ are centers and leaves diffeomorphic to $S^{n-1}$. This Foliation which is codimension one foliation with just center sigularities, will be called singular Seifert fibration.*

### Codimension one $C^\infty$ foliation in the Closed ball $\overline{\mathbb{B}^4}$

**Example 13.** *Here we give an example of codimension one $C^\infty$ foliation in the closed ball $\overline{\mathbb{B}^4}$, of radius one centered at $0 \in \mathbb{R}^4$ with only one singularity of saddle $2-2$ type at $0 \in \mathbb{B}^4$ and transverse to the boundary $S^3 = \partial\mathbb{B}^4$. Consider a function*

$$f(x) = -x_1^2 - x_2^2 + x_3^2 + x_4^2$$

*in $\mathbb{R}^4$. The level zero of this function is:*

$$C = f^{-1}(0),$$

*is a cone over two torus. This can easily be seen by taking intersection,*

$$C \cap S^3 = \mathbb{T}$$

*which is clearly a 2-torus ,intersection of the cylinders,*

$$x_1^2 + x_2^2 = \tfrac{1}{2} \text{ and } x_3^2 + x_4^2 = \tfrac{1}{2}$$

*For given $\epsilon > 0, f^{-1}([-\epsilon, \epsilon])$ is a neighborhood of $C$ and ,*

$$\mathbb{R}^4 \setminus f^{-1}([-\epsilon, \epsilon]) = R_1 \cup R_2 \ ,$$

*where $R_1$ and $R_2$ are two connected components diffeomorphic to $\mathbb{B}^4 \times S^1$ ,and $R_1 \cap \{x_3 = x_4 = 0\} \neq \emptyset, R_1 \cap \{x_1 = x_2 = 0\} \neq \emptyset$. For $\epsilon > 0$ small enough ,*

$$S^3 \setminus f^{-1}([-\epsilon, \epsilon]) = \mathbb{T}_1^2 \cup \mathbb{T}_2^2 \ ,$$

*where $\mathbb{T}_1^2$ and $\mathbb{T}_2^2$ are two solid tori i.e. diffeomorphic to $\mathbb{B}^2 \times S^1$. We define a new domain ,*

$$\mathbb{D} = f^{-1}([-\epsilon, \epsilon]) \cup S_1 \cup S_2 \ ,$$

*where* $S_1 \subset R_1$ *and* $S_2 \subset R_2$ *are diffeomorphic to* $\mathbb{B}^3 \times S^1$ *such that ,*

$$\partial S_1 \cap \mathbb{B}^4 = \partial R_1 \cap \mathbb{B}^4 \text{ and } \partial S_2 \cap \mathbb{B}^4 = \partial R_2 \cap \mathbb{B}^4.$$

*we define foliation* $\mathcal{F}$ *on* $\mathbb{D}$ *which has leaves as levels of* $f$ *on* $f^{-1}((-\epsilon, \epsilon))$*. on* $S_1$ *we introduce Reeb component on* $\mathbb{B}^3 \times S^1$ *whose axis is circle* $(x_3 = x_4 = 0) \cap S^3$*, having as sections on each* $\mathbb{B}^3 \times \{\theta\}$ *a foliation by spheres* $S^2$*. Similarly on* $S_2$ *we introduce Reeb component on* $\mathbb{B}^3 \times S^1$ *whose axis is circle* $(x_1 = x_2 = 0) \cap S^3$ *, Leaves of* $\mathcal{F}$ *are transverse to* $S^3$*. We finally take restriction* $\mathcal{F}|_{\overline{\mathbb{B}^4}}$ *.*

## 2.4 Dead branches, coupling and elimination of trivial center-saddle pairs of singularities

In this section we shall learn how to replace a non-trivial foliation having trivial center-saddle pairs of singularities by a trivial foliation after performing modifications under suitable conditions. By performing modifications under suitable conditions we can eliminate certain pairs of center-saddle singularities.

It is also possible to construct such pairings of two saddle singularities of complementary Hopf indices which are in stable connection, which we shall discuss in chapter four.

**Modification in dimension two**

Lets see first, the elimination of certain arrangements of singularities in dimension two. This elimination procedure may be seen as follows:

Vector field denote by $Z_\epsilon = (x_1^2 - \epsilon)\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$ , $\epsilon > 0$

This vector field has:

- A pair of saddle-source for $\epsilon > 0$
- A saddle node singularity for $\epsilon = 0$
- No singularity for $\epsilon < 0$

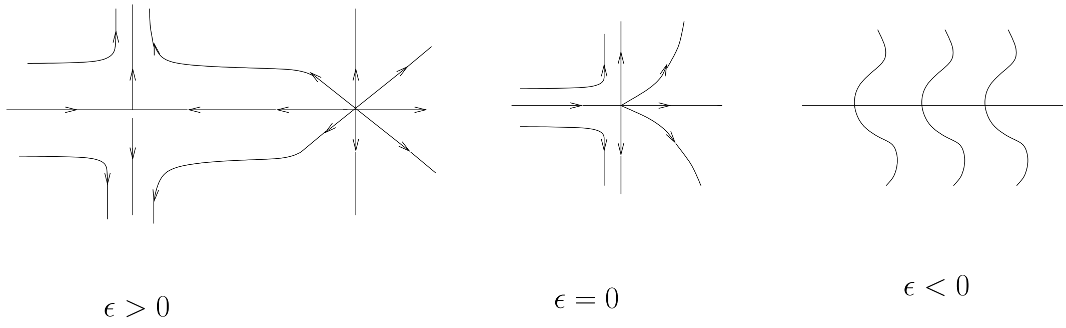


Figure 2.4:

By solving the equation ,

$$grad(\Omega_\epsilon) = Z_\epsilon$$

we obtain dual foliation. This gives $\Omega_\epsilon = df_\epsilon$ for the function ,

$$f_\epsilon = (\frac{x_1^3}{3} - \epsilon x_1) + \frac{x_2^2}{2} \ ,$$

whose level curves can be drawn by using figure **??**.

Thus the original non-trivial foliation having center-saddle pairing can be deformed by a trivial vertical foliation via passing through a saddle-node. So in dimension two our basic diagram is the following:

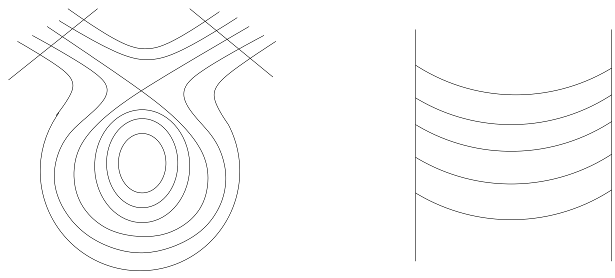

Figure 2.5:

We have a pair of center-saddle singularities which is replaced by a trivial foliation.

**Remark** The replacement of a center-saddle pairing as above does not change the holonomy of the foliation.

## Modification in dimension three

Let $\mathcal{F}_1 : d(-x_1^2+x_2^2+x_3^2)=0$ be foliation in a neighborhood of $0 \in \mathbb{R}^3$ having cone leaves and $\mathcal{F}_2 : d(x_1^2+X_2^2+x_3^2)=0$ be a foliation having spherical leaves. The tangency set between these two foliations is given by $\{x_1=0\} \cup \{x_2=x_3=0\}$.

We have therefore the following figure:

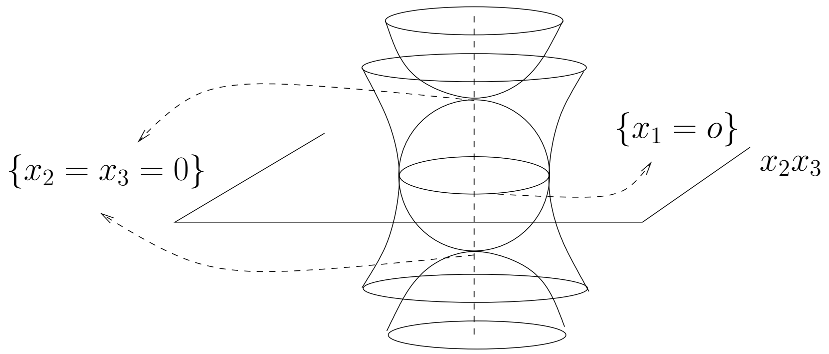


Figure 2.6:

We modify $\mathcal{F}_1$ replacing it by a foliation by concentric spheres centered at some point $\theta=(-b,0,0)$ in a region $\{x_1 \leq -a^2\}$ . The final result is center-saddle pair as depicted below:

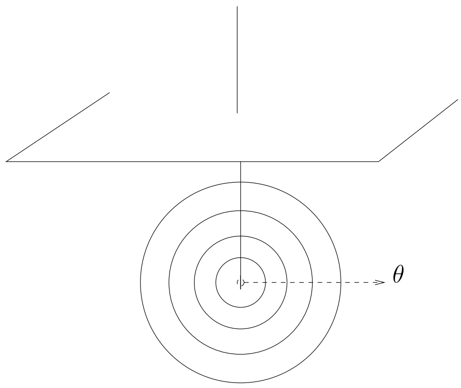


Figure 2.7:

The resulting foliation after modification is called trivial center-saddle pairing as depicted below:

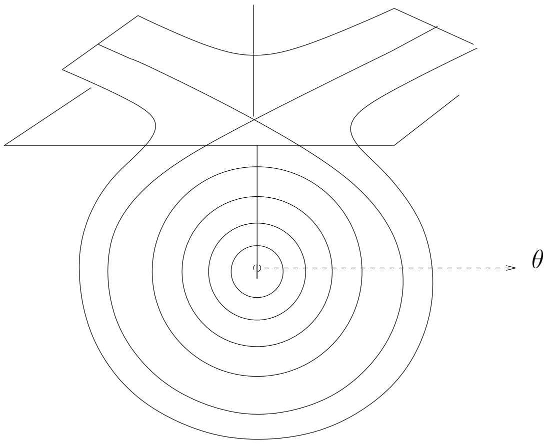


Figure 2.8:

## Dead Branches

**Definition 14.** *M be a Let $\mathcal{F}$ be a codimension one foliation with isolated singularities on a manifold $M^n$. By a dead branch of $\mathcal{F}$ we mean a region $R \subset M$ diffeomorphic to the product $B^{n-1} \times I$ of the closed unit ball $B^{n-1}$ by the interval $I = [0,1]$ as a manifold with corners and boundary. We assume sing$(\mathcal{F}) \cap R \neq \emptyset$ and boundary of the region,*

$$\partial R = B^{n-1} \times \partial I \cup \partial B^{n-1} \times I$$

*where $B^{n-1} \times \partial I$ is union of two connected invariant components (pieces of leaves of $\mathcal{F}$ ) say $L_1$ and $L_2$ and $\partial B^{n-1} \times I$ totally transverse curves (segments transverse to $\mathcal{F}$ ) say $\Sigma_1$ and $\Sigma_2$ (see figure 2.9), so*

$$\partial R = L_1 \cup L_2 \cup \Sigma_1 \cup \Sigma_2$$

It is clear that we can replace the foliation inside a given dead branch with the trivial foliation. A trivial center-saddle pairing is an example of dead branch.

Moreover we also assume that the holonomy from $\Sigma_1$ to $\Sigma_2$ is trivial in the sense that $\mathcal{F}|_{\Sigma_1}$ and $\mathcal{F}|_{\Sigma_2}$ are conjugated by a diffeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $L_{h(p)} = L_p, \forall p \in \Sigma_1$; except if $p$ belongs to a leaf containing a separatrix of some singularity of $\mathcal{F}$ in region $R$, in which case the image of $p$ will be another point $h(p)$ belonging to a leaf containing a separatrix of the same singularity of $\mathcal{F}$ in region $R$.

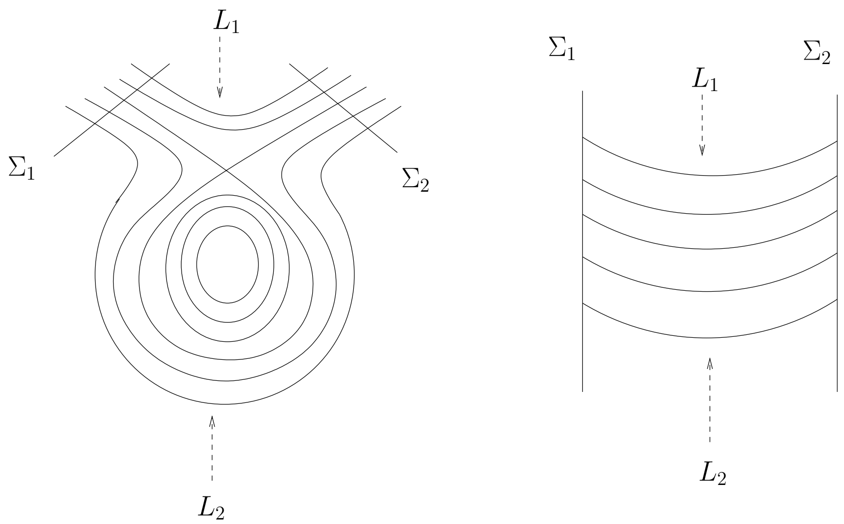


Figure 2.9:

**Definition 15.** *Two singularities $p, q$ of a foliation $\mathcal{F}$ on $M$ are said to be in trivial coupling or trivial pairing if they belong to a dead branch $R$ of $\mathcal{F}$ and $\mathcal{F}$ has no other singularities in $R$.*

**Proposition 5.** *Let $\mathcal{F}$ be a codimension one foliation with isolated singularities on $M$ having a dead branch $R \subset M$. Then there is a foliation $\tilde{\mathcal{F}}$ on $M$ such that:*

*(i) $\tilde{\mathcal{F}}$ and $\mathcal{F}$ agree on $M \setminus R$.*

*(ii) $\tilde{\mathcal{F}}$ is non singular in a neighborhood of $R$, indeed $\tilde{\mathcal{F}}|_R$ is conjugate to a trivial fibration.*

*(iii) The holonomy of $\tilde{\mathcal{F}}$ is conjugate to the holonomy of $\mathcal{F}$in the following sense:*

*Given any leaf $L$ of $\mathcal{F}$ such that $L \cap (M \setminus R) \neq \emptyset$ then corresponding leaf $\tilde{L}$ of $\tilde{\mathcal{F}}$ satisfies $Hol(\tilde{\mathcal{F}}, \tilde{L})$ is conjugate to $Hol(\mathcal{F}, L)$.*

**Definition 16.** *We shall call $\tilde{\mathcal{F}}$ a direct modification of $\mathcal{F}$ by elimination of dead branch. If a foliation $\mathcal{F}$ is obtained from a foliation $\tilde{\mathcal{F}}$ by introduction of a dead branch then we shall say $\mathcal{F}$ is an inverse modification of $\tilde{\mathcal{F}}$.*

**Example:** This is an example of combination of a center-saddle pairing where the saddle is accumulated by spherical leaves from a third center singularity. We begin with a foliation given by a center singularity and by an inverse modification we introduce in a regular part a pair center-saddle as depicted as:

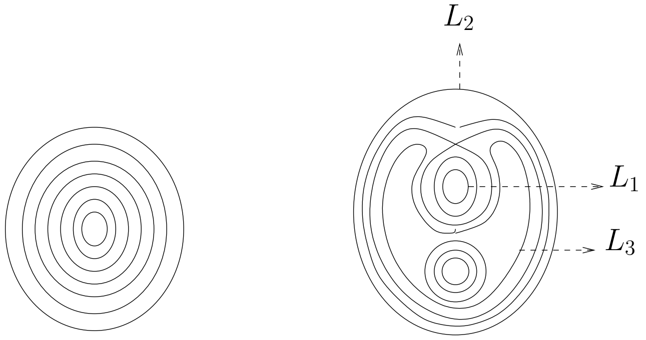


Figure 2.10:

The separatrix of the saddle gives a self connection and has the topology of two spheres with a unique intersection point. All leaves are diffeomorphic to spheres and if we consider only the annular region bounded by one internal leaf $L_1$ and one external leaf $L_2$ then we have non-trivial center-saddle pairing as depicted below:

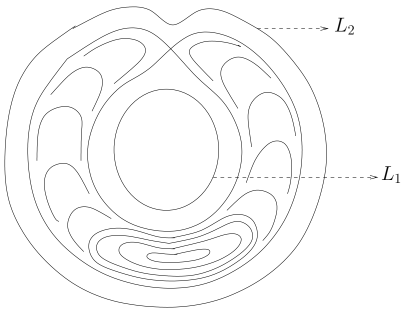


Figure 2.11:

By using of concept of dead branch we show how to transform a singular Reeb foliation in to a regular foliation. Given a singular Reeb foliation $\mathcal{F}$ lets assume that the center and saddle are close. In a small box $B$ around these two singularities we have the following diagram:

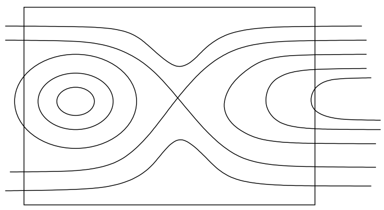

Figure 2.12:

We may therefore replace the foliation $\mathcal{F}$ in the box $B$ by a regular foliation as depicted below:

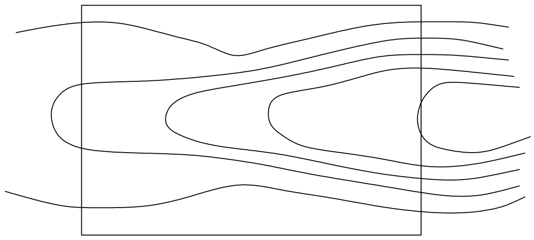

Figure 2.13:

Thus we can replace the above pairing center-saddle by a regular foliation in the box $B$ having some plane leaves and some cylindrical leaves as well.

## 2.5 Topology of separatrices

Let $\mathcal{F}$ be a Morse foliation on a compact n-dimensional manifold $M^n, n \geq 3$. We denote the set, $\mathcal{C}(\mathcal{F})$ = Union of all center singularities and leaves diffeomorphic to $S^{n-1}$ in $M^n$. For given center singularity $p \in \text{sing}(F)$ we denote the set, connected component of $\mathcal{C}(\mathcal{F})$ that contains singularity $p$, $\mathcal{C}_p(\mathcal{F})$.

**Remark:** (i) $\mathcal{C}(\mathcal{F})$ is open in $M$ as a consequence of Reeb local stability theorem.

(ii) $\mathcal{C}_p(\mathcal{F})$ is open in $M$ and $\mathcal{C}_p(\mathcal{F}) \cap \mathcal{C}_q(\mathcal{F}) \neq \emptyset$ if and only if $\mathcal{C}_p(\mathcal{F}) = \mathcal{C}_q(\mathcal{F})$.

(iii) If $q \in \text{sing}(\mathcal{F}) \cap \partial\mathcal{C}(\mathcal{F})$ then $q$ must be a saddle singularity.

(iv) Since $\mathcal{C}_p(\mathcal{F})$ is open in $M$ we have $\mathcal{C}_p(\mathcal{F}) = M$ if and only if $\partial\mathcal{C}_p(\mathcal{F}) = \emptyset$. In this case $\mathcal{F}$ is a foliation by center singularities and leaves diffeomorphic to $S^{n-1}$, therefore it defines a singular fibration $M \to S^1$ with fibers $S^{n-1}$. The foliation in this case will be called singular Seifert fibration.

We focus on 3-dimensional case. Let $p \in \text{sing}(\mathcal{F})$ be a center singularity and $q \in \text{sing}(\mathcal{F}) \cap \partial\mathcal{C}_p(\mathcal{F})$ be a saddle singularity. We denote a leaf of $\mathcal{F}$ that contains the separatrix of $\mathcal{F}$ through $q$, by $\Gamma_q$, which is accumulated by spherical leaves in $\mathcal{C}_p(\mathcal{F})$.

Since $\Gamma_q$ is accumulated by spheres, so for $\Gamma_q \cup \{q\}$ we have following possibilities:

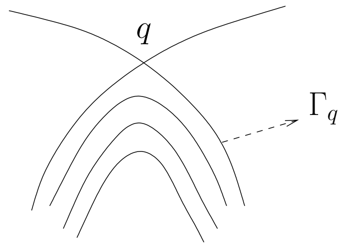


Figure 2.14:

(i) $\Gamma_q \cup \{q\}$ is homeomorphic to sphere $S^2$.

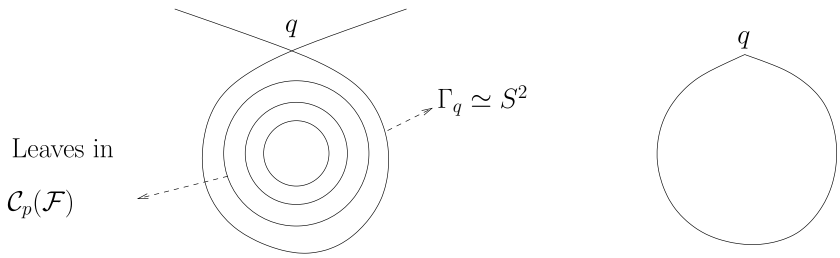


Figure 2.15:

(ii) $\Gamma_q \cup \{q\}$ is homeomorphic to a singular torus, which can be obtained by pinching a sphere at two different points and joining them.

By using this we obtain a lemma exclusively for dimension 3.

**Lemma 5.** *Let $\mathcal{F}$ be a Morse foliation on compact 3-dimensional manifold $M^3$. If $p \in$ sing($\mathcal{F}$) is a center singularity and $\partial\mathcal{C}_p(\mathcal{F}) \neq \emptyset$, then $\{q\} =$ sing($\mathcal{F}$) $\cap\, \partial\mathcal{C}_p(\mathcal{F})$ is a saddle and we have following possibilities for $\mathcal{C}_p(\mathcal{F})$ and $\partial\mathcal{C}_p(\mathcal{F})$ :*

*(i) $\partial\mathcal{C}_p(\mathcal{F}) \setminus \{q\}$ is connected. Then*

*(a) $\partial\mathcal{C}_p(\mathcal{F})$ is homeomorphic to a sphere $S^2$ with a pinch at $q$ and the pair $q - p$ belongs to a dead branch i.e. it can be modified to a trivial foliation; or*

*(b) $\partial\mathcal{C}_p(\mathcal{F})$ is homeomorphic to a singular torus obtained by pinching a sphere at two points and joining these points. So $\mathcal{C}_p(\mathcal{F})$ with $\partial\mathcal{C}_p(\mathcal{F})$ is singular Reeb component.*

*(ii) $\partial\mathcal{C}_p(\mathcal{F}) \setminus \{q\}$ has two connected components. Then $\partial\mathcal{C}_p(\mathcal{F})$ is union of two spheres $S^2$ with a common point $q$.*

# Chapter 3

# Applications of dead branches having trivial center-saddle pairings

In this chapter we shall discuss some results from Camacho , Scardua's papers [2],[3] to understand the application of dead branch. We shall discuss foliations transverse to spheres and a variant of Haefliger's theorem for $S^3$ and some results which have been proved in [2], [3] by Camacho, Scardua.

In 1978, E. Wagneur [44] generalized the Reeb sphere theorem to Morse foliations with saddles. He showed that the number of centers cannot be too much as compared with the number of saddles, notably, $c \geq s + 2$. So there are exactly two cases when $c > s$:
(1)$c = s + 2$
(2)$c = s + 1$

He obtained a description of the manifold admitting a foliation with singularities that satisfy (1).

Finally, in 2008, C. Camacho and B. Scardua considered the case (2), c=s+1. Interestingly, this is possible in a small number of low dimensions.

**Theorem.** Let $M^n$ be a compact connected manifold and F a Morse foliation on $M$. If $s = c + 1$, then:
(1)$n = 2, 4, 8$ or 16,
(2)$M^n$ is an Eells-Kuiper manifold.

## 3.1 Orientability of foliations and index lemma

A codimension one foliation $\mathcal{F}$ with isolated singularities on $M$ is called orientable if there exists $C^\infty$ integrable one-form $\Omega$ on $M$ such that sing$(\mathcal{F})$ = sing$(\Omega)$, the integrability of one-form $\Omega$ is in the sense that $\Omega \wedge (\dot{\Omega}) = 0$ everywhere, and foliation $\mathcal{F}$ coincide with the foliation defined by $\Omega = 0$ outside singular set.

Such integrable one form is called an orientation for $\mathcal{F}$ and two such integrable one-forms $\Omega$ and $\Omega'$ define the same orientation for $\mathcal{F}$ if

$$\Omega = h.\Omega'$$

for some positive function $h$ on $M$.

The foliation $\mathcal{F}$ is said to be locally orientable if each point (even singular) $p \in M$ admits a neighborhood where $\mathcal{F}$ is orientable i.e. $\mathcal{F}$ is defined in that neighborhood by an integrable one-form $\Omega_p$ as above. Basic example of an orientable foliation is a foliation with Morse type singularities on a simply connected manifold. Note that a $C^\infty$ oriented foliation $\mathcal{F}$ with isolated singularities is given by a $C^\infty$ integrable one-form $\Omega$ on $M$ with isolated singularities.

**Definition 17.** *Let $\overrightarrow{X}$ be vector field on $M \subset \mathbb{R}^n$ defined by*

$$\overrightarrow{X} : M \to TM$$

$$\overrightarrow{X}(x) = (x, v(x)), v(x) \in T_xM$$

*A singularity $p \in Sing(X)$ is called simple singularity of vector field $\overrightarrow{X}$ if*

$$Det\,(V'(x))_{x=p} \neq 0$$

*Since $M \subset \mathbb{R}^n$, $TM \cong M \times \mathbb{R}^n$ therefore $X(x) \cong V(x)$.*

**Definition 18.** *Let $p \in Sing(X)$ be a simple singularity of a vector field $\overrightarrow{X}$. We define the index of a vector field $\overrightarrow{X}$ at singularity $p$ as:*

$$I(X,p) = \begin{cases} +1, & if Det\,(DX(p)) > 0 \\ -1, & if Det\,(DX(p)) < 0 \end{cases}$$

**Definition 19.** *Let $f : M \to \mathbb{R}$ be a differentiable function and let $p \in M$ be a singularity of $f$ then the index of $f$ at $p$ is defined as follow:*

$$Ind\,(f,p) = max\{dim(E), E \subset T_pM, H(p).u < 0\}$$

*where $E$ is a subspace of $T_pM$, $H(p)$ is Hessian of $f$ at $p$ and $u \in E - \{0\}$.*

Now we correlate the index of a vector field with the index of integrable one-form which gives codimension one oriented foliation.

Let $p \in M$ and choose a local chart of $M$,

$$\phi : U \to \phi(U) \subset \mathbb{R}^n \quad such \quad that,$$

$$\phi(p) = 0, \quad p \in U$$

and $\text{sing}(\Omega) \cap U = \{p\}$. Let $\phi_*(\Omega) = \omega \in \Lambda(\phi(U))$. We write $\omega = \sum_{j=1}^{n} f_j dx_j$ with $C^\infty$ map, $f_j : \phi(U) \to \mathbb{R}$ and $f_j(0) = 0$, $j = 1, 2, ......n$ . Let $grad(\omega) = \sum_{j=1}^{n} f_j \frac{\partial}{\partial x_j}$ be the gradient vector field of $\omega$. We define the index of integrable one-form $\Omega$ at singular point $p$ by ,

$$\text{Index}(\Omega; p) = \text{Index}(\text{grad}(\omega); 0)$$

where $\text{Index}(\text{grad}(\omega); 0)$ is the ordinary poincare-Hopf index of smooth vector field $\text{grad}(\omega)$ at singular point $0 \in \mathbb{R}^n$ . The definition of $\text{Index}(\Omega; p)$ does not depend on the chart $\phi : U \to \phi(U) \subset \mathbb{R}^n$ . The $\text{Index}(\Omega; p) = 0$ if $p \notin \text{sing}(\Omega)$ .

We have then the following natural adaptation of Poincare-Hopf index theorem to foliations with isolated singularities :

**Lemma 6.** *Let $M^n$ be an oriented manifold and $D \subset M$ be a domain with connected regular boundary of class $C^2$. Let $\mathcal{F}$ be a codimension one foliation with isolated singularities on $M$ given by $C^\infty$ integrable one form such that $sing(\mathcal{F}) = sing(\Omega)$. Suppose $\mathcal{F}$ is either transverse to the boundary $\partial D$ or tangent to the boundary $\partial D$. Moreover in the tangent case, if $n$ is odd, suppose that $grad(\Omega)$ points outwards at the boundary . Then we have*

$$\sum_{p \in sing(\Omega) \cap D} Ind(\Omega; p) = \mathcal{X}(D),$$

*where $\mathcal{X}(D)$ is Euler characteristic of $D$.*

## Foliations transverse to spheres

Now we study the existence and properties of Morse foliations transverse to spheres. we begin with the simple situation:

Let $\mathcal{F}$ be a $C^\infty$ Morse foliation defined in a neighborhood $W$ of the closed ball $\overline{B}^n = \overline{B^n(0,1)}$ in $\mathbb{R}^n$ and transverse to the boundary spheres $S^{n-1}(0,1) = \partial\overline{B^n(0,1)}$. We can obtain a one-form $\Omega$ which defines $\mathcal{F}$ in $W$ because $\overline{B}^n$ is simply connected. We have local coordinates $(y_1, y_2, .......y_n) \in U_p \subset B^n$ for given singularity $p \in \mathrm{sing}(\mathcal{F}) \subset B^n$ such that ,

$$\Omega(y_1, .........y_n) = h_p d(-y_1^2...... - y_{r_p}^2 + y_{r_{p+1}}^2 + ........ + y_n^2)$$

for a $C^\infty$ function $h_p > 0$ in $U_p$ . We have defined the index of $\mathcal{F}$ at $p$ with respect to the orientation defined by $\Omega$ as $\mathrm{Index}(\mathcal{F}, p) = (-1)^{r_p} \in \{+1, -1\}$ . We have $\sum_{p \in \mathrm{sing}(\mathcal{F})} \mathrm{Ind}_\Omega(\mathcal{F}; p) = +1$ by the index theorem, in particular , $\mathrm{sing}(\mathcal{F}) \neq \emptyset$ and $\mathcal{F}$ has odd number of singularities in the ball . We have $\mathcal{X}(S^{n-1}) = 0$ , because boundary sphere admits a transverse foliation . Therefore $n$ is an even number . In this case $\mathrm{Index}_\Omega(\mathcal{F}; p)$ does not depend on the orientation fixed for $\mathcal{F}$ .

**Remark. (i):** A center singularity always has index $+1$ .**(ii):** A saddle singularity may have index $+1$ or $-1$ . **(iii):** By the index theorem $\sum_{p \in \mathrm{sing}(\mathcal{F})} \mathrm{Ind}_\Omega(\mathcal{F}; p) = +1$ and by definition of index $\mathrm{Index}(\mathcal{F}, p) = (-1)^{r_p} \in \{+1, -1\}$ , for $n = 2$ $\mathcal{F}$ has some center singularity because in dimension two a saddle singularity has index $-1$. **(iv):** For $n \geq 3$ the set $\mathrm{sing}(\mathcal{F})$ must contain a saddle singularity .

## 3.2 Haefliger's type theorem for $\mathbf{S^3}$

Before discussing Haefliger's type theorem for 3-sphere in [2] , first we would like to state classical Haefliger's theorem for the disc :

**Theorem.** Let $\overrightarrow{X}$ be a $C^1$ vector field defined in a neighborhood $U$ of disc $D^2 \subset \mathbb{R}^2$ such that $\overrightarrow{X} \pitchfork \partial\overline{D^2}$ points inward the disc satisfying the following conditions :

(i) $\overrightarrow{X}$ has only Morse singularities in disc $D^2$ .

(ii) $\overrightarrow{X}$ will be without saddle connections .

Then there exists a unilateral compact invariant one dimensional subset $\Gamma \subset D^2$.

Aim of discussing the Haefliger's theorem for $S^3$ is to understand the use of dead branch and modification inside dead branch to prove variant of Haefliger's theorem for foliations with Morse singularities.

**Theorem 11** (Camacho-Scardua, 2006)**.** *Let $\mathcal{F}$ be a $C^\infty$ Morse foliation on 3-sphere $S^3$ having $c$ centers and $s$ saddles satisfying the inequality $c \geq s$. Then $\mathcal{F}$ is an inverse modification of a Seifert fibration of $S^3$, i.e. a singular foliation of $S^3$ by Spheres $S^2$ and centers , or we have one of the following possibilities:*

*(i) There is a compact codimension one invariant subset whose holonomy is one-sided.*

*(ii) There is a singular Reeb component of $\mathcal{F}$ .*

**Proof.** We fix an orientation for $\mathcal{F}$. By hypothesis we have $c$ centers and $s$ saddles satisfying $c \geq s$. We will proceed by induction on saddle $s$.

(I) Consider the case $s = 0$, so $c \geq 0$. For $s = 0$ we further have two cases :

  (i) $c = 0$ ($s = 0$ already) , we have nonsingular codimension one foliation $\mathcal{F}$ on $S^3$, so by Novikov theorem $\mathcal{F}$ has some Reeb component and therefore $\mathcal{F}$ has a toral leaf $L \simeq S^1 \times S^1$ with one-side holonomy group.

  (ii) $c \geq 1$ ($s = 0$ already ), foliation $\mathcal{F}$ has only center singularities , therefore it is Seifert fibration by Reeb .

(II) Assume now $s \geq 1$, since by hypothesis we have $c \geq s$, so $c \geq s \geq 1$. Assume that result is true for $s - 1$ saddle singularities .

(III) Now suppose we have $c$ centers and $s$ saddles satisfying the inequality $c \geq s$, suppose a center $p_1$ in $S^3$. We denote $\mathcal{C}_{p_1}(\mathcal{F})$ connected component of $\mathcal{C}(\mathcal{F})$ which contains $p_1$, where $\mathcal{C}(\mathcal{F}) =$ Union of all centers and leaves diffeomorphic to $S^2$ of the foliation $\mathcal{F}$. Since we have $\mathcal{C}_{p_1}(\mathcal{F})$ so we have two cases for the boundary $\partial\mathcal{C}_{p_1}(\mathcal{F})$:

  (i) $\partial\mathcal{C}_{p_1}(\mathcal{F}) = \emptyset$ then $\mathcal{C}_{p_1}(\mathcal{F}) = S^3$, so all leaves of $\mathcal{F}$ are compact diffeomorphic to $S^2$ with trivial holonomy . In other words $\mathcal{F}$ is singular Seifert fibration of $S^3$.

  (ii) $\partial\mathcal{C}_{p_1}(\mathcal{F}) \neq \emptyset$ then by lemma 5 in section 2.5, $\partial\mathcal{C}_{p_1}(\mathcal{F}) \cap \text{sing}(\mathcal{F}) \neq \emptyset$, so any leaf $L \subset \partial\mathcal{C}_{p_1}(\mathcal{F})$ is separatrix of some saddle singularity $q_1$. This singularity is unique, because $\mathcal{F}$ has no saddle connections. On the other hand we can not have $\partial\mathcal{C}_{p_1}(\mathcal{F}) \subset \text{sing}\mathcal{F}$. Thus we can find a leaf $L_0$ of $\mathcal{F}$ such that $\Gamma_{q_1} = L_0 \cup$

$\{q_1\} \subset \partial\mathcal{C}_{p_1}(\mathcal{F})$, where $L_0$ is separatrix of saddle $q_1$. Since $\Gamma_{q_1}$ is accumulated by spherical leaves, so we have two possibilities for holonomy of $\Gamma_{q_1}$:

(a) $\Gamma_{q_1}$ has non-trivial holonomy and since $\Gamma_{q_1}$ is accumulated by spherical leaves, so $\Gamma_{q_1}$ has one-side holonomy and the result follows.

(b) $\Gamma_{q_1}$ has trivial holonomy then by lemma 5 in section 2.5, we have following three possibilities:

(i) We have trivial center-saddle pairing $p_1 - q_1$. We can eliminate both singularities $p_1$ and $q_1$ belonging to a dead branch by modification. After modification we obtain $\mathcal{F}_1$ in $S^3$ with same holonomy. This modified foliation $\mathcal{F}_1$ has one less center singularity and one less saddle singularity. But still we have $\#\{\text{centers of } \mathcal{F}_1\} \geq \#\{\text{saddles of } \mathcal{F}_1\}$. So by induction hypothesis, either $\mathcal{F}_1$ is an inverse modification of a Seifert fibration of $S^3$, or $\mathcal{F}_1$ satisfy one of the two possibilities mentioned in the statement of the theorem, and therefore we have the same possibilities for $\mathcal{F}$.

(ii) $\Gamma_{q_1}$ is homeomorphic to singular torus. Since we are in case $(b)$ in which $\Gamma_{q_1}$ has trivial holonomy, and $\Gamma_{q_1}$ is surrounded by leaves diffeomorphic to the torus. So we can isolate the region $R \subset S^3$ containing $\overline{\mathcal{C}_{p_1}(\mathcal{F})}$ invariant by $\mathcal{F}$ and diffeomorphic to solid torus where we have defined singular Reeb foliation.

(iii) The saddle $q_1$ is not self-connected. We have non-trivial center-saddle pairing $p_1 - q_1$ and $\Gamma_{q_1} \setminus \{q_1\}$ is diffeomorphic to sphere minus one point. In this case other separatrix of $q_1$ is also homeomorphic to a sphere with a pinch at $q_1$. As we are in case $(b)$ in which $\Gamma_{q_1}$ has trivial holonomy, these two separatrices are surrounded by spherical leaves. Thus we can fix an invariant region $R'$ which is bounded by $L_2$ and $L_3$, diffeomorphic to $S^2 \times [0, 1]$, containing the union of separatrices and with invariant boundary.

Inside region $R'$ we perform modification to $\mathcal{F}$. We obtain in this way a trivial foliation by spheres say $\mathcal{F}_1$ on $S^3$ with same holonomy as $\mathcal{F}$. $\mathcal{F}_1$ has one less center singularity and one less saddle singularity than $\mathcal{F}$. Again by induction hypothesis, either $\mathcal{F}_1$ is an inverse modification of a Seifert fibration of $S^3$, or $\mathcal{F}_1$ satisfy one of the two possibilities

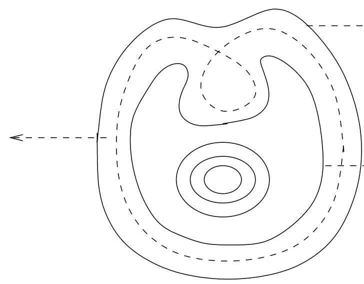


Figure 3.1:

mentioned in the statement of the theorem. Again we have the same possibilities for $\mathcal{F}$.

Hence the result follows.

## 3.3 Topology of $\mathbf{M^3}$ admitting Morse foliations

In this section we shall discuss a theorem which answers the question that; under what conditions a closed, connected and oriented 3-manifold $M^3$ which admits Morse foliation, will be diffeomorphic to 3-sphere $S^3$. The answer has been given by Camacho, Scardua in [2]. The aim of discussing this result is also to understand the concept of dead branch. Later we give some counter examples for general case. Those examples show the importance of hypothesis for the theorem. In the proof of the theorem we will need the following lemma:

**Lemma 7.** *Suppose an oriented* 3*-manifold* $M^3$ *which admits an oriented Morse foliation* $\mathcal{F}$*. Let* $q \in sing(\mathcal{F})$ *be a saddle singularity such that* $q \in \partial\mathcal{C}_{p_1}(\mathcal{F}) \cap \partial\mathcal{C}_{p_2}(\mathcal{F})$ *for two distinct centers* $p_1, p_2 \in sing(\mathcal{F})$*. Then union of separatrices of* $\mathcal{F}$ *through* $q$ *with* $\{q\}$ *is compact with each branch homeomorphic to* $S^2$ *and* $q$ *belongs to a dead branch with a pairing* $q - p_1$ *or* $q - p_2$*.*

**Theorem 12** (Camacho-Scaruda, 2006)**.** *Suppose a closed oriented* 3*-manifold* $M^3$ *which admits an oriented Morse foliation* $\mathcal{F}$ *having* $c$ *center singularities and* $s$ *saddle singularities satisfying* $c \geq s + 1$*. Then* $M^3$ *is diffeomorphic to* $S^3$*. Indeed* $\mathcal{F}$ *admits an isotopy to a Morse foliation having only two centers as singularities.*

**Proof.** By hypothesis $M^3$ admits an Morse foliation with $c$ center singularities and

$s$ saddle singularities, also satisfying $c \geq s+1$. We proceed by induction on saddle singularities $s$.

(I) If $s = 0$, then $\mathcal{F}$ has only center singularities and result follows from Reeb's thesis.

(II) Assume now that $s \geq 1$ and that result is true for foliations with at most $s-1$ singularities of saddle type.

(III) Now we will prove the result for $s$ saddle singularities. By hypothesis $\mathcal{F}$ has some center singularity, say $p_1 \in \text{sing}(\mathcal{F})$ and also some saddle singularity. Thus $\mathcal{C}_{p_1}(\mathcal{F}) \neq M$. Then by lemma 5 in section 2.5 we must have $\partial\mathcal{C}_{p_1}(\mathcal{F}) \cap \text{sing}(\mathcal{F}) \neq \emptyset$, and any leaf $L \subset \partial\mathcal{C}_{p_1}(\mathcal{F})$ must be a separatrix of some saddle singularity $q_1 \in \text{sing}(\mathcal{F})$. Since $\mathcal{F}$ has no saddle connections so the saddle singularity $q_1$ is unique. According to lemma 7, either $q_1 \notin \partial\mathcal{C}_{p_1'}(\mathcal{F})$ for any singularity $p_1\prime \neq p_1$, or $q_1$ belongs to a dead branch associated to a pairing $q_1 - p_1\prime$ for some center singularity $p_1\prime \in \text{sing}(\mathcal{F})$ possibly $p_1\prime = p_1$.

In the first case we call $p_1$ single. In the last case we can eliminate two singularities, one center and a saddle $q_1$ by modification of $\mathcal{F}$. Now we have modified foliation say $\mathcal{F}_1$ with one less center singularity and one less saddle singularity. Modified foliation $\mathcal{F}_1$ still satisfies, $\#\{\text{centers of } \mathcal{F}_1\} \geq \#\{\text{saddles of } \mathcal{F}_1\} + 1$. By induction hypothesis the result has been proved for $l-1$ saddle singularities, so result is true for $\mathcal{F}_1$. Therefore result is true for $\mathcal{F}$. By induction hypothesis the manifold $M^3$ is homeomorphic to $S^3$. This implies that indeed $M^3$ is diffeomorphic to $S^3$.

## Counter examples for general case

Here we give counter examples for the general case of the above result.

**Example 1:** In general the topology of a manifold $M^n$ is not determined by the $\#\{\text{center singularities}\} = \#\{\text{saddle singularities}\}$ for gievn foliation on $M$. Suppose $M^n$ be a compact manifold supporting a non-singular $C^\infty$ codimension one foliation $\mathcal{F}$ (e.g. if $M$ is an odd-dimensional). Then by inverse modification we can obtain a foliation $\tilde{\mathcal{F}}$ on $M$ having singular set of same number of center and saddle singularities. In spite of that $M^n$ is not necessarily homeomorphic to $S^n$.

**Example 2:** In general the inequality $\#\{\text{centers }\} \geq \#\{\text{saddles}\} + 1$ also does not imply that $M^n$ is homeomorphic to $n$-sphere. For example a manifold that admits a $C^\infty$ function $f : M \to \mathbb{R}$ having only three critical points of indices $0, 4$ and $2$ is the complex projective plane $\mathbb{C}P(2)$. Therefore $M^4 = \mathbb{C}P(2)$ admits a foliation with exactly two centers singularities and one saddle singularity, though $M^4$ is not homeomorphic to $S^4$.

## 3.4 Topology of $\mathrm{M}^{\mathrm{n}}$ admitting Morse foliations

A certain combination of the non-degenerate critical points of a real valued function of class $C^2$ defined on the closed manifold has an affect on the topology of that manifold. similar relation we can expect for foliated manifolds. For the first time this became evident that a foliated manifold with Morse type singularities can classify the topology of that manifold by te following result:

### Reeb sphere theorem

**Theorem.** Let $M^n$ be a closed, oriented and connected manifold of dimension $n \geq 2$, admits a $C^1$ transversally oriented codimension one foliation $\mathcal{F}$ with only center singularities. Then the singular set of $\mathcal{F}$ consists of two points and $M^n$ is homeomorphic to $n$-sphere.

Before discussing the general form of the theorem 12, we would like to mention here Eells-Kuiper manifolds.

### Eells-Kuiper manifold

**Definition 20.** *Let $M^n$ be a close connected manifold (not necessarily orientable) of dimension $n$. Suppose $M$ admits a Morse function $f : M \to \mathbb{R}$ of class $C^3$ with exactly three singular points. Then:*

*(a)* $n \in \{2, 4, 8, 16\}$

*(b) Manifold $M^n$ is a topologically a compactification of $\mathbb{R}^n$ by a $\frac{n}{2}$-sphere.*

*(c) The Eells-Kuiper manifold is diffeomorphic to real projective plane $\mathbb{R}P(2)$ for $n = 2$. For $n \geq 4$ it is simply connected and has an integral cohomology structure of the;*

*(i) Complex projective plane $\mathbb{C}P^2$ if $n = 4$*

*(ii) Quaternionic projective plane $\mathbb{H}P^2$ if $n = 8$*

*(iii) Cayley projective plane if $n = 16$*

The affect of Morse function of class $C^3$ with exactly three non-degenerate singular points on closed connected manifolds has been explained by Eells and Kuiper is the following:

**Theorem:** Let $M$ be a connected closed manifold (not necessarily orientable) of dimension $n$. Suppose $M$ admits a Morse function $f : M \to \mathbb{R}$ of class $C^3$ with exactly three singular points, then $M$ is an Eells-Kuiper manifold.

We have general form of the theorem 3.3 with some restriction on the combination between centers and saddles singularities, to prove that result we need following lemmas:

**Lemma 8.** *Let $\mathcal{F}$ be a Morse foliation on a manifold $M^n$ of dimension $n \geq 3$. Given centers $p, q \in sing(\mathcal{F})$ the sets $\mathcal{C}_p(\mathcal{F})$ and $\mathcal{C}_q(\mathcal{F})$ are open in $M$ and $\mathcal{C}_p(\mathcal{F}) \cap \mathcal{C}_q(\mathcal{F}) \neq \emptyset$ if and only if $\mathcal{C}_p(\mathcal{F}) = \mathcal{C}_q(\mathcal{F})$. Moreover we have $\mathcal{C}_p(\mathcal{F}) = M$ if and only if $\partial\overline{\mathcal{C}_p(\mathcal{F})} = \emptyset$, in this case $M^n$ is homeomorphic to $S^n$ provided that $M^n$ is orientable. In particular; either $M^n$ is homeomorphic to $S^n$, or $\partial\overline{\mathcal{C}_p(\mathcal{F})}$ contains some saddle singularity.*

For given two centers when the intersection of their basin is non-empty, the possibilities we have are given in the following lemma:

**Lemma 9.** *Suppose $p_1, p_2 \in sing(\mathcal{F})$ are distinct centers such that $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} \cap \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})} \neq \emptyset$. Then we have the following mutually exclusive possibilities:*

*(i) $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} = \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$, and so $M = \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cup \overline{\mathcal{C}_{p_2}(\mathcal{F})}$.*

*(ii) $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} \neq \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$, and there is a saddle point $q \in \partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} \cap \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$ with Morse index $1$ or $m - 1$, without self-connection.*

When a manifold is homeomorphic to an Eells-Kuiper manifold ?

Following proposition answers this question:

**Proposition 6.** *Suppose a closed connected manifold $M^n, n \geq 3$ which admits a Morse foliation $\mathcal{F}$. Assume that $\mathcal{F}$ has exactly two center singularities and one saddle singularity. Then $M^n$ is homeomorphic to an Eells-Kuiper manifold.*

The following lemma shows necessary condition for a center singularity and a saddle singularity form trivial pairing and belong to a dead branch:

**Lemma 10.** *Suppose $p_1, p_2 \in M$ are two different center singularities such that $q \in \partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} \cap \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$ be the saddle singularity. Assume that the index of $q$ is 1 and that there is no saddle self-connection at $q$. Then, either $p_1, q$ form trivial pairing, or $p_2, q$ form trivial pairing.*

Now we shall discuss the general case to the theorem 12 which classifies the topology of $n$-dimensional manifold for $n \geq 3$ which has been proved by Cesar Camacho and Bruno Scardua in [3] which is an extension of the E. Wagneur general form of Reeb sphere theorem having also saddle singularities in [44].

**Theorem 13** (Camacho-Scardua , 2008)**.** *Suppose we have a compact connected manifold $M^n$ admitting Morse foliation $\mathcal{F}$ with $c$ center and $s$ saddle singularities satisfying $c \geq s+1$. Then we have two possibilities:*

*(i) $c = s+2$, and $M^n$ is homeomorphic to $n$-sphere.*

*(ii) $c = s+1$, and $M^n$ is an Eells-Kuiper manifold.*

**Proof.** By hypothesis we have $c$ center and $s$ saddle singularities satisfying $c \geq s+1$, so we will proceed by induction on saddle singularities $s$.

(I) If $s = 0$ then $M^n$ is homeomorphic $S^n$ by Reeb's theorem.

(II) Suppose now that $s \geq 1$ and the result is true for foliations with at most $s-1$ singularities of saddle type.

(III) Now we will show the result for $s$ saddle singularities. By hypothesis we have $c \geq s+1$. Thus $c \geq 2$. Suppose $M^n$ is not homeomorphic to $S^n$, then by lemma

8 and lemma 9, for each center singularity say $p \in \text{sing}(\mathcal{F})$ there must be a saddle singularity $q_p \in \partial\overline{\mathcal{C}_p(\mathcal{F})}$. Since $c \geq s+1$ and $c \geq 2$, there are two center singularities $p_1, p_2$ such that $q_{p_1} = q_{p_2}$, i.e. there is a saddle $q$ such that $q \in \partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} \cap \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$ and by lemma 9 we have two cases, either $M = \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cup \overline{\mathcal{C}_{p_2}(\mathcal{F})}$, or $q$ has index 1 or $n-1$ and is not self-connected.

(i) If $M = \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cup \overline{\mathcal{C}_{p_2}(\mathcal{F})}$, clearly $\mathcal{C}_{p_i}(\mathcal{F}) \cap \text{sing}(\mathcal{F}) = \{p_i\}, i = 1, 2$. Thus $\text{sing}(\mathcal{F}) = \{p_1, p_2, q\}$ and by proposition 6, $M$ is an Eells-Kuiper manifold.

(ii) If $q$ has index 1 or $n-1$ and is not self-connected, then by lemma 10 we can eliminate one center singularity and one saddle singularity which form are in trivial pairing, replacing $\mathcal{F}$ by a Morse foliation $\mathcal{F}_1$ on $M$ with one less center singularity and one less saddle singularity. On $\mathcal{F}_1$ Number of center singularities is $c_1 = c - 1$ and number of saddle singularities is $s_1 = s - 1$, which still satisfy the condition $c_1 \geq s_1 + 1$. Observe also that $s > s_1 \geq 0$. By induction hypothesis $M^n$ is homeomorphic to $S^n$ or to an Eells-Kuiper manifold.

This proves the result.

# Chapter 4

# Coupling of two saddles and its applications

In previous chapter we discussed some results from Camacho-Scardua papers [2], [3] which has been proved by elimination of pair of center-saddle singularities in a dead branch. In this chapter we shall show how to combine two saddle singularities of complementary indices in a dead branch, and we shall extend some of those results by proving them through elimination of saddle-saddle singularities of complementary indices which are in stable connection ( stable manifold of one saddle singularity intersects transversally in a smooth connection the unstable manifold of another saddle singularity, which is of complementary index of the previous saddle ).

## 4.1 Coupling of two saddles

We show now how to combine in a dead branch two saddle singularities of complementary indices. We consider the function,

$$f_\epsilon : \mathbb{R}^3 \to \mathbb{R}, \text{ given by}$$

$$f_\epsilon = -\tfrac{x^2}{2} + (\tfrac{y^3}{3} - \epsilon y) + \tfrac{z^2}{2} \ , \ \epsilon \in \mathbb{R}$$

and consider deformation of the foliation given by $df_\epsilon = 0$ from $\epsilon > 0$ to $\epsilon < 0$ passing through $\epsilon = 0$. We focus only in the study of the function,

$$f = f_0 = -\frac{x^2}{2} + \frac{y^3}{3} + \frac{z^2}{2}$$

which gives a saddle-node singularity at the origin and whose phase portrait is given below:

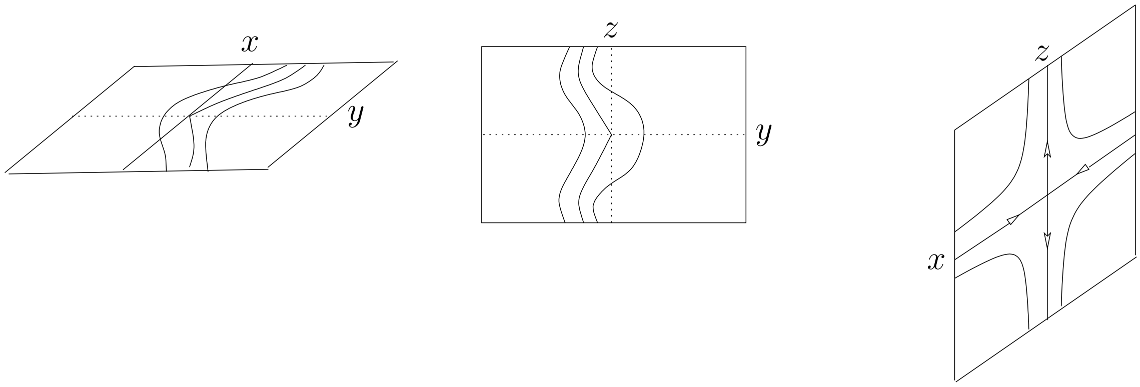


Figure 4.1:

We consider leaves $L_1$ and $L_2$ as in figure:

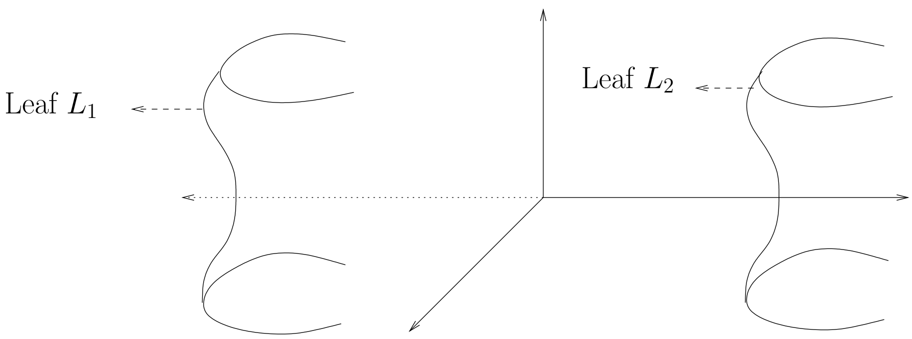


Figure 4.2:

where $L_1 = L_{(0,-1,0)}$ and $L_2 = L_{(0,1,0)}$ are planes.

The deformation takes place in the region laterally bounded by planes $L_1$ and $L_2$ by deforming the original foliation for $\epsilon > 0$ by a regular trivial foliation for $\epsilon < 0$ as the final stage, this passing through the saddle-node for $\epsilon = 0$ in figure 4.3. This procedure can be better understand in the two dimensional case as we have already seen.

In the 3-dimensional case we add one transverse axis to this original figure and proceed in a similar way.

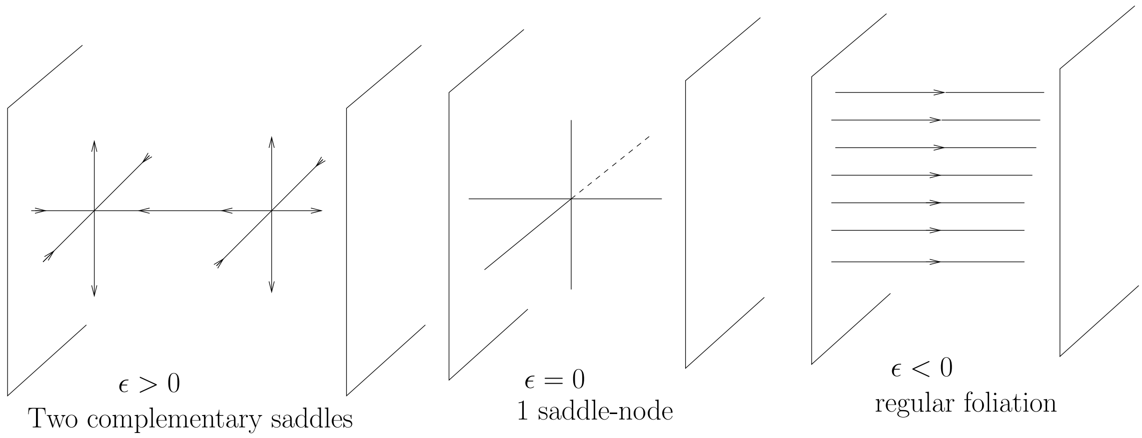


Figure 4.3:

Given two saddle singularities of complementary indices in dimension 3, to obtain an elimination procedure our arrows-scheme is the following:

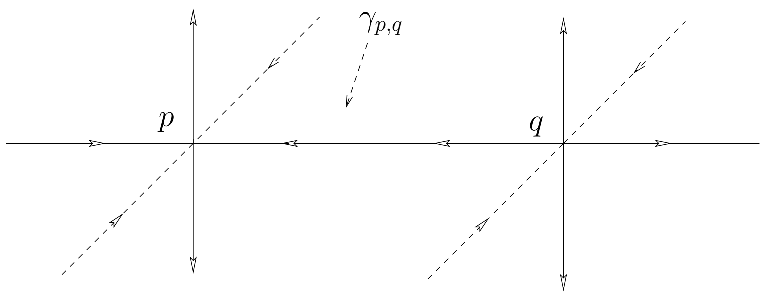


Figure 4.4:

$$\text{Index}(\Omega, p) = (-1)^2 = +1 \quad , \quad \text{Index}(\Omega, q) = (-1)^1 = -1$$

$$\text{Index}(\Omega, p) + \text{Index}(\Omega, q) = 0$$

For connection between two saddle singularities of complementary indices we shall make a natural hypothesis:

Suppose we have two saddle singularities $p$ and $q$ of complementary indices, such that, the stable manifold of $p$ intersects transversally the unstable manifold of $q$ in a smooth connection curve $\gamma_{p,q}$. Such a connection $\gamma_{p,q}$ will be called stable connection between $p$ and $q$.

By above construction in general we have the following proposition:

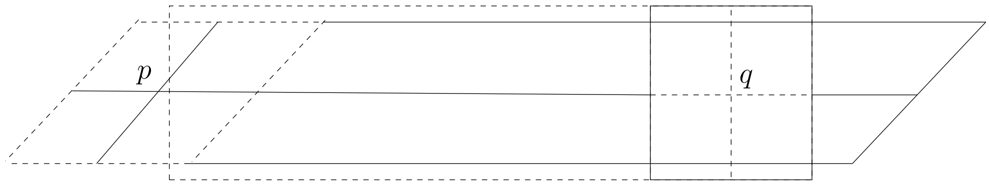


Figure 4.5:

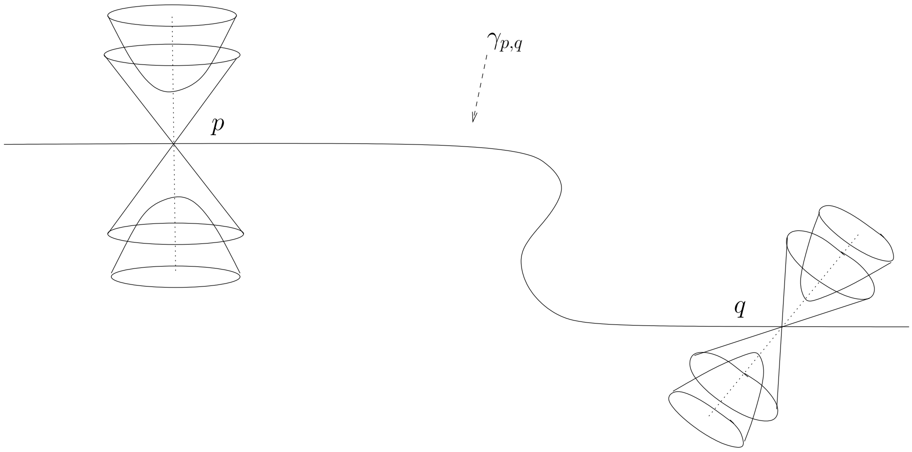


Figure 4.6:

**Proposition 7.** *Given a foliation $\mathcal{F}$ on n-dimensional manifold $M^n$. We can obtain a modification $\tilde{\mathcal{F}}$ of $\mathcal{F}$ on $M^n$ that exhibits two saddle singularities of complementary indices which are in strong stable connection.*

A converse of the above construction is given below, We begin with the following remark:

**Remark** Given a foliation $\mathcal{F}$ on $M^3$ with two complementary saddle singularities $p_1, p_2 \in sing(\mathcal{F})$ having a stable connection $\gamma$, there exists a neighborhood $U$ of $\gamma, p_1, p_2$ in $M^3$ and a coordinate system

$$\varphi : U \to \mathbb{R}^3, \quad \text{such that}$$
$$\varphi(p_1) = (0, 0, 0)$$
$$\varphi(p_2) = (0, 1, 0)$$

taking $\gamma$ onto the $y$-axis $\{x = z = 0\}$, and such that the stable manifold of $p_1$ is tangent to $\varphi^{-1}(\{z = 0\})$ at $p_1$ and the unstable manifold of $p_2$ is tangent to $\varphi^{-1}(\{x = 0\})$ at $p_2$.

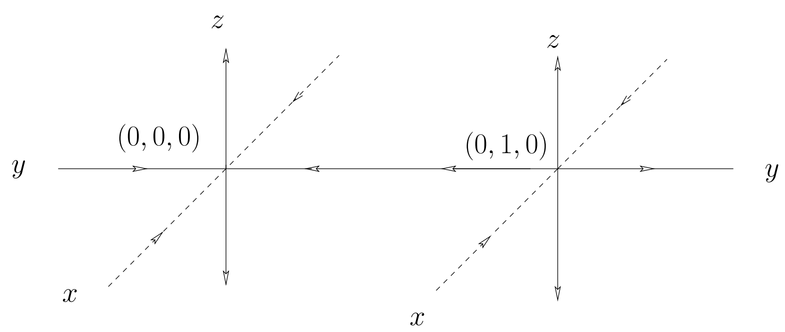


Figure 4.7:

The coupling of saddle and center in dimension two can be seen as follows:

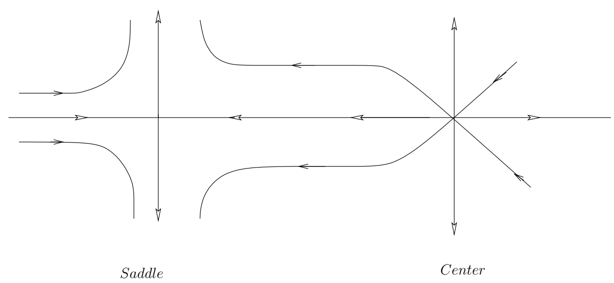


Figure 4.8: Orbit of grad(Ω)

We have two leaves $L_1$ and $L_2$, and $\Sigma_1, \Sigma_2$ represent the space of leaves of $\mathcal{F}$ in the region bounded by $L_1, L_2, \Sigma_1, \Sigma_2$. We can collapse the singularities into saddle-node and then deform this into a non-singular fibration by discs that is compatible with the leaves $L_1$ and $L_2$.

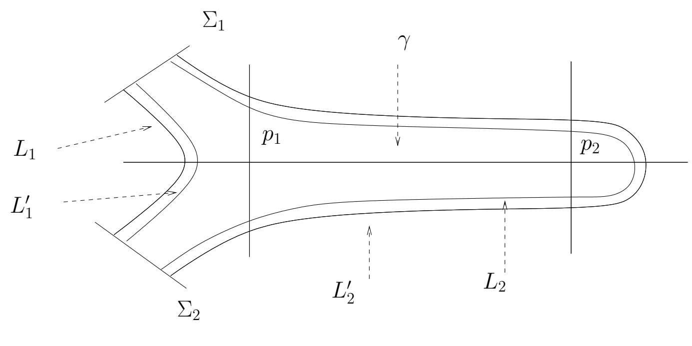


Figure 4.9:

We shall use the same idea for coupling of two saddle singularities in dimension three. We return to our 3-dimensional case in which we have two saddle singularity of complementary indices and exhibit a stable connection. Using the chart $\varphi : U \to \mathbb{R}^3$ of the remark, we may assume that we are on $\mathbb{R}^3$.

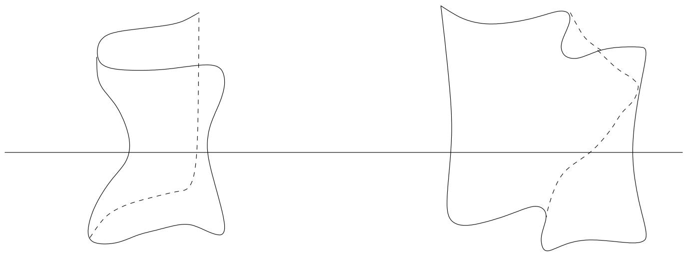

Figure 4.10:

Using the fact that the restriction of $\mathcal{F}$ to the stable manifold of $p_1$ (diffeomorphic to $\mathbb{R}^2$) we obtain a similar dynamical behaviour to the 2-dimensional case and the same way than before, we obtain leaves $L_1, L_1', L_2, L_2'$ and segments $\Sigma_1, \Sigma_2$ representing the leaf space of $\mathcal{F}$ as in the following figure, by using figure 4.9.

This corresponds to a cylindrical picture as shown in the following figure:

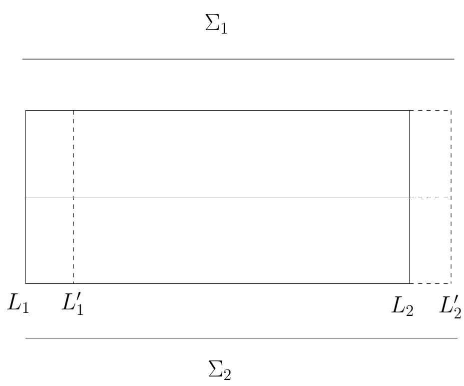


Figure 4.11:

We can therefore modify the foliation in the cylinder in order to obtain a non-singular fibration by discs transverse to the stable connection axis.

Now we make the inverse path. We shall see here how to obtain a singular foliation when we have non-singular Reeb component in 3-dimensional manifold. Given a foliation $\mathcal{F}$ without singularities assume that you have a transverse circle,

$$\gamma : S^1 \to M^3, \gamma \pitchfork \mathcal{F}$$

In a small tubular neighborhood of this curve we can construct a standard Reeb component.

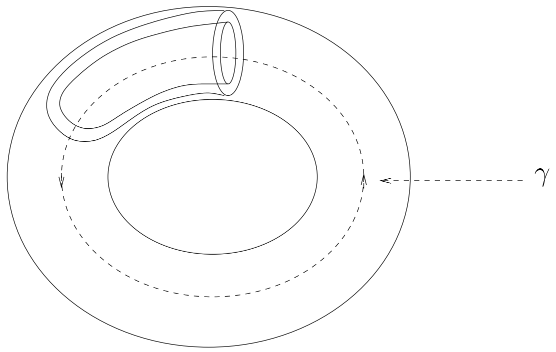


Figure 4.12:

Then, by standard modification we can replace this Reeb component by a singular Reeb foliation having $\gamma$ as central axis as shown in the following figure.

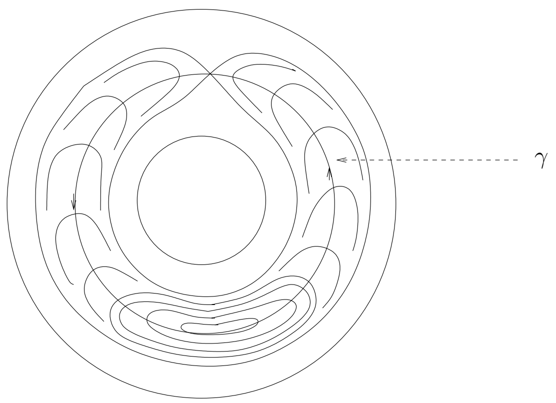


Figure 4.13:

This shows that once we have transverse circles to the foliation we can introduce couplings of center-saddle singularities. By using above construction we can replace a saddle singularity in $M^3$ by a center singularity of the same index. So Let now $\mathcal{F}$ be a foliation having a saddle singularity $p_1 \in M^3$ with two dimensional stable manifold $W^{ss}(p_1) \subset M^3$. Suppose there is a circle

$$\gamma : S^1 \to W^{ss}(p_1)$$

which is transverse to $\mathcal{F}$.

Then by the above procedure, we can obtain a coupling of center-saddle by the construction of a $\mathcal{F}$-compatible singular Reeb foliation having $\gamma(S^1)$ as axis. By construction we have saddle singularity $p_2$ and a center singularity $p_3$ in $W^{ss}(p_1)$ with the property that the stable manifold $W^{ss}(p_1)$ of $p_1$, meets transversely the unstable manifold $W^{uu}(p_2)$ of $p_2$ along a stable connection.

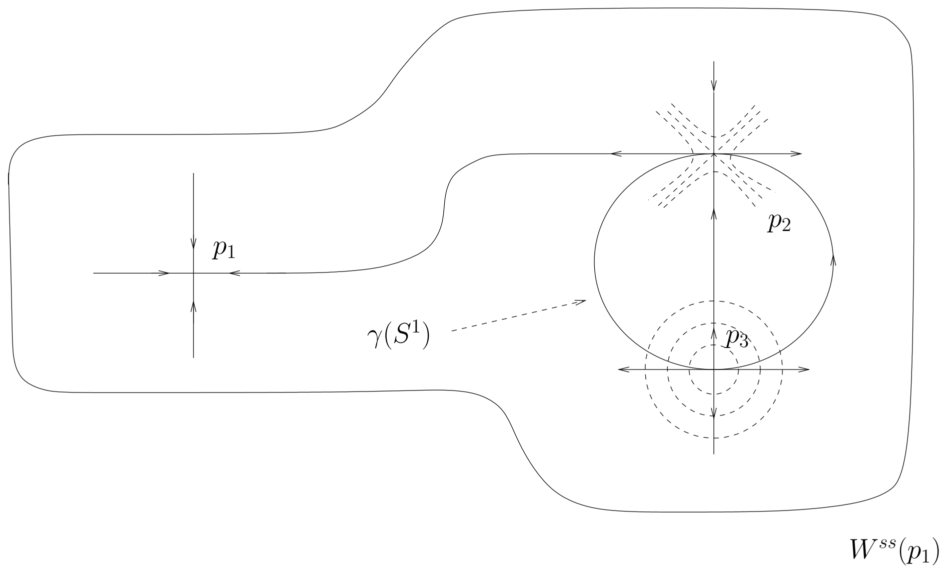


Figure 4.14:

The arrows indicate the vector field grad($\Omega$), where $\Omega$ is a fixed one form that defines $\mathcal{F}$. The curves indicate the leaves of the restriction $\mathcal{F}|_{W^{ss}(p_1)}$. Since the saddle $p_1$ and $p_2$ are in stable connection so we can eliminate them and obtain a modification of $\mathcal{F}$ where we have replaced the saddle $p_1$ by a compatible center $p_3$ of the same index.

As a particular case of the above construction we conclude that if the restriction $\mathcal{F}|_{W^{ss}(p_1)}$ has some limit cycle then we can, by a modification procedure, replace $p_1$ by a center singularity of same index. Indeed, if we have a limit cycle then, by radial component of the corresponding vector fields we can construct a transverse circle in the two dimensional manifold.

**Observation.** Let $\mathcal{F}$ be a foliation in 3-dimensional manifold $M^3$ having a saddle singularity $p_1$ with stable manifold $W^{ss}(p_1) \subset M^3$. Suppose there is a circle

$$\gamma : S^1 \to W^{ss}(p_1) \text{ such that } \gamma(S^1) \pitchfork \mathcal{F},$$

Since we have circle transverse to the foliation, so we can introduce coupling of center $p_3$ and saddle $p_2$ having $\gamma(S^1)$ as axis. By coupling of saddles, in which stable manifold of $p_1$, $W^{ss}(p_1)$ meets transversely the unstable manifold of $p_2$, $W^{uu}(p_2)$, we can eliminate saddles $p_1$ and $p_2$. So by modification procedure we can replace saddle singularity $p_1$ by a center singularity $p_3$ of same index.

## 4.2 Extension of the theorem 12 by coupling and elimination of pairs of saddles

Theorem 12 which says that, 3-dimensional manifold admitting Morse foliation having $c$ center and $s$ saddle singularities satisfying $c \geq s+1$, will be diffeomorphic to $S^3$ [2]. That result has been proved by elimination of trivial pair of center-saddle singularities belonging to a dead branch. Now we shall proof that result by eliminating the trivial pair of saddle-saddle singularities of complementary indices belonging to a dead branch.

**Theorem 14.** *Suppose a closed connected and oriented 3-manifold $M^3$ admitting an oriented Morse foliation $\mathcal{F}$ having c be the number of centers and s be the even number of saddle singularities in sing($\mathcal{F}$) satisfying $c \geq s+1$, and suppose that there are at least r pairs of saddles which are in stable connection, where $c > s-2r$. Then $M^3$ is diffeomorphic to $S^3$.*
*Indeed $\mathcal{F}$ admits an isotopy to a Morse foliation having only two centers as singularities.*

**Note** If $c > s$, i.e., if we have more centers than saddles, then the condition above is automatically satisfied.

**Proof .** Since by hypothesis we have $c$ center and an even number of saddle singularities $s$ satisfying:
$(a)$ $c \geq s+1$
We will proceed by induction on the number $s$ of saddle singularities.

(i) Let $s=0$, then we have just center singularities and result follows by Reeb's thesis.

(ii) Assume now that $s \geq 2$ and that result is true for foliations having at most $s-2$ saddle singularities.

(iii) Now we have to prove the result for $s$ saddles. Since by hypothesis we have foliation $\mathcal{F}$ with $s$ saddles and $c$ centers satisfying $c \geq s+1$. Also by hypothesis there are at least $r$ pairs of saddles which are in stable connection, where $c > s - 2r$. By proposition 7 we can obtain a modification $\tilde{\mathcal{F}}_1$ of $\mathcal{F}$ on $M^n$ that exhibits two saddle singularities say $q_1$ and $q_2$, of complementary indices which are in strong stable connection,i.e. stable manifold of $q_1$ intersects transversally the unstable manifold of $q_2$. We can eliminate saddle $q_1$ and $q_2$ which belong to a dead branch, by replacing $\tilde{\mathcal{F}}_1$ by a modified Morse foliation $\mathcal{F}_1$ on $M^n$ with a number $s_1$ of saddles given by $s_1 = s-2$ . Now we have two less saddle singularities $s-2$. By induction hypothesis the result is true for $s-2$ singularities of saddle type. So it has been proved for $s$ saddles. It proves our result.

(b) Now $s > c$ .

since we have saddle singularities, and by hypothesis there are at least $r$ pairs of saddles which are in stable connection, where $c > s - 2r$. So by proposition 7 we can obtain a modification $\tilde{\mathcal{F}}_2$ of $\mathcal{F}$ on $M^n$ that exhibits two saddle singularities say $q_1$ and $q_2$, of complementary indices which are in strong stable connection ,i.e. stable manifold of $q_1$ intersects transversally the unstable manifold of $q_2$. We can eliminate saddle $q_1$ and $q_2$ which belong to a dead branch, by replacing $\tilde{\mathcal{F}}_2$ by a modified Morse foliation $\mathcal{F}_2$ on $M^n$ with a number $s_2$ of saddles given by $s_2 = s - 2$ . By successive modifications of the foliation by elimination , a Morse foliation can be obtained as a final result satisfying $\{\#centers\} \geq \{\#saddles + 1\}$. We are again in case $(a)$ which has already been proved.

## 4.3 Extension of the theorem 13 by coupling and elimination of pairs of saddles

Theorem 13 is general case of the theorem 12 , which says that, $n$-dimensional manifold admitting Morse foliation with $c$ centers and $s$ saddle singularities satisfying $c \geq s+1$ is either homeomorphic to $S^n$ if $c = s+2$, or homeomorphic to an Eells-kuiper manifold if $c = s+1$, which has been proved by Cesar Camacho and Bruno Scardua in [3] by using the technique of elimination of trivial center-saddle singularities belonging to a dead branch. Here we shall extend theorem 13 by proving it with the help of coupling and elimination of two saddle singularities of complementary indices.

**Theorem 15.** *Let $\mathcal{F}$ be a Morse foliation on a compact connected manifold $M^n$ having $c$ centers and $s$ saddles in sing($\mathcal{F}$) satisfying $c \geq s+1$ or, more generally, there are at least $r$ pairs of saddles which are in stable connection, where $c > s-2r$. Then we have two possibilities:*

*(a) $c = s+2$ , and $M^n$ is homeomorphic to $S^n$.*

*(b) $c = s+1$ , and $M^n$ is an Eells-Kuiper manifold.*

**Proof.** For the case $c \geq s+1$ , we shall proceed by induction on number $s$ of saddle singularities .

(I) If $s = 0$ , then by Reeb's theorem $M^n$ is homeomorphic to $S^n$.

(II) Suppose now that $s \geq 1$ and the result is true for at most $s-1$ saddles.

(III) Now we shall show the result for $s$ saddles . Since by hypothesis we have $c \geq s+1$. Thus $c \geq 2$. Suppose that $M^n$ is not homeomorphic to $S^n$. Then by lemma 8 and lemma 9 for each center $p \in \text{sing}(\mathcal{F})$ there must be a saddle $q_{(p)} \in \partial\overline{\mathcal{C}_p(\mathcal{F})}$. Since $c \geq s+1$ and $c \geq 2$ , there are two centers $p_1, p_2$ such that $q_{p_1} = q_{p_2}$ , i.e. there is a saddle $q$ such that $q \in \partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} \cap \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$, and by lemma 9 we have two possibilities:

(i) $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} = \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$

(ii) $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} \neq \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$

(i) in the case $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} = \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$ we have $M = \overline{\mathcal{C}_{p_1}(\mathcal{F})} \cup \overline{\mathcal{C}_{p_2}(\mathcal{F})}$, so clearly $\mathcal{C}_{p_i} \cap \text{sing}(\mathcal{F}) = \{p_i\}, i = 1, 2$. Thus $\text{sing}(\mathcal{F}) = \{p_1, p_2, q\}$ ,which satisfies the condition $c = s+1$, and by proposition 6, $M$ is an Eells-Kuiper manifold .

(ii) In the case $\partial\overline{\mathcal{C}_{p_1}(\mathcal{F})} \neq \partial\overline{\mathcal{C}_{p_2}(\mathcal{F})}$ , saddle singularities satisfy $s \geq 2$ and by hypothesis we know that there are at least $r$ pairs of saddles which are in stable connection, where $c > s-2r$. Then by proposition 7, we obtain a modification $\tilde{\mathcal{F}}$ of $\mathcal{F}$ on $M^n$ which exhibits two saddle singularities of complementary indices say $q_3$ and $q_4$ which are in stable connection . We can eliminate these two saddles , replacing $\tilde{\mathcal{F}}$ by a Morse foliation $\mathcal{F}_1$ on $M^n$ with the same number of centers $c$ and two less saddles $s-2$ given by $s_1 = s-2$ . Therefore $c > s_1$

and $s > s_1 \geq 0$ . By induction hypothesis $M^n$ is homeomorphic to $S^n$ or to an Eells-Kuiper manifold .

Now if $s > c$ , by hypothesis we know that there are at least $r$ pairs of saddles which are in stable connection, where $c > s - 2r$. Then there are successive modifications of the foliation $\mathcal{F}$ by elimination of complementary saddles by proposition 7, we obtain a Morse foliation which satisfies $(a)$ which is $c = s + 2$ or $(b)$ which is $c = s + 1$ , and again the result follows by above demonstration .

## 4.4 Extension of the theorem 11 by coupling and elimination of pairs of saddles

Before discussing our desired result, which is Haefliger's type theorem for 3-sphere , first we would like to state classical Haefliger's theorem for the disc :

**Theorem :** Let $\overrightarrow{X}$ be a $C^1$ vector field defined in a neighborhood $U$ of disc $D^2 \subset \mathbb{R}^2$ such that $\overrightarrow{X} \pitchfork \partial\overline{D^2}$ points inward the disc satisfying the following conditions:

(i) $\overrightarrow{X}$ has only Morse singularities in disc $D^2$.

(ii) $\overrightarrow{X}$ will be without saddle connections .

Then there exists a unilateral compact invariant one dimensional subset $\Gamma \subset D^2$.

Here we shall learn the use of dead branch having two saddles of complementary indices and modification inside the dead branch to prove the extension of variant of Haefliger's theorem for foliations with singularities [2].

**Theorem 16.** *Let $\mathcal{F}$ be a $C^\infty$ Morse foliation on* 3*-sphere $S^3$ having c center and s saddle singularities, such that there are at least r pairs of saddles which are in stable connection, where $c > s - 2r$. Then $\mathcal{F}$ is an inverse modification of a Seifert fibration of $S^3$, i.e. a singular foliation of $S^3$ by Spheres $S^2$ and centers , or we have one of the following possibilities:*

*(a) There is a compact codimension one invariant subset whose holonomy is one-sided.*

*(b) There is a singular Reeb component of $\mathcal{F}$ .*

**Proof.** Lets first study the case $c \geq s$.
We fix an orientation for $\mathcal{F}$. By hypothesis we have $c$ centers and $s$ saddles satisfying $c \geq s$. We will proceed by induction on saddle $s$.

(I) Consider the case $s = 0$, so $c \geq 0$. For $s = 0$ we further have two cases:

  (i) $c = 0$ ($s = 0$ already) , we have nonsingular codimension one foliation $\mathcal{F}$ on $S^3$, so by Novikov theorem $\mathcal{F}$ has some Reeb component and therefore $\mathcal{F}$ has a toral leaf $L \simeq S^1 \times S^1$ with one-side holonomy group.

  (ii) $c \geq 1$ ($s = 0$ already ), foliation $\mathcal{F}$ has only center singularities , therefore it is Seifert fibration by Reeb .

(II) Assume now $s \geq 1$, since by hypothesis we have $c \geq s$, so $c \geq s \geq 1$. Assume that result is true for $s - 1$ saddle singularities

(III) Now suppose we have $c$ centers and $s$ saddles satisfying the inequality $c \geq s$, suppose a center $p_1$ in $S^3$. We denote $\mathcal{C}_{p_1}(\mathcal{F})$ connected component of $\mathcal{C}(\mathcal{F})$ which contains $p_1$, where $\mathcal{C}(\mathcal{F})$ = Union of all centers and leaves diffeomorphic to $S^2$ of the foliation $\mathcal{F}$. Since we have $\mathcal{C}_{p_1}(\mathcal{F})$, so we have two cases for the boundary $\partial\mathcal{C}_{p_1}(\mathcal{F})$:

  (i) $\partial\mathcal{C}_{p_1}(\mathcal{F}) = \emptyset$ then $\mathcal{C}_{p_1}(\mathcal{F}) = S^3$, so all leaves of $\mathcal{F}$ are compact diffeomorphic to $S^2$ with trivial holonomy . In other words $\mathcal{F}$ is singular Seifert fibration of $S^3$.

  (ii) $\partial\mathcal{C}_{p_1}(\mathcal{F}) \neq \emptyset$ then by lemma 5 , $\partial\mathcal{C}_{p_1}(\mathcal{F}) \cap \text{sing}(\mathcal{F}) \neq \emptyset$, so any leaf $L \subset \partial\mathcal{C}_{p_1}(\mathcal{F})$ is separatrix of some saddle singularity $q_1$. This singularity is unique, because $\mathcal{F}$ has no saddle connections. On the other hand we can not have $\partial\mathcal{C}_{p_1}(\mathcal{F}) \subset \text{sing}\mathcal{F}$. Thus we can find a leaf $L_0$ of $\mathcal{F}$ such that $\Gamma_{q_1} = L_0 \cup \{q_1\} \subset \partial\mathcal{C}_{p_1}(\mathcal{F})$, where $L_0$ is separatrix of saddle $q_1$. Since $\Gamma_{q_1}$ is accumulated by spherical leaves, so we have two possibilities for holonomy of $\Gamma_{q_1}$:

   (A) $\Gamma_{q_1}$ has non-trivial holonomy and since $\Gamma_{q_1}$ is accumulated by spherical leaves, so $\Gamma_{q_1}$ has one-side holonomy and the result follows.

   (B) $\Gamma_{q_1}$ has trivial holonomy , then by lemma 5 , we have following possibilities for $\partial\mathcal{C}_{p_1}(\mathcal{F})$:

(1) $\partial\mathcal{C}_{p_1}(\mathcal{F}) \setminus \{q_1\}$ is connected.Then

(i) $\partial\mathcal{C}_{p_1}(\mathcal{F})$ is homeomorphic to a sphere $S^2$ with a pinch at $q_1$. By hypothesis we know that there are at least $r$ pairs of saddles which are in stable connection, where $c > s - 2r$. Then by proposition 7 we obtain a modification $\tilde{\mathcal{F}}_1$ of $\mathcal{F}$ on $M^n$ which exhibits two saddle singularities of complementary indices $q_3$ and $q_4$ which are in stable connection . The foliation inside dead branch having pair of complementary saddles $q_3$ and $q_4$ which are in stable connection , can be modified to a trivial foliation by elimination of this pair . The modified foliation $\mathcal{F}_1$ has two less saddles given by $s_1 = s - 2$. By the induction hypothesis the modified foliation $\mathcal{F}_1$ is an inverse modification of a Seifert fibration of $S^3$ or $\mathcal{F}_1$ satisfy one of the two conditions mentioned in the statement , and therefore we have the same possibilities for $\mathcal{F}$.

(ii) $\partial\mathcal{C}_{p_1}(\mathcal{F}) \setminus \{q_1\}$ is homeomorphic to a singular torus . Since we are in case $(B)$ in which $\Gamma_{q_1}$ has trivial holonomy , and $\Gamma_{q_1}$ is surrounded by the leaves diffeomorphic to the torus. So we can isolate the region $R \subset S^3$ containing $\overline{\mathcal{C}_{p_1}(\mathcal{F})}$ invariant by $\mathcal{F}$ and diffeomorhic to the solid torus where we have defined singular Reeb foliation.

(2) $\partial\mathcal{C}_{p_1}(\mathcal{F}) \setminus \{q_1\}$ has two connected components . In this case the saddle $q_1$ is not self-connected and $\Gamma_{q_1}$ is homeomorphic to $S^2$ and $\Gamma_{q_1} \setminus \{q_1\}$ is diffeomorphic to a sphere minus one point . Again by hypothesis that there are at least $r$ pairs of saddles which are in stable connection, where $c > s - 2r$ and proposition 7 we modify foliation $\mathcal{F}$ . The modified foliation $\tilde{\mathcal{F}}_2$ of $\mathcal{F}$ will have two complementary saddles $q_4$ and $q_5$ in strong stable connection . By elimination of these two saddles we obtain foliation $\mathcal{F}_2$ which will have two less saddles . By induction hypothesis modified foliation $\mathcal{F}_2$ is an inverse modification of a Seifert fibration of $S^3$ or $\mathcal{F}_2$ satisfy one of the two conditions mentioned in the statement . It proves the result .

Now consider the case $s > c$ .
since we have saddle singularities , so by hypothesis we know that there are at least $r$ pairs of saddles which are in stable connection, where $c > s - 2r$, so by proposition 7 we

can obtain a modification $\tilde{\mathcal{F}}_3$ of $\mathcal{F}$ on $M^n$ that exhibits two saddle singularities say $q_5$ and $q_6$, of complementary indices which are in strong stable connection,i.e. stable manifold of $q_5$ intersects transversally the unstable manifold of $q_6$. We can eliminate saddle $q_5$ and $q_6$ which belong to a dead branch, by replacing $\tilde{\mathcal{F}}_3$ by a modified Morse foliation $\mathcal{F}_3$ on $M^n$ with a number $s_3$ of saddles given by $s_3 = s - 2$ .

By successive modifications of the foliation by elimination , a Morse foliation can be obtained as a final result satisfying $\#centers \geq \#saddles$ . We are again in first case in which centers are greater or equal to saddle singularities which has already proved.

# Chapter 5

# On the topology of compact foliations with singularities

One of the motivations to the study of foliations was the research of non homotopic invariant for the classification of 3-manifolds also as an attempt to prove the *Poincaré conjecture* for simply-connected closed three-manifolds. As general philosophy it was guessed that the existence (or nonexistence) of a suitable codimension one foliation might give information on the topology of the manifold itself. Because of some natural examples it is useful to consider also *foliations with singularities.* At this point the interplay between this new theory and Morse theory becomes more clear. The first natural examples of this are given by *Reeb's sphere recognition theorem* [17] and its celebrated extension due to J. Milnor theorem ([13], Chapter 6 or [12]). Motivated by this it is then natural to consider *foliations with Morse singularities* and, as a next step, to ask what information about the topology of the manifold we may infer if we admit the singular set of the foliation contains some saddle-type singularities.

In this chapter we essentially investigate a possible extension of the Reeb's sphere recognition theorem ([17]) in different contexts, where we admit different singular sets. We start with non-degenerate isolated singularities, but we also consider some cases of a degenerate but regular singular set, *i.e.*, a set with the property that its connected components have a fundamental system of compact "invariant" neighborhoods.

The main ingredients are classical Morse lemma [11] and Reeb stability theorems ([1], [4], [7]). The idea is that Reeb stability theorems may replace the Poincaré-Bendixson's theorem in higher dimensions. These theorems are a basic and recurrent tool in our research. In the last section we obtain, as a consequence of our results, the announced generalization of the Milnor-Reeb sphere recognition theorems.

## 5.1 Holonomy and (Reeb) stability

The notion of holonomy of a leaf of a foliation is originally found in the work of Ehresmann and Shih [6] (who already proved an stability result extending preliminary work of Reeb), and has been developed in the subsequent work of Reeb ([15]). Since it is classical and treated by several authors we shall not introduce it here, and we refer to [1], [4] or [7]. The precise statements given below are for future reference in the text:

**Theorem 17** (Reeb stability theorems [1, 4, 7]). *Let $\mathcal{F}$ be a $C^1$, codimension $k$ foliation of a manifold $M$:*

**(1)** (Local stability) *If $F$ is a compact leaf with finite holonomy group then there exists a fundamental system of neighborhoods $U$ of $F$ in $M$, such that each $U$ is saturated by $\mathcal{F}$, and in which all the leaves are compact with finite holonomy group. Furthermore we can define a retraction $\pi : U \to F$ such that, for every leaf $F' \subset U$, $\pi|_{F'} : F' \to F$ is a covering with a finite number of sheets and, for each $y \in F$, $\pi^{-1}(y)$ is homeomorphic to a disk of dimension $k$ and is transverse to $\mathcal{F}$. The neighborhood $U$ can be taken to be arbitrarily small.*

**(2)** (Global Stability Theorem) *Suppose $\mathcal{F}$ be a codimension one foliation and $M$ is a closed manifold. If there is a compact leaf $F$ with finite fundamental group then all the leaves of $\mathcal{F}$ are compact with finite fundamental group. If $\mathcal{F}$ is transversely orientable then every leaf of $\mathcal{F}$ is diffeomorphic to $F$; $M$ is the total space of a fibration $f : M \to S^1$ over $S^1$ with fibre $F$; and $\mathcal{F}$ is the fibre foliation $\{f^{-1}(\theta) | \, \theta \in S^1\}$.*

**(3)** (Local product structure) *Suppose $\mathcal{F}$ be a codimension one, class $C^r$ $(r \geq 1)$ and is transversely oriented foliation. If $F$ is a compact leaf of $\mathcal{F}$ with finite holonomy group then it has trivial holonomy and there exist an open neighborhood $V(F)$ of $F$ in $M$, saturated by $\mathcal{F}$, and a $C^r$ diffeomorphism $h : (-1,1) \times F \to V(F)$ such that the leaves of $\mathcal{F}$ in $V(F)$ are the sets $h(\{t\} \times F)$, $t \in (-1,1)$, where $F = h(\{0\} \times F)$. In particular $V(F) \setminus F$ has two connected components.*

In the course of the proof of Reeb stability theorems and in some of our arguments as well, the following fact is very useful.

**Proposition 8** (Stability lemma)**.** *Let $F$ be a compact leaf of a codimension one foliation $\mathcal{F}$ defined on a manifold $M$. Let $F_n$ be a sequence of compact leaves of $\mathcal{F}$ accumulating to a point in $F$. Then for all neighborhood $F \subset W \subset M$ one has $F_n \subset W$ for all $n$ large.*

Since a proof for it is not clearly stated in the basic references, we shall provide one. For this we shall need the following basic result of immediate proof:

**Lemma 11.** *Let $x \in \overline{\cup_{i\in I} F_i} \subset M$, where $\{F_i\}_{i\in I}$ is a sequence of leaves of a foliation $\mathcal{F}$ on a manifold $M$. Then for all $y \in L_x$ we have $y \in \overline{\cup_{i\in I} F_i}$.*

*Proof of Proposition 8.* Let $U_1, \ldots, U_k \subset W$ be a covering of $F$ with charts of $\mathcal{F}$ such that $U_i \cap F$ is a single plaque, $\alpha_i$ of $U_i, \forall i$. Since $F$ and $F_n$ $\forall n$ are compact they have the same transverse type, which is *discrete*. In particular $F \cap U_i$ and $F_n \cap U_i$ $\forall n$ contain a finite number of plaques of $U_i$. Moreover, as a consequence of the lemma stated above, $F \subset \overline{\cup_n F_n}$ and we can choose $n$ in a way that $F_n \cap U_i \neq \emptyset$ for all $1 \leq i \leq k$. We have to prove $F_n \subset U_1 \cup \cdots \cup U_k$. At this purpose suppose $U_1, \ldots, U_k$ is not a covering of $F_n$ ($n$ fixed). We may choose a finite set of foliated charts of $\mathcal{F}$, $U_{k+1}, \ldots, U_l$, in a way that $U_1, \ldots, U_l$ is a covering of $F_n$, $F_n \cap U_i$ contains a finite number of plaques of $U_i$ for all $i = 1, \ldots, l$ and $F \cap U_i = \emptyset$ $\forall i = k+1, \ldots, l$. For simplicity we suppose $l = k+1$. For all $y \in F_n \cap U_{k+1}$ we can find $1 \leq i_0 \leq k$ such that the projection along plaques of the plaque through $y$ intersect a plaque $\alpha(y) \subset U_{i_0}$. By construction $U_{i_0} \cap F \neq \emptyset$, then the space of plaques $\Sigma^{i_0}$ of $U_{i_0}$ is such that $\Sigma^{i_0} \cap F \neq \emptyset$ and $\Sigma^{i_0} \cap \alpha(y) \neq \emptyset$. We may identify $\Sigma^{i_0}$ with a suitable transverse section. Let $\Sigma_y$ a transverse at $y$. By classical Transverse uniformity lemma [1] there exists a $C^r$ diffeomorphism $f : \Sigma^{i_0} \to \Sigma_y$ such that $f(\Sigma^{i_0} \cap L) = \Sigma_y \cap L$ for any leaf $L$ of $\mathcal{F}$. In particular $F \cap \Sigma^{i_0} \neq \emptyset \Rightarrow F \cap \Sigma_y \neq \emptyset$, a contradiction. □

## 5.2 Compact foliations with singularities

Let $M$ be a compact connected manifold of dimension $n$, possibly with non-empty boundary, $\partial M$. Let $\mathcal{F}$ be a codimension one, $C^\infty$ foliation, with isolated singularities, on $M$ and, if $\partial M \neq \emptyset$, we suppose $\mathcal{F}$ is tangent or transverse to $\partial M$. We shall say [3] that the foliation is a *compact foliation with singularities* of $M$ if its leaves are compact. Notice that if the singularities of $\mathcal{F}$ are of Morse type, then there are only center type singularities and as in the original work of Reeb this imposes, in case $\mathrm{sing}(\mathcal{F}) \neq \emptyset$, severe constraints to the manifold. The classification of these manifolds is given below.

**Theorem 18.** *Let $\mathcal{F}$ be a transversely orientable compact foliation with nonempty singular set and having singularities all of center type on a compact connected oriented manifold $M$, tangent to the boundary $\partial M$ if non-empty. Then we have the following possibilities:*

**(1)** *$M$ is closed (empty boundary) and $\mathcal{F}$ has singularities. In this case there are exactly two singularities $\mathrm{sing}(\mathcal{F}) = \{p, q\}$ and there is a smooth function $f\colon M^n \to [0,1] \subset \mathbb{R}$, defining the foliation, such $\mathrm{sing}(f) = \mathrm{sing}(\mathcal{F})$, and such that the non-singular levels of $f$ are diffeomorphic to $S^{n-1}$. In particular, $M$ is homeomorphic to $S^n$.*

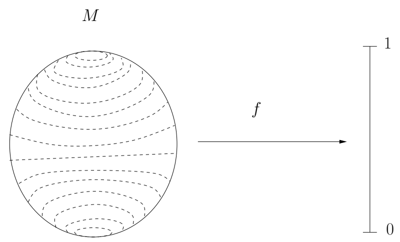


Figure 5.1:

**(2)** *The boundary of $M$ is not empty and $\mathcal{F}$ has singularities. In this case there is a single singularity $\mathrm{sing}(\mathcal{F}) = \{p\}$ which is a a center and there exists a smooth function $f\colon M^n \to [0,1] \subset \mathbb{R}$, which defines $\mathcal{F}$, with nonsingular levels diffeomorphic to $S^{n-1}$. Moreover, $\partial M$ has a single connected component and $M$ is homeomorphic to $\overline{B^n}$.*

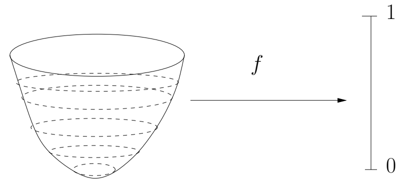


Figure 5.2:

**(3)** *$\mathcal{F}$ is nonsingular and $M$ has nonempty boundary. In this case $\partial M$ has two connected components, each diffeomorphic to $F$, the typical leaf of $\mathcal{F}$. $\mathcal{F}$ is given by a function $f : M^n \to [0,1] \subset \mathbb{R}$, with no critical points and levels diffeomorphic to $F$, and $M$ is homeomorphic to the product $F \times [0,1]$.*

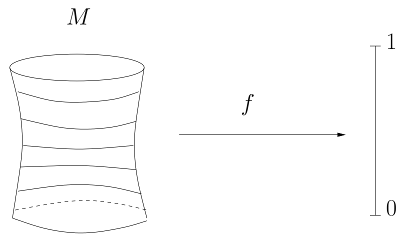


Figure 5.3:

**(4)** *If $\mathcal{F}$ nonsingular and $M$ is closed. Then $\mathcal{F}$ is given by a fibration $M^n \to S^1$, i.e., $M^n$ is a fiber bundle over $S^1$, with fiber a typical leaf $F$ of $\mathcal{F}$. In particular, if $\mathcal{F}$ has some leaf diffeomorphic to $S^{n-1}$ then $M$ is homeomorphic to $S^{n-1} \times S^1$ and $\mathcal{F}$ is the trivial foliation by spheres, $S^{n-1} \times \{y\}$, $y \in S^1$.*

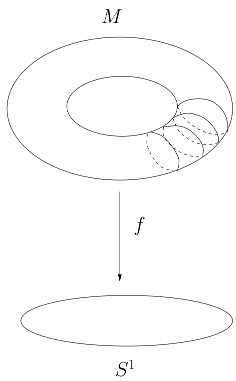


Figure 5.4:

Prior to the proof of Theorem 18 we need to recall a classical result and adapt it to our framework:

**Proposition 9** ([7] Corollary 2.19, page 103)**.** *Let $\mathcal{F}$ be a codimension one smooth (nonsingular) foliation on a connected oriented manifold $M^n$ tangent to the boundary of $M$ if nonempty. If $\mathcal{F}$ is transversely oriented and has all leaves compact then the leaf space $M/\mathcal{F}$ is a Hausdorff manifold of dimension one.*

Given now a compact foliation with singularities $\mathcal{F}$ on $M$ we put $\tilde{M} = M \setminus \text{sing}(\mathcal{F})$ and $\tilde{\mathcal{F}} = \mathcal{F}|_{\tilde{M}}$. By definition, the *space of leaves* $M/\mathcal{F}$ is defined as the quotient of $M$ by the equivalence relation: $x, y, \in M$, $x \simeq y \Leftrightarrow x = y \in \text{sing}(\mathcal{F})$ or $x, y \notin \text{sing}(\mathcal{F})$ and $L_x = L_y$, i.e, $x$ and $y$ belong to the same leaf of $\mathcal{F}$. Clearly, $M/\mathcal{F}$ contains as an open subset the leaf space $\tilde{M}/\tilde{\mathcal{F}}$ of the underlying nonsingular foliation. We have:

**Proposition 10.** *For a transversely oriented compact foliation with singularities $\mathcal{F}$ all of center type on a connected oriented manifold $M$ the space of leaves $M/\mathcal{F}$ is a Hausdorff one-dimensional manifold. In particular $M/\mathcal{F}$ is diffeomorphic to $S^1$ or* $[0, 1]$.

*Proof.* Let $p \in \text{sing}(\mathcal{F})$ be a singularity. As all leaves are compact, $p$ is a center. By the Morse Lemma, the foliation near $p$ is defined by the level sets of a real function. Right away, it follows that, near $p$, the space of leaves is (locally) Hausdorff. It remains to prove that $\tilde{M}/\tilde{\mathcal{F}}$ is Hausdorff. We adopt the following criterion: every point in $\tilde{M}/\tilde{\mathcal{F}}$ is closed and has a fundamental system of closed neighborhoods. This is equivalent to require that each leaf is closed and its neighborhoods, bounded by (neighbor) leaves, are closed. As $\tilde{\mathcal{F}}$ is a compact foliation with singularities, by the local product structure, the neighborhoods of each leaf $F$, bounded by neighbor leaves, are diffeomorphic to $[-\epsilon_1, \epsilon_2] \times F$, for some $0 < \epsilon_1, \epsilon_2 < 1$. □

*Proof of Theorem 18.* By hypothesis, we have a singular compact foliation with singularities $\mathcal{F}$, with $\text{sing}(\mathcal{F}) \neq \emptyset$, and/or $\partial M \neq \emptyset$ and $\mathcal{F}$ tangent to the boundary. Each leaf $F \subset \tilde{M} = M \setminus \text{sing}(\mathcal{F})$ of the foliation $\tilde{\mathcal{F}} = \mathcal{F}|_{\tilde{M}}$ is compact with trivial holonomy, then it has a neighborhood $V(F)$ diffeomorphic to the product $(-1, 1) \times F$, say through the diffeomorphism $h_F : (-1, 1) \times F \to V(F)$. Moreover, each leaf in $V(F)$ is the image of the set $\{t\} \times F$ for some $t \in (-1, 1)$ and the original leaf is the image of $\{0\} \times F$. This is a straightforward consequence of the Reeb local stability theorem.
If $M$ is a manifold with boundary and $F \subset \partial M$, for obvious reasons there exists a diffeomorphism of $V(F)$ with $(-1, 0] \times F$ or with $[0, 1) \times F$. In any case we will say that near $F$, $M$ has a *local product structure.* In $\tilde{M}$, as all leaves are compact and $M$ is always assumed to be Hausdorff, all leaves are closed.

**Claim 1.** *If* $\text{sing}(\mathcal{F}) \neq \emptyset$ *or* $\partial M \neq \emptyset$ *and* $\mathcal{F}$ *is tangent to the boundary then the space of leaves is homeomorphic to* $[0, 1]$.

*proof of Claim 1.* Let $\text{sing}(\mathcal{F}) \neq \emptyset$; then, as already noticed, $\forall p \in \text{sing}(\mathcal{F})$, $p$ is a center. Let $(U, \phi)$ be a local chart around $p$. By the Morse Lemma, it follows that $p$ is a point of local maximum or minimum for the function defining the foliation near $p$, $f = \pm \sum x_i^2 : \phi(U) \to \mathbb{R}$, a local first integral. This means that the image of $U$, by means of the projection onto the space of leaves, belongs to a left, respectively right, neighborhood of $f(p) = 0$ and this gives the space of leaves a boundary point, $\pi(p)$, determining the choice $M/\mathcal{F} \simeq [0, 1]$. At the same choice we are led if $\partial M \neq \emptyset$ and $\mathcal{F}$ is tangent to the boundary (or, in a more general situation, when $\#\{\text{connected components of } \partial M\} > 1$ and $\mathcal{F}$ is tangent to at least one). In fact every leaf in $\partial M$ is a boundary point of $M/\mathcal{F}$, as a consequence of the well known properties of the differential. As $\pi = cost$ on each connected component of $\partial M$. Let $c \in \partial M$, then, for $d(\pi|_{\partial M})_c : T_c\partial M \to T_{\pi(c)}M/\mathcal{F}$, we have

$$d(\pi|_{\partial M})_c : T_c\partial M \to \{0\}.$$

Then $d(\pi|_{\partial M})_c : T_c\partial M \to \partial M/\mathcal{F}$. □

As a consequence, when $(\text{sing}(\mathcal{F}) \neq \emptyset) \bigvee (\partial M \neq \emptyset)$, as $\partial(M/\mathcal{F}) \neq \emptyset$, then

$$\#\partial(M/\mathcal{F}) = 2 = \#\,\text{sing}(\mathcal{F}) + \#\{\text{connected components in } \partial M\}$$

and this gives the first characterization in cases (1), (2), (3).
Vice-versa, if $M/\mathcal{F} \simeq [0, 1]$, the projection can be seen as the restriction to the image of a map $M \to \mathbb{R}$,

$$\imath \circ \pi : M \to M/\mathcal{F} \hookrightarrow \mathbb{R}.$$

As $M$ is compact, this function has a maximum and a minimum ($\in \partial(M/\mathcal{F})$). Each of these points can be a regular or a critical point; so cases (1), (2), (3) get through the case $M/\mathcal{F} \simeq [0, 1]$ and in case 4, $M/\mathcal{F} \simeq S^1$.

In the following two claims we give an alternative proof of the fact that $\tilde{M}$ is a fiber bundle over $\tilde{M}/\tilde{\mathcal{F}}$.

**Claim 2.** *Let $F$ be a leaf of $\mathcal{F}$. Each leaf of $\mathcal{F}$ is diffeomorphic to $F$.*

*proof of Claim 2.* We have two cases: $\text{sing}(\mathcal{F}) \neq \emptyset$, $\text{sing}(\mathcal{F}) = \emptyset$. Let $p \in \text{sing}(\mathcal{F})$ be a center. By the Morse Lemma there exist leaves diffeomorphic to $S^{n-1}$. In this case we assume that $F$ is one of them and we consider the open sets $\mathcal{C}(\mathcal{F})$ and $\mathcal{C}_p(\mathcal{F})$. Otherwise,

if $\text{sing}(\mathcal{F}) = \emptyset$, let $F$ be a (any) leaf. We define the set $M(F) \subset M$ as the the union of leaves with trivial holonomy and diffeomorphic to $F$, and we consider $\mathcal{C}_F(\mathcal{F}) \subset M(F)$ as the connected component containing $F$. By the Reeb local stability theorem, also $\mathcal{C}_F(\mathcal{F})$ is open. Now we prove that $\mathcal{C}_p(\mathcal{F})$ or $\mathcal{C}_F(\mathcal{F})$ is closed in $M$. At this purpose, let $\{x_n\}$ be a sequence of points in $\mathcal{C}_p(\mathcal{F})$, resp. $\mathcal{C}_F(\mathcal{F})$ converging to $x_0 \in M$. We show that $x_0 \in \mathcal{C}_p(\mathcal{F})$, resp. $x_0 \in \mathcal{C}_F(\mathcal{F})$. In the first case it can happen $x_0 \in \text{sing}(\mathcal{F})$; in this case $x_0$ is a center and so $x_0 \in \mathcal{C}(\mathcal{F})$. But $x_0 \in \overline{\mathcal{C}_p(\mathcal{F})}$, by hypothesis, and $\mathcal{C}_p(\mathcal{F})$ is closed in $\mathcal{C}(\mathcal{F})$, so $x_0 \in \mathcal{C}_p(\mathcal{F})$.
So let $x_0$ be a regular point for the foliation. If the points $x_n, n > m_0 \in \mathbb{N}$ belong to the same leaf $L$, by compactness of $L$, $x_0 \in L$ too. Otherwise, let $L_x$ be the leaf through the point $x$. By hypothesis, $L_{x_n}$ is diffeomorphic to $F$, for all $n$; by the local product structure, all the leaves in $V(L_{x_0})$ are diffeomorphic to $L_{x_0}$ and $V(L_{x_0})$ contains leaves diffeomorphic to $F$. So $x_0 \in L_{x_0} \subset \mathcal{C}_p(\mathcal{F})$, resp. $\mathcal{C}_F(\mathcal{F})$.
As $M$ is connected we have $M = \mathcal{C}_p(\mathcal{F})$, resp. $M = \mathcal{C}_F(\mathcal{F})$. □

**Claim 3.** $\tilde{M} = M \setminus \text{sing}(\mathcal{F})$ *is a fiber bundle with base space given by the space of leaves* $\tilde{M}/\tilde{\mathcal{F}} = (M/\mathcal{F}) \setminus \pi(\text{sing}(\mathcal{F}))$ *with fiber* $F$*, a typical leaf of* $\mathcal{F}$*.*

*proof of Claim 3.* This is a consequence of the local product structure and so of the Reeb local stability theorem.
Set $B = \tilde{M}/\tilde{\mathcal{F}}$. $\forall x \in B$, $\pi^{-1}(x) = F_x$ a leaf of $\mathcal{F}$. We can find a neighborhood $V(F_x)$ defined by the local product structure. If it happens $x \in \partial B$ then $\pi^{-1}(x) = F_x \subset \partial\tilde{M}$, as we have excluded singularities. In this case, we know $V(F_x)$ is diffeomorphic to $[0,1) \times F_x$ or $(-1,0] \times F_x$. The set

$$\{\pi(V(F_x))\}_{x \in B}$$

is an open cover of $B = \tilde{M}/\tilde{\mathcal{F}}$. Let

$$\{U_i = \pi(V(F_i))\}_{i \in I}$$

a locally finite open subcover. $\pi^{-1}(U_i)$ is diffeomorphic to $(-1,1) \times F_i$, or to $[0,1) \times F_i$ or $(-1,0] \times F_i$ through the diffeomorphism $h_i = h_{F_i}$, i.e. to $(-1,1) \times F$, or $[0,1) \times F$ or $(-1,0] \times F$, as all leaves are diffeomorphic. Then, set $\phi_i = \pi_1(h_i(V(F_i))$, where $\pi_1$ is the projection on the first component, $\forall i \in I$ we have a diffeomorphism

$$\phi_i : U_i \to (-1,1) \text{ or } \phi_i : U_i \to (-1,0] \text{ or } \phi_i : U_i \to [0,1)$$

In this way $\{(U_i,\phi_i)\}_{i\in I}$ is a (locally finite) atlas on $\tilde{M}/\tilde{\mathcal{F}}$, where the change of coordinates, the composition of diffeomorphisms $\phi_j\circ\phi_i^{-1}:\phi_i(U_i\cap U_j)\to\phi_j(U_i\cap U_j)$, is clearly a diffeomorphisms. At last $\pi^{-1}(U_i)=V(F_i)$ are diffeomorphic to $U_i\times F$ and $\tilde{M}$ is given a fiber bundle structure on $\tilde{M}/\tilde{\mathcal{F}}$. □

In cases (1), (2) and (3) the manifold $\tilde{M}$ is a fiber bundle over an interval, so by classical reasons (cf. for example the book [8]) $\tilde{M}$ is the trivial bundle. In particular, in case (1), $\tilde{M}=M\setminus\{p,q\}\simeq(0,1)\times S^{n-1}$ (and $M$ is defined by a singular fibration $\pi:M\to[0,1]$); in case (2), $\tilde{M}=M\setminus\{p\}\simeq[0,1)\times S^{n-1}$ or $\simeq(-1,0]\times S^{n-1}$; in case (3), $M\simeq[0,1]\times F$. This completes case (3).
As for case (1), the conclusion $M\simeq S^n$ could be obtained with an application of classical Reeb's sphere recognition theorem (that we recall in the next pages), as we have proved that $\mathcal{F}$ has a first integral, the map $\imath\circ\pi:M\to\mathbb{R}$, or with Cantrell's theorems [22]. Alternatively, we can extend the homeomorphism $\tilde{h}$

$$M\setminus\{p,q\}\xrightarrow{\tilde{h}}(0,1)\times S^{n-1}\qquad\simeq\qquad\pi(M\setminus\{p,q\})\times S^{n-1}$$

to a homeomorphism $h$ on $M$

$$M\xrightarrow{h}[0,1]_t\times S_s^{n-1}/_\sim\qquad=\qquad\mathcal{S}(S^{n-1})\qquad\simeq\qquad S^n,$$

where $\mathcal{S}$ denote the suspension and $\sim$ is the equivalence relation

$$(t,s)\sim(t',s')\Leftrightarrow\begin{cases}t'=t, & s=s'\\ t=t'=0, & s,s'\in S^{n-1}\\ t=t'=1, & s,s'\in S^{n-1}\end{cases}$$

Then if $[\cdot]$ denote an equivalence class of the relation $\sim$, we may set $h(p)=h(\pi^{-1}(0))=[(0,s)]$ and $h(q)=h(\pi^{-1}(1))=[(1,s)]$. So $h$ is a bijection and it is continuous, for the continuity of $\pi$. It is also a homeomorphism, by the local description of the manifold around centers, given by the Morse Lemma.
Similarly we can complete case (2), extending the homeomorphism

$$\tilde{h}:\ M\setminus\{p\}\ \to\ (0,1]\times S^{n-1}$$

to a homeomorphism

$$h:\qquad M\qquad\to\ [0,1]_t\times S_s^{n-1}/_\sim\simeq\overline{B^n},$$

where $\sim$ is the equivalence relation

$$(t,s) \sim (t',s') \Leftrightarrow \begin{cases} t = t', & s = s' \\ t = t' = 0, & \forall s, s' \in S^{n-1}, \end{cases}$$

setting $h(p) = [(0,s)]$.
In case (4), $M$ is a fiber bundle over $S^1$ and the foliation is the trivial foliation $\pi^{-1}(x)$ (with leaves diffeomorphic to $\{x\} \times F$, for $x \in S^1$). In the case $F$ is diffeomorphic to the $(n-1)$-sphere, $S^{n-1}$, by classical theory, we know there exists only one non-trivial bundle, a sort of generalization of the Klein bottle. We can think at this bundle as the product $S^{n-1} \times [0,1]$ modulo an equivalence relation that identify the boundaries, by an orientation reversing diffeomorphism. The resulting bundle is always non-orientable (cf. [20]). In the orientable case, $M$ is the trivial bundle $S^{n-1} \times S^1$. This completes case (4).

□

## 5.3 On the sphere recognition theorem

In the classical version, Reeb's sphere recognition theorem gives the characterization of a compact $n$-dimensional manifold admitting a Morse foliation, whose singularities are centers and $\text{sing}(\mathcal{F}) \neq \emptyset$. In fact it states:

**Theorem 19** (Reeb's sphere recognition theorem, [17])**.** *Let $\mathcal{F}$ be a codimension one transversely orientable $C^\infty$ foliation with Morse singularities on a closed connected oriented manifold $M^n$, $n \geq 3$. Suppose* $\text{sing}(\mathcal{F}) \neq \emptyset$ *consists only of centers. Then $M^n$ is homeomorphic to the $n$-sphere $S^n$.*

In particular, if a closed connected oriented manifold $M$ admits a Morse function with exactly two critical points then this manifold is homeomorphic to $S^n$ ([11]). A highly nontrivial version is the celebrated result below due to Milnor:

**Theorem 20** (Milnor, [13] Chapter 6)**.** *Let $M^n$ be an $n$-dimensional compact connected oriented manifold admitting a $C^\infty$ function $f : M \to \mathbb{R}$ with exactly two (possibly degenerate) critical points. Then $M^n$ is homeomorphic to the sphere $S^n$.*

We shall give a generalization of Reeb sphere recognition theorem, after introducing a new concept.

**Definition 21.** We say that an isolated singularity $p$ of a $C^\infty$, codimension one foliation $\mathcal{F}$ on $M$ is a *stable singularity*, if there exists a neighborhood $U$ of $p$ in $M$ and a $C^\infty$ function $f : U \to \mathbb{R}$, defining the foliation in $U$, such that $f(p) = 0$ and $f^{-1}(a)$ is compact, for $|a|$ small.

By definition, the space of leaves near a stable singularity is a subset of the real line, and then (locally) it is Hausdorff. Right away, we may extend the result of Proposition 10.

**Proposition 11.** *In a compact foliation with singularities, with stable singularities, the space of leaves is Hausdorff.*

**Example 14.** The first example of stable singularities are centers, i.e. the origin for the foliation (locally) defined by the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f = \sum_i x_i^2$. Centers are the only examples of stable singularities which are also non-degenerate.

**Example 15.** The foliation defined by the function $f = \sum_j x_j^{m_j}$, with $2 \leq m_j$ even $\forall j$ and $m_j > 2$ for at least one $j$, has a stable singularity at the origin. In this case the stable singularity is degenerate, but the function $f$ is not flat.

**Example 16.** The foliation defined by the function $f = \exp(-\frac{1}{\sum x_j^2})$ has a stable singularity at the origin; it is degenerate and the Taylor polynomial of $f$ is identically zero at $x = 0$ ($f$ flat).

We can give a characterization of stable singularities [3].

**Lemma 12.** *An isolated singularity $p$ of a function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ defines a stable singularity for $df$ if and only if there exists a neighborhood $p \in V \subset U$ such that $\forall x \in V$ either we have $\omega(x) = \{p\}$ or $\alpha(x) = \{p\}$, where $\omega(x)$, resp. $\alpha(x)$, is the $\omega$-limit, resp. $\alpha$-limit, of the orbit of the vector field* $\operatorname{grad}(f)$ *through the point $x$.*

In particular it follows the well-known:

**Lemma 13.** *If a function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ has an isolated local maximum or minimum at $p \in U$ then $p$ is a stable singularity for $df$.*

It is also true the converse:

**Lemma 14.** *If $p$ is a stable singularity defined by the function $f$, then $p$ is a point of local maximum or minimum for $f$.*

*Proof.* By the characterization above there exists a neighborhood $p \in V \subset U$ such that $\forall x \in V$ either $\omega(x) = \{p\}$ or $\alpha(x) = \{p\}$. Let us suppose $\omega(x) = \{p\}$ (the other case is similar). Then $p$ is a point of maximum for $f$. In fact it is well known that $f$ is monotonous, strictly increasing, along the orbits of grad$(f)$. So if $x \in V$ and $\{\phi(t,x), t > 0\}$ is the positive semi-orbit through $x$, we have

$$\lim_{t \to +\infty} \phi(t,x) = p$$

So $f(\phi(t,x)) < f(p),\ t \geq 0$. In particular $f(x) < f(p)$. □

As a consequence, in the set $\mathcal{A} = \{f^{-1}(a) | a \in \mathrm{Ima}(f|_V)\}$ we can define a total order. We say that $f^{-1}(a) < f^{-1}(b)$ if and only if $R(f^{-1}(a)) \subseteq R(f^{-1}(b))$, where for each $a \in \mathrm{Ima}(f|_V)$, $R(f^{-1}(a))$ is the neighborhood of $p$ bounded by $f^{-1}(a)$. As $f^{-1}(a) \cap f^{-1}(b) = \emptyset$, if $a \neq b \in \mathrm{Ima}(f|_V)$, then the order is total.
The following example and the next lemma precise in what sense these singularities are stable.

**Example 17.** Let us consider the function $f_0 = x^3 - 3xy^2$ presenting one singularity at the origin, the Monkey saddle singularity. Now we perturb $f_0$, so we consider functions $f = f_0 + \epsilon(x^2 + y^2)$, $\epsilon \in \mathbb{R}$. Even for small $|\epsilon|$, $f$ presents four singularities: a center at $(0,0)$ and three saddles at resp. $(-2/3\epsilon, 0)$ and $(\epsilon/3, \pm\epsilon/\sqrt{3})$. We have $\mathrm{sing}(f_0) \subsetneq \mathrm{sing}(f)$.

This does not happen when we deal with stable singularities. In fact we have [3]:

**Lemma 15.** *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a $C^\infty$ function with a stable singularity at the point $p \in U$. Then we can perturb $f$ to obtain a function $\tilde{f}$ with a Morse center-type singularity at $p$ and no other singularity in a neighborhood of $p$.*

The likeness between stable singularities and and center-type singularities reflects heavily on the level hypersurfaces of the respective defining functions. In fact we have:

**Lemma 16.** *In a neighborhood of a stable singularity $p$ for a foliation $\mathcal{F}$, the leaves are diffeomorphic to spheres.*

*Proof.* Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a function defining the foliation in a neighborhood of $p$. Unless replacing $f$ with $-f$, by lemma 14 we can suppose $p$ is a local maximum for $f$. As a manifold is locally compact, the point $p \in M$ has a fundamental system of compact neighborhoods bounded by spheres $S^{n-1}(p, \epsilon)$. Choose an $\epsilon > 0$ such that $\overline{B^n}(p, \epsilon) \subset R(f^{-1}(a))$, where $a$ is such that $f^{-1}(a)$ is a leaf near $p$. We can also suppose that $R(f^{-1}(a)) \subset W$, the domain of a local chart around $p$. Consider the local flow of the vector field $\text{grad}(f)$, $\phi : \mathcal{D}(\phi) \subset \overline{R(f^{-1}(a))} \times \mathbb{R} \to \overline{R(f^{-1}(a))}$

$$\phi : (t, x) \xrightarrow{C^\infty} \phi(t, x) = \phi_t(x).$$

$\overline{R(f^{-1}(a))}$ is compact and so $\phi$ is globally defined, in particular we have $\mathcal{D}(\phi) \supset \mathcal{D} = (c, +\infty) \times f^{-1}(a)$ for some $c < 0$ and we restrict to such a domain. We define the map

$$F : (t, x) \longrightarrow (-t, \phi_t(x))$$

on the domain $\mathcal{D}$ and image $\text{Ima}(F) \subsetneq (-\infty, -c) \times \left(\overline{R(f^{-1}(a))} \setminus \{p\}\right)$. This is a $C^\infty$ map, as its components are, and it has an inverse whose expression coincides with $F$. Moreover $F$ is injective. In fact if $(t', x') \neq (t, x)$, we have:
if $t' \neq t \Rightarrow F(t', x') \neq F(t, x)$;
if $t' = t \Rightarrow x' \neq x$, then $F(t, x') \neq F(t, x)$ as $x \to \phi_t(x)$ is a diffeomorphism. Then $F$ is a diffeomorphism on its image.
We observe that $\pi_2(\text{Ima}(F)) = \overline{R(f^{-1}(a))} \setminus \{p\}$, where $\pi_2$ is the projection on the second set of the product. In particular

$$S^{n-1}(p, \epsilon) \subset \pi_2(\text{Ima}(F)). \tag{5.1}$$

Then we can define the open set

$$V = F^{-1}((-\infty, -c) \times S^{n-1}(p, \epsilon)),$$

where for each $x \in S^{n-1}(p, \epsilon)$, $V_x := F^{-1}((-\infty, -c) \times \{x\}) = \{y\}$ is a single point. In fact for 5.1 $V_x \neq \emptyset$. Now suppose $(-t, \phi(t, x)), (-t', \phi(t', x)) \in V_x$ with $t \neq t'$, say $t > t'$. We observe that both $\phi(t, x)$ and $\phi(t', x)$ belong to the same orbit through $x$ of the vector field $\text{grad}(f)$. As we know, $f$ is increasing along those orbits, so we have

$$f(\phi(t, x)) > f(\phi(t', x))$$

and the two points of the orbit cannot belong to the same leaf $\{f = a\}$. This means that $\forall x \in S^{n-1}(p, \epsilon)$ there exists a single $t_x$ such that $\phi(t_x, x) \in f^{-1}(a)$; in other words it is defined a map $x \to t_x$. As $V$ is open, the restriction $F|_V$ is a diffeomorphism

$$(t_x, x) \xleftarrow[F|_V]{} (-t_x, \phi_{t_x}(x)),$$

where $x \in S^{n-1}(p, \epsilon)$. Moreover as $\{p\} = \omega(x)$

$$\{\phi(t, x) | t \in [0, +\infty)\} \cap S^{n-1}(p, \epsilon) \neq \emptyset \qquad \forall x \in f^{-1}(a)$$

So $\pi_2(V) = f^{-1}(a)$ and $F|_V$, restricted to its components, gives, in particular, a $C^\infty$ bijection

$$\psi : f^{-1}(a) \to S^{n-1}(p, \epsilon).$$

For the same reasons its inverse is $C^\infty$, so $\psi$ is a diffeomorphism of $f^{-1}(a)$ with $S^{n-1}(p, \epsilon)$.

□

If a stable singularity is defined by a function which is not flat at the singularity, we can give an alternative *proof.*

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a non-flat function defining a stable singularity $p$; this means the Taylor polynomial of $f$ at 0 is non-trivial, but exhibits a first non-trivial jet of the form

$$\pm \sum_{j=1}^{n} a_j {x_j}^{2m_j}, \tag{5.2}$$

where $a_j > 0$ $\forall j \in \{1, \ldots, n\}$ and $m_j \in \mathbb{N}$. If $m_j = 1$ $\forall j \in \{1, \ldots, n\}$, we are already dealing with a center-type singularity and so the diffeomorphism exists by the Morse Lemma. So let $m_j > 1$ for some index $j \in \{1, \ldots, n\}$. Moreover let us suppose that $p$ is a minimum so that in the expression 5.2 we may select the positive sign (we can proceed in a similar way if $p$ is a maximum). First of all we observe that $W = \overline{R(f^{-1}(a))} = \{f^{-1}(\alpha)\}_{0 \leq \alpha \leq a}$ is homeomorphic to $\overline{B^n}$. This is a consequence of the fact that $W$ is star-shaped with respect to the origin, i.e. $x \in \partial W \Rightarrow tx \in W, 0 \leq t \leq 1$. This happens $\Leftrightarrow f(x) - f(tx) \geq 0$. In fact we have

$$\sum_{j=1}^{n} a_j {x_j}^{2m_j} - \sum_{j=1}^{n} a_j {t_j}^{2m_j} {x_j}^{2m_j} = \sum_{j=1}^{n} a_j (1 - {t_j}^{2m_j}) {x_j}^{2m_j} \geq 0$$

since it is a sum of products of non negative terms. Then $W \simeq \overline{B^n}$. Observe that a homeomorphism between the two sets is given by the $C^1$ function

$$\phi(x) = \begin{cases} f(x) \frac{x}{\|x\|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

that moves $x$ along the positive ray through it. Observe that $\phi(x)$ sends level sets of the function $f$, i.e. leaves of $\mathcal{F}$ in a neighborhood of $p$, into leaves of the singular trivial foliation of $\overline{B^n}$.

We can see that $\phi$ is surjective. In fact let $y \in \overline{B^n}$. Then there exists $x \in \phi^{-1}(y)$, given by the intersection of the level set $f^{-1}(\|y\|)$ with the ray $\lambda y, \lambda > 0$ ($\lambda < 0$ if $p$ is a local maximum).

We see also that $\phi$ is injective. Let $x_1 \neq x_2$ be two points of $W$. We have $\phi(x_1) = f(x_1)\frac{x_1}{\|x_1\|}$ and $\phi(x_2) = f(x_2)\frac{x_2}{\|x_2\|}$. If $f(x_1) \neq f(x_2) \Rightarrow \|\phi(x_1)\| \neq \|\phi(x_2)\|$ and so $\phi(x_1) \neq \phi(x_2)$. So let $f(x_1) = f(x_2)$. If $\phi(x_1) = \phi(x_2)$ then $\frac{x_1}{\|x_1\|} = \frac{x_2}{\|x_2\|}$ and so two points of the same leaf lie on the same ray, but this is a contradiction with the fact that $W$ is star-shaped with respect to the origin. So we have $\phi(x_1) \neq \phi(x_2)$.

At last we see that $\phi$ is an open map. Let $A \subset W$ be an open set, $x \in A$ and $y = \phi(x)$. We can find an open neighborhood of $y$ contained in $\phi(A)$. Recalling that $\phi$ is a bijection, it is enough to choose a neighborhood of $y$ given by the intersection of the annulus $D = \{||y|| - \epsilon < ||z|| < ||y|| + \epsilon\}$ with a little open cone $C$ with vertex at the origin, centered at $y$ and choose $\epsilon$ and the wideness of $C$ in a way that $\phi^{-1}(D \cap C) \subset A$.

For classical reasons $\phi$ is a homeomorphism between $(W, \partial W)$ and $(\overline{B^n}, S^{n-1})$. We consider the restriction

$$\psi = \phi|_{W\setminus\{0\}} : (W \setminus \{0\}, \partial W) \to (\overline{B^n} \setminus \{0\}, S^{n-1}).$$

$\psi$ is a diffeomorphism and $\psi|_{\partial W}$ is the diffeomorphism of $f^{-1}(a)$ with $S^{n-1}$.

□

With the result of Lemma 13 we have

**Theorem 21.** *Let $\mathcal{F}$ be a transversely orientable compact foliation with stable singularities on a closed connected oriented manifold $M$. Then $M$ admits a transversely orientable compact foliation with singularities all of center type. In particular $M$ must be as in cases* (1) *or* (4) *in Theorem 18.*

*Proof.* Indeed, given a singularity $p \in \text{sing}(\mathcal{F})$, according to Lemma 13 there is an invariant compact neighborhood $\overline{B}_p \subset M$ of $p$ diffeomorphic to the closed unit ball in $\mathbb{R}^m$, by a diffeomorphism that takes $\mathcal{F}|_{\overline{B}_p}$ into the foliation by concentric spheres $S^{m-1}(0; r), 0 < r \geq 1$ and the singularity $p$ into the origin $0 \in \mathbb{R}^m$.

□

**Corollary 4.** *Let $M^n$ be a closed $n$-dimensional manifold, $n \geq 3$. Suppose $M$ supports a $C^\infty$ codimension one transversely orientable foliation $\mathcal{F}$, with non-empty singular set, whose elements are all stable singularities. Then $M$ is homeomorphic to the sphere $S^n$*

*Proof.* Let $p_1, \ldots, p_k$, $k \geq 1$ be the stable singularities of the foliation and $f_i : U_i \ni p_i \to \mathbb{R}$, $f_i(p_i) = 0$ for all $i$, their defining functions. For $|a|$ small and $i = 1, \ldots, k$, by lemma 16, the compact leaf $f_i^{-1}(a)$ is diffeomorphic to a little sphere around $p_i$, $S^{n-1}(p_i, \epsilon)$, and bounds a region $R(f^{-1}(a))$ diffeomorphic to $B^n(p_i, \epsilon)$. From this point we can go on as in the proof of Theorem 19. In particular, as by Proposition 11 the space of leaves is Hausdorff, we may apply the classification Theorem 18 for singular Seifert fibrations. □

As a consequence of the last theorem we re-obtain the Milnor's version of Reeb's sphere recognition theorem. In fact we have:

*Proof of Theorem 20.* The function $f$ defines a singular foliation with compact leaves and having as singular set the two critical points of $f$, which according to Lemma 13 are stable singularities. We apply Corollary 4 above and conclude the proof. □

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