SEMILINEAR ELLIPTIC EQUATIONS

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P976 Semilinear Elliptic Equations / Jean-Pierre Puel.

Rio de Janeiro: UFRJ/IM, 2018.

124p.; 21cm.

Inclui Bibliografia / inclui índice remissivo.

ISBN: **978-85-87674-30-2**

1. Variational problems. 2. Second-order monotone nonlinear equations. 3. Singular solutions.

I. Universidade Federal do Rio de Janeiro. Instituto de Matemática.

II. Título.

Dedicated to Luiz Adauto Medeiros

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Preface

This book is based on the lecture notes for a graduate course taught at the Institute of Mathematics of the Federal University of Rio de Janeiro (IM-UFRJ) during the first semester of 2013. Its purpose is to present the classical methods and results for second order monotone nonlinear elliptic equations and for non monotone second order semilinear elliptic equations on a bounded domain, with a special focus on the problem $-\Delta u = \lambda e^u$.

This domain has been extensively studied and there are many books and articles treating different aspects of the problems, some of them are mentioned in the references but this list is from far not exhaustive. The methods and results presented here are not new, but this course gave me the opportunity to put together some known and less known results in a coherent text which is essentially self contained and with all the main proofs. Only some basic results of integration, functional analysis and of spectral decomposition are recalled without proof.

In Chapter 1, we give a quick review on distributions, L^p spaces, weak and weak-$*$ convergence, and the fundamental Sobolev spaces which will constitute the functional framework in these notes.

In Chapter 2 we recall the classical treatment of second order linear elliptic equations and some additional properties like the maximum principle, some regularity results and the spectral properties in the symmetric case.

Chapter 3 is devoted to the study of the minimisation of convex functionals and of second order semilinear and nonlinear monotone equations. For semilinear monotone equations we give a very general result based on integration techniques and for monotone operators we follow essentially the lines of the pioneering book by J.-L. Lions [15].

In Chapter 4, we consider second order (a priori non monotone) semilinear equations. In a first part we give a complete description of the methods based on maximum principle. Some of the results are well known (see for example [24]) but some additional properties are less known and will be of interest for specific applications. Then we study the variational methods with two cases: first of all we look for extrema of functionals on a manifold; then we prove and apply the celebrated Mountain Pass Theorem due to A. Ambrosetti and P. Rabinowitz [2].

In Chapter 5, we focus on the problem $-\Delta u = \lambda e^u$ with Dirichlet boundary conditions. This is a rich problem for which we use almost all methods described before successively. We give here all the results which are known on this problem, including in the last section the study of radial symmetric regular and singular solutions in a ball.

These notes have been written by Eleonora Moura and Rolci Cipolatti and I would like to express my warmest thanks to them for their very nice work and their constant tenacity.

I also want to thank all my colleagues from IM-UFRJ, in particular Flavio Dickstein and Rolci Cipolatti, for their hospitality and their constant help during my stay in Rio de Janeiro. We have made together a lot of mathematical work and we have also had many very interesting non mathematical discussions.

Acknowledgements

The author's visit was supported by the program *Ciência sem fonteiras* of the Brazilian government through CNPq.

Jean-Pierre Puel
Rio de Janeiro, 2018

Chapter 1

Preliminaries

In this chapter we recall some basic definitions and results concerning Distributions, L^p spaces and Sobolev spaces, that will be essential to the subsequent material.

We define the *support* of a function $u \in C(\mathbb{R}^N)$ as the set

$$\operatorname{supp} u := \overline{\{x \in \mathbb{R}^N \,;\, u(x) \neq 0\}}.$$

Notice that, since the support of u is a closed set, it may contain points where $u(x) = 0$.

We denote by $\mathcal{D}(\mathbb{R}^N)$ the vector space of $C^\infty(\mathbb{R}^N)$ with compact support, i.e.,

$$\mathcal{D}(\mathbb{R}^N) := \{u \in C^\infty(\mathbb{R}^N) \,;\, \operatorname{supp} u \text{ is compact}\}$$

As an example of a function in $\mathcal{D}(\mathbb{R}^N)$, consider

$$\rho(x) := \begin{cases} \exp\left(\dfrac{1}{|x|^2 - 1}\right) \text{ if } |x| < 1 \\ 0 \text{ if } |x| \geq 1 \end{cases} \tag{1.1}$$

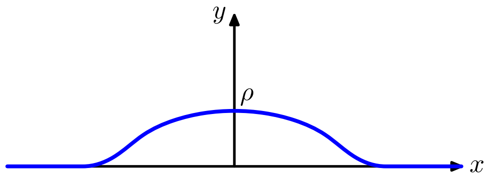

Figure 1.1. Graphic of the function (1.1).

It is clear that $\rho \in C^\infty(\mathbb{R}^N)$ and its support is the unit ball

$$\overline{B_1(0)} := \{x \in \mathbb{R}^N : |x| \leq 1\}.$$

If Ω is an open subset of $\mathbb{R}^N$, we denote

$$\mathcal{D}(\Omega) := \{u \in \mathcal{D}(\mathbb{R}^N) \,;\, \operatorname{supp} u \subset \Omega\}.$$

$\mathcal{D}(\Omega)$ is known as the space of *test functions* on Ω.

Note that $\operatorname{supp} u$ is a closed set contained in the open set Ω.

Example 1.1. Let Ω be the open interval $]a,b[$ and $\psi_1, \psi_2 : \Omega \to \mathbb{R}$ be the functions whose graphs are given by the following pictures:

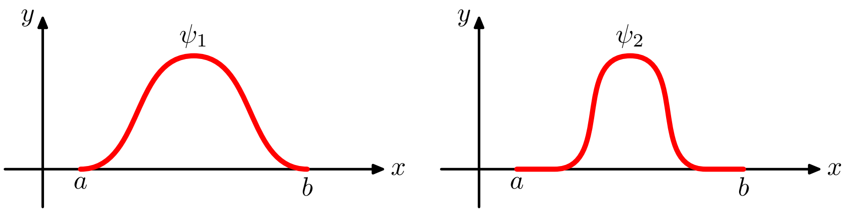

Figure 1.2. $\psi_1 \notin \mathcal{D}\big(]a.b[\big)$ but $\psi_2 \in \mathcal{D}\big(]a.b[\big)$.

Since $\operatorname{supp}\psi_1 = [a,b]$, it follows by definiton that $\psi_1 \notin \mathcal{D}(\Omega)$. On the other hand, $\operatorname{supp}\psi_2 \subset \Omega$ and so $\psi_2 \in \mathcal{D}(\Omega)$.

Example 1.2. For any compact $K \subset \mathbb{R}^N$, there exists a test function $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $\varphi(x) = 1$ for all $x \in K$. Indeed, let $\varepsilon > 0$ and consider the sets

$$K_\varepsilon := \big\{x \in \mathbb{R}^N \,;\, \operatorname{dist}(x, K) \leq \varepsilon\big\}, \quad F_\varepsilon := \big\{x \in \mathbb{R}^N \,;\, \operatorname{dist}(x, K) \geq 2\varepsilon.\big\}.$$

Then K_ε is compact, F_ε is closed and $K_\varepsilon \cap F_\varepsilon = \emptyset$.

Let $f : \mathbb{R}^N \to \mathbb{R}$ be the function defined by

$$f(x) := \frac{\operatorname{dist}(x, F_\varepsilon)}{\operatorname{dist}(x, K_\varepsilon) + \operatorname{dist}(x, F_\varepsilon)}.$$

It is easy to check that f is continuous, $0 \leq f \leq 1$ with support equals $K_{2\varepsilon}$.

To define the desired test function, we take $n \in \mathbb{N}$ with $n\varepsilon > 1$ and

$$\varphi(x) := (\phi_n * f)(x) = \int_{\mathbb{R}^N} \phi_n(x-y) f(y)\, dy,$$

where

$$\phi_n(x) := \frac{n^N}{\rho_0}\rho(nx), \quad \rho_0 := \left(\int_{\mathbb{R}^N} \rho(x)\, dx\right)^{-1},$$

with ρ the test function given in (1.1).

As the function φ inherits the regularity of ϕ, if follows that $\varphi \in C^\infty(\mathbb{R}^N)$. To show that φ has compact support, note that the support of the mapping $y \mapsto \phi(y-x)$ is the closed ball $\overline{B_{1/n}(x)}$. So, if $\operatorname{dist}(x, K) \geq 3\varepsilon$, $B_{1/n}(x) \subset F_\varepsilon$ and $\varphi(x) = 0$. Moreover, if $x \in K$, we have $B_{1/n}(x) \subset K_\varepsilon$ and consequently $\phi(x) = 1$.

Example 1.3. Given K compact and F closed, disjoint subsets of $\mathbb{R}^N$, we can construct a test function φ such that $\varphi|_K \equiv 1$ and $\varphi|_F \equiv 0$. Indeed, it suffices to consider $\varepsilon < \text{dist}(K, F)/4$, define

$$K_\varepsilon := \{x \in \mathbb{R}^N\,;\, \text{dist}(x, K) \leq \varepsilon\}, \quad F_\varepsilon := \{x \in \mathbb{R}^N\,;\, \text{dist}(x, F) \geq \varepsilon.\}$$

and repeat the arguments of the above example.

1.1 Distributions

Rigorously speaking, a *distribution* T in Ω is a continuous functional defined in $\mathcal{D}(\Omega)$ which is endowed with a (complicated) topology. More precisely, let $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ be a multi-index and $|\alpha| = \alpha_1 + \ldots + \alpha_N$. For every $\varphi \in \mathcal{D}(\Omega)$, we denote the *derivative of* φ *of order* α by

$$D^\alpha \varphi := \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}}.$$

If K is a compact subset of Ω, we define

$$\mathcal{D}_K(\Omega) := \{\psi \in \mathcal{D}(\Omega)\,;\, \text{supp}\, \psi \subset K\}.$$

On $\mathcal{D}_K(\Omega)$ we have the following family of seminorms $\{N_\alpha\,;\, \alpha \in \mathbb{N}^N\}$, where

$$N_\alpha(\psi) := \max_{x \in \Omega} |D^\alpha \psi(x)|.$$

Associated to this family we can introduce a topology on $\mathcal{D}(\Omega)$, namely *the inductive limit* of $\mathcal{D}_K(\Omega)$, $K \subset \Omega$, K compact (cf. Schwartz [26]). It is well known that this topology is not metrizable.

For our purposes, we can overcome these difficulties by introducing a notion of convergence in $\mathcal{D}(\Omega)$.

Definition 1.4. Let $\{\varphi_j\}_{j \in \mathbb{N}}$ be a sequence in $\mathcal{D}(\Omega)$. We say that $\varphi_j \to \varphi$ in $\mathcal{D}(\Omega)$ if:

(1) there exists a compact $K \subset \Omega$ such that $\text{supp}\, \varphi_j \subset K$ for every $j \in \mathbb{N}$;

(2) for all $\alpha \in \mathbb{N}^N$, $D^\alpha \varphi_j \to D^\alpha \varphi$ uniformly on K.

1.1.1 The space $\mathcal{D}'(\Omega)$ of distributions

The space of distributions, $\mathcal{D}'(\Omega)$, is the *topological dual* of $\mathcal{D}(\Omega)$, i.e., the space of linear continuous functional on $\mathcal{D}(\Omega)$ with values in $\mathbb{R}$ (or $\mathbb{C}$). For linear functionals on $\mathcal{D}(\Omega)$, to be continuous is equivalent to the fact that its restriction to $\mathcal{D}_K(\Omega)$ is continuous for the topology of $\mathcal{D}_K(\Omega)$. More precisely, a mapping $T \in \mathcal{D}'(\Omega)$ if, and only if,

(1) T is a linear mapping from $\mathcal{D}(\Omega)$ to $\mathbb{R}$ (or $\mathbb{C}$);

(2) T is continuous, i.e., if $\varphi_j \to \varphi$ in $\mathcal{D}(\Omega)$, then $\langle T, \varphi_j\rangle \to \langle T, \varphi\rangle$ in $\mathbb{R}$ (or $\mathbb{C}$).

Example 1.5. Let $f \in L^1_{\text{loc}}(\Omega)$, i.e., f is Lebesgue integrable on every compact $K \subset \Omega$. The mapping

$$\varphi \in \mathcal{D}(\Omega) \mapsto \int_\Omega f(x)\varphi(x)\,dx$$

is well defined and linear. It is easy to see that it is continuous. In fact, if $\varphi_j \to \varphi$ in $\mathcal{D}(\Omega)$ with $\operatorname{supp}\varphi_j \subset K$ for all j, $K \subset \Omega$ compact, then

$$\begin{aligned}\left|\int_\Omega f(x)\big(\varphi_j(x) - \varphi(x)\big)\,dx\right| &= \left|\int_K f(x)\big(\varphi_j(x) - \varphi(x)\big)\,dx\right| \\ &\le \max_{x\in\Omega}\big|\varphi_j(x) - \varphi(x)\big| \int_K |f(x)|\,dx.\end{aligned}$$

This is the basic example and, in this sense, the distributions in $\mathcal{D}'(\Omega)$ generalize the notion of functions of $L^1_{\text{loc}}(\Omega)$. So, it is usual to write T_f as the distribution associated to f:

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \langle T_f : \varphi\rangle = \int_\Omega f(x)\varphi(x)\,dx.$$

In the following we show an example of a distribution which is not an element of $L^1_{\text{loc}}(\Omega)$.

Example 1.6. Consider the mapping $\varphi \in \mathcal{D}(\Omega) \mapsto \varphi(x_0) \in \mathbb{R}$, where $x_0 \in \Omega$. Since it is linear and continuous, it defines a distribution δ_{x_0} (Dirac mass at point x_0):

$$\langle \delta_{x_0}, \varphi\rangle = \varphi(x_0).$$

It is an exercise to show that there cannot exist $f \in L^1_{\text{loc}}(\Omega)$ such that δ_{x_0} can be equal to T_f.

The following two properties are essential.

Lemma 1.7. *$\mathcal{D}(\Omega)$ is dense in $L^1(\Omega)$.*

Proof. Let $f \in L^1(\Omega)$ and $\varepsilon > 0$. Then, we can choose a compact $K \subset \Omega$ such that its restriction f_K to K extended to Ω by zero satisfies

$$\int_\Omega |f(x) - f_K(x)|\, dx \leq \frac{\varepsilon}{2}.$$

Take a regularizing sequence $\{\rho_n\}_{n\in\mathbb{N}}$, i.e., $\rho_n(x) = n^N \rho(nx)$, where

$$\rho \in C^\infty(\mathbb{R}^N), \quad \rho \geq 0, \quad \operatorname{supp} \rho \subset B_1(0), \quad \int_{\mathbb{R}^N} \rho(x)\, dx = 1.$$

It is clear that we can find n_0 such that $K + \overline{B_{1/n_0}(0)} \subset \Omega$. By taking the convolution $u_n = f_K * \rho_n$, $n \geq n_0$, it follows that $u_n \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp} u_n \subset K + \overline{B_{1/n_0}(0)} \subset \Omega$, which implies that $u_n \in \mathcal{D}(\Omega)$. Hence,

$$\int_\Omega |u_n(x) - f_K(x)|\, dx = \int_{\mathbb{R}^N} |(f_K * \rho_n)(x) - f_K(x)|\, dx \to 0, \text{ as } n \to +\infty.$$

So, we can find $n_1 \in \mathbb{N}$ such that

$$\int_\Omega |u_n(x) - f_K(x)|\, dx < \frac{\varepsilon}{2}, \quad \forall n \geq n_1,$$

which implies that

$$\int_\Omega |u_n(x) - f(x)|\, dx < \varepsilon, \quad \forall n \geq \max\{n_0, n_1\}$$

and the proof is complete. □

Proposition 1.8. *If $f \in L^1_{\mathrm{loc}}(\Omega)$ is such that $T_f = 0$ in $\mathcal{D}'(\Omega)$, then $f = 0$ a.e. in Ω.*

Proof. It suffices to prove that, for every open set Ω' such that $\overline{\Omega'}$ compact and $\overline{\Omega'} \subset \Omega$, we have $f = 0$ a.e. in Ω'.

Since $\mathcal{D}(\Omega') \subset \mathcal{D}(\Omega)$, it follows that

$$\langle T_f, \varphi\rangle = \int_{\Omega'} f(x)\varphi(x)\, dx = 0, \quad \forall \varphi \in \mathcal{D}(\Omega'). \tag{1.2}$$

By Lemma 1.7, given $\varepsilon > 0$, there exists $\phi \in \mathcal{D}(\Omega')$ such that

$$\int_{\Omega'} |f(x) - \phi(x)|\, dx < \varepsilon. \tag{1.3}$$

Therefore, from (1.2) and (1.3), we have

$$\left|\int_{\Omega'} \phi(x)\varphi(x)\,dx\right| = \left|\int_{\Omega'} \big(\phi(x) - f(x)\big)\varphi(x)\,dx\right| \le \varepsilon \max_{x\in\Omega'} |\varphi(x)|. \tag{1.4}$$

Let K_1 and K_2 be the sets defined by

$$K_1 := \big\{x \in \Omega' \,;\, \phi(x) \ge \varepsilon\big\}, \quad K_2 := \big\{x \in \Omega' \,;\, \phi(x) \le -\varepsilon\big\}.$$

It is clear that they are disjoints compact subsets of Ω'.

Let $\psi_i \in \mathcal{D}(\Omega')$, $0 \le \psi_i \le 1$, $i = 1, 2$, such that (see Example 1.3)

$$\psi_1(x) = \begin{cases} 1, & \text{if } \ x \in K_1, \\ 0, & \text{if } \ x \in K_2, \end{cases} \qquad \psi_2(x) = \begin{cases} 1, & \text{if } \ x \in K_2, \\ 0, & \text{if } \ x \in K_1, \end{cases}$$

By considering $\psi = \psi_1 - \psi_2$, we have $\psi \in \mathcal{D}(\Omega')$ satisfying $-1 \le \psi \le 1$ and

$$\psi(x) = \begin{cases} 1, & \text{if } \ x \in K_1, \\ -1, & \text{if } \ x \in K_2, \end{cases}$$

From (1.4), it follows that

$$\left|\int_{\Omega'} \phi(x)\psi(x)\,dx\right| \le \varepsilon,$$

and from the definition of K_i, $i = 1, 2$, we obtain

$$\left|\int_{K} \phi(x)\psi(x)\,dx\right| \le \left|\int_{\Omega'} \phi(x)\psi(x)\,dx\right| + \left|\int_{\Omega'\setminus K} \phi(x)\psi(x)\,dx\right| < \big(1+\text{meas}(\Omega')\big)\varepsilon,$$

where $K := K_1 \cup K_2$. Moreover, since $|\phi(x)| = \psi(x)\phi(x)$ for all $x \in K$ and $|\phi(x)| \le \varepsilon$ for all $x \in \Omega' \setminus K$, we have

$$\int_K |\phi(x)|\,dx = \int_K \phi(x)\psi(x)\,dx < \big(1 + \text{meas}(\Omega')\big)\varepsilon$$

Therefore,

$$\int_{\Omega'} |f(x)|\,dx \le \int_{\Omega'} |f(x) - \phi(x)|\,dx + \int_K |\phi(x)|\,dx + \int_{\Omega'\setminus K} |\phi(x)|\,dx < 2\big(1 + \text{meas}(\Omega')\big)\varepsilon.$$

and the proof is complete. □

1.1.2 Derivation and convergence in $\mathcal{D}'(\Omega)$

Let $T \in \mathcal{D}'(\Omega)$. The mapping $\varphi \in \mathcal{D}(\Omega) \mapsto -\left\langle T, \frac{\partial \varphi}{\partial x_i} \right\rangle$ is well defined, linear and continuous. It then defines a distribution denoted by $\frac{\partial T}{\partial x_i}$. More generally, for $\alpha \in \mathbb{N}^N$, we have:

Definition 1.9. The mapping $\varphi \in \mathcal{D}(\Omega) \mapsto (-1)^{|\alpha|}\langle T, D^\alpha \varphi\rangle$ is a distribution called *distributional derivative of* T *of order* α, denoted by $D^\alpha T$.

It is an elementary exercise to show that $\varphi \mapsto D^\alpha T$ is a distribution.

Remark 1.10. In this sense, we can define the derivative (of any order) of any function in $L^1_{\text{loc}}(\Omega)$; for $f \in L^1_{\text{loc}}(\Omega)$,

$$\langle D^{|\alpha|} f, \varphi\rangle = (-1)^{|\alpha|} \int_\Omega f(x) D^{|\alpha|}\varphi(x)\, dx.$$

Definition 1.11. A sequence $\{T_n\}_{n\in\mathbb{N}})$, $T_n \in \mathcal{D}'(\Omega)$, converges to T in $\mathcal{D}'(\Omega)$, if

$$\lim_{n\to\infty} \langle T_n, \varphi\rangle = \langle T, \varphi\rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

If the sequence $\{T_n\}_{n\in\mathbb{N}}$ converges to T in $\mathcal{D}'(\Omega)$, we write

$$T_n \to T \text{ in } \mathcal{D}'(\Omega).$$

Theorem 1.12. *Let* $\{T_n\}_{n\in\mathbb{N}}$ *be a sequence of distributions such that, for all* $\varphi \in D(\Omega)$, *the sequence* $\{\langle T_n, \varphi\rangle\}_{n\in\mathbb{N}}$ *converges in* $\mathbb{R}$ *to some limit called* $T(\varphi)$. *Then, the mapping* $\varphi \mapsto T(\varphi)$ *defines a distribution* T *and* $T_n \to T$ *in* $\mathcal{D}'(\Omega)$.

Corollary 1.13. *If* $T_n \to T$ *in* $\mathcal{D}'(\Omega)$, *then, for any* $\alpha \in \mathbb{N}^N$, $D^{|\alpha|}T_n \to D^{|\alpha|}T$ *in* $\mathcal{D}'(\Omega)$

Example 1.14. Let us consider the *Heaviside function* on $\mathbb{R}$, $H : \mathbb{R} \to \mathbb{R}$ defined by

$$H(x) = \begin{cases} 1 \text{ if } x > 0, \\ 0 \text{ if } x \leq 0, \end{cases}$$

Then $H \in L^1_{\text{loc}}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$. If $\varphi \in \mathcal{D}(\mathbb{R})$, then

$$\begin{aligned} \left\langle \frac{dH}{dx}, \varphi \right\rangle &= -\left\langle H, \frac{d\varphi}{dx} \right\rangle = -\int_{\mathbb{R}} H(x) \frac{d\varphi}{dx}(x)\, dx \\ &= -\int_0^{+\infty} \frac{d\varphi}{dx}(x)\, dx = \varphi(0) = \langle \delta_0, \varphi\rangle \end{aligned}$$

Since

$$\left\langle \frac{dH}{dx}, \varphi \right\rangle = \langle \delta_0, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}),$$

we have

$$\frac{dH}{dx} = \delta_0 \text{ in } \mathcal{D}'(\mathbb{R}).$$

Example 1.15. Let $f : \mathbb{R} \setminus \{x_0\} \to \mathbb{R}$ a C^1 function such that

$$\lim_{x \to x_0^-} f(x) = L_1, \quad \lim_{x \to x_0^+} f(x) = L_2,$$

where $L_1 \neq L_2$. As in the above example, $f \in L^1_{\text{loc}}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$. Therefore,

$$\begin{aligned} \left\langle \frac{df}{dx}, \varphi \right\rangle &= -\left\langle f, \frac{d\varphi}{dx} \right\rangle = -\int_{-\infty}^{x_0} f(x) \frac{d\varphi}{dx}(x)\, dx - \int_{x_0}^{+\infty} f(x) \frac{d\varphi}{dx}(x)\, dx \\ &= -L_1 \varphi(x_0) + \int_{-\infty}^{x_0} \frac{df}{dx}(x) \varphi(x)\, dx + L_2 \varphi(x_0) + \int_{x_0}^{+\infty} \frac{df}{dx}(x) \varphi(x)\, dx \end{aligned}$$

If we denote $g : \mathbb{R} \setminus \{x_0\} \to \mathbb{R}$ by

$$g(x) = \begin{cases} \dfrac{df}{dx}(x) & \text{if } x < x_0, \\ \dfrac{df}{dx}(x) & \text{if } x > x_0, \end{cases}$$

then $g \in L^1_{\text{loc}}(\mathbb{R})$ and

$$\frac{df}{dx} = [L_2 - L_1]\delta_{x_0} + g \quad \text{in} \quad \mathcal{D}'(\mathbb{R}).$$

1.2 L^p spaces

Let Ω be an open subset of $\mathbb{R}^N$. For $1 \leq p < +\infty$, we define

$$L^p(\Omega) := \left\{ f : \Omega \to \mathbb{R}\, ;\, f \text{ is measurable and } \int_\Omega |f(x)|^p\, dx < +\infty \right\},$$

and, for $p = +\infty$,

$$L^\infty(\Omega) := \left\{ f : \Omega \to \mathbb{R}\, ;\, f \text{ is measurable and essentially bounded} \right\},$$

For *essentially bounded* we mean that there exists $C > 0$ such that $|f(x)| \leq C$ for almost every $x \in \Omega$.

We set

$$\|f\|_p := \left[\int_\Omega |f(x)|^p \, dx \right]^{1/p} \quad \text{and} \quad \|f\|_\infty := \inf\big\{C \geq 0\,;\, |f(x)| \leq C, \ \text{a.e. in } \Omega\big\},$$

which are norms for $L^p(\Omega)$, $1 \leq p < +\infty$ and $L^\infty(\Omega)$, respectively.

We can prove that with these norms, $L^p(\Omega)$ $(1 \leq p \leq +\infty)$ is a *Banach Space*. Moreover, if $p = 2$, the space $L^2(\Omega)$ is a Hilbert space for the inner product

$$(f, g) = \int_\Omega f(x) g(x) \, dx.$$

Note that, if $f \in L^p(\Omega)$ and $g = f$ almost everywhere in Ω, they define the same element in $L^p(\Omega)$.

We write p' the *conjugate exponent* of $1 < p < +\infty$ by the relation $1/p + 1/p' = 1$. Or, more generally,

$$p' := \begin{cases} \dfrac{p}{p-1}, & \text{if } 1 < p < +\infty, \\ +\infty, & \text{if } p = 1, \\ 1, & \text{if } p = +\infty. \end{cases}$$

Theorem 1.16 (Hölder's inequality)**.** *Let $1 \leq p \leq +\infty$ and p' its conjugate exponent. If $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, then $fg \in L^1(\Omega)$ and*

$$\int_\Omega |f(x)\, g(x)| \, dx \leq \|f\|_p \|g\|_{p'}.$$

For $p = 2$, this inequality is called the Cauchy-Schwarz inequality.

Proof. The cases $p = +\infty$ and $p' = +\infty$ are trivial. So, let $1 < p < +\infty$. For all $a, b \geq 0$, we have (the Young inequality)

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Indeed, since $x \mapsto \log x$ is concave,

$$\log\left(\frac{a^p}{p} + \frac{b^{p'}}{p'}\right) \geq \frac{1}{p} \log a^p + \frac{1}{p'} \log b^{p'} = \log a + \log b = \log(ab).$$

Hence,

$$|f(x) g(x)| \leq \frac{1}{p} |f(x)|^p + \frac{1}{p'} |g(x)|^{p'},$$

which implies that

$$\int_\Omega |f(x)g(x)|\,dx \le \frac{1}{p}\int |f(x)|^p\,dx + \frac{1}{p'}\int |g(x)|^{p'}\,dx = \frac{1}{p}\|f\|_p^p + \frac{1}{p'}\|g\|_{p'}^{p'}.$$

If we replace f by λf, with $\lambda > 0$, we get

$$\lambda \int_\Omega |f(x)\,g(x)|\,dx \le \frac{1}{p}\lambda^p\|f\|_p^p + \frac{1}{p'}\|g\|_{p'}^{p'}.$$

So,

$$\int_\Omega |f(x)\,g(x)|\,dx \le \frac{1}{p}\lambda^{p-1}\|f\|_p^p + \frac{1}{p'\lambda}\|g\|_{p'}^{p'}, \qquad \lambda > 0,$$

which implies that

$$\int_\Omega |f(x)\,g(x)|\,dx \le \inf_{\lambda>0}\left\{\frac{1}{p}\lambda^{p-1}\|f\|_p^p + \frac{1}{p'\lambda}\|g\|_{p'}^{p'}\right\}.$$

We can see that the infimum is achieved when $\lambda = \|f\|_p^{-1}\|g\|_{p'}^{p'/p}$, and the result follows. □

Theorem 1.17 (Minkowski's inequality)**.** *Suppose* $1 \le p \le +\infty$. *If* $f, g \in L^p(\Omega)$, *then* $f + g \in L^p(\Omega)$ *and*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p. \tag{1.5}$$

Proof. If $f, g \in L^p(\Omega)$, then, by convexity or $s \mapsto |s|^p$,

$$|f(x) + g(x)|^p \le (|f(x)| + |g(x)|)^p \le 2^{p-1}(|f(x)|^p + |g(x)|^p).$$

Consequently, $f + g \in L^p(\Omega)$.

Now, if $p = 1$ or $p = \infty$, the inequality is trivial. Otherwise, we have

$$\|f + g\|_p^p = \int_\Omega |f + g||f + g|^{p-1} \le \int_\Omega |f||f + g|^{p-1} + \int_\Omega |g||f + g|^{p-1}.$$

Since $|f + g|^{p-1} \in L^{p'}(\Omega)$, it follows from Hölder's inequality that

$$\|f + g\|_p^p \le (\|f\|_p + \|g\|_p)\|f + g\|_p^{p-1}$$

and we obtain (1.5). □

Remark 1.18. As a consequence, we have that $f \mapsto \|f\|_p$ is a norm.

1.2.1 Essential convergence theorems

We present now three very important theorems and some additional properties without proof.

Theorem 1.19 (Beppo-Levi Monotone Convergence Theorem)**.** *Let* $\{f_n\}_{n\geq 1}$ *be an increasing sequence of non-negative functions in* $L^1(\Omega)$*, such that*

$$\sup_{n\geq 1} \int_\Omega f_n(x) < \infty.$$

Then, there exists $f \in L^1(\Omega)$ *such that*

(1) $f_n \to f$ *a.e. in* Ω*;*

(2) $\|f_n - f\|_1 \to 0$*, as* $n \to \infty$*.*

Theorem 1.20 (Lebesgue Dominated Convergence Theorem)**.** *Let* $\{f_n\}_{n\geq 1}$ *be a sequence of functions in* $L^1(\Omega)$*, such that*

(1) $f_n \to f$ *a.e. in* Ω*;*

(2) *there exists* $g \in L^1(\Omega)$ *such that, for all* $n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$*, a.e. in* Ω*.*

Then, $f \in L^1(\Omega)$ *and* $f_n \to f$ *in* $L^1(\Omega)$*.*

Theorem 1.21 (Vitali Theorem)**.** *Suppose that* Ω *is bounded. Let* $\{f_n\}_{n\geq 1}$ *be a sequence of functions in* $L^1(\Omega)$ *such that*

(1) $f_n \to f$ *a.e. in* Ω*;*

(2) (f_n) *is equi-uniformly integrable, i.e. for all* $\varepsilon > 0$*, there exists* $\delta > 0$*, such that, for any* $A \subset \Omega$*, with* $\mathrm{meas}(A) \leq \delta$*, we have*

$$\int_A |f_n(x)|\, dx \leq \varepsilon, \quad \forall n.$$

Then, $f \in L^1(\Omega)$ *and* $f_n \to f$ *in* $L^1(\Omega)$*.*

Lemma 1.22 (Fatou Lemma)**.** *Let* $\{f_n\}_{n\geq 1}$ *be a sequence of functions in* $L^1(\Omega)$ *such that*

(1) *for each* $n \in \mathbb{N}$, $f_n \geq 0$ *a.e. in* Ω*;*

(2) $\displaystyle\sup_{n\in\mathbb{N}} \int_\Omega f_n(x)\, dx < +\infty.$

Let $f(x) := \liminf_{n\to\infty} f_n(x)$*. Then,* $f \in L^1(\Omega)$ *and*

$$\int_\Omega f(x)\, dx \leq \liminf_{n\to\infty} \int_\Omega f_n(x)\, dx.$$

1.2.2 Additional properties:

In addition to the above results, it is worth mentioning the following.

Theorem 1.23 (Density). *If* $1 \leq p < \infty$, $\mathcal{D}(\Omega)$ *is dense in* $L^p(\Omega)$.

This result is not valid for L^∞, because the uniform limit of continuous functions are necessarily continuous.

Proposition 1.24 (Completeness). $L^p(\Omega)$ *is a Banach space, for all* $1 \leq p \leq +\infty$.

Theorem 1.25 (Topological Dual). *For* $1 < p < \infty$, *we have* $\big(L^p(\Omega)\big)' = L^{p'}(\Omega)$ *and* $\big(L^1(\Omega)\big)' = L^\infty(\Omega)$. *However, the topological dual of* $L^\infty(\Omega)$ *is not* $L^1(\Omega)$.

Remark. As a consequence, for $1 < p < +\infty$, $L^p(\Omega)$ is reflexive, i.e., it is isomorphic to its bidual. But $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive.

If $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$, with $1 \leq p \leq \infty$, we define the *convolution* of f and g by the function

$$(f * g)(x) := \int_{\mathbb{R}^N} f(x-y)g(y)dy = \int_{\mathbb{R}^N} f(y)g(x-y)dy.$$

Proposition 1.26. *If* $f \in L^1(\mathbb{R}^N)$ *and* $g \in L^p(\mathbb{R}^N)$, *with* $1 \leq p \leq \infty$, *then* $f * g \in L^p(\mathbb{R}^N)$ *and*

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Moreover, $\operatorname{supp}(f * g) \subset \operatorname{supp}(f) + \operatorname{supp}(g)$.

1.2.3 Weak and weak-$*$ convergence

Definition 1.27. Let E be a Banach space and E' its topological dual. We say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ in E *converges weakly* to $f \in E$ (and write $f_n \rightharpoonup f$ in E) if

$$\forall L \in E', \lim_{n\to\infty} \langle L, f_n\rangle = \langle L, f\rangle.$$

Remark. It is important to notice that strong convergence implies weak convergence, since

$$\big|\langle L, f_n - f\rangle\big| \leq \|L\|_{E'} \|f_n - f\|_E.$$

However, the converse is not true (in infinite dimension).

Example 1.28. For $1 \leq p < \infty$, $f_n \rightharpoonup f$ in $L^p(\Omega)$ weakly if

$$\lim_{n\to\infty} \int_\Omega g(x) f_n(x)\, dx = \int_\Omega g(x) f(x)\, dx, \quad \forall g \in L^{p'}(\Omega).$$

Example 1.29. In a Hilbert space H, as H' is isomorphic to H, we have $f_n \rightharpoonup f$ weakly in H if, and only if,

$$\lim_{n\to\infty} (g, f_n)_H = (g, f)_H, \quad \forall g \in H.$$

As a consequence of Banach-Steinhaus Theorem, we have

Proposition 1.30. *Every weakly convergent sequence in E is bounded in E.*

Theorem 1.31. *Let E be a Banach space and K a closed (for the strong topology) convex subset of E. Then K is closed for the weak topology of E.*

Proof. Let K be a closed convex subset of E. We denote by $\mathcal{H}_K$ the family of closed half-spaces which contain K, i.e.,

$$H \in \mathcal{H}_K \Leftrightarrow \exists L \in E',\, \alpha \in \mathbb{R} \ \text{ such that } H = \{v \in E\,;\, \langle L, v\rangle \geq \alpha\} \supset K.$$

We want to show that $K = \bigcap_{H\in\mathcal{H}_K} H$, i.e., K coincides with the intersection of all closed half-spaces containing K.

Of course, for all $H \in \mathcal{H}_K$, we have that $K \subset H$ which implies $K \subset \bigcap_{H\in\mathcal{H}_K} H$.

Now, suppose that exists $v_0 \in \bigcap_{H\in\mathcal{H}_K} H$, such that $v_0 \notin K$. Then, from Hahn-Banach Theorem, there exists a hyperplane $H_0 := \{v \in E\,;\, \langle L_0, v\rangle = \alpha_0\}$ separating K and v_0, i.e.,

$$\langle L_0, v\rangle > \alpha_0,\ \forall v \in K \ \text{ and } \ \langle L_0, v_0\rangle < \alpha_0.$$

Let $\widetilde{H_0} := \{v \in E\,;\, \langle L_0, v\rangle \geq \alpha_0\}$. Then $K \subset \widetilde{H_0}$ so that $\widetilde{H_0} \in \mathcal{H}_K$. Since $v_0 \notin \widetilde{H_0}$, we have a contradiction.

Now, if $L \in E'$, L is continuous for the weak topology, so that $\mathcal{H}_K$ is closed weakly. Since we proved that $K = \mathcal{H}_K$, the proof is complete. □

Proposition 1.32. *If $f_n \rightharpoonup f$ in E and if $L : E \to F$ is a continuous linear map, then*

$$L(f_n) \rightharpoonup L(f) \ \text{ in } \ F.$$

Proof. If $T \in F'$ and $L \in \mathcal{L}(E, F)$, we have $T \circ L \in E'$. Since $\langle T, L(f_n)\rangle_{F',F} = \langle T \circ L, f_n\rangle_{E',E}$, we have

$$\lim_{n\to\infty} \langle T, L(f_n)\rangle_{F',F} = \lim_{n\to\infty} \langle T \circ L, f_n\rangle_{E',E} = \langle T \circ L, f\rangle_{E',E} = \langle T, L(f)\rangle_{F',F}$$

□

Remark 1.33. It is noteworthy that, as $L^p(\Omega) \subset L^1_{\text{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega) \subset L^{p'}(\Omega)$, if $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$, we have

$$\int_\Omega f_n(x)\varphi(x)\,dx \to \int_\Omega f(x)\varphi(x)\,dx \quad \forall\varphi \in \mathcal{D}(\Omega),$$

which means that $T_{f_n} \to T_f$ in $\mathcal{D}'(\Omega)$, i.e., *weak convergence in $L^p(\Omega)$ implies convergence in the sense of distributions.*

In reflexive Banach spaces (in particular Hilbert spaces), weak convergence play an important role because of the following.

Theorem 1.34. *Let E be a reflexive Banach space. If $\{f_n\}_{n\in\mathbb{N}}$ is a bounded sequence in E, then $\{f_n\}_{n\in\mathbb{N}}$ is weakly relatively compact, i.e. there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and $f \in E$ such that $f_{n_k} \rightharpoonup f$ weakly in E.*

Notice that, if $1 < p < \infty$ and if (f_n) is bounded in $L^p(\Omega)$, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and $f \in L^p(\Omega)$ such that $f_{n_k} \rightharpoonup f$ weakly in $L^p(\Omega)$. This result is not valid in $L^1(\Omega)$ and $L^\infty(\Omega)$), because they are not reflexive.

We know that convergence in norm imply weak convergence, but the converse is not true, as we can see from the following examples.

Example 1.35 (Ω unbounded)**.** Let $\Omega = \mathbb{R}$, $1 < p < +\infty$ and $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 1 & \text{if } n \le x < n+1, \\ 0 & \text{otherwise.} \end{cases}$$

First of all, notice that $\|f_n\|_p = 1$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\varphi(x)\,dx = \lim_{n\to\infty} \int_n^{n+1} \varphi(x)\,dx = 0, \quad \forall\varphi \in \mathcal{D}(\mathbb{R})$$

Take $g \in L^{p'}(\mathbb{R})$. Since $\mathcal{D}(\mathbb{R})$ is dense in $L^{p'}(\mathbb{R})$ (see Thm 1.23), for $\varepsilon > 0$ given, we can find $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\|g - \varphi\|_{p'} < \varepsilon$. Therefore, from Hölder's inequality,

$$\left|\int_{\mathbb{R}} g(x) f_n(x)\,dx\right| \le \|f_n\|_p \|g-\varphi\|_{p'} + \int_n^{n+1} \varphi(x)\,dx \le \varepsilon + \int_n^{n+1} \varphi(x)\,dx,$$

which implies that $f_n \rightharpoonup 0$ in $L^p(\mathbb{R})$ weakly. However, f_n does not converge strongly in $L^p(\mathbb{R})$, because $\|f_n\|_p = 1$ forall $n \in \mathbb{N}$.

Example 1.36 (Ω bounded)**.** Let $\Omega =]0,1[$ and $f_n :]0,1[\to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 1 & \text{if } \dfrac{2k}{2n} \le x < \dfrac{2k+1}{2n}, \quad k = 0,\ldots,n-1. \\ -1 & \text{if } \dfrac{2k+1}{2n} \le x < \dfrac{2k+2}{2n}, \quad k = 0,\ldots,n-1. \end{cases}$$

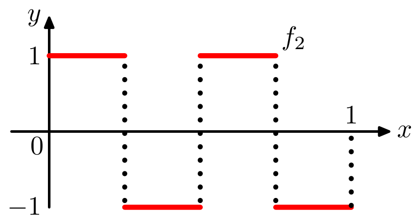

Figure 1.3. The graphic of function f_n.

To show that $f_n \rightharpoonup 0$ in $L^2(]0,1[)$, we proceed in four steps.
Step 1: Notice that, by definition,

$$\int_{k/n}^{(k'+1)/n} f_n(x)\,dx = 0, \quad 0 \le k \le k' \le 1. \tag{1.6}$$

Step 2: For any $0 < a < b < 1$, we have

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = 0.$$

Indeed, for $n \in \mathbb{N}$ such that $1/n < b - a$, consider

$$k_1(n) := \max\left\{k \in \mathbb{N}\,;\ \frac{k}{n} < a\right\}, \quad k_2(n) := \min\left\{k' \in \mathbb{N}\,;\ \frac{k'+1}{n} > b\right\}.$$

Then, it is clear that $k_1(n) < k_2(n)$ and following the previous step,

$$\begin{aligned}\left|\int_a^b f_n(x)\,dx\right| &\le \left|\int_{k_1(n)/n}^a f_n(x)\,dx\right| + \left|\int_b^{(k_2(n)+1)/n} f_n(x)\,dx\right| \\ &\le \left|a - \frac{k_1(n)}{n}\right| + \left|b - \frac{(k_2(n)+1)}{n}\right| < 2\varepsilon\end{aligned}$$

if we choose $n \in \mathbb{N}$ such that $1/n < \varepsilon$.
Step 3: As an immediate consequence of (1.6), we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\psi(x)\,dx = 0, \quad \text{for all step function } \psi. \tag{1.7}$$

Step 4: Let $\varepsilon > 0$ and $g \in L^2(]0,1[)$. From density of $\mathcal{D}(]0,1[) \subset L^2(]0,1[)$, we can choose a function $\varphi \in \mathcal{D}(]0,1[)$ such that $\|g - \varphi\|_2 < \varepsilon$. Hence,

$$\begin{aligned}\left|\int_0^1 g(x)f_n(x)\,dx\right| &\le \left|\int_0^1 [g(x) - \varphi(x)]f_n(x)\,dx\right| + \left|\int_0^1 f_n(x)\varphi(x)\,dx\right| \\ &\le \varepsilon\|f_n\|_2 + \left|\int_0^1 f_n(x)\varphi(x)\,dx\right|\end{aligned}$$

Since the space of step functions is dense in $L^1(]0,1[)$, there exists a step function ψ such that $\|\varphi-\psi\|_1<\varepsilon$. Therefore, from Hölder's inequality and the fact that $\|f_n\|_\infty=1$ for all $n\in\mathbb{N}$, we have

$$\left|\int_0^1 g(x)f_n(x)\,dx\right| < 2\varepsilon + \left|\int_0^1 f_n(x)\psi(x)\,dx\right|$$

from which we conclude that $f_n \rightharpoonup 0$ weakly in $L^2(]0,1[)$.

However, f_n does not converge strongly in $L^2(\mathbb{R})$, because $\|f_n\|_2=1$ for all $n\in\mathbb{N}$.

Remark 1.37. A natural question is what happens concerning bounded sequences in non reflexive Banach spaces. Two cases must be considered, depending wether E is the dual of another Banach spaces F or not.

Let us consider the first case where F is a Banach space and $E=F'$.

Definition 1.38. We say that $\{g_n\}_{n\in\mathbb{N}}$ *converges weakly-$*$* to g in F' (and we write $g_n \overset{*}{\rightharpoonup} f$ in F'), if

$$\forall f\in F,\quad \langle g_n, f\rangle_{F',F}\to\langle g,f\rangle_{F',F},\quad \text{as } n\to\infty.$$

Example 1.39. Let $E=L^\infty(\Omega)=(L^1(\Omega))'$. Then $g_n \overset{*}{\rightharpoonup} g$ weak-$*$ in $L^\infty(\Omega)$, if

$$\forall f\in L^1(\Omega),\quad \int_\Omega g_n(x)f(x)\,dx \to \int_\Omega g(x)f(x)\,dx.$$

In this case, we have a positive answer to our question, namely:

Theorem 1.40. *Let E be the dual space of a Banach space F ($E=F'$). If $\{g_n\}_{n\in\mathbb{N}}$ is a bounded sequence in E, then there exists a subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ and $g\in E$ such that $g_{n_k}\overset{*}{\rightharpoonup} g$ weak-$*$ in E.*

Remark 1.41. If $g_{n_k}\overset{*}{\rightharpoonup} g$ weak-$*$ in $L^\infty(\Omega)$, then $g_{n_k}\to g$ in $\mathcal{D}'(\Omega)$.

Unfortunately, in the other case (for example, in $L^1(\Omega)$), we don't have analogous property and we cannot say anything important for bounded sequences.

1.3 Sobolev spaces

In this section we introduce the Sobolev spaces and we present their principal properties. As before, Ω will denote an open subset of $\mathbb{R}^n$.

For $m \in \mathbb{N}$, we define

$$H^m(\Omega) := \big\{ v \in L^2(\Omega) \,;\, D^\alpha v \in L^2(\Omega),\, \forall \alpha \in \mathbb{N}^N,\, |\alpha| \leq m \big\},$$

where the derivative D^α is in the sense of distribution. We endow $H^m(\Omega)$ with the norm

$$\|v\|^2_{H^m(\Omega)} := \|v\|^2_{L^2(\Omega)} + \sum_{1 \leq |\alpha| \leq m} \|D^\alpha v\|^2_{L^2(\Omega)}$$

For example,

$$H^1(\Omega) = \Big\{ v \in L^2(\Omega) \,;\, \frac{\partial v}{\partial x_i} \in L^2(\Omega),\, \forall i = 1, \dots, N \Big\},$$

$$\|v\|^2_{H^1(\Omega)} = \|v\|^2_{L^2(\Omega)} + \sum_{i=1}^N \left\| \frac{\partial v}{\partial x_i} \right\|^2_{L^2(\Omega)}.$$

Proposition 1.42. *$H^m(\Omega)$ is a Hilbert space for the scalar product*

$$(v, w)_{H^m(\Omega)} := (v, w)_{L^2(\Omega)} + \sum_{1 \leq |\alpha| \leq m} (D^\alpha v, D^\alpha w)_{L^2(\Omega)}.$$

Proof. The only thing that is not evident is that it is complete. To prove this fact, let $\{v_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $H^m(\Omega)$. Then, it is a Cauchy sequence in $L^2(\Omega)$ and the same is true for the sequence $\{D^\alpha v_n\}_{n \in \mathbb{N}}$, for all $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq m$.

As $L^2(\Omega)$ is complete, there exist v and v_α in $L^2(\Omega)$ such that

$$\begin{cases} v_n \to v & \text{in } L^2(\Omega), \\ D^\alpha v_n \to v_\alpha & \text{in } L^2(\Omega). \end{cases}$$

Then,

$$\begin{cases} v_n \to v & \text{in } \mathcal{D}'(\Omega), \\ D^\alpha v_n \to v_\alpha & \text{in } \mathcal{D}'(\Omega). \end{cases}$$

But we know from Corollary 1.13 that $v_n \to v$ in $\mathcal{D}'(\Omega)$ implies that $D^\alpha v_n \to D^\alpha v$ in $\mathcal{D}'(\Omega)$. So, we have $D^\alpha v = v_\alpha$. This shows that v and $D^\alpha v$ belong to $L^2(\Omega)$ for $1 \leq |\alpha| \leq m$, so that $v \in H^m(\Omega)$.

Now, for a given $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$\|v_n - v\|^2_{L^2(\Omega)} < \varepsilon^2/C,$$
$$\|D^\alpha v_n - D^\alpha v\|^2_{L^2(\Omega)} < \varepsilon^2/C,$$

where $C = \sharp\{\alpha \in \mathbb{N}^N\,;\, 1 \leq |\alpha| \leq m\}$, from which we conclude that

$$\|v_n - v\|^2_{H^m(\Omega)} < \varepsilon^2, \quad \forall\, n \geq n_0,$$

and v_n converges to v in $H^m(\Omega)$, □

Definition 1.43. For $m \in \mathbb{N}$ we define $H_0^m(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$, i.e., $H_0^m(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^m(\Omega)}$.

$H_0^m(\Omega)$ is a closed subspace of $H^m(\Omega)$ and therefore is a Hilbert space (for the same scalar product as for $H^m(\Omega)$).

Proposition 1.44. *The map $v \mapsto \widetilde{v}$ defined by*

$$\widetilde{v}(x) := \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega, \end{cases}$$

(called extension by zero) is a linear continuous operator from $H_0^m(\Omega)$ to $H^m(\mathbb{R}^N)$. In particular, it is continuous from $H_0^1(\Omega)$ to $L^2(\mathbb{R}^N)$.

Proof. It is clear that if $\varphi \in \mathcal{D}(\Omega)$, then $\widetilde{\varphi} \in \mathcal{D}(\mathbb{R}^n)$, $D^\alpha \widetilde{\varphi} = \widetilde{D^\alpha \varphi}$ for all $\alpha \in \mathbb{N}^N$ and $\|\widetilde{\varphi}\|_{H^1(\mathbb{R}^N)} = \|\varphi\|_{H_0^1(\Omega)}$. So, the conclusion follows by continuity, since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$. □

Remark 1.45. The extension by zero is not a continuous linear operator from $H^1(\Omega)$ to $L^2(\mathbb{R}^N)$. Indeed, let $\Omega = (0,1)$ and $f :]0,1[\to \mathbb{R}$ defined by $f(x) = 1$ for all $x \in]0,1[$. Then

$$\widetilde{f}(x) = \begin{cases} 1 & \text{if } \ 0 < x < 1, \\ 0 & \text{if } \ |x| \geq 1. \end{cases}$$

Since

$$\frac{d\widetilde{f}}{dx} = \delta_1 - \delta_0 \notin L^2(]0,1[).$$

Theorem 1.46 (Poincaré's inequality)**.** *If Ω is an open set bounded in one direction, there exists a constant $C > 0$ depending only on Ω such that*

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Proof. We can assume without lost of generality that Ω be bounded in the x_1-direction, i.e., $\Omega \subset [a,b] \times \mathbb{R}^{N-1}$. By denoting $x = (x_1, x')$, $x' \in \mathbb{R}^{N-1}$, we have for every $\varphi \in \mathcal{D}(\Omega)$,

$$\varphi(x_1, x')^2 = \int_a^{x_1} \frac{\partial}{\partial s}\varphi(s,x')^2\, ds = 2\int_a^{x_1} \varphi(s,x')\frac{\partial \varphi}{\partial s}(s,x')\, ds.$$

Now, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\int_\Omega \varphi(x_1,x')^2\, dx_1 dx' &= \int_a^b dx_1 \int_{\mathbb{R}^{N-1}} \varphi(x_1,x')^2\, dx' \\
&= 2\int_a^b dx_1 \int_{\mathbb{R}^{N-1}} \left[\int_a^{x_1} \varphi(s,x')\frac{\partial \varphi}{\partial s}(s,x') ds\right] dx' \\
&\leq 2\int_a^b dx_1 \left[\int_\Omega |\varphi(s,x')| \left|\frac{\partial \varphi}{\partial s}(s,x')\right| dx' ds\right] \\
&\leq 2(b-a)\|\varphi\|_{L^2(\Omega)} \left\|\frac{\partial \varphi}{\partial x_1}\right\|_{L^2(\Omega)}.
\end{aligned}$$

Hence, for every $\varphi \in \mathcal{D}(\Omega)$,

$$\|\varphi\|_{L^2(\Omega)} \leq 2(b-a)\left\|\frac{\partial \varphi}{\partial x_1}\right\|_{L^2(\Omega)} \leq 2(b-a)\|\nabla\varphi\|_{L^2(\Omega)}.$$

Therefore, by density, for every $u \in H_0^1(\Omega)$,

$$\|u\|_{L^2(\Omega)} \leq 2(b-a)\|\nabla u\|_{L^2(\Omega)},$$

which implies that

$$\begin{aligned}
\|u\|^2_{H^1(\Omega)} &= \|u\|^2_{L^2(\Omega)} + \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(\Omega)} \\
&\leq (1+C^2)\|\nabla u\|^2_{L^2(\Omega)} \leq (1+C^2)\,\|u\|^2_{H^1(\Omega)}
\end{aligned}$$

and the proof is complete. □

As an immediate consequence of the Poincaré inequality, we obtain.

Corollary 1.47. *If Ω is an open set bounded in one direction, then, the map*

$$u \mapsto \|\nabla u\|_{L^2(\Omega)}$$

is a norm on $H_0^1(\Omega)$ equivalent to the $H^1(\Omega)$-norm.

It is important to characterize the topological dual of $H_0^1(\Omega)$, which is denoted by $H^{-1}(\Omega)$. This is done by the following result.

Theorem 1.48. *The space $H^{-1}(\Omega)$ consists of all distributions $T \in \mathcal{D}'(\Omega)$ of the form*

$$T = f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}, \quad \text{where } f_0, \dots, f_N \in L^2(\Omega).$$

Moreover,

$$\inf \left\{ \left(\|f_0\|_{L^2}^2 + \sum_{i=1}^{n} \|f_i\|_{L^2}^2 \right)^{1/2} ; T = f_0 + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \right\}.$$

is a norm on $H^{-1}(\Omega)$,

Proof. The Sobolev space $H_0^1(\Omega)$ can be viewed as a closed subspace of $L^2(\Omega)^{N+1}$. Indeed, the maping

$$H_0^1(\Omega) \to \left(L^2(\Omega)\right)^{N+1}, \quad v \mapsto \left(v, \frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_N}\right)$$

is linear and continuous. We know that every linear continuous form T on $H_0^1(\Omega)$ can be extended to a linear continuous form $\widetilde{T}$ on $L^2(\Omega)^{N+1}$. So, by identifying $L^2(\Omega)^{N+1}$ with its dual, it follows that there exist $f_0, f_1, \dots, f_N \in L^2(\Omega)$ such that

$$\langle \widetilde{T}, v \rangle = (f_0, v)_{L^2(\Omega)^{N+1}} - \sum_{i=1}^{N} \left(f_i, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)^{N+1}}, \quad \forall v \in H_0^1(\Omega).$$

In particular, for every $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned} \langle \widetilde{T}, \varphi \rangle &= (f_0, \varphi)_{L^2(\Omega)} - \sum_{i=1}^{N} \left(f_i, \frac{\partial \varphi}{\partial x_i} \right)_{L^2(\Omega)} \\ &= (f_0, \varphi)_{L^2(\Omega)^{N+1}} + \sum_{i=1}^{N} \left\langle \frac{\partial f_i}{\partial x_i}, \varphi \right\rangle \end{aligned}$$

which means that $\widetilde{T} = f_0 + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$ □

1.3.1 Additional results

We can extend the definition of Sobolev spaces of order m based on the L^p spaces, for $1 \leq p \leq +\infty$.

Definition 1.49. Let $m \in \mathbb{N}$ and p such that $1 \leq p \leq \infty$. We call

$$W^{m,p}(\Omega) = \big\{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega),\, 1 \leq |\alpha| \leq m\big\},$$

which is endowed with the norm

$$\|v\|_{W^{m,p}(\Omega)} = \left(|v|^p_{L^p(\Omega)} + \sum_{1\leq|\alpha|\leq m} |D^\alpha v|^p_{L^p(\Omega)} \right)^{1/p},$$

By adapting the same arguments in the the proof of Propositions 1.42 and 1.44, we have:

Proposition 1.50. *$W^{m,p}(\Omega)$ is a Banach space. Moreover, $W_0^{m,p}(\Omega)$ defined as the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$, i.e.,*

$$W_0^{m,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{W_0^{m,p}(\Omega)},$$

is a closed subspace of $W^{m,p}(\Omega)$ and, therefore, is also a Banach space.

Proposition 1.51. *The extension by zero $v \mapsto \widetilde{v}$ is a linear continuous operator from $W_0^{m,p}(\Omega)$ to $W^{m,p}(\mathbb{R}^N)$*

The case of $\Omega = \mathbb{R}^N$ deserves more details, because some important consequences can be obtained.

Theorem 1.52 (Density of $\mathcal{D}(\mathbb{R}^N)$ in $H^m(\mathbb{R}^N)$)**.** *For every $v \in H^m(\mathbb{R}^N)$, there exists $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R}^N)$ such that $\varphi_n \to v$ in $H^m(\mathbb{R}^N)$, as $n \to \infty$.*

Proof. Let ψ be a regular cutoff function, i.e., $\psi(x) = \theta(|x|)$, where $\theta \in \mathbb{C}^\infty(\mathbb{R})$ is such that

$$\begin{cases} 0 \leq \theta(s) \leq 1, & \text{for all } s \in \mathbb{R}, \\ |\theta'(s)| \leq 2 & \text{for all } s \in \mathbb{R}, \\ \theta(s) \equiv 1, & \text{for all } |s| < 1, \\ \theta(s) \equiv 0, & \text{for all } |s| \geq 2. \end{cases}$$

Then, $\psi \in \mathcal{D}(\mathbb{R}^N)$. For each $R > 0$, denote $\phi_R(x) = \psi(x/R)$.

Given $v \in H^m(\mathbb{R}^N)$, we consider $v_R(x) := \psi_R(x)v(x)$. So, it is easy to show that

(1) $v_R \in H^m(\mathbb{R}^N)$,

(2) $\operatorname{supp} v_R$ is compact and contained in $B_{2R}(0)$,

(3) $v_R \to v$ strongly in $H^m(\mathbb{R}^N)$ as $R \to \infty$.

Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a regularizing sequence. For every $n \in \mathbb{N}$, define $\varphi_n = \phi_n * v_R$. Since $\varphi_n \in C^\infty(\mathbb{R}^N)$ and $\operatorname{supp}\varphi_n \subset B_{2R}(0) + \operatorname{supp}\phi_n$, we have that $\varphi_n \in \mathcal{D}(\mathbb{R}^N)$.

For every $w \in L^2(\mathbb{R}^N)$, $\rho_n * w \to w$ in $L^2(\mathbb{R}^N)$. Hence, for fixed R,

$$\begin{aligned}\|\varphi_n - v_R\|^2_{H^m(\mathbb{R}^N)} &= \sum_{|\alpha|\le m} \|D^\alpha(\rho_n * v_R) - D^\alpha v_R\|^2_{L^2(\mathbb{R}^N)}\\ &= \sum_{|\alpha|\le m} \|\rho_n * (D^\alpha v_R) - D^\alpha v_R\|^2_{L^2(\mathbb{R}^N)} \to 0,\end{aligned}$$

as $n \to \infty$.

Therefore, the sequence $\{\phi_n * \psi_n v\}_{n\in\mathbb{N}}$ has the desired properties. □

It is known from the classical theory of Fourier transform that the mapping $u \mapsto \widehat{u}$ given by

$$\widehat{u}(\xi) := \int_{\mathbb{R}^N} \exp(-2\,\mathrm{i}\pi x\cdot\xi)u(x)\,dx \tag{1.8}$$

is well defined for $u \in L^1(\mathbb{R}^N, \mathbb{C})$ (which, for simplicity, we write $L^1(\mathbb{R}^N)$)

It can be shown that it is an isometry on $\mathcal{S}(\mathbb{R}^N)$ with the L^2 scalar product, where

$$\mathcal{S}(\mathbb{R}^N) := \left\{u \in C^\infty(\mathbb{R}^N)\,;\, |x|^\beta D^\alpha u \in L^\infty(\mathbb{R}^N),\, \forall\alpha,\beta \in \mathbb{N}^N\right\}$$

is the usual *Schwartz space*, where the Parserval-Plancherel Formula hold, i.e.,

$$\int_{\mathbb{R}^N} u(x)\overline{v(x)}\,dx = \int_{\mathbb{R}^N} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,d\xi, \quad \forall u, v \in \mathcal{S}(\mathbb{R}^N). \tag{1.9}$$

Since $\mathcal{D}(\mathbb{R}^N) \subset \mathcal{S}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$, it follows from density that (1.8) can be extended by continuity on $L^2(\mathbb{R}^N)$ conserving the same properties, i.e., the mapping $u \in L^2(\mathbb{R}^N) \mapsto \widehat{u} \in L^2(\mathbb{R}^N)$ satisfies

$$\int_{\mathbb{R}^N} u(x)\overline{v(x)}\,dx = \int_{\mathbb{R}^N} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,d\xi$$

and, in particular,

$$\int_{\mathbb{R}^N} |u(x)|^2\,dx = \int_{\mathbb{R}^N} |\widehat{u}(\xi)|^2\,d\xi.$$

On the other hand, it can be shown that

$$\widehat{\frac{\partial u}{\partial x_j}}(\xi) = -2\,\mathrm{i}\pi\xi_j\widehat{u}(\xi),$$

from what we can deduce that

$$H^1(\mathbb{R}^N) = \big\{u \in L^2(\mathbb{R}^N)\,;\ \big(1+|\xi|^2\big)^{1/2}\widehat{u} \in L^2(\mathbb{R}^N)\big\}.$$

and

$$\|u\|^2_{H^1(\mathbb{R}^N)} \sim \int_{\mathbb{R}^N} \big(1+|\xi|^2\big)^{1/2}|\widehat{u}(\xi)|^2 d\xi.$$

This suggests a natural extension for "*fractional Sobolev spaces*" on $\mathbb{R}^N$. More precisely, for $s \in \mathbb{R}$, $s \geq 0$, se define

$$H^s(\mathbb{R}^N) := \big\{u \in L^2(\mathbb{R}^N)\,;\ \big(1+|\xi|^2\big)^{s/2}\widehat{u} \in L^2(\mathbb{R}^N)\big\},$$
$$\|u\|^2_{H^s(\mathbb{R}^N)} := \big\|\big(1+|\xi|^2\big)^{s/2}|\widehat{u}|\big\|^2_{L^2(\mathbb{R}^N)}.$$

It can be shown that $\mathcal{D}(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$.

For the case $\Omega \neq \mathbb{R}^N$, we define

$$H^s(\Omega) := \big\{u|_\Omega\,;\ u \in H^s(\mathbb{R}^N)\big\}.$$

For $s \in \mathbb{N}$ and Ω sufficiently regular, all definitions are equivalent.

Theorem 1.53. *If Ω is bounded, the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Moreover, if Ω is regular, also the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*

Proof. Let $u \in H^1_0(\Omega)$ and $\widetilde{u} \in H^1(\mathbb{R}^N)$ its extension by zero. Since Ω is bounded, we can choose $R > 0$ such that $\operatorname{supp}\widetilde{u} \subset \overline{\Omega} \subset B_R(0)$. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $H^1_0(\Omega)$ such that $u_n \rightharpoonup u$ in $H^1_0(\Omega)$ weakly. So, $v_n := u_n - u \rightharpoonup 0$ in $H^1_0(\Omega)$ weakly and is bounded in $H^1_0(\Omega)$. Therefore, $\widetilde{v_n} \in H^1(\mathbb{R}^N)$ with $\operatorname{supp}\widetilde{v_n} \subset B_R(0)$ and

$$\begin{aligned}
\|\widetilde{v_n}\|_{L^2(\mathbb{R}^N)} = \|\widehat{\widetilde{v_n}}\|_{L^2(\mathbb{R}^N)} &= \int_{B_M(0)} \big|\widehat{\widetilde{v_n}}(\xi)\big|^2 d\xi + \int_{\mathbb{R}^N\setminus B_M(0)} \big|\widehat{\widetilde{v_n}}(\xi)\big|^2 d\xi \\
&\leq \int_{B_M(0)} \big|\widehat{\widetilde{v_n}}(\xi)\big|^2 d\xi + \int_{\mathbb{R}^N\setminus B_M(0)} \left(\frac{1+|\xi|^2}{1+M^2}\right) \big|\widehat{\widetilde{v_n}}(\xi)\big|^2 d\xi \\
&\leq \int_{B_M(0)} \big|\widehat{\widetilde{v_n}}(\xi)\big|^2 d\xi + \frac{1}{1+M^2}\|\widetilde{v_n}\|^2_{H^1(\mathbb{R}^N)}.
\end{aligned}$$

Let $\varepsilon > 0$. As $\|\widetilde{v_n}\|^2_{H^1(\mathbb{R}^N)} = \|v_n\|^2_{H^1_0(\Omega)} \leq C$, there exists $M_0 > 0$ such that

$$\frac{1}{1+M_0^2}\|\widetilde{v_n}\|^2_{H^1(\mathbb{R}^N)} < \frac{\varepsilon}{2}.$$

Now,

$$\widehat{\widetilde{v_n}}(\xi) = \int_{\mathbb{R}^N} \mathrm{e}^{-2\,\mathrm{i}\pi x\cdot\xi} v_n(x)\,dx = \int_{B_R(0)} \mathrm{e}^{-2\,\mathrm{i}\pi x\cdot\xi} v_n(x)\,dx$$
$$= \int_{\mathbb{R}^N} v_n(x)\left(\mathrm{e}^{-2\,\mathrm{i}\pi x\cdot\xi}\chi_{B_R(0)}(x)\right)\,dx$$

Note that, for ξ given, the map $x \mapsto \mathrm{e}^{-2\,\mathrm{i}\pi x\cdot\xi}\chi_{B_R(0)}(x)$ is a function of $L^2(\mathbb{R}^N)$. As $\widetilde{v_n} \rightharpoonup 0$ in $L^2(\mathbb{R}^N)$ weakly, we have, for $n \to +\infty$,

$$\widehat{\widetilde{v_n}}(\xi) = \int_{\mathbb{R}^N} v_n(x)\left(\mathrm{e}^{-2\,\mathrm{i}\pi x\cdot\xi}\chi_{B_R(0)}(x)\right)\,dx \to 0.$$

On the other hand,

$$|\widehat{\widetilde{v_n}}(\xi)| \le \int_{B_R(0)} |v_n(x)|\,dx \le C(R)\|v_n\|_{L^2(\Omega)} \le C'(R).$$

From Lebesgue's Theorem 1.20,

$$\int_{B_{M(0)}} |\widehat{\widetilde{v_n}}(\xi)|^2\,d\xi \to 0 \quad \text{as } n \to \infty.$$

Therefore, $\|\widetilde{v_n}\|_{L^2(\mathbb{R}^N)} \to 0$, which proves the compactness of the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

If Ω is bounded and regular, it is possible to define an extension (but not by zero) from $H^1(\Omega)$ to $H_0^1(\Omega')$, where Ω' is bounded such that $\Omega' \supset \Omega$ and the previous argument can be applied. □

Proposition 1.54. *Let $C_b(\mathbb{R}^N)$ the space of bounded continuous functions. If $s > N/2$, we have $H^s(\mathbb{R}^N) \hookrightarrow C_b(\mathbb{R}^N)$.*

Proof. Let $u \in H^s(\mathbb{R}^n)$. By the inverse Fourier transform $\mathcal{F}^{-1}$, we have

$$u(x) = \int_{\mathbb{R}^N} \mathrm{e}^{2\,\mathrm{i}\pi x\cdot\xi}\widehat{u}(\xi)\,d\xi = \int_{\mathbb{R}^N} \frac{\mathrm{e}^{2\,\mathrm{i}\pi x\cdot\xi}}{(1+|\xi|^2)^{s/2}}\left(1+|\xi|^2\right)^{s/2}\widehat{u}(\xi)\,d\xi. \tag{1.10}$$

We know that $g(\xi) := \left(1+|\xi|^2\right)^{s/2}\widehat{u}(\xi)$ is a function of $L^2(\mathbb{R}^N)$. If the map $\xi \mapsto \left(1+|\xi|^2\right)^{-s/2}$ also belongs to $L^2(\mathbb{R}^N)$, Eq. (1.10) shows that u is the inverse Fourier transform of the function $\xi \mapsto g(\xi)\left(1+|\xi|^2\right)^{-s/2}$ which belongs to $L^1(\mathbb{R}^N)$. Hence, the conclusion follows, because the inverse Fourier transform maps $L^1(\mathbb{R}^n)$ into $C_b(\mathbb{R}^N)$.

Note that $\xi \mapsto \left(1+|\xi|^2\right)^{-s/2}$ belongs to $L^2(\mathbb{R}^N)$ if, and only if $\xi \mapsto \left(1+|\xi|^2\right)^{-s}$ belongs to $L^1(\mathbb{R}^N)$, which is equivalent to $r \mapsto r^{N-1}(1+r^2)^{-s}$ be integrable on $(0,+\infty)$. It is easy to see that this conditions is satisfied if, and only if, $2s-(N-1) > 1$, i.e., $s > N/2$. □

Corollary 1.55. *If $s > N/2$, we have $H_0^s(\Omega) \hookrightarrow C_b(\overline{\Omega})$. Moreover, if Ω is regular enough, $H^s(\Omega) \hookrightarrow C_b(\overline{\Omega})$.*

We can also prove the following embedding theorem.

Theorem 1.56. (1) *If $p < N$, $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, where $p^* = Np/(N-p)$ and $\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^N)}, \quad \forall u \in W^{1,p}(\mathbb{R}^n)$.*
(2) *Moreover, $W^{1,N}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^N)$, $\forall q \in [N, +\infty[$ and*
(3) *if $p > N$, $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^N)$.*

By iteration we obtain

Theorem 1.57.

(1) *If* $\dfrac{1}{p} - \dfrac{m}{N} > 0, \quad W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^N)$. *for* $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m}{N}$.

(2) *If* $\dfrac{1}{p} - \dfrac{m}{N} = 0, \quad W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^N)$. *for* $q \in [p, +\infty[$.

(2) *If* $\dfrac{1}{p} - \dfrac{m}{N} < 0, \quad W^{m,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^N)$.

1.3.2 The Trace Operator on $H^1(\Omega)$

Let Ω be an open subset of $\mathbb{R}^N$, whose boundary will be denoted by Γ. We assume that Ω be regular in the following sense: for each point $\sigma \in \Gamma$ there exist $R > 0$ and a diffeomorphism $h : B_R(\sigma) \cap \Omega \to B^+$, where B^+ is the half unit ball of $\mathbb{R}^N$, i.e.,

$$B^+ := \big\{x = (x', x_N)\,;\, x' \in \mathbb{R}^{N-1},\ |x| < 1 \text{ and } x_N \geq 0\big\}.$$

With this assumption, we can proceed as a first step by considering $\Omega = \mathbb{R}^N_+$.

Theorem 1.58. *The classical trace defined on $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ by $\gamma_0\varphi(x') = \varphi(x', 0)$ $\forall\, x' \in \mathbb{R}^{N-1}$ and $\forall\, \varphi \in \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ can be extended by continuity to a continuous linear map from $H^1(\mathbb{R}^N_+)$ to $L^2(\mathbb{R}^{N-1})$.*

Proof. For each $\varphi \in \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$, we have

$$\gamma_0\varphi(x')^2 = -\int_0^{+\infty} \frac{\partial}{\partial x_N}\big(\varphi(x', s)\big)^2\, ds = -2\int_0^{+\infty} \varphi(x', s)\frac{\partial \varphi}{\partial x_N}(x', s)\, ds.$$

Hence,

$$\|\gamma_0\varphi\|^2_{L^2(R^{N-1})} \leq 2\|\varphi\|_{L^2(R^N_+)} \left\|\frac{\partial \varphi}{\partial x_N}\right\|_{L^2(R^N_+)}$$

and the conclusion follows by continuity. □

Remark 1.59. In fact, we can prove that $\gamma_0 u \in H^{1/2}(\mathbb{R}^{N-1})$ for all $u \in H^1(\mathbb{R}^N_+)$ and the mapping $u \mapsto \gamma_0 u$ is linear continuous and surjective from $H^1(\mathbb{R}^N_+)$ to $H^{1/2}(\mathbb{R}^{N-1})$. Conversely, it can be proved that there exists a linear continuous lifting $R : H^{1/2}(\mathbb{R}^{N-1}) \to H^1(\mathbb{R}^N_+)$ such that $\gamma_0 \circ R$ is the identity operator on $H^{1/2}(\mathbb{R}^{N-1})$.

Using local charts, these results can be extended to the case of Ω regular, i.e., there exist $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ linear continuous that generalizes the usual traces for $\varphi \in \mathcal{D}(\overline{\Omega})$, and we also have

$$\|\gamma_0 v\|^2_{L^2(\Gamma)} \leq C\|v\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega)$$

and it can also be proved that there exists a linear and continuous lifting from $H^{1/2}(\Gamma)$ to $H^1(\Omega)$

Remark 1.60. Also noteworthy is the fact that

$$H^1_0(\Omega) = \{v \in H^1(\Omega)\,;\, \gamma_0 v = 0\}.$$

Indeed, it is clear that $H^1_0(\Omega) \hookrightarrow \{v \in H^1(\Omega)\,;\, \gamma_0 v = 0\}$, as we can approximate $v \in H^1_0(\Omega)$ by functions in $\mathcal{D}(\Omega)$, which have a zero trace. The reverse inclusion is more complicate (for details, see [8]).

It is also important to take in account the following properties:

- $\mathcal{D}(\mathbb{R}^N_+)$ is not dense in $H^m(\mathbb{R}^N_+)$, $(m \geq 1)$, but it can proved that $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is dense in $H^m(\mathbb{R}^N_+)$, where $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is the space of restrictions to $\mathbb{R}^N_+$ of functions of $\mathcal{D}(\mathbb{R}^N)$.

- If Ω is regular enough, this implies that $\mathcal{D}(\overline{\Omega})$ is dense in $H^m(\Omega)$.

Chapter 2

Second order variational problems

In this chapter we consider boundary value problems for linear elliptic partial differential equations of divergent form

$$\sum_{i,j=1}^{N} -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu = f, \qquad \text{in } \Omega,$$

where c and a_{ij}, $i, j = 1, \dots, N$ are given functions defined on Ω.

Note that we can write this equation in the following concise form

$$-\operatorname{div}(A\nabla u) + cu = f,$$

where div is the divergent operator and A is the matrix with entries a_{ij}.

2.1 Lax-Milgram Theorem

Concerning the existence and uniqueness of solutions, the following result is fundamental.

Theorem 2.1 (Lax-Milgram)**.** *Let V be a Hilbert space for the scalar product $((\cdot,\cdot))$ associated with the norm $\|.\|$ and let $a(\cdot,\cdot)$ be a bilinear form on $V \times V$ such that:*

(1) *$a(\cdot,\cdot)$ is continuous, i.e., there exists $M > 0$ such that*

$$|a(v,w)| \leq M\|v\|\|w\|, \quad \forall v, w \in V,$$

(2) *$a(\cdot,\cdot)$ is coercive, i.e., there exists $\alpha > 0$ such that*

$$a(v,v) \geq \alpha\|v\|^2, \quad \forall v \in V. \tag{2.1}$$

Then, for every $L \in V'$, there exists a unique $u \in V$ satisfying

$$\begin{cases} a(u,v) = \langle L, v\rangle, & \forall v \in V, \\ u \in V. \end{cases} \tag{2.2}$$

Moreover, the mapping $L \in V' \mapsto u \in V$ is linear and continuous with

$$\|u\| \leq \frac{1}{\alpha}\|L\|_{V'}.$$

Proof. Let $L \in V'$. For each fixed $v \in V$, the mapping $w \mapsto a(v,w)$ is a continuous linear form on V. Therefore, there exists an element $A_v \in V'$ such that

$$a(v,w) = \langle A_v, w\rangle_{V',V}, \quad \forall w \in V.$$

Then, the problem (2.2) is equivalent to determine the unique solution $u \in V$ of the equation

$$A_u = L \text{ in } V', \quad u \in V.$$

From Riesz Theorem, V' is isomorphic to V, so that if $J : V \to V'$ is this isomorphism, we define $\mathcal{A} : V \to V$, $\mathcal{A}v := J^{-1}A_v \in V$.

It is clear that the map $v \mapsto A_v$ is linear from V to V', so that $\mathcal{A} : V \to V$ is also linear. Let us show that $\mathcal{A}$ is continuous. In fact, for all $v, w \in V$ we have

$$\left|(\!(\mathcal{A}v, w)\!)\right| = \left|\langle A_v, w\rangle_{V',V}\right| = |a(v,w)| \le M\|v\|\|w\|.$$

which implies $\|\mathcal{A}v\|^2 \le M\|\mathcal{A}v\|\|v\|$ for all $v \in V$. Consequently, $\|\mathcal{A}v\| \le M\|v\|$ for all $v \in V$, which means that $\mathcal{A}$ is continuous.

Let us call $F = J^{-1}L \in V$, so that the problem (2.2) is equivalent to

$$\mathcal{A}u = F \in V, \quad u \in V. \tag{2.3}$$

For a given $\rho > 0$, the problem (2.3) is equivalent to determine a solution $u \in V$ of $u - \rho(\mathcal{A}u - F) = u$, or equivalently, a fixed point $u \in V$ of $S : V \to V$, $S(v) = v - \rho(\mathcal{A}v - F)$.

If we take $\rho := \alpha/M^2$, we obtain

$$\begin{aligned}\|S(v) - S(w)\|^2 &= \|v - w\|^2 - 2\rho\big(\!\big(\mathcal{A}(v-w), v-w\big)\!\big) + \rho^2\|\mathcal{A}(v-w)\|^2 \\ &= \|v-w\|^2 - 2\rho a(v-w, v-w) + \rho^2\|\mathcal{A}(v-w)\|^2 \\ &\le \left[1 - 2\rho\alpha + \rho^2M^2\right]\|v-w\|^2 = \left[1 - \frac{\alpha^2}{M^2}\right]\|v-w\|^2.\end{aligned}$$

Since we can assume (without loss the generality) that $\alpha < M$, it follows that $0 < 1 - \alpha^2/M^2 < 1$, so that S is a strict contraction and therefore has a (unique) fixed point. So, we have a unique solution of (2.3) or (2.2).

Moreover, $\alpha\|u\|^2 \le a(u,u)$ and $\langle L, u\rangle_{V',V} \le \|L\|_{V'}\|u\|$, from which we have $\|u\| \le \frac{1}{\alpha}\|L\|_{V'}$ and the proof is complete. □

2.2 Applications

2.2.1 The Dirichlet problem

Consider an open subset $\Omega \subset \mathbb{R}^N$ and let $V = H_0^1(\Omega)$. We define

$$a(u,v) := \int_\Omega A(x)\nabla u(x) \cdot \nabla v(x)\,dx + \int_\Omega c(x)u(x)v(x)\,dx$$
$$= \sum_{i,j=1}^N \int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\frac{\partial v}{\partial x_i}(x)\,dx + \int_\Omega c(x)u(x)v(x)\,dx, \quad \forall u,v \in V,$$

where $a_{ij}, c \in L^\infty(\Omega)$, $(i,j = 1,N)$, and we assume that

(1) there exists $\alpha > 0$ such that, for every $\xi \in \mathbb{R}^N$,

$$A(x)\xi \cdot \xi = \sum_{i,j=1}^N a_{ij}(x)\xi_j\xi_i \geq \alpha|\xi|^2, \quad \text{a.e. in } \Omega. \tag{2.4}$$

(2) there exists $\beta > 0$ such that $c(x) \geq \beta$, a.e. in Ω.

Then, $a(\cdot,\cdot)$ is a continuous and coercive bilinear form in $H_0^1(\Omega)$.

Given $f_0 \in L^2(\Omega)$, $f_1,\ldots,f_N \in L^2(\Omega)$, the mapping

$$v \in H_0^1(\Omega) \mapsto \int_\Omega f_0(x)v(x)\,dx - \sum_{j=1}^N \int_\Omega f_j(x)\frac{\partial v}{\partial x_j}(x)\,dx$$

is obviously a continuous linear form on $H_0^1(\Omega)$. Therefore, by Theorem 2.1, there exists a unique $u \in H_0^1(\Omega)$ such that (omitting the variable x if there is no risk of confusion)

$$a(u,v) = \int_\Omega f_0 v\,dx - \sum_{j=1}^N \int_\Omega f_j \frac{\partial v}{\partial x_j}\,dx, \quad \forall v \in H_0^1(\Omega). \tag{2.5}$$

In particular, for any $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int_\Omega A\nabla u \cdot \nabla\varphi\,dx + \int_\Omega cu\varphi\,dx = \int_\Omega f_0\varphi\,dx - \sum_{j=1}^N \int_\Omega f_j\frac{\partial\varphi}{\partial x_j}\,dx \tag{2.6}$$

and this is equivalent to (2.5) by density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$. Since all the functions $a_{ij}\frac{\partial u}{\partial x_j}$, cu, f_0 and f_i belongs to $L^2(\Omega) \subset \mathcal{D}'(\Omega)$, Eq. (2.6) can be regarded in the sense of distributions, i.e.,

$$\left\langle -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu, \varphi\right\rangle = \langle f_0,\varphi\rangle + \left\langle \sum_{i=1}^N \frac{\partial f_i}{\partial x_i}, \varphi\right\rangle, \quad \forall\varphi \in \mathcal{D}(\Omega),$$

which means that $u \in H_0^1(\Omega)$ is a unique function that satisfies the equation

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu = f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \quad \text{in} \quad \mathcal{D}'(\Omega).$$

Moreover, from Theorem 1.48, it follows that the above equations is in the sense of $H^{-1}(\Omega)$.

Remark 2.2. Also important is the particular case where $a_{ij} = \delta_{ij}$ (the Kronecker's symbol). In this case, we have a unique solution $u \in H_0^1(\Omega)$ such that

$$-\Delta u + cu = f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}.$$

Remark 2.3. It follows from Poincaré inequality that the result is true for $\beta = 0$ or $c \geq 0$ if Ω is bounded.

2.3 Case of $a(\cdot, \cdot)$ symmetric

We go back to Lax-Milgram Theorem in the case where $a(\cdot, \cdot)$ is symmetric, i.e.,

$$a(v, w) = a(w, v), \quad \forall v, w \in V.$$

Let us define the functional $J : V \to \mathbb{R}$ by

$$J(v) = \frac{1}{2}a(v, v) - \langle L, v\rangle_{V',V}.$$

Lemma 2.4. *J is Fréchet-differentiable and, with the hypothesis (1) and (2) in Theorem 2.1, J is convex, continuous and coercive, i.e.,*

$$\lim_{\|v\|_V \to \infty} J(v) = +\infty.$$

Proof. To see that J is differentiable, we remark that for $v, w \in V$,

$$\begin{aligned} J(v + w) &= \frac{1}{2}\big(a(v, v) + 2a(v, w) + a(w, w)\big) - \langle L, v\rangle_{V',V} - \langle L, w\rangle_{V',V} \\ &= J(v) + a(v, w) - \langle L, w\rangle_{V',V} + \frac{1}{2}a(w, w). \end{aligned} \tag{2.7}$$

Since that, for any $v \in V$, the mapping $w \mapsto a(v,w) - \langle L, w\rangle_{V',V}$ is linear and continuous and

$$0 \le \alpha\|w\| \le \frac{a(w,w)}{\|w\|} \le M\|w\|, \quad \forall w \in V,$$

it follows that J is differentiable in V and

$$\langle J'(v), w\rangle = a(v,w) - \langle L, w\rangle_{V',V}.$$

In particular, J is continuous.

Since J is differentiable in V, we know that it is convex if, and only if, its differential $J' : V \to V'$ is monotone (semipositive definite in the context of matrix), i.e.,

$$\left\langle J'(v) - J'(w), v - w\right\rangle_{V',V} \ge 0, \quad \forall v, w \in V.$$

Indeed,

$$\left\langle J'(v) - J'(w), v - w\right\rangle_{V',V} = a(v, v-w) - a(w, v-w) = a(v-w, v-w) \ge 0.$$

Moreover, it is coercive because

$$J(v) \ge \frac{\alpha}{2}\|v\|^2 - \|L\|_{V'}\|v\| \quad \Rightarrow \quad J(v) \to +\infty \text{ as } \|v\| \to +\infty$$

and the proof is complete. □

Remark 2.5. In the present case, as J has a particular form (quadratic plus linear), the Eq. (2.7) can be written as

$$J(v) - J(u) = a(u, v-u) - \langle L, v-u\rangle_{V',V} + \frac{1}{2}a(v-u, v-u)$$

Therefore, we see that u is a solution of (2.2) if, and only if, $J(u) = \min_{v\in V} J(v)$. Moreover, J' is strictly monotone, since for $u \ne w$,

$$\left\langle J'(v) - J'(w), v - w\right\rangle_{V',V} = a(v-w, v-w) \ge \alpha\|v-w\|^2 > 0.$$

The previous situation is a particular case of a general result.

Theorem 2.6. *Let E be a reflexive Banach space and $J : E \to \mathbb{R}$ be a convex, continuous and coercive functional. Then, there exists $u \in E$ such that*

$$J(u) = \min_{v\in E} J(v).$$

(1) *If J is strictly convex, then u is unique.*

(2) *If J is Gâteaux-differentiable at u, i.e. there exists $D_G J(u) \in V'$ such that, for all $v \in V$,*

$$\lim_{t \to 0} \frac{J(u + tv) - J(u)}{t}$$

exists and is equal to $\langle D_G J(u), v \rangle_{E',E}$, then $u \in E$ is characterized by the equation

$$\langle D_G J(u), v \rangle = 0, \quad v \in E.$$

To prove the above theorem, we need the following result.

Lemma 2.7. *Let E be a Banach space and $J : E \to \mathbb{R}$ be a convex functional and continuous for the strong topology of E. Then, J is lower semi-continuous for the weak topology of E, i.e.,*

$$v_n \rightharpoonup v \text{ weakly in } E \quad \Rightarrow \quad J(v) \leq \liminf_{n \to +\infty} J(v_n).$$

Proof of Lemma 2.7. If the mapping $J : E \to \mathbb{R}$ is convex and continuous, then for every $\lambda \in \mathbb{R}$, the set $\{v \in E\,;\, J(v) \leq \lambda\}$ is closed and convex. Therefore, it follows from Theorem 1.31 that it is also closed for the weak topology.

Let $\{v_n\}_{n \in \mathbb{N}}$ be a sequence in E such that $v_n \rightharpoonup v$ weakly in E and suppose (by contradiction) that $J(v) > \liminf_{n \to +\infty} J(v_n)$. Then, there exist $\varepsilon > 0$, $k_0 \in \mathbb{N}$ and a subsequence $\{v_{n_k}\}_{k \in \mathbb{N}}$ such that $J(v_{n_k}) < J(v) - \varepsilon$, $\forall k \geq k_0$.

Now take $\lambda = J(v) - \varepsilon/2$. For $k \geq k_0$,

$$v_{n_k} \in \{w \in E,\ J(w) \leq \lambda\}.$$

Since this set is weakly closed and $v_{n_k} \rightharpoonup v$ weakly in E, we have necessarily $v \in \{w \in E,\ J(w) \leq \lambda\}$, which is a contradiction. □

Proof of Theorem 2.6. Let us define $\beta := \inf_{v \in E} J(v)$, $(-\infty \leq \beta < +\infty)$ and let $\{v_n\}_{n \in \mathbb{N}}$ be a sequence in E such that $J(v_n) \to \beta$, as $n \to +\infty$. The sequence $\{v_n\}_{n \in \mathbb{N}}$ is bounded because, otherwise there exists a subsequence $\{v_{n_k}\}_{k \in \mathbb{N}}$ such that $\|v_{n_k}\| \to +\infty$. But J is coercive, so that $J(v_{n_k}) \to +\infty$, which is in contradiction with the fact that $J(v_{n_k}) \to \beta$.

Since E is reflexive, there exists a subsequence $\{v_{n_k}\}_{k \in \mathbb{N}}$ and $u \in E$ such that $v_{n_k} \rightharpoonup u$ weakly in E. Now, since J is a convex and continuous functional, it follows from Lemma 2.7 that J is lower semi-continuous for the weak topology of E, and then

$$\liminf_{n \to +\infty} J(v_{n_k}) \geq J(u), \text{ as } v_{n_k} \rightharpoonup u \text{ weakly in } E.$$

However, $J(v_{n_k}) \to \beta$ implies that $J(u) \le \beta$. So $\beta > -\infty$ and $J(u) = \beta$. Therefore,

$$J(u) = \min_{v\in E} J(v).$$

(1) If J is strictly convex and we assume that there exist two solutions $\bar{u}$ and $\hat{u}$, $\bar{u} \ne \hat{u}$, then for $0 < t < 1$,

$$J\big((1-t)\bar{u} + t\hat{u}\big) < (1-t)J(\bar{u}) + tJ(\hat{u}) = (1-t)\beta + t\beta = \beta$$

and we have a contradiction. Hence, the minimum is unique.
(2) If J is Gateaux-differentiable at u, then, for all $v \in E$, $u + tv \in E$ and $J(u+tv) \ge J(u)$. Hence,

$$\lim_{\substack{t\to 0\\ t>0}} \frac{J(u+tv) - J(u)}{t} = \langle D_G J(u), v\rangle \ge 0, \quad \forall\, v \in E,$$

which implies that $\langle D_G J(u), v\rangle = 0$ for all $v \in E$.

Conversely, if $\langle D_G J(u), v\rangle = 0$ for all $v \in E$, as J is convex, it follows that $J(v) - J(u) \ge \langle D_G J(u), v-u\rangle = 0$ and we conclude that $J(u) \le J(v)$ for all $v \in E$. □

Remark 2.8. The boundary value problem treated in subsection 2.2.1 was homogenous on the boundary. The question now is how to solve the problem in the case of a non homogenous boundary condition. More precisely, with the same conditions as before for c, a_{ij} and $f \in L^2(\Omega)$, we want to solve the problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu = f \text{ in } \Omega, \\ \gamma_0 u = g \text{ on } \Gamma, \end{cases} \tag{2.8}$$

where $g : \Gamma \to \mathbb{R}$ is a given function.

Assuming Ω regular, we know that the trace γ_0 maps $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$. Hence, if $g \in H^{1/2}(\Gamma)$, there exist $G \in H^1(\Omega)$ such that $\gamma_0 G = g$. So, by considering $u = \widetilde{u} + G$, and substituting in (2.8), we have formally

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial \widetilde{u}}{\partial x_j}\right) + c\widetilde{u} = f + \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial G}{\partial x_j}\right) - cG \text{ in } \Omega, \\ \gamma_0 \widetilde{u} = 0 \text{ on } \Gamma. \end{cases}$$

This is a homogeneous Dirichlet problem and if we denote $f_0 = f - cG$ and $f_i = \sum_{j=1}^{N} a_{ij}\frac{\partial G}{\partial x_j}$, $i = 1, \ldots, N$, it follows that f_0 and f_i are in $L^2(\Omega)$ and we can find a solution $\widetilde{u} \in H_0^1(\Omega)$. Therefore $u := \widetilde{u} + G \in H^1(\Omega)$ is a solution of (2.8)

2.4 Regularity and the Maximum Principle

In the previous section, we gave sufficient conditions for the existence of solutions $u \in H^1(\Omega)$ of the Dirichlet problem for second order elliptic equations. In this section we focus on the question of regularity and we establish the (weak and strong) maximum principle.

2.4.1 Regularity

Let Ω be an open subset of $\mathbb{R}^N$ and consider the Dirichlet problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu = f \text{ in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{2.9}$$

where a_{ij}, c satisfy the conditions assumed in subsection 2.2.1.

Theorem 2.9. *If Ω is of class $C^{2,\alpha}(\Omega)$, $\alpha > 0$, $a_{ij} \in C_b^1(\overline{\Omega})$ and $f \in L^2(\Omega)$, then the solution u of (2.9) satisfies $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and there exists $C > 0$ such that*

$$\|u\|_{H^2(\Omega)} \le C\,\|f\|_{L^2(\Omega)}.$$

Remark 2.10. Once we have the existence of a (unique) solution $u \in H_0^1(\Omega)$, the term cu can be incorporated as part of the right hand side of the equation, so that, for the analysis of regularity, we can suppose that $c = 0$.

Proof. Let us restrict to the case $\Omega = \mathbb{R}^N_+$, since the general situation can be proved by local charts. So, let $u \in H_0^1(\mathbb{R}^N_+)$ satisfying the equation

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) = f \text{ in } \mathbb{R}^N_+. \tag{2.10}$$

If we derive both sides of the above equation in the direction x_k, for $k \in \{1, \ldots, N-1\}$, we have formally the following equation.

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v_k}{\partial x_j}\right) = \frac{\partial f}{\partial x_k} + \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial a_{ij}}{\partial x_k}\frac{\partial u}{\partial x_j}\right) \text{ in } \mathbb{R}^N_+, \tag{2.11}$$

where $v_k = \dfrac{\partial u}{\partial x_k} \in L^2(\mathbb{R}^N_+)$. This suggests us to consider look for the solution of the following Dirichlet problem.

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v_k}{\partial x_j}\right) = \frac{\partial f}{\partial x_k} + \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial a_{ij}}{\partial x_k}\frac{\partial u}{\partial x_j}\right) \text{ in } \mathbb{R}^N_+, \\ v_k \in H_0^1(\mathbb{R}^N_+). \end{cases} \tag{2.12}$$

Since $\dfrac{\partial a_{ij}}{\partial x_k} \in L^\infty$ and $\dfrac{\partial u}{\partial x_k} \in L^2$, it follows that

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial a_{ij}}{\partial x_k}\frac{\partial u}{\partial x_j}\right) \in H^{-1}(\mathbb{R}^N_+).$$

Hence, by Lax-Milgram Theorem 2.1, $v_k \in H^1_0(\Omega)$, which implies that

$$\frac{\partial^2 u}{\partial x_i \partial x_k} \in L^2(\Omega), \quad i = 1, \dots N,\ k = 1, \dots, N-1.$$

For the missing term $\dfrac{\partial^2 u}{\partial x_N^2}$, we get from the equation,

$$\begin{aligned}\frac{\partial}{\partial x_N}\left(a_{NN}\frac{\partial u}{\partial x_N}\right) &= -\sum_{\substack{i,j=1\\ i \text{ or } j \neq N}} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - f \\ &= -\sum_{\substack{i,j=1\\ i \text{ or } j \neq N}} \left(a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j}\right) - f.\end{aligned} \tag{2.13}$$

Since the right hand side of (2.13) is a function of $L^2(\mathbb{R}^N_+)$, we obtain (formally)

$$a_{NN}\frac{\partial^2 u}{\partial x_N^2} + \frac{\partial a_{NN}}{\partial x_N}\frac{\partial u}{\partial x_N} \in L^2(\mathbb{R}^N_+), \tag{2.14}$$

which implies that

$$a_{NN}\frac{\partial^2 u}{\partial x_N^2} \in L^2(\mathbb{R}^N_+).$$

But we know that $a_{NN} \geq \alpha > 0$ a.e. in $\overline{\mathbb{R}^N_+}$, so that $\dfrac{\partial^2 u}{\partial x_N^2} \in L^2(\mathbb{R}^N_+)$. Therefore, $u \in H^2(\mathbb{R}^N_+)$ and the proof is complete. □

Remark 2.11. There are two formal steps in the above proof, namely, in the expressions (2.11) and (2.14); a more rigorous argument (see [11]) is by employing the finite difference operators D^k_h defined by

$$D^k_h v(x) = \frac{v(x + he_k) - v(x)}{h}.$$

We have a more general result due to Agmon-Douglis-Nirenberg [1] using Calderon-Zygmund's singular integrals.

Theorem 2.12. *Let us assume that $a_{ij} \in C^1_b(\Omega)$ and satisfy the coercivity hypothesis* (2.1) *and that Ω is of class $C^{2,\alpha}$, $\alpha > 0$. Then, if $f \in L^p(\Omega)$, $1 < p < +\infty$, we have $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$.*

2.4.2 Maximum principle

If $u \in L^2(\Omega)$, we define respectively the *positive part* and the *negative part* of u by

$$u^+(x) := \operatorname*{ess\,sup}_{x\in\Omega}\{u(x), 0\}, \quad u^-(x) := \operatorname*{ess\,sup}_{x\in\Omega}\{-u(x), 0\}.$$

It is easy to see that $u = u^+ - u^-$ and $|u| = u^+ + u^-$.

Theorem 2.13. *The mapping $u \mapsto u^+$ maps $H_0^1(\Omega)$ (respectively $H^1(\Omega)$) to $H_0^1(\Omega)$ (respectively $H^1(\Omega)$) and is continuous. Moreover, if $u \in H^1(\Omega)$, we have*

$$\frac{\partial u^+}{\partial x_i} = \frac{\partial u}{\partial x_i}\chi_{\{u>0\}}.$$

An analogous result for u^- is also valid.

Proof. For $\varepsilon > 0$, we define

$$\varphi_\varepsilon(s) := \begin{cases} (s^2+\varepsilon^2)^{1/2} - \varepsilon & \text{if } s > 0, \\ 0 & \text{if } s \le 0, \end{cases}$$

It is clear that $\varphi_\varepsilon \in C^1(\mathbb{R})$, $|\varphi_\varepsilon(s)| \le |s|$ and $\varphi_\varepsilon(s) \to s^+$, as $\varepsilon \to 0$. By classical density arguments, we can show that, if $u \in H^1(\Omega)$, then $\varphi_\varepsilon \circ u \in H^1(\Omega)$ and

$$\frac{\partial}{\partial x_i}(\varphi_\varepsilon \circ u) = \varphi_\varepsilon'(u)\frac{\partial u}{\partial x_i} = \frac{u}{(u^2+\varepsilon^2)^{1/2}}\frac{\partial u}{\partial x_i}, \quad \text{if } u > 0.$$

So, for $u \in H^1(\Omega)$ fixed, we pass to the limit when $\varepsilon \to 0$ to obtain, by application of the Lebesgue's theorem, $\varphi_\varepsilon(u) \to u^+$ in $L^2(\Omega)$, because $\varphi_\varepsilon(u) \to u^+$ a.e. and $|\varphi_\varepsilon(u)| \le |u|$.

Again, since

$$\frac{\partial}{\partial x_i}\varphi_\varepsilon(u) \to \begin{cases} \dfrac{u}{|u|}\dfrac{\partial u}{\partial x_i} & \text{if } u > 0, \\ 0 & \text{if } u \le 0 \end{cases} \quad \text{and} \quad \left|\frac{\partial}{\partial x_i}\varphi_\varepsilon(u)\right| \le \left|\frac{\partial u}{\partial x_i}\right|$$

a.e. in Ω, we have also by the Lebesgue's theorem,

$$\frac{\partial}{\partial x_i}\varphi_\varepsilon(u) \to \frac{\partial u}{\partial x_i}\chi_{\{u>0\}} \ \text{ in } \ L^2(\Omega).$$

Hence, $u^+ \in H^1(\Omega)$ and

$$\frac{\partial u^+}{\partial x_i} = \frac{\partial u}{\partial x_i}\chi_{\{u>0\}}. \tag{2.15}$$

In the same way, we have $u^- \in H^1(\Omega)$ and

$$\frac{\partial u^-}{\partial x_i} = -\frac{\partial u}{\partial x_i}\chi_{\{u<0\}}. \tag{2.16}$$

Moreover, since $u = u^+ - u^-$, we have

$$\frac{\partial u}{\partial x_i} = \frac{\partial u^+}{\partial x_i} - \frac{\partial u^-}{\partial x_i} = \frac{\partial u}{\partial x_i}\left[\chi_{\{u>0\}} + \chi_{\{u<0\}}\right] = \frac{\partial u}{\partial x_i}\chi_{\{u\neq 0\}}.$$

from where we deduce that

$$\frac{\partial u}{\partial x_i}\chi_{\{u=0\}} = 0. \tag{2.17}$$

To show that the mapping $u \mapsto u^+$ is continuous on $H^1(\Omega)$, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $H^1(\Omega)$ such that $u_n \to u$ in $H^1(\Omega)$ and (after extracting a subsequence if necessary)

$$u_n \to u \quad \text{and} \quad \frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \text{ a.e. in } \Omega, \quad i = 1, \ldots, N.$$

We know that $u_n^+ \to u^+$ in $L^2(\Omega)$, because the mapping $u \mapsto u^+$ is Lipschitz in $L^2(\Omega)$. On the other hand, we have from (2.15),

$$\frac{\partial u_n^+}{\partial x_i} = \frac{\partial u_n}{\partial x_i}\chi_{\{u_n>0\}} = \frac{\partial u_n}{\partial x_i}\chi_{\{u_n>0\}}\left[\chi_{\{u>0\}} + \chi_{\{u<0\}} + \chi_{\{u=0\}}\right].$$

It is clear that, almost everywhere in Ω, we have

$$\begin{aligned}
&\frac{\partial u_n}{\partial x_i}\chi_{\{u_n>0\}}\chi_{\{u>0\}} \to \frac{\partial u}{\partial x_i}\chi^2_{\{u>0\}} = \frac{\partial u}{\partial x_i}\chi_{\{u>0\}},\\
&\frac{\partial u_n}{\partial x_i}\chi_{\{u_n>0\}}\chi_{\{u<0\}} \to \frac{\partial u}{\partial x_i}\chi_{\{u>0\}}\chi_{\{u<0\}} = 0,\\
&\frac{\partial u_n}{\partial x_i}\chi_{\{u_n>0\}}\chi_{\{u=0\}} \to \frac{\partial u}{\partial x_i}\chi_{\{u>0\}}\chi_{\{u=0\}} = 0,
\end{aligned}$$

the last one following from (2.17). Then

$$\frac{\partial u_n^+}{\partial x_i} \to \frac{\partial u}{\partial x_i}\chi_{\{u>0\}} = \frac{\partial u^+}{\partial x_i} \quad \text{a.e. in } \Omega.$$

Since

$$\left|\frac{\partial u_n^+}{\partial x_i}\right| \le \left|\frac{\partial u_n}{\partial x_i}\right|$$

and as we are assuming that $\dfrac{\partial u_n}{\partial x_i}$ converges strongly in $L^2(\Omega)$, it follows from Lebesgue Theorem that

$$\frac{\partial u_n^+}{\partial x_i} \to \frac{\partial u^+}{\partial x_i} \quad \text{in} \quad L^2(\Omega)$$

and $u_n^+ \to u$ in $H^1(\Omega)$. □

We are now in condition to prove the weak maximum principle. Let $a_{ij} \in L^\infty(\Omega)$ satisfying the coercivity condition (2.4) and $c \in L^\infty(\Omega)$ $c \geq \beta > 0$ a.e. Let us denote

$$\mathcal{A}u := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu.$$

Theorem 2.14 (Weak maximum principle)**.** *If $u \in H^1(\Omega)$ is a solution of*

$$\mathcal{A}u \geq 0 \text{ in } H^{-1}(\Omega) \quad \text{and} \quad \gamma_0 u \geq 0 \text{ a.e. on } \Gamma,$$

then $u \geq 0$ almost everywhere in Ω.

We recall that $L \geq 0$ in $H^{-1}(\Omega)$ means that $\langle L, v\rangle \geq 0$ for every $v \in H_0^1(\Omega)$ such that $v \geq 0$ a.e. in Ω.

Proof. Let

$$a(u,v) := \int_\Omega A\nabla u \cdot \nabla v \, dx + \int_\Omega cu \cdot v \, dx,$$

where A is the matrix with entries a_{ij}. From (2.15) and (2.16), it follows that

$$a(u,u^+) = a(u^+,u^+) \quad \text{and} \quad a(u,u^-) = -a(u^-,u^-).$$

Let $u \in H^1(\Omega)$ such that $Au \geq 0$ and $\gamma_0 u \geq 0$. Then, $u^- \in H_0^1(\Omega)$ and $u^- \geq 0$, So,

$$0 \leq \langle Au, u^-\rangle = a(u,u^-) = -a(u^-,u^-),$$

which implies that $a(u^-,u^-) \leq 0$ and consequently $u^- = 0$ and we conclude that $u \geq 0$ a.e. in Ω. □

Corollary 2.15. *Let $u, v \in H^1(\Omega)$ such that*

$$\mathcal{A}u \leq \mathcal{A}v \quad \text{in} \quad H^{-1}(\Omega) \quad \text{and} \quad \gamma_0 u \leq \gamma_0 v \quad \text{a.e. on} \quad \Gamma.$$

Then, $u \leq v$ a.e. in Ω.

Theorem 2.16 (Strong maximum principle)**.** *Let $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ such that $\Delta u \in L^1_{\rm loc}(\Omega)$ and*

$$-\Delta u \geq 0 \ \ in \ \ \Omega, \quad and \quad u|_\Gamma \geq 0.$$

If u is not identically zero and Ω is connex, then $u(x) > 0$, $\forall x \in \Omega$.

The proof of Theorem 2.16 is a consequence of the mean value property of subharmonic functions.

Lemma 2.17. *Let u be a function satisfying the hypothesis of Theorem 2.16. Then, for any ball $B = B_R(y) \subset \Omega$, we have*

$$u(y) \geq \fint_{\partial B} u(\sigma)\, dS_R = \fint_B u(x)\, dx.$$

Proof of Lemma 2.17. Let $B_R(y) \subset \Omega$ a ball of radius $R > 0$ centered at y. For each $0 < \rho < R$, we have from Gauss' Theorem,

$$\int_{\partial B_\rho(y)} \nabla u(\sigma) \cdot \nu(\sigma)\, dS_\rho = \int_{B_\rho(y)} \Delta u(x)\, dx \leq 0. \tag{2.18}$$

If we denote $\sigma = y + \rho\omega$ with $\omega \in \mathbb{S}^{N-1}$, it follows that $dS_\rho = \rho^{N-1} dS_1$ and

$$\begin{aligned} \int_{\partial B_\rho(y)} \nabla u(\sigma) \cdot \nu(\sigma)\, dS_\rho &= \rho^{N-1} \int_{\partial B_1(0)} \nabla u(y + \rho\omega) \cdot \omega\, dS_1 \\ &= \rho^{N-1} \frac{\partial}{\partial \rho} \int_{\partial B_1(0)} u(y + \rho\omega)\, dS_1 \\ &= \rho^{N-1} \frac{\partial}{\partial \rho} \left[\rho^{1-N} \int_{\partial B_\rho(y)} u(\sigma)\, dS_\rho \right] \end{aligned}$$

So, by (2.18) we see that the mapping

$$\rho \mapsto \rho^{1-N} \int_{\partial B_\rho(y)} u(\sigma)\, dS_\rho$$

is not increasing and

$$\rho^{1-N} \int_{\partial B_\rho(y)} u(\sigma)\, dS_\rho \geq R^{1-N} \int_{\partial B_R(y)} u(\sigma)\, dS_R.$$

But u being continuous in Ω, we know that

$$\lim_{\rho \to 0^+} \rho^{1-N} \int_{\partial B_\rho(y)} u(\sigma)\, dS_\rho = N\omega_N u(y),$$

where $\omega_N = \dfrac{2\pi^{N/2}}{\Gamma(N/2)}$ is the superficial measure of the unit sphere in $\mathbb{R}^N$. So, we conclude that

$$u(y) \geq \frac{1}{N\omega_N R^{N-1}} \int_{\partial B_R(y)} u(\sigma)\, dS_R = \fint_{\partial B_R(y)} u(\sigma)\, dS_R.$$

Now, we can write

$$N\omega_N \rho^{N-1} u(y) \geq \int_{\partial B_\rho(y)} u(\sigma)\, dS_\rho.$$

Then, by integration in ρ from 0 to R, we get

$$u(y) \geq \frac{1}{\omega_N R^N} \int_{B_R(y)} u(x)\, dx = \fint_{B_R(y)} u(x)\, dx.$$

□

Proof of Theorem 2.16. Let $\Omega_0 := \{x \in \Omega, u(x) = 0\}$. Since u is continuous, Ω_0 is closed (in Ω). For every $y \in \Omega_0$, we have from Lemma 2.14,

$$0 = u(y) \geq \fint_{B_R(y)} u(x)\, dx,$$

for R such that $B_R \subset \Omega$. Since $u \geq 0$, it follows that $u = 0$ on B_R, which implies that Ω_0 is open in Ω. Therefore, $\Omega_0 = \Omega$ and $u \equiv 0$. Hence, if u is not identically zero, we have necessarily $\Omega_0 = \Omega$ and $u > 0$. □

Lemma 2.18 (Hopf's Lemma)**.** *Let $\Omega \subset \mathbb{R}^N$ be of class C^2 and a function $u \in C^1(\overline{\Omega})$, $u \not\equiv 0$, satisfying*

$$-\Delta u \geq 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u|_\Gamma \geq 0.$$

If $u(x_0) = 0$ for some $x_0 \in \Gamma$, then

$$\frac{\partial u}{\partial \nu}(x_0) < 0,$$

where ν is the normal exterior to Ω at x_0.

Proof. From Theorem 2.16, we know that $u(x) > 0$ for all $x \in \Omega$. Let $x_0 \in \Gamma$ such that $u(x_0) = 0$. Since $\Gamma \in C^2$, there exist $y \in \Omega$ and $\rho > 0$ such that $B_\rho(y) \subset \Omega$ with $x_0 \in \partial B_\rho$. Without loss of generality we can assume that $y = 0$. Let us consider the (barrier) function

$$w(x) = e^{-\alpha|x|^2} - e^{-\alpha\rho^2}, \quad x \in B_\rho(0),$$

for some $\alpha > 0$ to be chosen later. Since w is radial, we have

$$-\Delta w = -\left(\frac{\partial^2 w}{\partial r^2} + \frac{N-1}{r}\frac{\partial w}{\partial r}\right) = 2\alpha e^{-\alpha r^2}(N - 2\alpha r^2).$$

In the open annular region $B_\rho \setminus B_{\rho/2}$, we have

$$-\Delta w \leq 2\alpha e^{-\alpha r^2}\left(N - \frac{\alpha\rho^2}{2}\right),$$

since $\rho^2/4 < r^2 < \rho^2$.

By choosing $\alpha > 0$ large enough, we obtain

$$\begin{cases} -\Delta w \leq 0 \quad \text{in} \quad B_\rho \setminus B_{\rho/2}, \\ \quad w = 0 \quad \text{on} \quad \partial B_\rho, \\ \quad w = e^{-\alpha\rho^2/4} - e^{-\alpha\rho^2} > 0 \quad \text{on} \quad \partial B_{\rho/2}, \end{cases}$$

Since $u(x) > 0$ on $\partial B_{\rho/2}$, there exists $\varepsilon > 0$ such that

$$\begin{cases} -\Delta(u - \varepsilon w) \geq 0 \quad \text{in} \quad B_\rho \setminus B_{\rho/2}, \\ \qquad u - \varepsilon w = 0 \quad \text{on} \quad \partial B_\rho, \\ \qquad u - \varepsilon w \geq 0 \quad \text{on} \quad \partial B_{\rho/2}, \end{cases}$$

From Theorem 2.14 it follows that $u - \varepsilon w \geq 0$ in $B_\rho \setminus B_{\rho/2}$ and $(u - \varepsilon w)(x_0) = 0$. Therefore

$$\frac{\partial u}{\partial \nu}(x_0) \leq \varepsilon \frac{\partial w}{\partial \nu}(x_0) = -2\varepsilon\alpha\rho e^{-\alpha\rho^2} < 0$$

and the proof is complete. □

2.5 Eigenvalues and eigenfunctions

In view of the next applications, we recall some important facts from Functional Analysis.[1] Let H be an infinite-dimensional Hilbert space equipped with the scalar product $(\cdot,\cdot)$ and $T : H \to H$ be a continuous linear operator.

Definition 2.19. The *spectrum* of T is the set $\sigma(T)$ of all scalars μ such that $(\mu I - T)$ is not invertible.

It follows that $(\mu I - T)^{-1}$ is a continuous linear operator, if $\mu \notin \sigma(T)$.

[1] A good reference for all these results is the Brezis' book on Function Analysis (see [4]).

Definition 2.20. A linear operator $T : H \to H$ is *compact* if it maps bounded sets into relatively compact sets.

Clearly, every compact operator is bounded.

Lemma 2.21. *If $T : H \to H$ is a compact linear operator, then $0 \in \sigma(T)$.*

Proof. If $0 \notin \sigma(T)$, then T^{-1} is continuous. Since T^{-1} sends bounded sets in bounded sets, $T \circ T^{-1} = I$ should be compact, which is a contradiction if H is infinite-dimensional. □

Theorem 2.22. *Let $T : H \to H$ be a compact linear operator.*

(1) *If $\mu \in \sigma(T)$, $\mu \neq 0$, then μ is an eigenvalue of T, i.e., there exists $w \in H$, $w \neq 0$, such that $Tw = \mu w$, in which case w is called an eigenvector of T.*

(2) *Each eigenvalue $\mu \neq 0$ is associated to a finite-dimensional subspace of H called eigenspace.*

(3) *Each eigenvalue $\mu \neq 0$ is isolated and 0 is an accumulation point of the spectrum.*

Let $T : H \to H$ be a continuous linear operator. Given $v \in H$, the linear mapping $u \in H \mapsto (Tu, v) \in \mathbb{R}$ is obviously linear and continuous. So, it defines an element $T^*v \in H$ such that $(Tu, v) = (u, T^*v)$.

Definition 2.23. The mapping $v \in H \mapsto T^*v \in H$ is linear and continuous. We call T^* the *adjoint operator* of T. We say T is *self-adjoint* if $T^* = T$, i.e.,

$$(Tu, v) = (u, Tv), \quad \forall u, v \in H.$$

Theorem 2.24. *Let $T : H \to H$ be a compact self-adjoint linear operator. If H is separable (i.e. contains a countable dense set), then $\sigma(T) \setminus \{0\}$ is a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ satisfying $\mu_n \to 0$ Each μ_n is associated to an eigenvector w_n which can be chosen such that $\{w_n\}_{n\in\mathbb{N}}$ is a Hilbert basis of H, i.e.,*

(1) *$(w_n, w_m) = \delta_{nm}$ for all $n, m \in \mathbb{N}$;*

(2) *the linear space spanned by $(w_n)_{n\in\mathbb{N}}$ is dense in H.*

• Applications.

Let Ω be a bounded open set of $\mathbb{R}^N$. Consider the operator

$$Au := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\big(a_{ij}\frac{\partial u}{\partial x_j}\big) + cu, \tag{2.19}$$

where $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$, $\forall i, j = 1, \ldots, N$, $c \in L^\infty(\Omega)$, $c \geq 0$ and assume that there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_j\xi_i \geq \alpha|\xi|^2, \tag{2.20}$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$.

Associated to this operator, we consider the bilinear form: for every $u, v \in H_0^1(\Omega)$,

$$a(u,v) := \sum_{i,j=1}^{N} \int_\Omega a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx + \int_\Omega cu \cdot v\,dx,$$

Since $a_{ij} = a_{ji}$, it is clear that a is symmetric: $a(u, v) = a(v, u)$.

As we have seen, for any $L \in H^{-1}(\Omega)$, there exists a unique function $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \langle L, v\rangle, \quad \text{for all } v \in H_0^1(\Omega),$$

and we can write

$$\begin{cases} Au = L \text{ in } H^{-1}(\Omega), \\ u \in H_0^1(\Omega). \end{cases}$$

The operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism, so that we can consider the continuous linear operator

$$v \in L^2(\Omega) \mapsto Tv := A^{-1}(v) = u \in H_0^1(\Omega).$$

In view of the fact that the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, it follows that $T : L^2(\Omega) \to L^2(\Omega)$ is a compact operator.

In order to show that T is self-adjoint, take $u, v \in L^2(\Omega)$ and let $Tv = \overline{v}$ and $Tu = \overline{u}$. Then $\overline{u}, \overline{v} \in H_0^1(\Omega)$ and

$$(Tu, v)_{L^2(\Omega)} = (\overline{u}, v)_{L^2(\Omega)} = a(\overline{v}, \overline{u}) = a(\overline{u}, \overline{v}) = (u, \overline{v})_{L^2(\Omega)} = (u, Tv).$$

It follows form the uniqueness of solution that $\mathrm{Ker}\,(T) = \{0\}$. Moreover, T has a countable set of eigenfunctions $w_n \in H_0^1(\Omega)$ which is a Hilbert basis for $L^2(\Omega)$, where the corresponding eigenvalues satisfy $\mu_n \to 0$ as $n \to \infty$, Let $\mu \neq 0$ be an eigenvalue of T and $\lambda = \mu^{-1}$. If w is a corresponding eigenvector, then $\lambda T w = w$, which implies that

$$\begin{cases} Aw = \lambda w, \\ w \in H_0^1(\Omega), \end{cases}$$

i.e., λ is an eigenvalue of A.

Since

$$\alpha \|\nabla w\|_{L^2(\Omega)}^2 \leq \langle Aw, w \rangle = a(w,w) = \lambda \|w\|_{L^2}^2,$$

it follows from Poincaré's inequality (see Theorem 1.46) that

$$\alpha \|\nabla w\|_{L^2(\Omega)}^2 \leq \lambda \|w\|_{L^2}^2 \leq C\lambda \|\nabla w\|_{L^2(\Omega)}^2,$$

which implies that $\lambda > 0$. Thus, the eigenvalues of A are a sequence of positive numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ that can be ordered,

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots,$$

with $\lambda_n \to +\infty$, and the corresponding eigenfunctions $\{w_n\}_{n\in\mathbb{N}}$ can be chosen to be a Hilbert basis of $L^2(\Omega)$.

So, as $(w_n, w_m)_{L^2(\Omega)} = \delta_{nm}$, it follows that, for every $v \in L^2(\Omega)$,

$$v = \sum_{i=1}^{\infty} a_i w_i,$$

where

$$a_n = (v, w_n)_{L^2(\Omega)}, \quad \text{and} \quad \sum_{n=1}^{\infty} a_n^2 < +\infty.$$

Now, as in the present situation the bilinear form a defines a scalar product in $H_0^1(\Omega)$ and $a(w_n, w_m) = (Aw_n, w_m)_{L^2(\Omega)} = \lambda_n \delta_{nm}$, it follows that the sequence $\{w_n/\sqrt{\lambda_n}\}_{n\in\mathbb{N}}$ is orthonormal in $H_0^1(\Omega)$ for this scalar product. So, if $v \in H_0^1(\Omega) \subset L^2(\Omega)$, we have

$$v = \sum_{n=1}^{\infty} (v, w_n)_{L^2(\Omega)} = \sum_{n=1}^{\infty} \sqrt{\lambda_n} (v, w_n)_{L^2(\Omega)} \frac{w_n}{\lambda_n} = \sum_{n=1}^{\infty} b_n \frac{w_n}{\lambda_n}$$

and we conclude that

$$\sum_{n=1}^{\infty} \lambda_n \left| (v, w_n)_{L^2(\Omega)} \right|^2 < +\infty.$$

This way we have the following characterisation of $H_0^1(\Omega)$ as

$$\begin{aligned} H_0^1(\Omega) &= \left\{ v \in L^2(\Omega)\,;\ \sum_{n=1}^{\infty} \lambda_n \left|(v, w_n)_{L^2(\Omega)}\right|^2 < +\infty \right\} \\ &= \left\{ v = \sum_{i=1}^{\infty} \alpha_i w_i;\ \sum_{i=1}^{\infty} \lambda_i |\alpha_i|^2 < +\infty \right\} \end{aligned}.$$

By the same way, we can characterize

$$D(A) := \left\{ v \in H_0^1(\Omega),\ Av \in L^2(\Omega) \right\} = \left\{ v = \sum_{i=1}^{\infty} \alpha_i w_i,\ \sum_{i=1}^{\infty} \lambda_i^2 |\alpha_i|^2 < +\infty \right\}.$$

as well as $D(A^2)$, $D(A^3)$, etc.

It is noteworthy that if $v \in H_0^1(\Omega)$,

$$a(v,v) = \sum_{i=1}^{\infty} \lambda_i \|(v, w_i)_{L^2}\|^2 \geq \lambda_1 \|v\|^2_{L^2(\Omega)}.$$

which implies that

$$\lambda_1 \leq \frac{a(v,v)}{\|v\|^2_{L^2}}, \quad \forall v \in H_0^1(\Omega), v \neq 0.$$

Since $a(w_1, w_1) = \lambda_1 \|w_1\|^2$, the lower bound is attained at $v = w_1$, and we have the following variational characterization

$$\lambda_1 = \min_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{a(v,v)}{\|v\|^2_{L^2(\Omega)}}.$$

This ratio is called the *Rayleigh quotient* of A.

Proposition 2.25. *The first eigenvalue λ_1 is simple. It is associated with an eigenfunction w_1 that does not change sign in Ω (and therefore can be taken positive in Ω).*

Proof. Let w_1 be the eigenfunction associated to λ_1. We have

$$\lambda_1 = \frac{a(w_1, w_1)}{\|w_1\|^2_{L^2(\Omega)}} = \min_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{a(v,v)}{\|v\|^2_{L^2(\Omega)}},$$

As we know that $|w_1| = |w_1^+| + |w_1^-| \in H_0^1(\Omega)$, $a(|w_1|, |w_1|) = a(w_1, w_1)$ and $\big\||w_1|\big\|_{L^2(\Omega)} = \|w_1\|_{L^2(\Omega)}$, it follows that $|w_1|$ also realizes the minimum and is an eigenfunction corresponding to λ_1.

This is clear because, for u and v two linearly independent functions of $H_0^1(\Omega)$, the mapping

$$s \in \mathbb{R} \mapsto f(s) := \frac{a(u+sv, u+sv)}{\|u+sv\|^2_{L^2(\Omega)}}$$

is well defined and differentiable, satisfying $f'(0) = 0$ if u realizes the minimum. The fact that $\|u\|^4_{L^2(\Omega)} f'(0) = 2a(u,v)\|u\|^2_{L^2(\Omega)} - 2a(u,u)(u,v)_{L^2(\Omega)}$ and $a(u,u) = \lambda_1 \|u\|^2_{L^2(\Omega)}$ implies that $a(u,v) = \lambda_1 (u,v)_{L^2(\Omega)}$ for all $v \in H_0^1(\Omega)$, since this is obvious if u and v are linearly dependent.

So, as we stated before, $|w_1|$ is an eigenfunction corresponding to λ_1. Moreover, as $w_1^+ = (|w_1| + w_1)/2$, it follows that w_1^+ is also an eigenfunction corresponding to λ_1, i.e.,

$$\begin{cases} Aw^+ = \lambda_1 w^+, \\ w^+ \in H_0^1(\Omega), \end{cases}$$

Since $\lambda_1 w^+ \geq 0$ and we may assume that $w^+ \not\equiv 0$ (otherwise $w = -w^-$), the strong maximum principle imply that $w^+ > 0$ in Ω. So, $w^- = 0$ and $w = w^+$.

In order to conclude the proof, suppose that we had two linearly independent eigenfunctions corresponding to λ_1. Since we can assume them orthogonal, we have a contradiction with the fact that they have constant sign. So, λ_1 is a simple eigenvalue and the proof is complete. □

Note: If the operator A is no more of order 2, the former result is false.

As the last result of this chapter, we present (without proof) the Fredholm alternative, which comes from the following general statement: *If T is a compact operator, then*

$$\mathrm{R}(\mu I - T) = \mathrm{Ker}\,(\mu I - T^*)^\perp.$$

Theorem 2.26 (Fredholm alternative). *Let λ_i be an eigenvalue of A defined in* (2.19) *and E_i the corresponding eigenspace. If $w_i^1, \ldots, w_i^k$ are linearly independent eigenfunctions corresponding to λ_i which generate E_i, then the equation*

$$Au - \lambda_i u = f$$

has a solution if, and only if, $(f, w_i^j)_{L^2(\Omega)} = 0, \quad \forall j = 1, \ldots, k.$

Chapter 3

Second-order monotone nonlinear equations

In this chapter we are interested on boundary value problems for nonlinear partial differential equations. In the first section, we consider semilinear partial differential equations of the form

$$-\operatorname{div}(A\nabla u) + f(u) = g,$$

where $A(x)$ is the matrix with entries $a_{ij}(x)$ and $f : \mathbb{R} \to \mathbb{R}$.

In Section 3.2 we deal with nonlinear problems involving minimisation of convex functionals and in Section 3.3 we present some important properties of monotone operators.

3.1 Semilinear monotone equations

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and non decreasing function, i.e., . $s, s' \in \mathbb{R}$, $s \leq s'$ implies that $f(s) \leq f(s')$. For simplicity and without loss of generality, we can assume $f(0) = 0$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $g \in H^{-1}(\Omega)$. We are interested in the following boundary value problem

$$\begin{cases} -\Delta u + f(u) = g, \\ \quad u \in H_0^1(\Omega). \end{cases} \tag{3.1}$$

Theorem 3.1. *There exists a unique solution u of* (3.1) *such that*

$$\int_\Omega f\big(u(x)\big)u(x)\,dx < +\infty.$$

Proof. We define the truncation operator: for $k \in \mathbb{R}_+$, $T_k : \mathbb{R} \to \mathbb{R}$,

$$T_k(s) = \begin{cases} -k, \text{ if } s \leq -k, \\ \quad s, \text{ if } -k \leq s \leq k, \\ \quad k, \text{ if } s \geq k, \end{cases}$$

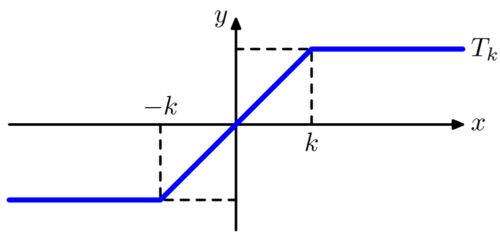

Figure 3.1. The truncation function T_k.

In order to prove the existence of solution of (3.1), let us consider the approximate problem

$$\begin{cases} -\Delta w + (f \circ T_k)(w) = g, \\ \quad w \in H_0^1(\Omega). \end{cases}$$

It is clear that, for every $v \in L^2(\Omega)$, $-k \le T_k(v) \le k$ a.e. in Ω and as f is continuous, $\left|f\big(T_k(v)\big)\right| \le M_k$, for some $M_k > 0$.

Since Ω is bounded, for every $v \in L^2(\Omega)$, the boundary value problem

$$\begin{cases} -\Delta w_k + (f \circ T_k)(v) = g, \\ \quad w_k \in H_0^1(\Omega). \end{cases}$$

has a unique solution $w_k = S_k(v) \in H_0^1(\Omega)$, and we have from the Hölder and Poincaré inequalities,

$$\begin{aligned} \int_\Omega |\nabla w_k(x)|^2\,dx &\le \big|\langle g, w_k\rangle\big| + \int_\Omega M_k |w_k(x)|\,dx \\ &\le \big(\|g\|_{H^{-1}} + M_k \text{meas}(\Omega)^{1/2}\big)\|w_k\|_{L^2(\Omega)} \\ &\le C\|w_k\|_{H_0^1(\Omega)} \end{aligned}$$

which implies $\|w_k\|_{H_0^1(\Omega)} \le C$.

If we denote by $B_C(0)$ the ball of radius C and center at the origin in the space $H_0^1(\Omega)$, the former inequality says that

$$S_k : L^2(\Omega) \to \overline{B_C(0)}. \tag{3.2}$$

But we know that $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. Hence,

$$S_k : L^2(\Omega) \to L^2(\Omega) \quad \text{and} \quad \overline{S_k\big(L^2(\Omega)\big)} \text{ is compact in } L^2(\Omega).$$

Moreover, S_k is continuous. Indeed, if $v_n \to v$ in $L^2(\Omega)$ as $n \to +\infty$, we can find a subsequence such that $v_{n_i} \to v$ a.e. in Ω. So,

$$f\big(T_k(v_{n_i})\big) \to f\big(T_k(v)\big) \quad \text{a.e. in } \Omega \quad \text{and} \quad \big|f\big(T_k(v_{n_i})\big)\big| \le M_k \quad \text{a.e.}$$

From the Lebesgue Convergence Theorem,

$$f\big(T_k(v_{n_i})\big) \to f\big(T_k(v)\big) \quad \text{in } L^2(\Omega),$$

which implies that $S_k(v_{n_i}) \to S_k(v)$ in $H_0^1(\Omega)$.

Since the limit point $S_k(v)$ is unique, the whole sequence $\{S_k(v_n)\}_{n\in\mathbb{N}}$ converges to $S_k(v)$. So, S_k is continuous from $L^2(\Omega)$ to $L^2(\Omega)$ and $\overline{S_k\big(L^2(\Omega)\big)}$ is compact. From Schauder Fixed Point Theorem, it follows that S_k has a fixed point u_k, i.e., $u_k = S_k(u_k) \in H_0^1(\Omega)$. Hence, from (3.2), $\|u_k\|_{H_0^1(\Omega)} \le C$.

Since

$$\int_\Omega |\nabla u_k(x)|^2\,dx + \int_\Omega f\big(T_k(u_k)\big)u_k\,dx = \langle g, u_k\rangle$$

and

$$\int_\Omega f\big(T_k(u_k)\big)u_k\,dx \ge 0,$$

it follows that $\|u_k\|_{H_0^1(\Omega)} \le \|g\|_{H^{-1}(\Omega)}$, from which we get

$$\int_\Omega f\big(T_k(u_k)\big)u_k\,dx \le \|g\|_{H^{-1}(\Omega)}\|u_k\|_{H_0^1(\Omega)} \le \|g\|^2_{H^{-1}(\Omega)}.$$

Therefore, we can extract a subsequence (still called $\{u_k\}_{k\in\mathbb{N}}$) such that

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in } H_0^1(\Omega),\\ u_k \to u & \text{strongly in } L^2(\Omega),\\ u_k \to u & \text{a.e. in } \Omega,\end{cases}$$

from which we also get $f\big(T_k(u_k)\big) \to f(u)$ a.e. in Ω.

Let A be a measurable subset of Ω and let $\varepsilon > 0$. Then, because

$$\int_\Omega f\big(T_k(u_k)\big)u_k\,dx = \int_\Omega \big|f\big(T_k(u_k)\big)\big|\,|u_k|\,dx$$

we have for $M := \|g\|^2_{H^{-1}(\Omega)}$,

$$\begin{aligned}\int_A \big|f\big(T_k(u_k)\big)\big|\,dx &\le \int_{A\cap\{|u_k|\ge R\}} \big|f\big(T_k(u_k)\big)\big|\frac{|u_k|}{R}\,dx + \int_{A\cap\{|u_k|\le R\}} \big|f\big(T_k(u_k)\big)\big|\,dx\\ &\le \frac{M}{R} + \int_A \max\big\{|f(R)|, |f(-R)|\big\}\,dx.\end{aligned}$$

Take $R > 0$ such that $M/R < \varepsilon/2$ and $\delta > 0$ satisfying $\max\{|f(R)|, |f(-R)|\}\delta < \varepsilon/2$. With these choices, if $\operatorname{meas}(A) \leq \delta$, we have

$$\int_A \Big| f\big(T_k(u_k)\big)\Big| \, dx < \varepsilon.$$

Therefore, from Vitali's Theorem 1.21, we conclude that

$$f\big(T_k(u_k)\big) \to f(u) \quad \text{in} \quad L^1(\Omega)$$

and we have

$$\begin{cases} -\Delta u + f(u) = g \text{ in } \mathcal{D}'(\Omega), \\ \quad u \in H_0^1(\Omega). \end{cases}$$

On the other hand, as we have

$$\begin{cases} f\big(T_k(u_k)\big)u_k \geq 0 \quad \text{in } \Omega, \\ f\big(T_k(u_k)\big)u_k \to f(u)u \;\; \text{a.e. in } \Omega, \\ \displaystyle\int_\Omega f\big(T_k(u_k)\big)u_k \, dx \leq M, \end{cases}$$

it follows from Fatou's Lemma 1.22 that

$$\int_\Omega f\big(u(x)\big)u(x) \, dx \leq M.$$

For the uniqueness, if u and $\overline{u}$ are solutions, we have

$$-\Delta(u - \overline{u}) + f(u) - f(\overline{u}) = 0.$$

Multiplying this equation by $u - \overline{u}$, we obtain

$$\int_\Omega |\nabla(u - \overline{u})|^2 \, dx + \int_\Omega \big(f(u) - f(\overline{u})\big)(u - \overline{u}) \, dx = 0.$$

and we have the conclusion, because f is monotone. □

It is noteworthy that, in this proof and except for uniqueness, the only requirements on f for the existence are the continuity and the property $f(s)s \geq 0$, which means that $f(s)$ has the same sign of s.

Remark 3.2. The same arguments applied in the previous proof can be used to solve the problem

$$\begin{cases} Au + f(u) = g, \\ u \in H_0^1(\Omega). \end{cases}$$

where A is the operator defined in (2.19) under the condition (2.20), but not necessarily symmetric.

3.2 Minimisation of convex functional

Let E be a reflexive Banach space, and let $J : E \to \mathbb{R}$ be a functional which is convex, coercive and lower semi continuous for the weak topology of E. As stated in Theorem 2.6, the minimization problem

$$J(u) = \min_{v \in E} J(v).$$

has a solution $u \in E$, which is unique if J is strictly convex and is solution of the equation

$$D_G J(u) = 0 \text{ in } E',$$

if J is Gateaux-differentiable.

Example 3.3. Let Ω be a bounded subset of $\mathbb{R}^N$, $E = W_0^{1,p}(\Omega)$, $1 < p < +\infty$ and $f \in L^{p'}(\Omega)$, $1/p + 1/p' = 1$. We define

$$J(v) := \frac{1}{p} \int_\Omega |\nabla v(x)|^p \, dx - \int_\Omega f(x) v(x) \, dx.$$

So, it is clear that J is a continuous and strictly convex functional. Moreover, from Poincaré Inequality (see Theorem 1.46),

$$\begin{aligned} J(v) &\geq \frac{1}{p} \|v\|^p_{W_0^{1,p}(\Omega)} - \|f\|_{L^{p'}(\Omega)} \|v\|_{L^p(\Omega)} \\ &\geq \frac{1}{p} \|v\|^p_{W_0^{1,p}(\Omega)} - C \|f\|_{L^{p'}(\Omega)} \|v\|_{W_0^{1,p}(\Omega)} \to +\infty, \end{aligned}$$

if $\|v\|_{W_0^{1,p}(\Omega)} \to +\infty$, which means that J is coercive in $W_0^{1,p}(\Omega)$. Hence, there exists a unique $u \in W_0^{1,p}(\Omega)$ such that

$$J(u) = \min_{v \in W_0^{1,p}(\Omega)} J(v).$$

Moreover, J is Gateaux-differentiable in $W_0^p(\Omega)$ and

$$\big\langle D_G(u), v \big\rangle = \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) \, dx - \int_\Omega f(x) v(x) \, dx.$$

Note that $|\nabla u|^{p-2} \nabla u \in L^{p'}(\Omega)$ if $u \in W_0^{1,p}(\Omega)$. In fact, since the mapping $s \mapsto |s|^p$ is of classe C^1 for $p > 1$, it is easy to show that J is Fréchet-differentiable in $W_0^{1,p}(\Omega)$.

Therefore, as a consequence of Theorem 2.6, we have

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big) = f \quad \text{in} \quad \mathcal{D}'(\Omega), \\ \quad u \in W_0^{1,p}(\Omega). \end{cases}$$

As a usual notation in the world of nonlinear partial differential equations, we have the *p-laplacian operator* Δ_p defined by $\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$. With this notation, the former boundary value problem is usually written as

$$\begin{cases} -\Delta_p u = f \quad \text{in} \quad \mathcal{D}'(\Omega), \\ \quad u \in W_0^{1,p}(\Omega). \end{cases}$$

Example 3.4. Consider the reflexive Banach space $E = H_0^1(\Omega) \cap L^p(\Omega)$ with $1 < p < +\infty$ endowed with the norm $\|\cdot\|_E = \|\cdot\|_{H_0^1} + \|\cdot\|_{L^p}$. Let $f \in L^2(\Omega)$ (or $f \in H^{-1}(\Omega) \cap L^{p'}(\Omega)$). The functional

$$J(v) := \frac{1}{2}\int_\Omega |\nabla v(x)|^2\,dx + \frac{1}{p}\int_\Omega |v(x)|^p\,dx - \int_\Omega f(x)v(x)\,dx$$

is strictly convex and continuous. As before, applying Poincaré's inequality, we see that J is coercive in E. Hence, there exists a unique $u \in E$ such that

$$J(u) = \min_{v\in E} J(v).$$

Moreover, J is Gateaux-differentiable in E (in fact, Fréchet-differentiable) and

$$\big\langle D_G(v), w\big\rangle = \int_\Omega \nabla v(x)\cdot\nabla w(x)\,dx + \int_\Omega |v(x)|^{p-2}v(x)w(x)\,dx - \int_\Omega f(x)w(x)\,dx$$

for all v and $w \in E$. So, the minimizing function $u \in E$ is solution of

$$\begin{cases} -\Delta u + |u|^{p-2}u = f \quad \text{in} \quad \mathcal{D}'(\Omega), \\ \quad u \in H_0^1(\Omega) \cap L^p(\Omega). \end{cases}$$

Example 3.5. The following is an important example related to the problem of minimal surfaces, but it is out of range of this notes and cannot be treated here.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $E = W_0^{1,1}(\Omega)$. We consider the functional $J : E \to \mathbb{R}$ defined by

$$J(v) = \int_\Omega \sqrt{1 + |\nabla v(x)|^2}\,dx - \int_\Omega f(x)v(x)\,dx,$$

where $f \in L^\infty(\Omega)$.

It is clear that J is strictly convex and we can prove that J is Fréchet-differentiable in E with

$$J'(v) = -\operatorname{div}\left((1 + |\nabla v|^2)^{-1/2}\nabla v\right), \quad \forall v \in W_0^{1,1}(\Omega).$$

If u is the solution of the variational problem

$$J(u) = \min_{v \in E} J(v),$$

then u satisfies the equation of minimal surfaces

$$\begin{cases} -\operatorname{div}\left(\dfrac{1}{\sqrt{1+|\nabla v|^2}}\nabla v\right) = f \quad \text{in} \quad \mathcal{D}'(\Omega), \\ u \in W_0^{1,1}(\Omega). \end{cases}$$

In this example we have some difficulties related to the fact that $W_0^{1,1}(\Omega)$ is not reflexive. Moreover, Poincaré's inequality does not hold in this space. However, if $f = 0$, it is clear that J is coercive in $W_0^{1,1}(\Omega)$.

3.3 Monotone operators

Let E be a real Banach space and E' its topological dual space.

Definition 3.6. An operator $A : E \to E'$ is called *monotone* if

$$\langle A(u) - A(v), u - v\rangle \geq 0, \quad \forall u, v \in E.$$

One says that A is *strictly monotone* if

$$\langle A(u) - A(v), u - v\rangle > 0, \quad \forall u, v \in E, u \neq v.$$

As simple examples of strictly monotone operators, we have $A = -\Delta_p$ and $E = W_0^{1,p}(\Omega)$, for $1 < p < +\infty$ and Ω a bounded domain of $\mathbb{R}^N$.

Proposition 3.7. *If $J : E \to \mathbb{R}$ is a convex functional which is Gâteaux differentiable, then its derivative $u \mapsto D_G J(u)$ is a monotone operator from E to E'.*

Proof. For every $v, w \in E$ and for every $t \in [0, 1]$, we have from convexity

$$J\big(v + t(w - v)\big) \leq J(v) + t\big[J(w) - J(v)\big]$$

which implies that

$$\frac{J\big(v + t(w - v)\big) - J(v)}{t} \leq J(w) - J(v).$$

Passing to the limit as $t \to 0$, we obtain that

$$\langle D_G J(v), w - v \rangle \leq J(w) - J(v).$$

By the same way, we obtain

$$\langle D_G J(w), v - w \rangle \leq J(v) - J(w),$$

and we get by addition

$$\langle D_G J(v) - D_G J(w), v - w \rangle \geq 0$$

as we wanted to prove. □

The former result enables us to solve equations of the form

$$\begin{cases} A(u) = 0, \\ u \in E, \end{cases}$$

if A is the Gateaux-derivative of a coercive convex functional and E is a reflexive Banach space.

Unfortunately, there exist monotone operators which are not derivative of convex functionals. As examples, we can mention the following defined in $H_0^1(\Omega)$, where Ω is a bounded domain of $\mathbb{R}^N$:

$$Au = -\operatorname{div}(M\nabla u), \quad M \text{ is a non symmetric positive matrix},$$

and also

$$Au = -\Delta u + b \cdot \nabla u, \quad b \in \mathbb{R}^N.$$

Definition 3.8. Let $A : E \to E'$ be an operator.

(1) We say that A is *bounded* if it maps bounded sets of E into bounded sets of E'.

(2) It is said to be *hemicontinuous* if, for every $u, v, w \in E$, the real-valued function $t \in \mathbb{R} \mapsto \langle A(u + tv), w \rangle \in \mathbb{R}$ is continuous on $\mathbb{R}$.

Theorem 3.9 (Main Theorem on Monotone Operators)**.** *Let E be a reflexive and separable Banach space and let $A : E \to E'$ be a monotone, bounded and hemicontinuous operator which is coercive in the following sense*

$$\lim_{\|v\|_E \to \infty} \frac{\langle A(v), v \rangle}{\|v\|} = +\infty.$$

Then A is surjective, i.e., for every $f \in E'$, there exists $u \in E$ such that $A(u) = f$. Moreover, if A is strictly monotone, the solution u is unique.

For the proof of Theorem 3.9, we need to use the following lemma.

Lemma 3.10. *Let $P : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous mapping satisfying the following property: there exists $\rho > 0$ such that, for every $\xi \in \mathbb{R}^m$ with $|\xi| = \rho$, $P(\xi) \cdot \xi \geq 0$. Then there exists $\xi_0 \in \mathbb{R}^m$, $|\xi_0| \leq \rho$, such that $P(\xi_0) = 0$.*

Proof of Lemma 3.10. Let $B_\rho = \big\{\xi \in \mathbb{R}^m,\ |\xi| \leq \rho\big\}$. Suppose that $P(\xi) = 0$ has no solution in B_ρ. Then,

$$\xi \in B_\rho \mapsto -\frac{P(\xi)}{|P(\xi)|}\rho$$

maps B_ρ into B_ρ and it is continuous.

From Brouwer's fixed point theorem, this mapping has a fixed point $\xi^* \in B_\rho$, i.e.,

$$\xi^* = -\frac{P(\xi^*)}{|P(\xi^*)|}\rho.$$

Therefore, $|\xi^*| = \rho$ and $P(\xi^*) \cdot \xi^* = -\rho|P(\xi^*)| < 0$, which is a contradiction. □

Proof of Theorem 3.9. Since E is separable, there exists a countable set of linearly independent functions $\{w_n\}_{n\in\mathbb{N}}$ such that

$$E_m := \operatorname{span}\{w_1, \ldots, w_m\} \quad \text{and} \quad E = \overline{\bigcup_m E_m}.$$

We proceed in two steps.

Step 1 - Approximation: For each $m \in \mathbb{N}$, we look for a solution $u \in E_m$ of the following system of m nonlinear equations in m unknowns,

$$\langle A(u), w_j\rangle = \langle f, w_j\rangle, \quad j = 1, \ldots, m. \tag{3.3}$$

For each $v \in E_m$, there exists $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}_m$ such that $v = \sum_{i=1}^m \xi_i w_i$. So, we can consider the mapping $P : \mathbb{R}^m \to \mathbb{R}^m$, $P(\xi) = \big(P(\xi)_1, \ldots, P(\xi)_m\big)$, defined as

$$P(\xi)_j := \big\langle A(v), w_j\big\rangle - \langle f, w_j\rangle, \quad j = 1, \ldots, m.$$

Then, from coercivity we have

$$P(\xi) \cdot \xi = \big\langle A(v), v\big\rangle - \langle f, v\rangle \geq \|v\|_E \left[\frac{\big\langle A(v), v\big\rangle}{\|v\|} - \|f\|_{E'}\right] > 0,$$

if $\|v\|_E$ is large enough.

Moreover, the mapping P is continuous in $\mathbb{R}^m$. Indeed, it suffices to show that $v \in E_m \mapsto \big\langle A(v), w_j \big\rangle$ is continuous in E_m for each $j = 1, \ldots, m$.

Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence in E_m such that v_n converges to $v \in E_m$. Then, $\{v_n\}_{n\in\mathbb{N}}$ is bounded and therefore $\{A(v_n)\}_{n\in\mathbb{N}}$ is bounded in E', which implies that $\big\{\big\langle A(v_n), w_j \big\rangle\big\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$.

So, we can extract a subsequence still denoted by $\big\{\big\langle A(v_n), w_j \big\rangle\big\}_{n\in\mathbb{N}}$ such that

$$\big\langle A(v_n), w_j \big\rangle \underset{n\to+\infty}{\longrightarrow} \mathcal{X}_j, \quad \forall j = 1, \ldots, m$$

and to prove the continuity, we just have to show that $\mathcal{X}_j = \big\langle A(v), w_j \big\rangle$.

By hypothesis, $v_n = \sum_{j=1}^m \xi_j^n w_j$ and $v = \sum_{j=1}^m \nu_j w_j$, where $\xi_j^n \to \nu_j$ as $n \to +\infty$, for all $j = 1, \ldots, m$. Then,

$$\big\langle A(v_n), v_n \big\rangle = \sum_{j=1}^m \xi_j^n \big\langle A(v_n), w_j \big\rangle \underset{n\to+\infty}{\longrightarrow} \sum_{j=1}^m \nu_j \mathcal{X}_j.$$

Now, if $w \in E_m$, $w := \sum_{j=1}^m \alpha_j w_j$, we also have

$$\big\langle A(v_n), w \big\rangle = \sum_{j=1}^m \alpha_j \big\langle A(v_n), w_j \big\rangle \underset{n\to+\infty}{\longrightarrow} \sum_{j=1}^m \alpha_j \mathcal{X}_j.$$

As we are assuming that A is monotone, we have

$$0 \leq \big\langle A(v_n) - A(w), v_n - w \big\rangle = \sum_{j=1}^m (\xi_j^n - \alpha_j) \big\langle A(v_n), w_j \big\rangle - \big\langle A(w), v_n - w \big\rangle$$

$$\underset{n\to+\infty}{\longrightarrow} \sum_{j=1}^m (\nu_j - \alpha_j) \mathcal{X}_j - \big\langle A(w), v - w \big\rangle,$$

so that

$$\sum_{j=1}^m (\nu_j - \alpha_j) \mathcal{X}_j - \big\langle A(w), v - w \big\rangle \geq 0.$$

Let us consider $w := v + tu$, with $t > 0$ and $u \in E_m$, $u := \sum_{j=1}^m \delta_j w_j$. Then,

$$-t \sum_{j=1}^m \delta_j \mathcal{X}_j + t \big\langle A(v + tu)), u \big\rangle \geq 0.$$

Dividing by t and letting $t \to 0^+$, we obtain from the fact that A is hemi-continuous,

$$\sum_{j=1}^m \big(\mathcal{X}_j - \big\langle A(v), w_j \big\rangle\big) \, \delta_j \leq 0.$$

Since the choice of δ_j, $j = 1, \dots, m$ is arbitrary, we conclude that $\mathcal{X}_j = \big\langle A(v), w_j \big\rangle$, which proves that $v \in E_m \mapsto \big\langle A(v), w_j \big\rangle$ is continuous.

As consequence of Lemma 3.10, there exists $\xi^* \in \mathbb{R}^m$ such that $P(\xi^*) = 0$. This means that, if we define $u_m \in E_m$ by $u_m := \sum_{j=1}^m \xi_j^* w_j$, we have

$$\big\langle A(u_m), w_j \big\rangle - \langle f, w_j \rangle = 0, \quad j = 1, \dots, m.$$

which means that u_m is a required solution of (3.3).

Step 2 - Passage to the limit: Since we have

$$\big\langle A(u_m), u_m \big\rangle = \langle f, u_m \rangle \le \|f\|_{E'} \|u_m\|_E,$$

it follows from the coercivity of A that there exist $C > 0$ such that $\|u_m\|_E \le C$. Hence, $\|A(u_m)\|_{E'} \le C'$ for some $C' > 0$.

Therefore, we can extract a subsequence (still denoted with m as index) such that

$$\begin{aligned} u_m &\rightharpoonup u \quad \text{in } E \text{ weakly}, \\ A(u_m) &\rightharpoonup \mathcal{X} \quad \text{in } E' \text{ weakly}, \end{aligned}$$

For any $j \in \mathbb{N}$ and for $m \ge j$ we have $\big\langle A(u_m), w_j \big\rangle = \langle f, w_j \rangle$. Letting $m \to +\infty$, we have $\big\langle \mathcal{X}, w_j \big\rangle = \langle f, w_j \rangle$, for all $j \in \mathbb{N}$, from which we conclude that $\mathcal{X} = f$ in E', because $E = \overline{\bigcup_m E_m}$.

To finish the proof, we have to show that $\mathcal{X} = A(u)$. We know that

$$\begin{cases} \big\langle A(u_m), u_m \big\rangle = \langle f, u_m \rangle \underset{m \to +\infty}{\to} \langle f, u \rangle = \langle \mathcal{X}, u \rangle, \\ \big\langle A(u_m) - A(v), u_m - v \big\rangle \ge 0 \Rightarrow \big\langle \mathcal{X} - A(v), u - v \big\rangle \ge 0, \quad \forall v \in E. \end{cases}$$

So, by taking $v := u - tw$, with $t > 0$ and $w \in E$, we get

$$t \big\langle \mathcal{X} - A(u - tw), w \big\rangle \ge 0, \quad \forall w \in E, \ \forall t > 0.$$

Dividing by t and letting $t \to 0^+$, we obtain

$$\big\langle \mathcal{X} - A(u), w \big\rangle \ge 0, \quad \forall w \in E,$$

which means that $\mathcal{X} = A(u)$ and the proof is finished. □

Remark 3.11. If A is strictly monotone, we have uniqueness of the solution $A(u) = f$ in the previous result.

Exercise 3.1. Show that the proof of Theorem 3.9 works if we replace monotonicity by the following property: *if $u_m \rightharpoonup u$ weakly in E and $A(u_m) \rightharpoonup \mathcal{X}$ weakly in E' with $\limsup_{m \to +\infty} \langle A(u_m), u_m \rangle \le \langle \mathcal{X}, u \rangle$, then $\mathcal{X} = A(u)$ in E'.*

Remark 3.12. There are a lot of generalizations of the previous result for nondifferentiable convex functionals, subdifferentiable functionals, maximal monotone operators , etc., that will not be considered in these notes. But, as an example, we only mention the case of pseudomonotone operators.

Definition 3.13. An operator $A : E \to E'$ is said to be *pseudomonotone* if A is bounded and satisfies the following property: *if* $u_m \rightharpoonup u$ *in* E *weakly and*

$$\limsup_{m\to+\infty}\langle A(u_m), u_m - u\rangle \le 0,$$

then

$$\langle A(u), u - v\rangle \le \liminf_{m\to+\infty}\langle A(u_m), u_m - v\rangle, \quad \forall v \in E.$$

We can show that if A is *pseudomonotone* and E is reflexive, then A is continuous from E with the strong topology into E' endowed with the weak topology. Moreover, we can also prove the following result.

Theorem 3.14 (Main Theorem on Pseudomonotone Operators)**.** *Let E be a reflexive Banach space and $A : E \to E'$ be a pseudomonotone coercive operator. Then A is surjective, i.e., for every $f \in E'$, there exists $u \in E$ such that $A(u) = f$.*

Chapter 4

Some semilinear non monotone equations

In this case of non monotone equations, there is no general theory, but we have some methods to solve a few examples of semilinear equations, as we will see in this chapter.

4.1 Methods based on maximum principle

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ and consider the bilinear form on $H_0^1(\Omega)$,

$$a(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_j}(x) \frac{\partial v}{\partial x_i}(x)\, dx + \int_\Omega c(x)u(x)v(x)\, dx,$$

where $a_{ij} \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$ and

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_j\xi_i \geq \alpha|\xi|^2, \quad \alpha > 0, \quad \forall \xi \in \mathbb{R}^N, \quad \text{a.e. in } \Omega.$$

Then, for $c_0 > \|c\|_{L^\infty}$ we have, for every $v \in H_0^1(\Omega)$,

$$a(v,v) + c_0\|v\|^2_{L^2(\Omega)} \geq \alpha\|v\|^2_{H_0^1(\Omega)}.$$

As usual, we denote by A the corresponding differential operator

$$A := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu.$$

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a *Caratheodory function*, i.e., a function satisfying the following properties:

$$\begin{cases} \text{for a.e. } x \in \Omega, \text{ the mapping } \ s \mapsto f(x,s) \text{ is continuous;} \\ \forall s \in \mathbb{R}, \text{ the mapping } \ x \mapsto f(x,s) \text{ is measurable.} \end{cases}$$

We are interested in the following boundary value problem

$$\begin{cases} Au + f(\cdot, u) = 0 \ \text{ in } \ \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{4.1}$$

To simplify the notation, we will write in the sequel $f(u)$ instead of $f(\cdot, u)$, or to be more precise, $f(u)(x) := f\big(x, u(x)\big)$.

It is noteworthy to remark that the former problem in not necessarily homogeneous as f may depend explicitly on x. For exemple, $f(u)(x) := |u(x)|^2 u(x) + g(x)$.

Definition 4.1. We say that φ is a subsolution of problem (4.1) if $\varphi \in H^1(\Omega)$, $f(\varphi) \in L^2(\Omega)$ and

$$\begin{cases} A\varphi + f(\varphi) \le 0 \ \text{ in } \ H^{-1}(\Omega), \\ \varphi \le 0 \ \text{ on } \ \Gamma. \end{cases}$$

Definition 4.2. We say that ψ is a supersolution of problem (4.1) if $\psi \in H^1(\Omega)$, $f(\psi) \in L^2(\Omega)$ and

$$\begin{cases} A\psi + f(\psi) \ge 0 \ \text{ in } \ H^{-1}(\Omega), \\ \psi \ge 0 \ \text{ on } \ \Gamma. \end{cases}$$

It is clear that $\varphi \in H^1(\Omega)$ is a subsolution of the problem (4.1) if $\varphi|_\Gamma \le 0$ and

$$a(\varphi, v) + \int_\Omega f(\varphi)(x) v(x)\, dx \le 0, \quad \forall v \in H_0^1(\Omega), \ v \ge 0.$$

and the analogous if ψ is a supersolution.

4.1.1 Existence Result

Sub and supersolutions, together with the classical maximum principle, are powerful tools to prove existence results for semilinear non monotone boundary value problems. This is that asserts the following theorem that we will prove now.

Theorem 4.3. *Let φ and ψ be respectively a subsolution and a supersolution of (4.1) with $\varphi \le \psi$ a.e. in Ω, and f be a Caratheodory function. Let $m_0 = \operatorname{ess\,inf} \varphi$ and $m_1 = \operatorname{ess\,sup} \psi$ (not necessarily finite). We assume that there exists $\mu \ge c_0$ such that the mapping $s \mapsto f(\cdot, s) - \mu s$ is decreasing for $m_0 < s < m_1$, a.e. in Ω. Then there exists a solution u of (4.1) with*

$$\varphi \le u \le \psi \quad \text{a.e. in } \Omega.$$

Moreover, there exist a minimal solution $\underline{u}$ *and a maximal solution* $\overline{u}$, $\underline{u} \le \overline{u}$, *such that*

$$\varphi \le \underline{u} \le \bar{u} \le \psi \quad \text{a.e. in } \Omega,$$

i.e., if u *is a solution of* (4.1) *with* $\varphi \le u \le \psi$, *a.e. in* Ω, *then we have*

$$\varphi \le \underline{u} \le u \le \overline{u} \le \psi \quad \text{a.e. in } \Omega.$$

Proof. If $\mu \ge c_0$, we know that $A + \mu I$ is coercive. Then, let us consider the following iterative scheme: we take $u_0 = \varphi$, $v_0 = \psi$ and define sequences $\{u_n\}_{n\ge 0}$ and $\{v_n\}_{n\ge 0}$ in $H_0^1(\Omega)$ by

$$\begin{cases} Au_{n+1} + \mu u_{n+1} + f(u_n) - \mu u_n = 0, \\ Av_{n+1} + \mu v_{n+1} + f(v_n) - \mu v_n = 0, \\ u_{n+1}, v_{n+1} \in H_0^1(\Omega). \end{cases} \tag{4.2}$$

Then, we have the following properties.

$$\begin{cases} Au_1 + \mu u_1 + f(\varphi) - \mu\varphi = 0, \\ A\varphi + \mu\varphi + f(\varphi) - \mu\varphi \le 0, \\ u_1|_\Gamma = 0, \;\; \varphi|_\Gamma \le 0. \end{cases} \qquad \begin{cases} Av_1 + \mu v_1 + f(\psi) - \mu\psi = 0, \\ A\psi + \mu\psi + f(\psi) - \mu\psi \ge 0, \\ v_1|_\Gamma = 0, \;\; \psi|_\Gamma \ge 0, \end{cases}$$

From the weak maximum principle, we have $u_1 \ge \varphi$ and $v_1 \le \psi$ a.e. in Ω.

Notice that we have $\varphi \le \psi$, from which we can write

$$f(\varphi) - \mu\varphi \ge f(\psi) - \mu\psi \quad \Rightarrow \quad \begin{cases} Au_1 + \mu u_1 \le Av_1 + \mu v_1, \\ u_1|_\Gamma = 0, \;\; v_1|_\Gamma = 0. \end{cases}$$

and we obtain $u_1 \le v_1$ a.e. in Ω.

Arguing by induction, let us assume that we have constructed $u_1, \dots, u_n$ and $v_1, \dots, v_n$ such that

$$\varphi \le u_1 \le \cdots \le u_n \le v_n \le \cdots \le v_1 \le \psi, \quad \text{a.e. in } \Omega.$$

Since $u_{n-1} \le u_n$ implies that $f(u_n) - \mu u_n \le f(u_{n-1}) - \mu u_{n-1}$ and since

$$\begin{cases} Au_{n+1} + \mu u_{n+1} + f(u_n) - \mu u_n = 0, \\ Au_n + \mu u_n + f(u_{n-1}) - \mu u_{n-1} = 0, \\ u_{n+1}, u_n \in H_0^1(\Omega). \end{cases}$$

we obtain as before, $u_{n+1} \ge u_n$ a.e. in Ω. By the same way we get $v_{n+1} \le v_n$ and the same arguments give

$$u_n \le v_n \quad \Rightarrow \quad f(v_n) - \mu v_n \le f(u_n) - \mu u_n \quad \Rightarrow \quad u_{n+1} \le v_{n+1}.$$

So, by induction, we have constructed an increasing sequence $\{u_n\}_{n\in\mathbb{N}}$ and a decreasing sequence $\{v_n\}_{n\in\mathbb{N}}$ such that

$$\varphi \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq \psi, \quad \text{a.e. in } \Omega.$$

Claim 1 : There exist $\underline{u}$ and $\overline{u}$ in $L^2(\Omega)$ such that $\varphi \leq \underline{u} \leq \overline{u} \leq \psi$ a.e., in Ω and

$$u_n \underset{n\to+\infty}{\longrightarrow} \underline{u}, \quad v_n \underset{n\to+\infty}{\longrightarrow} \overline{u} \quad \text{in} \quad L^2(\Omega). \tag{4.3}$$

In fact, there exists $Z \subset \Omega$ with $\text{meas}(Z) = 0$ such that, for $x \in \Omega \setminus Z$,

$$\varphi(x) \leq u_1(x) \leq \cdots \leq u_n(x) \leq \cdots \leq v_n(x) \leq \cdots \leq v_1(x) \leq \psi(x).$$

Therefore, $\underline{u}(x) := \lim_{n\to+\infty} u_n(x)$ and $\overline{u}(x) := \lim_{n\to+\infty} v_n(x)$ satisfy $\varphi(x) \leq \underline{u}(x) \leq \overline{u}(x) \leq \psi(x)$. This means that we can define the functions $\underline{u}, \overline{u} : \Omega \to \mathbb{R}$, which are the pointwise limits of the respective sequences. In order to prove (4.3), it suffices to notice that we have $\varphi, \psi \in L^2(\Omega)$ and $\varphi \leq u_n \leq v_n \leq \psi$ a.e. Therefore, the result follows from Lebesgue Theorem. Moreover, because $u_n \leq v_n$ for all $n \in \mathbb{N}$, we have that $\underline{u} \leq \overline{u}$ and the Claim 1 is proved.

Now, we consider the operator $F : X \to L^2(\Omega)$, where $X = \big\{w \in L^2(\Omega)\,;\, \varphi \leq w \leq \psi\big\}$, defined by $F = f - \mu I$, or more precisely, $F(w) = f(w) - \mu w$ for all $w \in X$. Then, we have,

Claim 2 : F is continuous.

Let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence of X such that $w_n \to w$ in $L^2(\Omega)$. Passing to a subsequence if necessary, we have

$$\begin{cases} w_n \to w \ \text{ a.e. in } \Omega, \\ F(w_n) \to F(w) \ \text{ a.e. in } \Omega, \end{cases}$$

But $F(\psi) \leq F(w_n) \leq F(\varphi)$ and the Lebesgue Theorem implies that $F(w_n) \to F(w)$ in $L^2(\Omega)$. So, the proof of Claim 2 is finished.

Since $f = F + \mu I$, f is continuous as an operator from X into $L^2(\Omega)$. In particular, we have

$$\begin{cases} f(u_n) \to f(\underline{u}) \\ f(v_n) \to f(\overline{u}) \end{cases} \text{in} \quad L^2(\Omega). \tag{4.4}$$

From (4.4) and (4.2), we have

$$\begin{cases} u_n \to \underline{u} \\ v_n \to \overline{u} \end{cases} \text{in} \quad H_0^1(\Omega). \tag{4.5}$$

So, passing to the limit as $n \to +\infty$ in (4.2), we get

$$A\underline{u} + f(\underline{u}) = 0, \quad \text{and} \quad A\overline{u} + f(\overline{u}) = 0.$$

Finally, let u be a solution of (4.1) such that $\varphi \le u \le \psi$. Then, by the maximum principle, $u \ge u_1$ (respectively $u \le v_1$) and, by the same argument, $u \ge u_2$ (respectively $u \le v_2$), and so on. Hence, $\underline{u} \le u \le \overline{u}$. $\square$

The fact that avery solution u belonging to X satisfies necessarily the condition $\underline{u} \le u \le \overline{u}$ means that $\underline{u}$ and $\overline{u}$ are the "minimal" and "maximal" solutions, respectively; or more exactly, the smallest and largest solutions in X.

We may have $\underline{u} = \overline{u}$ and, in that case, there is uniqueness of solution in X. We note also that the argument in the proof is constructive (and could be implemented in numerical calculations).

Remark 4.4. Let us mention a very general result which includes the previous theorem. First of all, notice that in Theorem 4.3, if we define

$$T(v) := (A + \mu I)^{-1}\big[\mu v - f(v)\big],$$

then T is *increasing* in the sense that $v \le w$ implies that $T(v) \le T(w)$ and

$$\varphi \le T(\varphi), \quad \psi \ge T(\psi), \quad \varphi \le \psi.$$

We consider H an ordered space in which the order has the following property P: *every non empty family that is totally ordered and bounded from above (respectively bounded from below) admits a least upper bound (respectively a largest lower bound).*

Theorem 4.5. *Let $T : H \to H$ be an increasing mapping, where H is an ordered space satisfying the property P. Assume that T has a lower fixed point φ, i.e., $\varphi \le T(\varphi)$, and an upper fixed point ψ, i.e., $\psi \ge T(\psi)$ such that $\varphi \le \psi$. Then T has a minimal fixed point $\underline{u}$ and a maximal fixed point $\overline{u}$ in the set $X := \big\{w \in H\,;\, \varphi \le w \le \psi\big\}$.*

Before proceeding with the proof, it is interesting to remark that no continuity and no topology is assumed in Theorem 4.5. Moreover, concerning the Remark 4.4, we notice that the usual order in the space $H = L^2(\Omega)$ has the mentioned property P.

Proof of Theorem 4.5. Let $\mathcal{U}$ and $\mathcal{V}$ be the following sets:

$$\mathcal{U} := \big\{u \in H\,;\, \varphi \le u \le \psi,\ u \le T(u)\big\}$$
$$\mathcal{V} := \big\{v \in H\,;\, \varphi \le v \le \psi,\ T(v) \le v\big\}.$$

Note that $\mathcal{U} \ne \emptyset$ and $\mathcal{V} \ne \emptyset$, because $\varphi \in \mathcal{U}$ and $\psi \in \mathcal{V}$.

Now, let $\mathcal{W}$ be the following set

$$\mathcal{W} := \big\{ v \in \mathcal{V} \,;\; v \geq u,\, \forall u \in \mathcal{U} \big\}.$$

Again, $\mathcal{W} \neq \emptyset$, because $\psi \in \mathcal{W}$.

We claim that $\mathcal{W}$ is *inductive*, i.e., each non empty totally ordered set $B \subset \mathcal{W}$ has a largest lower bound which belongs to $\mathcal{W}$.

Let $\{w_\alpha\}$ be a non empty family of $\mathcal{W}$ which is totally ordered. Of course, this family is bounded from below by φ and then, it possesses a larger lower bound $\underline{w} \in H$, i.e.,

$$\underline{w} \leq w_\alpha, \quad \forall \alpha. \tag{4.6}$$

As $\mathcal{W} \subset \mathcal{V}$, we have

$$T(w_\alpha) \leq w_\alpha, \quad \forall \alpha. \tag{4.7}$$

From the fact that T is increasing, it follows from (4.6) that $T(\underline{w}) \leq T(w_\alpha)$ for all α. So, $T(\underline{w})$ is a lower bound of $\{w_\alpha\}$ and we have $T(\underline{w}) \leq \underline{w}$. This means that $\underline{w} \in \mathcal{V}$.

On the other hand, we know that if $u \in \mathcal{U}$, then $u \leq w_\alpha$ for every α, which implies that $u \leq \underline{w}$. So, we have shown that

$$\underline{w} \in \mathcal{V} \quad \text{and} \quad u \leq \underline{w},\ \forall u \in \mathcal{U}.$$

i.e., $\underline{w} \in \mathcal{W}$ and so $\mathcal{W}$ is inductive.

Hence, from Zorn's Lemma, $\mathcal{W}$ possesses a minimal element w_0.

As $\mathcal{W} \subset \mathcal{V}$ and T is increasing, we have

$$\begin{cases} w_0 \in \mathcal{V} \;\Rightarrow\; T(w_0) \leq w_0, \\ w_0 \geq \varphi \;\Rightarrow\; T(w_0) \geq T(\varphi) \geq \varphi, \end{cases} \quad\Longrightarrow\quad \varphi \leq T(w_0) \leq w_0 \leq \psi.$$

Since $T^2(w_0) \leq T(w_0)$, we have

$$T(w_0) \in \mathcal{V}. \tag{4.8}$$

On the other hand, as $w_0 \geq u$ for all $u \in \mathcal{U}$, it follows from the definition of $\mathcal{U}$ that $T(w_0) \geq T(u) \geq u$. So,

$$T(w_0) \geq u,\ \forall u \in \mathcal{U}. \tag{4.9}$$

From (4.8) (4.9), we have $T(w_0) \in \mathcal{W}$. As w_0 is the largest lower bound of $\mathcal{W}$, we have necessarily $w_0 \leq T(w_0)$. So, we conclude that $T(w_0) = w_0$.

Moreover, if u is a fixed point of T, i.e., $u = T(u)$, then $u \in \mathcal{U}$ and so $u \leq w_0$. This means that w_0 is the largest fixed point of T.

With the same arguments we can prove that T possesses a smallest fixed point, and the proof is finished. □

Remark 4.6. In the last two results, the subsolution (or the lower fixed point) φ must be smaller then the supersolution (or the upper fixed point) ψ. This is essential, as we can see by the following counter example.

We consider the spectral problem

$$\begin{cases} -\phi'' - \lambda\phi = 0 \text{ in }]0,1[, \\ \phi(0) = \phi(1) = 0. \end{cases}$$

It is well known that all eigenvalues are positive and simple, i.e., $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$, where $\lambda_k = k^2\pi^2$ with the corresponding eigenfunctions given by $\phi_k(x) = \sin(k\pi x)$.

Let us consider the problem

$$\begin{cases} -u'' - \lambda_2 u = f \text{ in }]0,1[, \\ u(0) = u(1) = 0, \end{cases} \tag{4.10}$$

where $f \in C^1\big([0,1]\big)$ with $f(0) = f(1) = 0$. It is clear that there exists $\alpha > 0$ such that

$$-\alpha(\lambda_2 - \lambda_1)\phi_1 \le f \le \alpha(\lambda_2 - \lambda_1)\phi_1.$$

So, by considering $\varphi_\alpha := \alpha\phi_1$ and $\psi_\alpha := -\varphi_\alpha$, we see that

$$\begin{cases} -\varphi_\alpha'' - \lambda_2\varphi_\alpha f = -\alpha(\lambda_2 - \lambda_1)\phi_1 - f \le 0 \text{ in }]0,1[, \\ \varphi_\alpha(0) = \varphi_\alpha(1) = 0, \end{cases}$$

which says that φ_α is a subsolution for the problem (4.10). By the same way we can see that ψ_α is a supersolution for (4.10).

Moreover, it is clear that

$$\psi_\alpha(x) \le 0 \le \varphi_\alpha(x), \quad \forall x \in [0,1]$$

and if we take f such that

$$\int_0^1 f(x)\phi_2(x)\,dx \neq 0,$$

the problem does not have solution, as we can see by multiplying both sides of (4.10) by ϕ_2 and taking the integral on $[0,1]$.

4.1.2 Example

As an application of the Theorem 4.3, let us consider the following example. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the (semilinear) boundary value problem.

$$\begin{cases} -\Delta u + u^3 - \lambda u = 0 \text{ in } \Omega, \ \lambda > 0, \\ u \in H_0^1(\Omega). \end{cases} \tag{4.11}$$

It is clear that, for all $\lambda \in \mathbb{R}$ the null function is a solution. But for $\lambda \leq \lambda_1$, where λ_1 is the first eigenfuncion of $-\Delta$ in Ω, the null function is the unique solution. Indeed,

$$\langle -\Delta u - \lambda u, u\rangle + \int_\Omega u^4(x)\, dx = \int_\Omega \left(|\nabla u(x)|^2 - \lambda |u(x)|^2 + |u(x)|^4\right)\, dx = 0.$$

But if $\lambda \leq \lambda_1$, it follows from the variational characterization of λ_1 that

$$\int_\Omega \left(|\nabla u(x)|^2 - \lambda |u(x)|^2\right)\, dx \geq 0,$$

which implies that $\|u\|_L^4(\Omega) = 0$ and $u = 0$.

For $\lambda > \lambda_1$, let ϕ_1 be the (positive) eigenfunction associated to λ_1, i.e.,

$$\begin{cases} -\Delta \phi_1 = \lambda_1 \phi_1 \text{ in } \Omega, \\ \phi_1 \in H_0^1(\Omega), \ \phi_1 > 0 \text{ in } \Omega. \end{cases}$$

We assume that $\sup\{\phi_1(x)\,;\, x \in \Omega\} = 1$ and we define $\varphi_\alpha := \alpha\phi_1$, $\alpha > 0$. Then,

$$-\Delta\varphi_\alpha + \varphi_\alpha^3 - \lambda\varphi_\alpha = \alpha\lambda_1\phi_1 - \alpha\lambda\phi_1 + \alpha^3\phi_1^3 = \alpha\phi_1\left[(\lambda_1 - \lambda) + \alpha^2\phi_1^2\right].$$

If $\alpha \leq \sqrt{\lambda - \lambda_1}$, we have that φ_α is a subsolution. On the other hand, $\psi := \sqrt{\lambda}$ is a supersolution and

$$\varphi_\alpha(x) = \alpha\phi_1(x) \leq \sqrt{\lambda - \lambda_1} \leq \sqrt{\lambda} = \psi(x), \quad \forall x \in \Omega.$$

From Theorem 4.3, there exists a solution u of (4.11) such that $\sqrt{\lambda - \lambda_1}\phi_1 \leq u \leq \sqrt{\lambda}$. So, this problem admits also a positive solution u.

Remark 4.7. In the former example, we have $m_0 = 0$ and $m_1 = \sqrt{\lambda}$. Therefore, to have $s \mapsto s^3 - \lambda s - \mu s$ decreasing in $[0, \sqrt{\lambda}]$, it suffices to take $0 < \mu < 2\lambda$.

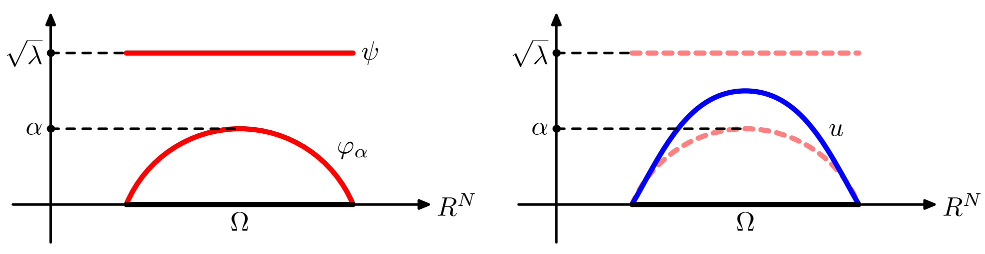

Figure 4.1. At left, the graphics of subsolution ψ and supersolution φ_α; at right, a solution u such that $\psi \leq u \leq \varphi_\alpha$.

4.1.3 The symmetric case; more properties

We return to the problem (4.1) assuming the hypothesis of Theorem 4.3, but now with A a symmetric operator, i.e.,

$$Au = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\big(a_{ij}\frac{\partial u}{\partial x_j}\big) + cu,$$

where $a_{ij} = a_{ji}$. In this case, the associated bilinear form $a(u, v)$ is also symmetric. By introducing the parameter μ if necessary, we can assume that A is coercive and $s \mapsto f(\cdot, s)$ is decreasing in $[m_0, m_1]$.

We define

$$G(x,s) := \int_0^s f(x,\tau)\,d\tau, \quad K := \big\{v \in\ L^2(\Omega)\,;\, \varphi \le v \le \psi \ \text{ a.e. in } \Omega\big\}.$$

Note that

$$\big|G(\cdot,s)\big| \le \max\big[f(\varphi), f(\psi)\big]s, \quad \forall s \in [m_0, m_1], \quad \text{a.e. in } \Omega.$$

So, with the notation $G(v)(x) := G\big(x, v(x)\big)$, we have

$$\big|G(v)\big| \le \max\big[f(\varphi), f(\psi)\big]|v| \ \text{ a.e. in } \Omega,$$

which implies that $G(v) \in L^1(\Omega)$ if $v \in L^2(\Omega)$, because the function $\max\big[f(\varphi), f(\psi)\big] \in L^2(\Omega)$.

Let us consider the functional

$$J(v) := \frac{1}{2}a(v,v) + \int_\Omega G(v)(x)\,dx,$$

which is well defined for $v \in H_0^1(\Omega) \cap K$.

We claim that J is Gateaux-differentable in $K \cap H_0^1(\Omega)$. Indeed, since the mapping $s \mapsto f(x, s)$ is continuous for almost $x \in \Omega$, then $s \mapsto G(x, s)$ is differentiable and

$$\lim_{\substack{t\downarrow 0\\ 0<t<1}} \frac{G\big(u+tw\big)(x) - G(u)(x)}{t} = f\big(x, u(x)\big), \qquad \text{a.e. } x \in \Omega.$$

Since K is convex, $u + t(v-u) \in K$ if $u, v \in K$ and $0 \le t \le 1$. So, for almost every $x \in \Omega$,

$$\begin{aligned}\left|\frac{G\big(u+t(v-u)\big)(x) - G(u)(x)}{t}\right| &\le \frac{1}{t}\left|\int_{u(x)}^{u(x)+t(v(x)-u(x))} f(x,s)\,ds\right| \\ &\le \frac{1}{t}\int_{u(x)}^{u(x)+t(v(x)-u(x))} \big|f(x,s)\big|\,ds \\ &\le \max\big[|f(x,\varphi(x)|, |f(x,\psi(x)|\big]|v(x)-u(x)|.\end{aligned}$$

Hence, from the Lebesgue Theorem, we obtain, for all $u, v \in H_0^1(\Omega) \cap K$,

$$\big\langle J'(u), v-u\big\rangle = a(u, v-u) - \big\langle f(u), v-u\big\rangle. \tag{4.12}$$

Let us consider the set

$$K_{\underline{u}} := \big\{v \in L^2(\Omega)\,;\, \varphi \le v \le \underline{u}\big\},$$

where $\underline{u}$ is the minimal solution of (4.1). Then we have the following characterization.

Lemma 4.8. $J(\underline{u}) = \min\limits_{v \in K_{\underline{u}} \cap H_0^1(\Omega)} J(v)$.

Proof. Firstly we notice that $v \mapsto \int_\Omega G(v)(x)\,dx$ is bounded on K, because

$$\left|\int_\Omega G(v)\,dx\right| \le \int_\Omega \max\big\{\big|f(\varphi)\big|, \big|f(\psi)\big|\big\}\max\big\{|\varphi|, |\psi|\big\}\,dx$$

and it is obviously continuous on K with the topology of $L^2(\Omega)$. Since $K \cap H_0^1(\Omega)$ is convex and closed in $H_0^1(\Omega)$, it is also closed for the weak topology of $H_0^1(\Omega)$ and we conclude from the compactness of the injection $H_0^1(\Omega) \to L^2(\Omega)$ that $v \mapsto \int_\Omega G(v)(x)\,dx$ is compact in $K \cap H_0^1(\Omega)$ for the weak topology of $H_0^1(\Omega)$.

As the same properties hold for $K_{\underline{u}} \cap H_0^1(\Omega)$, it follows that J achieves its minimum on this set, i.e., there exists $u \in K_{\underline{u}} \cap H_0^1(\Omega)$ such that

$$J(u) = \min_{v \in K_{\underline{u}} \cap H_0^1(\Omega)} J(v).$$

Therefore, from (4.12), we obtain $\big\langle J'(u), v-u\big\rangle \ge 0$ $\forall v \in K_{\underline{u}} \cap H_0^1(\Omega)$, i.e.,

$$a(u, v-u) + \int_\Omega f\big(x, u(x)\big)\big(v(x) - u(x)\big)\,dx \ge 0, \quad \forall v \in K_{\underline{u}} \cap H_0^1(\Omega)$$

In particular, for $v = \underline{u} \in K_{\underline{u}} \cap H_0^1(\Omega)$, we have

$$a(u, \underline{u} - u) + \int_\Omega f\big(x, u(x)\big)\big(\underline{u}(x) - u(x)\big)\,dx \ge 0. \tag{4.13}$$

But we know that

$$a(\underline{u}, \underline{u} - u) + \int_\Omega f\big(x, \underline{u}(x)\big)\big(\underline{u}(x) - u(x)\big)\,dx = 0. \tag{4.14}$$

As $u \le \underline{u}$, we have $f(\underline{u}) \le f(u)$ and

$$\int_\Omega \big(f(\underline{u}) - f(u)\big)(\underline{u} - u)\,dx \le 0$$

and we obtain by subtracting (4.13) from (4.14), $a(\underline{u} - u, \underline{u} - u) \le 0$ and so $\overline{u} = u$. □

Theorem 4.9. *Consider the hypotheses of Theorem 4.3 and assume that $\varphi \in C^0(\overline{\Omega})$ is a strict subsolution, i.e., a subsolution but not a solution. Suppose that $s \mapsto f(x,s)$ is C^1 a.e. in Ω is such that $f'_s(\underline{u}) \in L^\infty(\Omega)$. Then we have*

$$a(v,v) + \int_\Omega f'(\underline{u}) v^2 \, dx \geq 0, \quad \forall v \in H^1_0(\Omega),$$

which implies that the linearized operator at $\underline{u}$, $A + f'(\underline{u})I$, has a non negative first eigenvalue.

Proof. By hypothesis, we have

$$\begin{cases} Au + f(\underline{u}) = 0, \text{ a.e. in } \Omega \\ A\varphi + f(\varphi) \leq 0, \text{ a.e. in } \Omega \text{ (but not identically 0)} \\ (u - \varphi)|_\Gamma \geq 0, \end{cases}$$

By the strong maximum principle, $\underline{u} > \varphi$ in Ω. So, for every $\phi \in \mathcal{D}^+(\Omega)$, there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then $\underline{u} - \varepsilon\phi \geq \varphi$. Therefore,

$$J(\underline{u}) \leq J(\underline{u} - \varepsilon\phi).$$

On the other hand, we have

$$J(\underline{u} - \varepsilon\phi) = J(\underline{u}) - \varepsilon\big\langle J'(\underline{u}), \phi\big\rangle + \frac{\varepsilon^2}{2}\big\langle J''(\underline{u})\phi, \phi\big\rangle + \mathrm{o}(\varepsilon^2), \quad \forall \phi \in \mathcal{D}^+(\Omega).$$

As we know that $\big\langle J'(\underline{u}), \phi\big\rangle = 0$, it follows that

$$\big\langle J''(\underline{u})\phi, \phi\big\rangle \geq 0, \quad \forall \phi \in \mathcal{D}^+(\Omega).$$

Taking the closure of $\mathcal{D}^+(\Omega)$ in $H^1_0(\Omega)$, we obtain

$$a(v,v) + \int_\Omega f'_s(\underline{u}) v^2 \, dx \geq 0, \quad \forall v \geq 0 \text{ in } H^1_0(\Omega). \tag{4.15}$$

Since we know that $\forall v \in H^1_0(\Omega)$, $v = v^+ - v^-$, with $v^+, v^- \in H^1_0(\Omega)$, we have

$$\begin{cases} a(v^+, v^+) + \displaystyle\int_\Omega f'_s(\underline{u})(v^+)^2 \, dx \geq 0, \\ a(v^-, v^-) + \displaystyle\int_\Omega f'_s(\underline{u})(v^-)^2 \, dx \geq 0, \end{cases}$$

and we conclude that (4.15) holds for all $v \in H^1_0(\Omega)$. This finishes the proof. □

Remark 4.10. After an obvious adaptation of Lemma 4.8, we can prove an analogous result for $\overline{u}$.

4.1.4 Uniqueness results

Theorem 4.11. *Under the hypotheses of Theorem 4.9 and assuming that $\varphi, \psi \in C^0(\overline{\Omega})$ are sub and supersolutions, but not solutions, then the solution u in $K = \{v \in L^2(\Omega),\ \varphi \le v \le \psi \text{ a.e.}\}$ is unique if one of the following conditions is satisfied:*

(1) $s \mapsto f(x,s)$ *is strictly convex on* $[m_0, m_1]$;

(2) $s \mapsto f(x,s)$ *is strictly concave on* $[m_0, m_1]$;

(3) $s \mapsto \frac{\partial f}{\partial s}(x,s)$ *is strictly concave on* $[m_0, m_1]$.

Proof. (1) Since $\langle J'(\underline{u}), \overline{u} - \underline{u}\rangle = \langle J'(\overline{u}), \overline{u} - \underline{u}\rangle = 0$, we have

$$\langle J'(\overline{u}) - J'(\underline{u}), \overline{u} - \underline{u}\rangle = 0.$$

So, if $w = \overline{u} - \underline{u}$,

$$a(w,w) + \int_\Omega \big[f(\overline{u}) - f(\underline{u})\big] w\, dx = 0. \tag{4.16}$$

But we know that (see (4.15))

$$a(w,w) + \int_\Omega \frac{\partial f}{\partial s}(\underline{u}) w^2 \ge 0. \tag{4.17}$$

Subtracting (4.17) from (4.16) we get

$$\int_\Omega \left[f(\overline{u}) - f(\underline{u}) - \frac{\partial f}{\partial s}(\underline{u}) w \right] w\, dx \le 0. \tag{4.18}$$

On the other hand, as $w \ge 0$ and f is convex, we have $f(\overline{u}) - f(\underline{u}) \ge \frac{\partial f}{\partial s}(\underline{u})(\overline{u} - \underline{u})$ a.e. in Ω, which implies that $\big[f(\overline{u}) - f(\underline{u}) - \frac{\partial f}{\partial s}(\underline{u})w\big]w \ge 0$ a.e. in Ω and

$$\int_\Omega \left[f(\overline{u}) - f(\underline{u}) - \frac{\partial f}{\partial s}(\underline{u}) w \right] w\, dx \ge 0. \tag{4.19}$$

From (4.18) from (4.19) we conclude that

$$\int_\Omega \left[f(\overline{u}) - f(\underline{u}) - \frac{\partial f}{\partial s}(\underline{u}) w \right] w\, dx = 0.$$

Bus f is supposed to be strictly convex and so we have $\overline{u} = \underline{u}$.

(2) Repeating the initial arguments of item (1), we obtain the same inequality (4.18). But now, as f is supposed to be concave, we have

$$f(\underline{u}) - f(\overline{u}) \le f'_s(\overline{u})(\underline{u} - \overline{u}) = -\frac{\partial f}{\partial s}(\overline{u}) w \text{ a.e. in } \Omega.$$

So, $f(\overline{u}) - f(\underline{u}) - \frac{\partial f}{\partial s}(\overline{u})w \geq 0$ a.e. in Ω and the result follows as before.
(3) Again we write $w = \overline{u} - \underline{u}$, so that (4.16) holds. Now, as $s \mapsto f(\cdot, s)$ is of class C^1, we have for almost every $x \in \Omega$,

$$f(\overline{u}) - f(\underline{u}) = \int_0^1 \frac{\partial f}{\partial s}(u + tw)w^2\, dt.$$

From Fubini Theorem and the concavity of $\frac{\partial f}{\partial s}$, we obtain

$$\begin{aligned}\int_\Omega \big[f(\overline{u}) - f(\underline{u})\big] w\, dx &= \int_0^1 \left[\int_\Omega \frac{\partial f}{\partial s}(\underline{u} + tw)w^2\, dx\right] dt \\ &\geq \int_0^1 \left[(1-t)\int_\Omega \frac{\partial f}{\partial s}(\underline{u})w^2 dx + t\int_\Omega \frac{\partial f}{\partial s}(\overline{u})w^2 dx\right] dt \\ &= \frac{1}{2}\int_\Omega \frac{\partial f}{\partial s}(\underline{u})w^2 dx + \frac{1}{2}\int_\Omega \frac{\partial f}{\partial s}(\overline{u})w^2 dx.\end{aligned}$$

Adding $a(w,w)$ in both sides of the previous inequality, we get

$$\begin{aligned}0 = a(w,w) + \int_\Omega \big[f(\overline{u}) - f(\underline{u})\big] w\, dx \geq \frac{1}{2}\left[a(w,w) + \int_\Omega \frac{\partial f}{\partial s}(\underline{u})w^2 dx\right] \\ + \frac{1}{2}\left[a(w,w) + \int_\Omega \frac{\partial f}{\partial s}(\overline{u})w^2 dx\right] \geq 0.\end{aligned}$$

Therefore,

$$\int_0^1 \left[\int_\Omega \Big\{\frac{\partial f}{\partial s}(\underline{u} + tw) - (1-t)\frac{\partial f}{\partial s}(\underline{u}) - t\frac{\partial f}{\partial s}(\overline{u})\Big\} w^2\, dx\right] dt = 0$$

and the conclusion follows because $\frac{\partial f}{\partial s}$ is strictly concave. □

Example 4.12. As an application of this last result, let us return to the problem (4.11)

$$\begin{cases} -\Delta u + u^3 - \lambda u = 0 \ \text{ in } \ \Omega, \\ u \in H_0^1(\Omega). \end{cases}$$

We have seen that for $\lambda > \lambda_1$, $\varphi_\alpha ;= \alpha\phi$ is subsolution if $0 < 0 \leq \sqrt{\lambda - \lambda_1}$, $\psi := \sqrt{\lambda}$ is a supersolution and, as a consequence of Theorem 4.3, there exists a positive solution u on $K := \{v \in L^2(\Omega)\,;\, \varphi_\alpha \leq v \leq \psi\}$. But $f(s) := s^2 - \lambda s$ is strictly convex in $\mathbb{R}^+$. Then, this solution is unique in K.

4.2 Variational Methods

In this section we focus on two type of equations in a given Banach space V. The first one is equations of the form

$$u \in V, \quad A(u) = 0,$$

where A is the Fréchet derivative of a C^1 functional $J : V \to R$, i.e., $A(v) = J'(v)$, for all $v \in V$.

We will also consider equations like

$$u \in V, \quad A(u) = \lambda B(u), \quad \lambda \in \mathbb{R},$$

where A and B are respectively Fréchet derivatives of C^1 functionals $J, H : V \to R$, i.e., $A(v) = J'(v)$ and $B(v) = H'(v)$, for all $v \in V$.

4.2.1 Extreme values of functionals on manifolds

Lemma 4.13 (Lagrange multiplier)**.** *Let J and H be two C^1 functionals from V to $\mathbb{R}$. Suppose the v_0 is a extremum of J on the manifold*

$$S := \big\{v \in V\,;\, H(v) = c \in \mathbb{R}\big\}$$

such that $H'(v_0) \neq 0$. Then, there exists $\lambda \in \mathbb{R}$ such that $J'(v_0) = \lambda H'(v_0)$.

Proof. Since H is smooth, we know that $H'(v_0)$ is normal to S_c at v_0, i.e., for every sequence $\{v_n\}_{n\in\mathbb{N}}$, $v_n \in S_c$, $v_n \neq v_0$ such that $v_n \to v_0$, we have

$$\lim_{n\to+\infty} \left\langle H'(v_0), \frac{v_n - v_0}{\|v_n - v_0\|_V} \right\rangle = 0.$$

Let $\Pi(v_0)$ the tangent hyperplane to S_c at v_0, i.e.,

$$\Pi(v_0) := \big\{w \in V\,;\, \big\langle H'(v_0), w\big\rangle = 0\big\}.$$

If $u_0 \in V$ is such that $\big\langle H'(v_0), u_0\big\rangle \neq 0$ and $v \in V$, we define

$$\tau(v) := \frac{\big\langle H'(v_0), v\big\rangle}{\big\langle H'(v_0), u_0\big\rangle}.$$

Then $\big\langle H'(v_0), v - \tau(v)u_0\big\rangle = 0$ and so $v - \tau(v)u_0 \in \Pi(v_0)$, which implies that

$$v = w + \tau(v)u_0, \quad w \in \Pi(v_0).$$

Note that if $v = 0$, then $\tau(v) = 0$ and $w = 0$, i.e., $V = \Pi(v_0) \oplus \mathbb{R}u_0$, or more precisely, $\forall v \in V$, there exist $w \in \Pi(v_0)$ and $\tau \in \mathbb{R}$ unique such that $v = w + \tau u_0$.

We claim that locally, in a neighborhood of v_0, S_c is defined by the equation

$$v = v_0 + w + \tau u_0, \quad \tau = \Phi(w),\ w \in \Pi(v_0).$$

Indeed, let us write

$$F(\tau, w) := H(v_0 + w + \tau u_0) - c, \quad (\tau, w) \in \mathbb{R} \times \Pi(v_0).$$

It is clear that F is C^1 from $\mathbb{R} \times \Pi(v_0)$ to $\mathbb{R}$, $F(0,0) = 0$ and

$$S_c = \big\{(\tau, w)\,;\, F(\tau, w) = 0\big\}.$$

From the definition of F, we have

$$\begin{cases} D_w F(0,0)[w] = \big\langle H'(v_0), w\big\rangle = 0, \quad \forall w \in \Pi(v_0), \\ \quad D_\tau F(0,0) = \big\langle H'(v_0), u_0\big\rangle \neq 0. \end{cases}$$

So, by the Implicit Function Theorem, there exists a neighborhood of $(0,0)$ in $\mathbb{R} \times \Pi(v_0)$ and a C^1 function Φ defined on a neighborhood of 0 in $\Pi(v_0)$ such that $\Phi(0) = 0$, $D_w\Phi(0) = 0$, $\tau = \Phi(w)$ and $F(w, \Phi(w)) = 0$.

Then, for $v \in S_c$, v in a neighborhood of v_0, $v = v_0 + w + \Phi(w)u_0$, we can write

$$J(v) = J\big(v_0 + w + \Phi(w)u_0\big) =: \widetilde{J}(w), \quad w \text{ in a neighborhood of 0 in } \Pi(v_0).$$

As v_0 is an extremum of J in S_c, we have $D_w\widetilde{J}(0) = 0$. Moreover, as

$$D_w\widetilde{J}(0)[w] = \big\langle J'(v_0), w\big\rangle + \Big\langle J'(v_0), \big\langle D_w\Phi(0), w\big\rangle u_0\Big\rangle$$

and $\big\langle D_w\Phi(0), w\big\rangle = 0$ for all $w \in \Pi(v_0)$, we have

$$D_w\widetilde{J}(0)[w] = \big\langle J'(v_0), w\big\rangle = 0, \quad \forall w \in \Pi(v_0).$$

So, $J'(v_0)$ is normal to S_c at the point v_0 and then is colinear to $H'(v_0)$, i.e., there exists $\lambda \in \mathbb{R}$ such that $J'(v_0) = \lambda H'(v_0)$. □

Example 4.14 (Pohozaev)**.** Let Ω be a bounded domain of $\mathbb{R}^N$, We want to show that the following semilinear Dirichlet problem

$$\begin{cases} -\Delta u - |u|^{p-2}u = 0 \ \text{ in } \Omega,\ 1 < p < \dfrac{2N}{(N-2)},\ p \neq 2, \\ u \in H_0^1(\Omega), \end{cases} \tag{4.20}$$

has a positive solution.

Let $V = H_0^1(\Omega)$, H and J the following functionals defined on V as

$$H(v) := \frac{1}{2}\int_\Omega |\nabla v(x)|^2\,dx,\ \ J(v) := \frac{1}{p}\int_\Omega |v(x)|^p\,dx,\ \ 1 < p < \frac{2N}{N-2}.$$

Then the embedding $H_0^1(\Omega) \subset L^p(\Omega)$ is compact. Let S_1 be the unit sphere of V, i.e.,

$$S_1 := \big\{v \in V\,;\, H(v) = 1\big\}.$$

Since J is bounded on S_1, we look for $v_0 \in S_1$ such that

$$J(v_0) = \max\{J(v)\,;\, v \in S_1\}.$$

Let us consider a maximizing sequence $\{v_n\}_{n\in\mathbb{N}}$ in S_1, i.e.,

$$\lim_{n\to\infty} J(v_n) = \sup_{v\in S_1} J(v).$$

After extracting a subsequence if necessary, we have

$$\begin{aligned} v_n &\rightharpoonup v_0 \ \text{ in } H_0^1(\Omega) \text{ weakly},\\ v_n &\to v_0 \ \text{ in } L^p(\Omega) \text{ strongly}. \end{aligned}$$

It is clear that

$$J(v_n) \underset{n\to\infty}{\longrightarrow} J(v_0) = \sup_{v\in S_1} J(v)$$

but S_1 is not weakly compact in $H_0^1(\Omega)$.

To prove that $v_0 \in S_1$, we proceed as follows: we know that

$$H(v_0) \le \liminf_{n\to\infty} H(v_n) = 1.$$

So, there exists $t \ge 1$ such that $tv_0 \in S_1$. If $t > 1$ we would have $J(tv_0) = t^p J(v_0) > J(v_0)$, which is a contradiction. So, $t = 1$ and $v_0 \in S_1$. Therefore, we have

$$\begin{cases} v_0 \in S_1,\\ J(v_0) = \max\limits_{v\in S_1} J(v). \end{cases}$$

Note that $H'(v_0) = -\Delta v_0 \neq 0$, because otherwise we would have $v_0 = 0$. Then, from Lemma 4.13, the exists $\lambda \in \mathbb{R}$ such that $J'(v_0) = \lambda H'(v_0)$, i.e.,

$$|v_0|^{p-2} v_0 = -\lambda \Delta v_0, \quad v_0 \in H_0^1(\Omega), \quad \frac{1}{2}\int_\Omega |\nabla v_0(x)|^2\,dx = 1.$$

Multiplying the above equation by v_0 and integrating on Ω, we obtain

$$\int_\Omega |v_0(x)|^p\,dx = \lambda \int_\Omega |\nabla v_0(x)|^2\,dx = 2\lambda.$$

So, $2\lambda = pJ(v_0) > 0$

By considering $\mu = 1/\lambda$, we see that we have found $v_0 \in H_0^1(\Omega)$ satisfying

$$\begin{cases} -\Delta v_0 = \mu |v_0|^{p-2} v_0 & \text{in } \Omega, \\ v_0 \in H_0^1(\Omega), \quad H(v_0) = 1. \end{cases}$$

Now, we use the homogeneity of $s \mapsto |s|^{p-2}s$ to eliminate the parameter μ; let $u := \alpha v_0$, $\alpha > 0$. Then,

$$-\Delta u = \frac{\alpha\mu}{\alpha^{p-1}} |v_0|^{p-2} v_0 = \frac{\mu}{\alpha^{p-2}} |u|^{p-2} u.$$

If $p \neq 2$, we can choose $\alpha > 0$ such that $\alpha^{p-2} = \mu$ to obtain a solution u of (4.20). Moreover, since $J(v_0) = J(|v_0|)$ and $|v_0| \in S_1$, it follows that $|v_0|$ maximizes J and, as consequence, $|u_0|$ is also a solution of (4.20).

Remark 4.15. We can also obtain a (positive) solution of (4.20) as a minimizing variational problem. Indeed, let us consider the same notation as before, but now we introduce the manifold

$$\Sigma_1 := \{v \in V\,;\, J(v) = 1\}$$

and the variational problem

$$H(v) = \min_{v \in \Sigma_1} H(v). \tag{4.21}$$

Let $\{v_n\})n \in \mathbb{N}$ a minimizing sequence, $v_n \in \Sigma_1$, i.e.,

$$\lim_{n\to\infty} H(v_n) = \inf_{v \in S_1} H(v).$$

Since $\{v_n\}_{n\in\mathbb{N}}$ is bounded in V, there exists a subsequence (still denoted by v_n) such that

$$\begin{aligned} v_n &\rightharpoonup v_0 \quad \text{in } H_0^1(\Omega) \text{ weakly}, \\ v_n &\to v_0 \quad \text{in } L^p(\Omega) \text{ strongly}. \end{aligned}$$

So, $v_0 \in \Sigma_1$ and

$$H(v_0) \le \liminf_{n\to\infty} H(v_n).$$

Hence, v_0 is the solution of (4.21) and there exists $\lambda \in \mathbb{R}$ such that $H'(v_0) = \lambda J'(v_0)$, i.e.,

$$-\Delta v_0 = \lambda |v_0|^{p-2} v_0.$$

Repeating the arguments used before, we show that $\lambda > 0$ and, by homogeneity, that there exists $\alpha > 0$ such that $u = \alpha|v_0|$ is a positive solution of (4.20).

Remark 4.16. There are many other solutions for (4.20), which can be proved using topological methods such that Ljusternik-Schnirelmann categories (see [16]).

Moreover, there are other important problems that can be treated with theses methods as Von Karman equations, among other.

4.2.2 The mountain pass theorem

This is an important theorem due to A. Ambrosetti and P. Rabinowitz in 1973 (see [2]).

Let V be a Banach space and $F \in C^1(V;\mathbb{R})$. We are going to give conditions which imply the existence of non trivial critical points of F. To proceed, we need the following compactness condition, introduced by Palais and Smale [20].

Definition 4.17 (Palais-Smale Condition)**.** We say that F satisfies *Palais-Smale condition* (PS) if from every sequence $\{v_n\}_{n\in\mathbb{N}}$ in V such that

$$\begin{aligned} &(1)\ \ \{F(v_n)\}_{n\in\mathbb{N}} \text{ is bounded,}\\ &(2)\ \ F'(v_n) \underset{n\to\infty}{\longrightarrow} 0 \text{ in } V', \end{aligned}$$

we can extract a convergent subsequence.

When this condition is satisfied in the region $\big\{v \in V\,;\, F(v) \geq \alpha\big\}$ (respectively $\big\{v \in V\,;\, F(v) \leq -\alpha\big\}$) for every $\alpha > 0$, we say that F satisfies (PS$^+$) (respectively (PS$^-$)).

Theorem 4.18 (Mountain Pass Theorem)**.** *Let $F \in C^1(V;\mathbb{R})$ which satisfies (PS) and suppose that*

$$\begin{aligned} &(1)\ \ F(0) = 0 \text{ and there exist } \rho, \alpha > \text{ such that } F|_{\partial B_\rho(0)} \geq \alpha,\\ &(2)\ \ \exists v_0 \in V \setminus B_\rho(0) \text{ such that } F(v_0) < \alpha. \end{aligned}$$

Then there exists a critical value c of F, $c \geq \alpha$, which can be characterized as follows

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} F\big(\gamma(t)\big),$$

where

$$\Gamma := \Big\{\gamma \in C\big([0,1];V\big)\,;\, \gamma(0) = 0,\ \gamma(1) = v_0\Big\}.$$

This means that there exists $u \in V$ such that $F(u) = c$ (so that $u \neq 0$) and $F'(u) = 0$.

Remark 4.19. There has been many extensions of this theorem. The one presented here is the basic version. In fact, it is sufficient that F satisfy (PS^+).

A key point in the proof of the Mountain Pass Theorem is the following *Deformation Lemma*. Given a differentiable functional $F : V \to \mathbb{R}$ and $d \geq 0$, we set

$$A_d := \{v \in V\,;\, F(v) \leq d\},$$
$$K_d := \{v \in V\,;\, F(v) = d,\ F'(v) = 0\}.$$

Lemma 4.20 (Deformation Lemma)**.** *Let $F \in C^1(V;\mathbb{R})$ which satisfies (PS). If $c \in \mathbb{R}$ and if $\mathcal{N}$ is a neighborhood of K_c, then there exists a deformation $\eta \in C\big([0,1]\times V; V\big)$, $(t,x) \mapsto \eta_t(x) := \eta(t,x)$ and a constant $\overline{\varepsilon} > 0$ such that, for every ε, $0 < \varepsilon < \overline{\varepsilon}$,*

(1) $\eta_0(v) = v$, $\forall v \in V$;
(2) $\eta_t(v) = v$, $\forall v \in V$ *such that* $F(v) \notin [c-\overline{\varepsilon}, c+\overline{\varepsilon}]$, $\forall t \in [0,1]$;
(3) $\forall\, t \in [0,1], v \mapsto \eta_t(v)$ *is a homeomorphism*;
(4) $\|\eta_t(v) - v\| \leq 1$, $\forall v \in V$ *and* $\forall t \in [0,1]$;
(5) $\eta_1(A_{c+\varepsilon} \setminus \mathcal{N}) \subset A_{c-\varepsilon}$;
(6) *If* $K_c = \emptyset$, *then* $\eta_1\big(A_{c+\varepsilon}\big) \subset A_{c-\varepsilon}$;
(7) *If F is even, then η_t is odd in V.*

Note that we would like to use the gradient of F in order to decrease strictly the "altitude" of a point which is away from a neighborhood of critical points. In fact, we would like to solve the following differential equation: $\forall v \in V$,

$$\begin{cases} \dfrac{d\eta}{dt}(t,v) = -F'\big(\eta(t,v)\big), & t \in [0,1], \\ \eta(0,v) = v. \end{cases}$$

Then we would have

$$\frac{d}{dt}F\big(\eta(t,v)\big) = -\|F'\big(\eta(t,v)\big)\|^2,$$

so that if $\|F'\|$ is bounded from below by a positive number, F could decrease strictly.

But, first of all we would need F' Lipschitz which is very difficult to assume in infinite dimension for interesting cases. Second of all, we have $F'(v) \in V'$ and we want a differential equation with values in V. To overcome these difficulties, we will construct a *pseudo-gradient* (an idea of R. Palais and D.C. Clark)

Definition 4.21. Let E be a real Banach space and $\Phi \in C^1(E;\mathbb{R})$. We say that $v \in E$ is a *pseudo-gradient* for Φ at $u \in E$ if

$$\begin{aligned}&(1) \quad \|v\| \leq 2\|\Phi'(u)\|_{E'},\\ &(2) \quad \langle \Phi'(u), v\rangle \geq \|\Phi'(u)\|_{E'}^2.\end{aligned}$$

If $\widetilde{E} := \{u \in E\,;\ \Phi'(u) \neq 0\}$, then the mapping $x \mapsto v(x)$ is a *pseudo-gradient field* for Φ if

$$\begin{aligned}&(1) \quad v : E \to E \text{ is locally Lipschitz},\\ &(2) \quad \forall x \in \widetilde{E},\ v(x) \text{ is a pseudo-gradient for } \Phi \text{ at } x.\end{aligned}$$

Lemma 4.22 (Pseudo-gradient Lemma)**.** *If $\Phi \in C^1(E;\mathbb{R})$, then there exists a pseudo-gradient field for Φ on $\widetilde{E}$.*

Proof. Let $u \in \widetilde{E}$ and $w \in E$ with $\|w\| = 1$ such that

$$\big\langle \Phi'(u), w\big\rangle > \frac{2}{3}\|\Phi'(u)\|_{E'}.$$

Then, $z := \frac{3}{2}\|\Phi'(u)\|_{E'}w$ is a pseudo-gradient for Φ at u satisfying the strict inequalities

$$\|z\| < 2\|\Phi'(u)\|_{E'} \quad \text{and} \quad \big\langle \Phi'(u), z\big\rangle > \|\phi'(u)\|_{E'}^2.$$

As Φ' is continuous, there exists an open neighborhood $\mathcal{N}_u$ of u such that z is also a pseudo-gradient for Φ at v, with $v \in \mathcal{N}_u$. So, $\big\{\mathcal{N}_u\big\}_{u\in\widetilde{E}}$ is an open covering of $\widetilde{E}$. Since $\widetilde{E}$ is a metric space, it is paracompact (cf. for exemple [23]), i.e., every open covering of $\widetilde{E}$ has a *locally finite*[1] subcovering. Hence, we can extract a locally finite subcovering $\{\mathcal{N}_{u_i}\}_{i\in I}$ of $\widetilde{E}$ associated to pseudo-gradients z_i.

Let $\rho_i(x) := \operatorname{dist}(x, \mathcal{N}_{u_i}^c)$, i.e., the distance of x to the complement of $\mathcal{N}_{u_i}$. Then the mapping $x \mapsto \rho_i(x)$ is Lipschitz and vanishes outside $\mathcal{N}_{u_i}$. Let

$$\beta_i(x) := \frac{\rho_i(x)}{\displaystyle\sum_{k\in I}\rho_k(x)}.$$

Then, $\{\beta_i\}_{i\in I}$ is a partition of the unity associated to $\{\mathcal{N}_{u_i}\}_{i\in I}$ Therefore, the convex combination

$$v(x) := \sum_{i\in I}\beta_i(x)z_i$$

is a pseudo-gradient for Φ at $x \in \widetilde{E}$ and $v : E \to E$ is locally Lipschitz. This completes the proof. □

[1] A collection $\mathcal{A}$ of subsets of a topological space X is said to be *locally finite* in X if every point of X has a neighbourhood that intersects only finitely many elements of $\mathcal{A}$.

We are now in position to prove the Deformation Lemma 4.20.

Proof. We assume $K_c \neq \emptyset$ (otherwise the proof is simpler). It follows from the Palais-Smale condition that K_c is compact. Let us call $\mathcal{N}_\delta(K_c)$ the δ-neighborhoods of K_c. For δ small enough, $\mathcal{N}_\delta(K_c) \subset \mathcal{N}$.

Hence, there exist $\overline{\varepsilon} > 0$ and $b > 0$ such that

$$\|F'(x)\|_{V'} \geq b, \quad \forall x \in A_{c+\overline{\varepsilon}} \setminus A_{c-\overline{\varepsilon}} \setminus \mathcal{N}_{\delta/8}.$$

Indeed, otherwise we could construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ with

$$F(x_n) \underset{n\to\infty}{\longrightarrow} c, \; F'(x_n) \underset{n\to\infty}{\longrightarrow} 0, \quad x_n \notin \mathcal{N}_{\delta/8}.$$

From (PS) there would be a subsequence converging to x with

$$F(x) = c, \quad F'(x) = 0, \quad x \notin \mathcal{N}_{\delta/8},$$

which is impossible.

Of course this remains valid if we decrease $\overline{\varepsilon}$. So, we can assume that

$$0 < \overline{\varepsilon} < \min\left\{\frac{b\delta}{32}, \frac{b^2}{8}, \frac{1}{8}\right\}.$$

Let $0 < \varepsilon < \overline{\varepsilon}$ and

$$\begin{aligned} A &:= \big\{x \in V \,;\, F(x) \geq c + \overline{\varepsilon} \;\text{ or }\; F(x) \leq c - \overline{\varepsilon}\big\}, \\ B &:= \big\{x \in V \,;\, c - \varepsilon \leq F(x) \leq c + \varepsilon\big\}. \end{aligned}$$

Since $A \cap B = \emptyset$, we define

$$g(x) := \operatorname{dist}(x, A)\big[\operatorname{dist}(x, A) + \operatorname{dist}(x, B)\big]^{-1}.$$

It is clear that g is Lipschitz, $g \equiv 0$ on A, $g \equiv 1$ on B and $0 \leq g \leq 1$. By the same way we can construct a Lipschitz function $\overline{g}$ such that

$$\begin{cases} \overline{g} \equiv 1 \;\text{ on }\; V \setminus \mathcal{N}_{\delta/4}, \\ \overline{g} \equiv 0 \;\text{ on }\; \mathcal{N}_{\delta/8}, \\ 0 \leq \overline{g} \leq 1. \end{cases}$$

Let $h : \mathbb{R}^+ \to \mathbb{R}$ be the Lipschitz function defined by

$$h(s) := \begin{cases} 1 \text{ if } s \in [0, 1], \\ \dfrac{1}{s} \text{ if } s \geq 1. \end{cases}$$

As $F \in C^1(V;\mathbb{R})$, there exists a pseudo-gradient v for F on $\widetilde{V}$, where $\widetilde{V} = \{x \in V\,;\, F'(x) \neq 0\}$. Finally we set

$$\Phi(x) := -g(x)\overline{g}(x)h\big(\|v(x)\|\big)v(x).$$

As $\overline{g} \equiv 0$ on $\mathcal{N}_{\delta/8}$, Φ can be extended by 0 on $\mathcal{N}_{\delta/8}$ and therefore, is defined on the whole space V. So, Φ is locally Lipschitz and $0 \le \|\Phi(x)\| \le 1$.

For each $u \in V$ we consider the differential equation

$$\begin{cases} \dfrac{d\eta}{dt}(t,u) = \Phi\big(\eta(t,u)\big), \\ \eta(0,u) = u. \end{cases}$$

Since Φ is locally Lipschitz, this equations admits a unique solution $\eta(t,u)$ for $t \in \big[0, t^+(u)\big)$ and as Φ is bounded, we have $t^+(u) = +\infty$ for each $u \in V$.

In order to verify that η satisfies the conditions (1)—(7), we begin by remarking that $\eta_t(v) := \eta(t,v) \in C^1([0,1] \times V;V\big)$ and $\eta_0(v) = v$ for all $v \in V$ and (1) holds. Moreover, as $\Phi = 0$ on A, it follows that $\eta_t(v) = v$, $\forall v \in A$, $\forall t \in [0,1]$ and we have (2).

Using the backward differential equation and the uniqueness for both solutions, we see that for every $t \in [0,1]$, $\eta_t(\cdot)$ is a homeomorphism from V to V. So the condition (3) is satisfied.

Now, as $\|\Phi\| \le 1$ and $0 \le t \le 1$, we have $\|\eta_t(v) - v\| \le 1$ and (4) is verified.

Let us show (5), i.e., that $\eta_1(A_{c+\varepsilon} \setminus \mathcal{N}_\delta) \subset A_{c-\varepsilon}$.

$$\begin{aligned} \frac{d}{dt}F\big(\eta_t(x)\big) &= \Big\langle F'\big(\eta_t(x)\big), \frac{d}{dt}\eta_t(x)\Big\rangle = \Big\langle F'\big(\eta_t(x)\big), \Phi\big(\eta_t(x)\big)\Big\rangle \\ &= -g\big(\eta_t(x)\big)\overline{g}\big(\eta_t(x)\big)h\Big(\big\|v\big(\eta_t(x)\big)\big\|\Big)\Big\langle F'\big(\eta_t(x)\big), v\big(\eta_t(x)\big)\Big\rangle \le 0. \end{aligned}$$

Then F decreases along the orbits $t \mapsto \eta_t(x)$. So, if $x \in A_{c-\varepsilon}$, we have

$$F\big(\eta_1(x)\big) \le F\big(\eta_t(x)\big) \quad \Rightarrow \quad \eta_1(x) \in A_{c-\varepsilon}.$$

In order to verify the condition (6), let $Y := A_{c+\varepsilon} \setminus A_{c-\varepsilon} \setminus \mathcal{N}_\delta$. We have to show that $\eta_1(Y) \subset A_{c-\varepsilon}$.

Let $x \in Y$ and $\phi_x(t) := F\big(\eta_t(x)\big)$. Then we have that $\frac{d}{dt}\phi_x(t) \le 0$. To show that $\phi_x(1) \le c - \varepsilon$, we observe that as $\Phi = 0$ on $A_{c-\overline{\varepsilon}}$, the orbit $\eta_t(x)$ cannot enter $A_{c-\overline{\varepsilon}}$. Then,

$$0 \le \phi_x(0) - \phi_x(t) \le 2\overline{\varepsilon}.$$

By continuity, as $x \in A_{c+\varepsilon} \setminus A_{c-\varepsilon} \setminus \mathcal{N}_\delta$, for t small enough and for all $s \in [0,t]$,

$$\eta_s(x) \in A_{c+\varepsilon} \setminus A_{c-\varepsilon} \setminus \mathcal{N}_{\delta/2}.$$

So, $\eta_s(x) \in \widetilde{V}$, which implies that

$$g\big(\eta_s(x)\big) = \overline{g}\big(\eta_s(x)\big) = 1, \quad \forall s \in [0,t].$$

Let us define $Z := A_{c+\varepsilon} \setminus A_{c-\varepsilon} \setminus \mathcal{N}_{\delta/2}$. Then,

$$\begin{aligned}
2\overline{\varepsilon} &\geq -\int_0^t \frac{d}{dt}\phi_x(s)\,ds = \int_0^t h\Big(\big\|v\big(\eta_s(x)\big)\big\|\Big)\Big\langle F'\big(\eta_s(x)\big), v\big(\eta_s(x)\big)\Big\rangle\,ds \\
&\geq \int_0^t h\Big(\big\|v\big(\eta_s(x)\big)\big\|\Big)\Big\|F'\big(\eta_s(x)\big)\Big\|^2\,ds \geq b\int_0^t h\Big(\big\|v\big(\eta_s(x)\big)\big\|\Big)\Big\|F'\big(\eta_s(x)\big)\Big\|\,ds \\
&\geq \frac{b}{2}\int_0^t h\Big(\big\|v\big(\eta_s(x)\big)\big\|\Big)\Big\|v\big(\eta_s(x)\big)\Big\|\,ds \geq \frac{b}{2}\left\|\int_0^t h\Big(\big\|v\big(\eta_s(x)\big)\big\|\Big)v\big(\eta_s(x)\big)\,ds\right\| \\
&= \frac{b}{2}\left\|\int_0^t \Phi\Big(v\big(\eta_s(x)\big)\Big)\right\| = \frac{b}{2}\|\eta_t(x) - x\|
\end{aligned}$$

Therefore,

$$\|\eta_t(x) - x\| \leq \frac{4\overline{\varepsilon}}{b} < \frac{\delta}{8}.$$

In particular, the orbit cannot enter $\mathcal{N}_{\delta/2}$ and it cannot leave Z without entering $A_{c-\varepsilon}$.

Let us show that this happens actually for $t \in [0,1]$ (otherwise we have for all $t \in [0,1]$, $\eta_t(x) \in Z$). Since,

$$\frac{d\phi_x}{dt} \leq -h\Big(\big\|v\big(\eta_t(x)\big)\big\|\Big)\Big\|F'\big(\eta_t(x)\big)\Big\|^2, \tag{4.22}$$

we have two possibilities: either $\|v\| \leq 1$ in which case the right hand side of (4.22) is less than or equal to $-b^2$ (because $h(\|v\|) = 1$ and $\|F'\| \geq b^2$), or $\|v\| \geq 1$ and the right hand side of (4.22) is less than or to equal $-1/4$ (because $-h(\|v\|)\|F'\|^2 = -\|F'\|^2/\|v\| \leq -\|v\|/4 \leq -1/4$).

Then we have,

$$\frac{d\phi_x}{dt} \leq -\min\left\{b^2, \frac{1}{4}\right\}.$$

and so

$$\min\left\{b^2, \frac{1}{4}\right\} \leq \phi_x(0) - \phi_x(1) \leq 2\overline{\varepsilon},$$

which is a contradiction. This completes the proof of the deformation lemma. □

With these last two results, we are able to prove the Mountain Pass Theorem.

Proof of the Mountain Pass Theorem. First of all, we note that for all $\gamma \in \Gamma$, there exists $t_\gamma \in [0,1]$ such that $\|\gamma(t_\gamma)\| = \rho$, so that $F\big(\gamma(t_\gamma)\big) \geq \alpha$. Hence

$$\max_{t\in[0,1]} F\big(\gamma(t)\big) \geq \alpha, \quad \forall \gamma \in \Gamma.$$

If

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} F\big(\gamma(t)\big),$$

then $c \geq \alpha$. We have to show that c is a critical value of F, i.e., there exists $u \in V$ such that $F'(u) = 0$ and $F(u) = c$. So, u will be a critical point of F which is different from zero.

From the definition of c, we know that for each $\varepsilon > 0$, there exists $\gamma \in \Gamma$ such that

$$c \leq \max_{t\in[0,1]} F\big(\gamma(t)\big) < c + \varepsilon.$$

Let $\overline{\varepsilon} > 0$ as in the Deformation Lemma, but chosen such that $c - \overline{\varepsilon} > \max\{0, F(v_0)\}$ and suppose that $K_c = \emptyset$. From the Deformation Lemma, there exists $\eta_1 : V \to V$ continuous such that, for all $x \in A_{c-\overline{\varepsilon}}$, $\eta_1(x) = x$ and $\eta_1\big(A_{c+\varepsilon}\big) \subset A_{c-\varepsilon}$.

Let $\widetilde{\gamma} := \eta_1 \circ \gamma$. Then $\widetilde{\gamma} \in C^1\big([0.1]; V\big)$ and

$$\begin{cases} \widetilde{\gamma}(0) = \eta_1\big(\gamma(0)\big) = \eta_1(0) = 0 \text{ (because } 0 \in A_{c-\overline{\varepsilon}}), \\ \widetilde{\gamma}(1) = \eta_1\big(\gamma(1)\big) = \eta_1(v_0) = v_0 \text{ (because } v_0 \in A_{c-\overline{\varepsilon}}). \end{cases} \tag{4.23}$$

This means that $\widetilde{\gamma} \in \Gamma$. Now, as

$$\begin{cases} F\big(\gamma(t)\big) \leq c + \varepsilon, & \forall t \in [0,1], \\ F\big(\widetilde{\gamma}(t)\big) \leq c - \varepsilon, & \forall t \in [0,1], \end{cases}$$

it follows that $\max_{t\in[0,1]} F\big(\widetilde{\gamma}(t)\big) \leq c - \varepsilon$. This leads to a contradiction and proves the theorem. □

4.2.3 Application - Example

Let us consider the following boundary value problem

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ \quad u = 0 \ \text{ on } \ \Gamma, \end{cases} \tag{4.24}$$

where Ω is a bounded domain with boundary Γ and $f \in C(\overline{\Omega} \times \mathbb{R}; \mathbb{R})$ satisfying the following condition: there exist positive constants a_1 and a_2 such that

$$\big|f(x,\xi)\big| \leq a_1|\xi|^s + a_2, \quad 0 \leq s < \frac{N+2}{N-2}, \ \text{ if } \ N > 2, \tag{4.25}$$

and $s \geq 0$ if $N = 2$.

We denote by F the primitive of f, i.e.,

$$F(x,\xi) := \int_0^\xi F(x,t)\,dt.$$

Then we define the functional $J : H_0^1(\Omega) \to \mathbb{R}$ by

$$J(u) := \frac{1}{2}\int_\Omega \big|\nabla u(x)\big|^2 dx - \int_\Omega F\big(x,u(x)\big)\,dx.$$

It is clear that J is well defined because, from (4.25), we can choose positive constants A and B such that $\big|F(x,\xi)\big| \leq A|\xi|^{s+1} + B$, where

$$s + 1 < 2N/(N-2).$$

So, if we denote

$$H(u) := \int_\Omega F\big(x,u(x)\big)\,dx,$$

the embedding $H_0^1(\Omega) \hookrightarrow L^{2N/(N+2)}$ ensures that $H(u)$ is well defined for $u \in H_0^1(\Omega)$. Moreover, it is classical that $J \in C^1\big(H_0^1(\Omega);\mathbb{R}\big)$ and, for all $u, v \in H_0^1(\Omega)$,

$$\big\langle J'(u), v\big\rangle = \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx - \int_\Omega f\big(x,u(x)\big)v(x)\,dx.$$

Let us show that H is weakly continuous, i.e.,

$$u_m \rightharpoonup u \text{ in } H_0^1(\Omega) \text{ weakly } \;\Rightarrow\; H(u_m) \to H(u). \tag{4.26}$$

We know from the Rellich-Kondrachov theorem that $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ with $1 \leq q < 2N/(N-2)$, the injection being compact. If $u_m \rightharpoonup u$ in $H_0^1(\Omega)$ weakly, then $u_m \to u$ in $L^{s+1}(\Omega)$ strongly, because we are assuming that $s + 1 < 2N/(N-2)$. For a subsequence, $u_{m_k} \to u$ a.e. in Ω, which implies $F\big(\cdot,u_{m_k}\big) \to F(\cdot,u)$ a.e. in Ω. Since $|F(\cdot,u_{m_k})| \leq A|u_{m_k}|^{s+1} + B$ and $|u_{m_k}|^{s+1}$ converges strongly in $L^1(\Omega)$, we have (4.26) as a consequence of the Lebesgue Theorem. As this is valid for any subsequence, we have convergence of the whole sequence $\{H(u_m)\}_{m\in\mathbb{N}}$.

Now, let us prove that the mapping $u \mapsto H'(u)$ is completely continuous, i.e.,

$$u_m \rightharpoonup u \text{ in } H_0^1(\Omega) \text{ weakly } \;\Rightarrow\; H'(u_m) \to H'(u) \text{ in } H^{-1}(\Omega) \text{ strongly.} \tag{4.27}$$

Repeating the above arguments, if $u_m \rightharpoonup u$ in $H_0^1(\Omega)$ weakly, then $u_m \to u$ in $L^p(\Omega)$ strongly, for all $1 \leq p < 2N/(N-2)$. So, we can extract a subsequence u_{m_k} such that $u_{m_k} \to u$ a.e. in Ω, which implies that $f(\cdot, u_{m_k}) \to f(\cdot, u)$ a.e., in Ω.

Since $|f(\cdot, u_{m_k})| \leq a_1 |u_{m_k}|^s + a_2$ and $|u_{m_k}|^s$ converges strongly in $L^p(\Omega)$ if $sp \leq 2N/(N-2)$, the Lebesgue Theorem implies that $f(\cdot, u_{m_k}) \to f(\cdot, u)$ in $L^p(\Omega)$. But we are assuming that $s < (N+2)/(N-2)$, so that $sp < 2N/(N-2)$ if and only if $p < 2N/(N+2)$.

We know that

$$H_0^1(\Omega) \hookrightarrow L^{2N/(N-2)}(\Omega) \;\Rightarrow\; L^{2N/(N+2)}(\Omega) = \big(L^{2N/(N-2)}(\Omega)\big)' \hookrightarrow H^{-1}(\Omega).$$

Therefore, we have $f(\cdot, u_{m_k}) \to f(\cdot, u)$ in $L^{2N/(N+2)}(\Omega)$ and consequently $f(\cdot, u_{m_k}) \to f(\cdot, u)$ in $H^{-1}(\Omega)$.

As we will see below, J satisfies (PS) under additional hypotheses on f, which assures the existence of a non trivial solution of (4.24).

Theorem 4.23. *Let $f \in C(\overline{\Omega} \times \mathbb{R}; \mathbb{R})$ satisfying* (4.25) *and also*

$$\begin{cases} (1) \quad f(x,\xi) = o\big(|\xi|\big) \text{ when } \xi \to 0; \\ (2) \quad \text{there exist } 0 \leq \theta < \dfrac{1}{2} \text{ and } r > 0 \text{ such that} \\ \qquad\qquad 0 < F(x,\xi) \leq \theta \xi f(x,\xi) \ \text{ if } \ |\xi| \geq r. \end{cases} \tag{4.28}$$

Then the problem (4.24) *possesses a non trivial solution.*

Remark 4.24. The hypothesis (1) implies that $u = 0$ is a solution of (4.28). The hypothesis (2) is satisfied if f is a polynomial (or a comparable function) in ξ of degree greater than of equal to 2.

Proof. Let $\mu = 1/\theta$. Then $\mu > 2$ and

$$\frac{F'(\cdot,\xi)}{F(\cdot,\xi)} \geq \frac{\mu}{\xi} \quad \text{for } |\xi| \geq r.$$

So, $\ln\big(F(\cdot,\xi)\big) \geq \ln\big(a_3 |\xi|^\mu\big)$ for $|\xi| \geq r$, for some $a_3 > 0$ and we obtain by continuity $a_4 > 0$ such that $0 \leq F(\cdot,\xi)| \leq a_4$ for $|\xi| \leq r$, so that

$$-F(\cdot,\xi) \leq -a_3 |\xi|^\mu + a_4, \quad \forall |\xi| \geq r.$$

Let us take $t > 0$ and $u_0 \in H_0^1(\Omega)$. Then, if

$$\widetilde{\Omega} = \big\{x \in \Omega\,;\, |u_0(x)| \geq b > 0\big\} \neq \emptyset,$$

we have

$$J(tu_0) \leq \frac{t^2}{2} \int_\Omega \|u_0\|^2_{H_0^1(\Omega)} - t^\mu \int_\Omega a_3 |u_0(x)|^\mu \, dx \longrightarrow -\infty, \quad t \to +\infty.$$

Therefore, $J(0) = 0$, $J(tu_0) < 0$ if t is large enough and for u_0 suitably choosen.

From (2), for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|\xi| \le \delta$ we get

$$|f(x,\xi) \le \varepsilon \quad \text{and} \quad |F(x,\xi)| \le \frac{1}{2}\varepsilon|\xi|^2.$$

Moreover, there exists A_ε such that $|F(x,\xi)| \le A_\varepsilon|\xi|^{s+1}$ for $|\xi| \ge \delta$.

So, we have

$$|F(x,\xi)| \le \frac{1}{2}\varepsilon|\xi|^2 + A_\varepsilon|\xi|^{s+1}, \quad \forall \xi \in \mathbb{R}. \tag{4.29}$$

The estimate (4.29) let us assert that, for some $C > 0$,

$$\big|H(v)\big| \le C\left(\frac{\varepsilon}{2} + A_\varepsilon\|v\|^{s-1}\right)\|v\|^2_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

As ε is arbitrary, $H(v) = o\big(\|v\|^2_{H_0^1(\Omega)}\big)$ if $v \to 0$ in $H_0^1(\Omega)$. Therefore, for $\rho > 0$ small enough and $\|v\|_{H_0^1(\Omega)} = \rho$, we have $J(v) \ge \alpha > 0$.

We claim that J satisfies (PS). Indeed, let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence such that

$$|J(u_m)| \le M \quad \text{and} \quad J'(u_m) \to 0 \ \text{ in } H^{-1}(\Omega). \tag{4.30}$$

For each $v \in H_0^1(\Omega)$ we have

$$J(v) - \theta\big\langle J'(v), v\big\rangle = \left(\frac{1}{2} - \theta\right)\|v\|_{H_0^1(\Omega)^2} + \int_\Omega \Big(\theta f\big(x, v(x)\big)v(x) - F\big(x, v(x)\big)\Big)\,dx.$$

So, for m large enough,

$$\left(\frac{1}{2} - \theta\right)\|u_m\|^2_{H_0^1(\Omega)} \le \Big|J(u_m) - \theta\big\langle J'(u_m), u_m\big\rangle\Big| + M_1 \le M_2 + \widetilde{\theta}\|u_m\|_{H_0^1(\Omega)},$$

which implies that $\{u_m\}_{m\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$.

After extraction a subsequence we have $u_m \rightharpoonup u$ in $H_0^1(\Omega)$ weakly and $H'(u_m) \to$ in $H^{-1}(\Omega)$ strongly.

But (4.30) means that $-\Delta u_m - H'(u_m) \to 0$ in $H^{-1}(\Omega)$, so that $\{-\Delta u_m\}_{m\in\mathbb{N}}$ converges strongly in $H^{-1}(\Omega)$ and so $\{u_m\}_{m\in\mathbb{N}}$ converges strongly in $H_0^1(\Omega)$.

Then we can apply the Mountain Pass Theorem to complete the proof. □

Remark 4.25. Let $f \in C(\mathbb{R};\mathbb{R})$ satisfying (4.25) and $u \in H_0^1(\Omega)$ a solution of

$$-\Delta u = f(u) \ \text{ in } \Omega,$$

where Ω is a regular star shaped (with respect to the origin) bounded domain of $\mathbb{R}^N$, $N \geq 1$. Then, u satisfies the *Pohozaev Identity*, i.e.,

$$2N\int_\Omega F\big(u(x)\big)\,dx + (2-N)\int_\Omega f\big(u(x)\big)u(x)\,dx = \int_\Gamma \big(\sigma\cdot\nu(\sigma)\big)|\nabla u(\sigma)|^2 dS(\sigma),$$

where $\nu(\sigma)$ denotes the unitary normal vector to the exterior of Ω at $\sigma \in \Gamma$. As $\sigma \cdot \nu(\sigma) \geq 0$, it follows that

$$\int_\Omega F\big(u(x)\big)\,dx \geq \frac{(N-2)}{2N}\int_\Omega f\big(u(x)\big)u(x)\,dx.$$

If $N > 2$ and $f(\xi) = |\xi|^s$, the above inequality implies that

$$s \leq (N+2)/(N-2).$$

So, this implies conditions on the growth of f for the existence of non trivial solutions.

Chapter 5

Study of the problem $-\Delta u = \lambda e^u$

In this chapter we will study the following boundary value problem in details

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^N$ is a regular bounded domain.

5.1 Preliminaries

First of all, let us mention that the case $\lambda < 0$ enters in the context of monotone operators as treated before and if $\lambda = 0$ the unique solution is $u = 0$. So, the only interesting case is $\lambda > 0$, for which we have to make precise what we mean by a solution, i.e., we must establish the good functional framework. But at least formally, for $\lambda > 0$, we can say that

(1) $u = 0$ is not a solution of the problem,

(2) $e^u > 0 \Rightarrow u > 0$ in Ω.

Second of all, if $u \in L^\infty(\Omega)$, then $e^u \in L^\infty(\Omega)$ and so $u \in W^{2,p}(\Omega)$ for all $p < \infty$. Hence, $u \in C^1(\Omega)$ and by a bootstrap argument we conclude that $u \in C^\infty(\Omega)$. This was the class considered by several authors.

Here we will prove that there may exist solutions which are not in $L^\infty(\Omega)$. More precisely, in [18] we considered the case of $u \in H_0^1(\Omega)$ such that $e^u \in L^1(\Omega)$.

Remark 5.1. Notice that as $u \geq 0$ and $e^u = 1 + u + \frac{u^2}{2!} + \cdots$, if $e^u \in L^1(\Omega)$ then $u \in L^p(\Omega)$ for all $p < \infty$. But this does not imply that $u \in L^\infty(\Omega)$.

In [5] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa considered *ultra weak solutions*. More precisely, they look for solutions u such that

$$u \operatorname{dist}(\cdot, \Gamma) \in L^1(\Omega) \quad \text{and} \quad e^u \operatorname{dist}(\cdot, \Gamma) \in L^1(\Omega),$$

where $\operatorname{dist}(x, \Gamma)$ is the distance of $x \in \Omega$ to the boundary Γ.

They found essentially the same results but they did not show that their solutions did not satisfy $e^u \in L^1(\Omega)$.

From now on we refer to problem (5.1) as $(5.1)_\lambda$ when it is important to emphasize the dependence on λ.

So, we will consider here two classes of solutions, namely

$$\begin{aligned} \mathcal{R} &:= \big\{u \in L^\infty(\Omega) \cap H_0^1(\Omega)\,;\ \ u \text{ is a solution of } (5.1)_\lambda\big\}; \\ \mathcal{S} &:= \big\{u \in H_0^1(\Omega)\,;\, e^u \in L^1(\Omega) \text{ and } u \text{ is a solution of } (5.1)_\lambda\big\}; \end{aligned} \tag{5.2}$$

We say that $\mathcal{R}$ and $\mathcal{S}$ are respectively the classes of *regular solutions* and *singular solutions*.

5.2 Solutions of $(5.1)_\lambda$ for $\lambda > 0$ near 0

First of all, notice that $\varphi = 0$ is a subsolution of $(5.1)_\lambda$. On the other hand, if $\psi \in H_0^1(\Omega)$ satisfies $-\Delta\psi = 1$ in Ω, then $\psi \in L^\infty(\Omega)$, $\psi > 0$ in Ω and $0 \le \psi \le M$ for some constant $M > 0$. So,

$$-\Delta\psi - \lambda e^\psi = 1 - \lambda e^\psi \ge 1 - \lambda e^M.$$

By choosing $\lambda > 0$ such that $1 - \lambda e^M \ge 0$, ψ is a supersolution of $(5.1)_\lambda$. This proves that there exists a minimal solution $\underline{u}_\lambda$ with $0 \le \underline{u}(\lambda) \le \psi$ and $\underline{u}(\lambda) \in \mathcal{R}$, if $\lambda > 0$ is small enough.

In fact, we can be much more precise concerning regular solutions, as it is asserted in the following theorem.

Theorem 5.2. *Let $p > N/2$ and $V := W^{2,p}(\Omega) \cap H_0^1(\Omega)$. Then there exists $\delta > 0$ and an increasing C^1 mapping $\lambda \mapsto \underline{u}(\lambda)$ from $[0,\delta)$ to V, where $\underline{u}(\lambda) \in \mathcal{R}$ is the minimal solution of $(5.1)_\lambda$. Moreover, there exists a neighborhood $\mathcal{O}$ of 0 in V in which $\underline{u}(\lambda)$ is the unique solution of $(5.1)_\lambda$.*

To proceed the proof, we need the following elementary result.

Lemma 5.3. *if $p > N/2$ the mapping $v \mapsto e^v$ is continuously Fréchet-differentiable from $W^{2,p}(\Omega)$ to $L^p(\Omega)$.*

Proof. First of all notice that $e^t = 1 + t + e^\xi t^2/2$ for some $|\xi| \le |t|$. If $|t| + |s| \le M$ we have

$$e^s\big[e^t - 1 - t\big] \le \frac{e^{2M}}{2} t^2.$$

Since $W^{2,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, it follows that, if $p > N/2$ and $u, v \in W^{2,p}(\Omega)$, e^u, $e^v \in C^0(\overline{\Omega})$. Then, for each $x \in \Omega$ we can write

$$\big|e^{u(x)+v(x)} - e^{u(x)} - e^{u(x)}v(x)\big| \le \frac{e^{\|u\|_\infty} e^{\|v\|_\infty}}{2} \big|v(x)\big|^2.$$

This implies that

$$\left(\int_\Omega \left|e^{u(x)+v(x)} - e^{u(x)} - e^{u(x)}v(x)\right|^p dx\right)^{1/p} \leq \frac{e^{\|u\|_\infty} e^{\|u\|_\infty}}{2} \left(\int_\Omega |v(x)|^{2p}\, dx\right)^{1/p}.$$

Therefore, if $M > 0$ and $u, v \in W^{2,p}(\Omega)$ such that $\|u\|_\infty, \|u\|_\infty \leq M$, we have

$$\|e^{u+v} - e^u - e^u v\|_{L^p(\Omega)} \leq C_1 \|v^2\|_{L^p(\Omega)} \leq C_2 \|v\|_\infty^2 \leq C_3 \|v\|^2_{W^{2,p}(\Omega)},$$

where C_i are constants that depend only on M and Ω. This proves that the mapping $F : W^{2,p}(\Omega) \to L^p(\Omega)$, $F(v) := e^v$ is Fréchet-differentiable with derivative at $u \in W^{2,p}(\Omega)$ given by $F'(u)[w] = e^u w$.

It is clear that $F' : W^{2,p}(\Omega) \to \mathcal{L}\big(W^{2,p}(\Omega), L^p(\Omega)\big)$ is continuous because, for $u, v, w \in W^{2,p}(\Omega)$ with $\|w\|_{W^{2,p}(\Omega)} = 1$, we have

$$\big\|(F'(u)[w] - F'(v))[w]\big\|_p = \big\|(e^u - e^v)w\big\|_p \leq \|e^u - e^v\|_p \|w\|_\infty \leq C\|e^u - e^v\|_p,$$

where $C > 0$ depends only on Ω. So, we have

$$\begin{aligned}\big\|F'(u) - F'(v)\big\|_{\mathcal{L}} &= \sup\Big\{\big\|(F'(u)[w] - F'(v))[w]\big\|_p \,;\ \|w\|_{W^{2,p}(\Omega)} = 1\Big\}\\ &\leq C\|e^u - e^v\|_p = C\|F(u) - F(v)\|_p\end{aligned}$$

and the conclusion follows because F is necessarily continuous. □

Proof ot Theorem 5.2. We consider the function $F : V \times \mathbb{R} \to L^p(\Omega)$ defined by

$$F(v, \lambda) := -\Delta v - \lambda e^v.$$

By Lemma 5.3 we know that F is C^1, $F(0,0) = 0$ and $D_v F(0,0) = -\Delta$, which is invertible as a function from V to $L^p(\Omega)$. Indeed, we know that the problem

$$\begin{cases} -\Delta w = f & \text{in } \Omega,\\ w \in V, \end{cases}$$

has a unique solution for $f \in L^p(\Omega)$ and $\|v\|_V \leq C\|f\|_{L^p(\Omega)}$.

From the Implicit Function Theorem, it follows that

(1) there exists a neighborhood $\mathcal{O}$ of 0 in V,

(2) there exists a neighborhood $(-\delta, \delta)$ of 0 in $\mathbb{R}$

and a C^1 function $\lambda \mapsto \underline{u}(\lambda)$ from $(-\delta, \delta)$ to $\mathcal{O}$ with $\underline{u}(0) = 0$ such that

$$F\big(\lambda, \underline{u}(\lambda)\big) = 0, \quad \forall \lambda \in (-\delta, \delta)$$

and $\underline{u}(\lambda)$ is the unique solution of $(5.1)_\lambda$ in $\mathcal{O}$.

Discarding the interval $(-\delta, 0)$ which is not of our interest here, we have a C^1 curve $\lambda \mapsto \underline{u}(\lambda)$ defined in $[0, \delta)$ such that $\underline{u}(\lambda)$ is a solution of $(5.1)_\lambda$.

Moreover, by differentiating on λ, we have for $\underline{u}'(\lambda) := \frac{d\underline{u}}{d\lambda}(\lambda)$,

$$\begin{cases} -\Delta \underline{u}'(\lambda) - \lambda e^{\underline{u}(\lambda)} \underline{u}'(\lambda) = e^{\underline{u}(\lambda)}, \\ \underline{u}'(\lambda) \in H_0^1(\Omega). \end{cases}$$

As for $\lambda > 0$ small, $\underline{u}(\lambda) \approx 0$, the operator $-\Delta - \lambda e^{\underline{u}(\lambda)}$ is invertible and coercive, we have from the strong maximum principle, $\underline{u}'(\lambda) > 0$ in Ω. Therefore, we can find $\delta_1 \leq \delta$ such that the mapping $\lambda \mapsto \underline{u}(\lambda)$ is increasing for $\lambda \in (0, \delta_1)$, i.e.,

$$\forall \lambda, \lambda' \in (0, \delta_1), \ \ \lambda < \lambda' \ \Rightarrow \ \underline{u}(\lambda) \leq \underline{u}(\lambda').$$

Now, we see that $\underline{u}(\lambda')$ is a supersolution for $(5.1)_\lambda$ and 0 is a subsolution with $0 \leq \underline{u}(\lambda')$. So, there exists a minimal solution u of $(5.1)_\lambda$ with $0 \leq u \leq \underline{u}(\lambda')$.

But, as $u \mapsto \lambda e^u$ is strictly convex, Theorem 4.11 assures the uniqueness of solution in the interval $0 \leq v \leq \underline{u}(\lambda')$. But we already know that $\underline{u}(\lambda)$ is a solution. Then, $\underline{u}(\lambda)$ is the minimal solution of $(5.1)_\lambda$.

In short, we have a branch of minimal solutions $\lambda \in [0, \delta_1] \mapsto \underline{u}(\lambda) \in V$ (which implies that $\underline{u}(\lambda) \in \mathcal{R}$), which is increasing and C^1. □

Remark 5.4. Let us consider the linearized operator at $\underline{u}(\lambda)$, i.e.,

$$v \in H_0^1(\Omega) \mapsto -\Delta v - \lambda e^{\underline{u}(\lambda)} v \in H^{-1}(\Omega).$$

For λ small enough we have

$$\int_\Omega |\nabla v(x)|^2 \, dx - \lambda \int_\Omega e^{\underline{u}(\lambda)(x)} |v(x)|^2 \, dx \geq 0.$$

But we also have

$$\int_\Omega |\nabla v(x)|^2 \, dx - \lambda' \int_\Omega e^{\underline{u}(\lambda')(x)} |v(x)|^2 \, dx \geq 0.$$

So, if $\lambda < \lambda'$ we have $e^{\underline{u}(\lambda')} > e^{\underline{u}(\lambda)}$ and

$$\int_\Omega |\nabla v(x)|^2 \, dx - \lambda \int_\Omega e^{\underline{u}(\lambda)(x)} |v(x)|^2 \, dx > 0 \quad \forall v \in H_0^1(\Omega), \ v \neq 0,$$

which means that the linearized operator at $\underline{u}(\lambda)$ is coercive.

Remark 5.5. It is important to note that the result stated in Theorem 5.2 does not say that there are not other solutions far from 0, or even in another class of solutions.

5.3 Range of λ for the existence of a solution

Let $u \in \mathcal{S}$ be a solution of $(5.1)_\lambda$, where $\mathcal{S}$ is defined in (5.2). If λ_1 and φ_1 are respectively the first eigenvalue and eigenfunction of $-\Delta$, i.e.,

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1 \quad \text{in } \Omega. \\ \varphi_1 \in H_0^1(\Omega), \ \varphi_1 > 0 \text{ in } \Omega \text{ and } \|\varphi_1\|_{L^2(\Omega)} = 1, \end{cases}$$

then φ_1 is regular and bounded.

If we multiply both sides of $(5.1)_\lambda$ by φ_1, it comes

$$\lambda_1 \int_\Omega u(x)\varphi_1(x)\,dx = \lambda \int_\Omega e^{u(x)}\varphi_1(x)\,dx.$$

Since $0 < u < e^u$ and $\varphi_1 > 0$ in Ω, we obtain

$$\lambda \int_\Omega e^{u(x)}\varphi_1(x)\,dx = \lambda_1 \int_\Omega u(x)\varphi_1(x)\,dx \le \lambda_1 \int_\Omega e^{u(x)}\varphi_1(x)\,dx,$$

from which we conclude that $\lambda \le \lambda_1$. So, we cannot have a solution of $(5.1)_\lambda$ for $\lambda > \lambda_1$.

Theorem 5.6. *Suppose that there exists a solution $u_0 \in \mathcal{S}$ of $(5.1)_{\lambda_0}$. Then, for each $\lambda \in [0, \lambda_0)$ there exists a solution $u \in \mathcal{R}$.*

Proof. let $\lambda < \lambda_0$. By hypothesis,

$$\begin{cases} -\Delta u_0 = \lambda_0 e^{u_0} \quad \text{in} \quad \Omega, \\ u_0 \in H_0^1(\Omega), \end{cases} \tag{5.3}$$

with $e^{u_0} \in H^{-1}(\Omega) \cap L^1(\Omega)$.

Then u_0 is a supersolution for $(5.1)_\lambda$ and $u_0 \ge 0$. Let us consider the sequence $\{u_n\}_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$ defined by

$$-\Delta u_{n+1} = \lambda e^{u_n}, \quad n \ge 0.$$

For $n = 0$ we know that $e^{u_0} \in H^{-1}(\Omega)$, so that u_1 is well defined and

$$\begin{cases} -\Delta u_1 = \lambda e^{u_0} \le \lambda_0 e^{u_0} = -\Delta u_0, \\ u_1 - u_0 \in H_0^1(\Omega), \end{cases} \tag{5.4}$$

By the maximum principle, $u_1 \le u_0$ and so $0 \le e^{u_1} \le e^{u_0}$ a.e. in Ω, from which we conclude that $e^{u_1} \in L^1(\Omega)$. Moreover, since $0 \le e^{u_1} \le e^{u_0}$ in the sense of distributions, it follows that $e^{u_1} \in H^{-1}(\Omega)$ and by iteration we conclude that $\{u_n\}_{n\in\mathbb{N}}$ and $\{e^{u_n}\}_{n\in\mathbb{N}}$ are respectively well defined in $H_0^1(\Omega)$ and $H^{-1}(\Omega)\cap L^1(\Omega)$ and they are decreasing sequences of positive functions. Therefore, $u_n \to u$ in $H_0^1(\Omega)$ with $e^u \in H^{-1}(\Omega)\cap L^1(\Omega)$, and u is a solution of $(5.1)_\lambda$ such that $u \in \mathcal{S}$.

In order to prove that $u \in \mathcal{R}$, note that $u_1, u_0 \in H_0^1(\Omega)$ satisfy

$$-\Delta u_1 = \lambda e^{u_0} = \left(\frac{\lambda}{\lambda_0}\right)\lambda_0 e^{u_0} = -\left(\frac{\lambda}{\lambda_0}\right)\Delta u_0. \tag{5.5}$$

Then we have

$$u_1 = \left(\frac{\lambda}{\lambda_0}\right)u_0 \;\Rightarrow\; e^{u_1} = e^{(\lambda/\lambda_0)u_0} \;\Rightarrow\; (e^{u_1})^{\lambda_0/\lambda} = e^{u_0} \in L^1(\Omega).$$

Therefore, as $\lambda_0 > \lambda$, $e^{u_1} \in L^{\lambda_0/\lambda}(\Omega)$ which implies that $u_2 \in W^{2,\lambda_0/\lambda}(\Omega)$.

Now, for $\theta > 0$, the mapping $f(s) := e^{s\theta}$ is convex and differentiable. So, $f(b) - f(a) \geq f'(a)(b-a)$. Taking $b := 1$ and $a := \lambda/\lambda_0$ we have

$$e^{\frac{\lambda}{\lambda_0}\theta} + \left(1 - \frac{\lambda}{\lambda_0}\right)\theta e^{\frac{\lambda}{\lambda_0}\theta} \leq e^{\theta}.$$

If $\theta := u_0(x)$, we obtain

$$e^{u_1(x)} \leq e^{u_0(x)} - \left(1 - \frac{\lambda}{\lambda_0}\right)u_0(x)e^{u_1(x)}.$$

As $u_2 \leq u_0$, we can write

$$\lambda e^{u_1} \leq \lambda e^{u_0} - \left(1 - \frac{\lambda}{\lambda_0}\right)u_2\lambda e^{u_1} \quad \text{in } \Omega. \tag{5.6}$$

On the other hand we have

$$-\Delta\left(\frac{u_2^2}{2}\right) = -u_2\Delta u_2 - |\nabla u_2|^2 \leq u_2\lambda e^{u_1}. \tag{5.7}$$

From (5.6) and (5.7) we get

$$-\Delta u_2 \leq -\Delta u_1 + \left(1 - \frac{\lambda}{\lambda_0}\right)\Delta\left(\frac{u_2^2}{2}\right)$$

or

$$-\Delta\left[u_2 + \left(1 - \frac{\lambda}{\lambda_0}\right)\left(\frac{u_2^2}{2}\right)\right] \leq -\Delta u_1.$$

From maximum principle, we have

$$u_2 + \left(1 - \frac{\lambda}{\lambda_0}\right)\left(\frac{u_2^2}{2}\right) \le u_1,$$

which implies that

$$\mathrm{e}^{u_2 + \left(1 - \frac{\lambda}{\lambda_0}\right)\left(\frac{u_2^2}{2}\right)} \le \mathrm{e}^{u_1} \in \ L^1(\Omega).$$

It follows that $\mathrm{e}^{pu_2} \in L^1(\Omega)$ $\forall p < +\infty$. Therefore, $\mathrm{e}^{u_2} \in L^p(\Omega)$, $\forall p < +\infty$, which implies that $u_3, u_4, \ldots \in L^\infty(\Omega)$. Hence, $u \in L^\infty(\Omega)$ so that $u \in \mathcal{R}$ and the proof is complete. □

Remark 5.7. It follows from Theorem 5.6 that the set of λ for which there exists a solution to $(5.1)_\lambda$ (in $\mathcal{R}$ or $\mathcal{S}$) is an interval. This interval is bounded, so it has the form $[0, \lambda^*[$, where λ^* is less than or equal to the first eigenvalue λ_1 of $-\Delta$ and we have the same critical value λ^* for solutions in $\mathcal{S}$ and $\mathcal{R}$. Moreover, if $\lambda > \lambda^*$, there is no solution in $\mathcal{S}$ (of course not in $\mathcal{R}$) and if $\lambda < \lambda^*$, there exists at least a solution in $\mathcal{R}$, Therefore, if $\lambda \in [0, \lambda^*[$, there exists a minimal solution $\underline{u}(\lambda) \in V$ and as we have seen, the linearized operator at $\underline{u}(\lambda)$ is coercive. So, locally we can apply again the implicit function theorem which shows that the curve $\lambda \mapsto \underline{u}(\lambda)$ is increasing and C^1 on $[0, \lambda^*[$.

The following picture illustrates what we have proven until now.

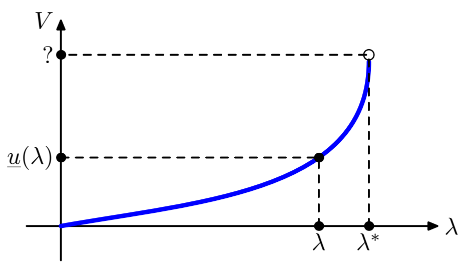

Figure 5.1. The graph of the mapping $\lambda \in [0.\lambda^*[\mapsto \underline{u}(\lambda) \in \mathcal{R}$.

5.4 What happens for $\lambda = \lambda^*$?

The answer for the question in the title of this section is given by the following theorem.

Theorem 5.8. *There exists $u^* \in \mathcal{S}$ such that $\lim_{\lambda \uparrow \lambda^*} \underline{u}(\lambda) = u^*$ in $H_0^1(\Omega)$. Moreover, if $N \le 9$, then $u^* \in \mathcal{R}$.*

Proof. For $\lambda < \lambda^*$ we know that

$$\int_\Omega \left|\nabla \underline{u}(\lambda)\right|^2 dx = \lambda \int_\Omega e^{\underline{u}(\lambda)} \underline{u}(\lambda)\, dx. \tag{5.8}$$

On the other hand we know that the linearized operator at $\underline{u}(\lambda)$ is coercive, so that

$$\int_\Omega \left|\nabla v\right|^2 dx \geq \lambda \int_\Omega e^{\underline{u}(\lambda)} |v|^2\, dx, \quad \forall v \in H_0^1(\Omega) \tag{5.9}$$

If we choose $v = \underline{u}(\lambda)$ in (5.9) and taking into acount (5.8), we obtain

$$\int_\Omega e^{\underline{u}(\lambda)} |\underline{u}(\lambda)|^2\, dx \leq \int_\Omega e^{\underline{u}(\lambda)} \underline{u}(\lambda)\, dx \tag{5.10}$$

We note that $se^s \leq \frac{1}{2} s^2 e^s$ for every $s \geq 2$. So,

$$\int_{\{\underline{u}(\lambda)\geq 2\}} e^{\underline{u}(\lambda)} \underline{u}(\lambda)\, dx \leq \frac{1}{2} \int_\Omega e^{\underline{u}(\lambda)} |\underline{u}(\lambda)|^2\, dx$$

and we get from (5.10)

$$\int_\Omega e^{\underline{u}(\lambda)} |\underline{u}(\lambda)|^2\, dx \leq \int_{\{u(\lambda)\leq 2\}} e^{\underline{u}(\lambda)} \underline{u}(\lambda)\, dx + \frac{1}{2} \int_\Omega e^{\underline{u}(\lambda)} |\underline{u}(\lambda)|^2\, dx$$

Then,

$$\frac{1}{2} \int_\Omega e^{\underline{u}(\lambda)} |\underline{u}(\lambda)|^2\, dx \leq 2e^2 \text{meas}(\Omega)$$

In the same way we obtain

$$\begin{aligned}\int_\Omega e^{\underline{u}(\lambda)} \underline{u}(\lambda)\, dx &\leq \int_{\{u(\lambda)\leq 1\}} e^{\underline{u}(\lambda)} \underline{u}(\lambda)\, dx + \int_\Omega e^{\underline{u}(\lambda)} |\underline{u}(\lambda)|^2\, dx \\ &\leq (e + 4e^2)\text{meas}(\Omega) =: C_1.\end{aligned}$$

Since $\underline{u}(\lambda)$ is solution of $(5.1)_\lambda$, we have

$$\int_\Omega \left|\nabla \underline{u}(\lambda)\right|^2 dx \leq \lambda^* \int_\Omega e^{\underline{u}(\lambda)} \underline{u}(\lambda)\, dx \leq \lambda^* C_1.$$

So, we have

$$\begin{cases} (1) \ \ \underline{u}(\lambda) \text{ is bounded in } H_0^1(\Omega); \\ (2) \ \ e^{\underline{u}(\lambda)} \text{ is bounded in } L^1(\Omega); \\ (3) \ \ e^{\underline{u}(\lambda)}\underline{u}(\lambda) \text{ is bounded in } L^1(\Omega); \\ (4) \ \ e^{\underline{u}(\lambda)}|\underline{u}(\lambda)|^2 \text{ is bounded in } L^1(\Omega). \end{cases} \tag{5.11}$$

But we know that $\underline{u}(\lambda)$ is increasing in λ. So, there exists $u^* \in H_0^1(\Omega)$ such that $\underline{u}(\lambda) \rightharpoonup u^*$ in $H_0^1(\Omega)$ weakly and $\underline{u}(\lambda) \to u^*$ in $L^2(\Omega)$ strongly. Since $\underline{u}(\lambda)$ is also increasing in $L^2(\Omega)$ and converges a.e. in Ω, we have

$$\mathrm{e}^{\underline{u}(\lambda)} \underset{\lambda\uparrow\lambda^*}{\longrightarrow} \mathrm{e}^{u^*} \quad \text{a.e. in } \Omega.$$

On the other hand, if $A \subset \Omega$ is a measurable set, we have.

$$\int_A \mathrm{e}^{\underline{u}(\lambda)} dx \le \int_{A\cap\{\underline{u}(\lambda)\le R\}} \mathrm{e}^{\underline{u}(\lambda)} dx + \frac{1}{R}\int_{A\cap\{\underline{u}(\lambda)\ge R\}} \mathrm{e}^{\underline{u}(\lambda)}\underline{u}(\lambda)\, dx \le \mathrm{e}^R \mathrm{meas}(A) + \frac{C_1}{R}.$$

If we choose $R > 0$ such that $C_1/R < \varepsilon/2$ and $\delta < \varepsilon \mathrm{e}^{-R}/2$, we see that

$$\mathrm{meas}(A) < \delta \quad \Rightarrow \quad \int_A \mathrm{e}^{\underline{u}(\lambda)} dx < \varepsilon.$$

So, by Vitali's Theorem we have $\mathrm{e}^{\underline{u}(\lambda)} \to \mathrm{e}^{u^*}$ in $L^1(\Omega)$. Therefore, it is clear that u^* is a solution of $(5.1)_{\lambda^*}$ and $u^* \in \mathcal{S}$

Now we assume that $N \le 9$. For $q > 1$, let

$$v_q := \mathrm{e}^{\frac{(q-1)}{2}\underline{u}(\lambda)} - 1 \quad \text{and} \quad w_q := \mathrm{e}^{(q-1)\underline{u}(\lambda)} - 1.$$

Then $v_q, w_q \in H_0^1(\Omega)$ and

$$\begin{cases} \nabla v_q = \dfrac{q-1}{2}\mathrm{e}^{\frac{q-1}{2}\underline{u}(\lambda)}\nabla\underline{u}(\lambda), \\ \nabla w_q = (q-1)\mathrm{e}^{(q-1)\underline{u}(\lambda)}\nabla\underline{u}(\lambda), \end{cases}$$

Hence, from (5.9) with $v = v_q$, we obtain

$$\lambda \int_\Omega \mathrm{e}^{\underline{u}(\lambda)} \left[\mathrm{e}^{\frac{q-1}{2}\underline{u}(\lambda)} - 1\right]^2 dx \le \frac{(q-1)^2}{4}\int_\Omega \mathrm{e}^{(q-1)\underline{u}(\lambda)}|\nabla\underline{u}(\lambda)|^2\, dx \tag{5.12}$$

Now, multiplying both sides of $(5.1)_\lambda$ by w_q and taking the integral on Ω, we obtain

$$\begin{aligned} (q-1)\int_\Omega \mathrm{e}^{(q-1)\underline{u}(\lambda)}|\nabla\underline{u}(\lambda)|^2 dx &= \lambda\int_\Omega \mathrm{e}^{\underline{u}(\lambda)}\left[\mathrm{e}^{(q-1)\underline{u}(\lambda)} - 1\right] dx \\ &= \lambda\int_\Omega \left[\mathrm{e}^{q\underline{u}(\lambda)} - \mathrm{e}^{\underline{u}(\lambda)}\right] dx. \end{aligned} \tag{5.13}$$

Let us rewrite the left hand side of (5.12) as

$$\int_\Omega \mathrm{e}^{\underline{u}(\lambda)}\left[\mathrm{e}^{\frac{q-1}{2}\underline{u}(\lambda)} - 1\right]^2 dx = \int_\Omega \left[\mathrm{e}^{q\underline{u}(\lambda)} - \mathrm{e}^{\underline{u}(\lambda)} + 2\big(\mathrm{e}^{\underline{u}(\lambda)} - \mathrm{e}^{\frac{q+1}{2}\underline{u}(\lambda)}\big)\right] \tag{5.14}$$

So, from (5.12), (5.13) and (5.14), we obtain

$$\int_\Omega \Big[e^{q\underline{u}(\lambda)} - e^{\underline{u}(\lambda)} + 2\big(e^{\underline{u}(\lambda)} - e^{\frac{q+1}{2}\underline{u}(\lambda)}\big)\Big] \leq \frac{q-1}{4}\int_\Omega \big[e^{q\underline{u}(\lambda)} - e^{\underline{u}(\lambda)}\big]\, dx$$

or equivalently

$$(5-q)\int_\Omega \big[e^{q\underline{u}(\lambda)} - e^{\underline{u}(\lambda)}\big]\, dx \leq 8\int_\Omega \big[e^{\frac{q+1}{2}\underline{u}(\lambda)} - e^{\underline{u}(\lambda)}\big]\, dx$$

Since $q > (q+1)/2$, this last inequality implies that $e^{\underline{u}(\lambda)}$ is bounded in $L^q(\Omega)$ if $q \leq 5$ and, consequently, $\underline{u}(\lambda)$ is bounded in $W^{2,q}(\Omega)$.

But we know that $W^{2,q}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ if $N/q < 2$. This shows that $u^* \in C^0(\overline{\Omega})$ if $N < 10$ and consequently, $u^* \in \mathcal{R}$. This finishes the proof. □

Remark 5.9. It is noteworthy that the estimate $N \leq 9$ is sharp, as we will see later on.

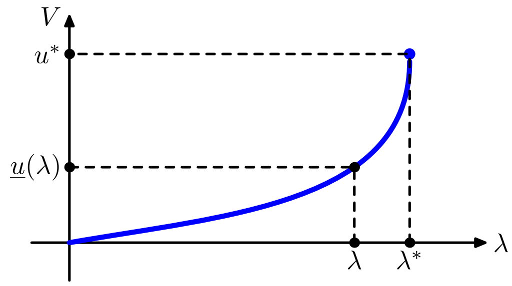

Figura 5.2. The graph of the mapping $\lambda \in [0.\lambda^*]$ including the singular solution at λ^*.

The previous theorem uses the fact that $\underline{u}(\lambda)$ is the minimal solution of $(5.1)_\lambda$ (see (5.9)). In fact, if Ω is star shaped and $N \geq 3$, we can obtain estimates on all solutions $u \in \mathcal{R}$.

Theorem 5.10. *Let Ω be a star shaped bounded domain of $\mathbb{R}^N$ with $N \geq 3$. For simplicity we assume that it is star shaped at the origin. Then, there exists $M > 0$ such that, for all $\lambda \in [0, \lambda^*[$ and for all $u(\lambda) \in \mathcal{R}$, we have*

$$\begin{cases} \|u(\lambda)\|_{H_0^1(\Omega)} \leq M\lambda^{1/2} \\ \|e^{u(\lambda)}\|_{L^1(\Omega)} \leq M \\ \|e^{u(\lambda)}u(\lambda)\|_{L^1(\Omega)} \leq M \end{cases} \tag{5.15}$$

Proof. It is based on the Pohozaev identity, which we obtain by multiplying $(5.1)_\lambda$ by $x \cdot \nabla u(x)$. More precisely, if $u = u(\lambda)$ is a solution of $(5.1)_\lambda$ and recalling that u is regular, we obtain

$$\int_\Omega -\Delta u(x)\big(x\cdot\nabla u(x)\big)\,dx = \lambda\int_\Omega \mathrm{e}^{u(x)}\big(x\cdot\nabla u(x)\big)\,dx = \lambda\int_\Omega x\cdot\nabla\big(\mathrm{e}^{u(x)}-1\big)\,dx,$$

from which we obtain after integrating by parts,

$$\frac{N-2}{2}\int_\Omega |\nabla u(x)|^2 dx + \frac{1}{2}\int_\Gamma |\nabla u(\sigma)|^2\big(\sigma\cdot\nu(\sigma)\big)\,d\sigma = N\lambda\int_\Omega \big(\mathrm{e}^{u(x)}-1\big)\,dx,$$

Now, by multiplying $(5.1)_\lambda$ by u, we obtain

$$\int_\Omega |\nabla u(x)|^2 dx = \lambda\int_\Omega \mathrm{e}^{u(x)}u(x)\,dx.$$

Since we are assuming that Ω is star shaped at the origin, it follows that $\sigma\cdot\nu(\sigma)\geq 0$ for $\sigma\in\Gamma$, and we obtain

$$\int_\Omega \mathrm{e}^{u(x)}u(x)\,dx \leq \frac{2N}{N-2}\int_\Omega \big(\mathrm{e}^{u(x)}-1\big)\,dx.$$

This implies that there exists $M>0$ (which depends only on N and on Ω) such that

$$\int_\Omega \mathrm{e}^{u(x)}u(x)\,dx \leq M$$

which gives the estimate (5.15). □

Remark 5.11. Since we know that $\lambda\leq\lambda^*$ is a necessary condition for existence of solutions to $(5.1)_\lambda$, Theorem 5.10 implies that $\mathcal{R}$ is a bounded subset of $H_0^1(\Omega)$ and also that the set $\{\mathrm{e}^u\,;\,u\in\mathcal{R}\}$ is bounded in $L^1(\Omega)$. So, we can summarise these facts by saying that $\mathcal{R}$ is bounded in the class $\mathcal{S}$.

As a consequence of Theorem 5.10 we have the following result on a relative compactness of $\mathcal{R}$ in the class $\mathcal{S}$ of singular solutions.

Corollary 5.12. *For each sequence $\{u(\lambda_n)\}_{n\in\mathbb{N}}$, $\lambda_n\in[0,\lambda^*[$ of regular solutions (in $\mathcal{R}$) of $(5.1)_{\lambda_n}$ we can extract subsequences $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and $\{u(\lambda_{n_k})\}_{k\in\mathbb{N}}$ such that*

(1) $\lambda_{n_k}\to\lambda_0$ *in* $\mathbb{R}$;

(2) $u(\lambda_{n_k})\rightharpoonup u_0$ *in* $H_0^1(\Omega)$ *weakly;*

(3) $\mathrm{e}^{u(\lambda_{n_k})}\to e^{u_0}$ *in* $L^1(\Omega)$ *strongly,*

where $u_0 \in \mathcal{S}$ *is a singular solution of* $(5.1)_{\lambda_0}$.

Proof. Since $\{\lambda_n\}_{n\in\mathbb{N}}$ is bounded, we can extract a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ such that $\lambda_{n_k} \to \lambda_0$, for some λ_0. From (5.15) and the same subsequent arguments in the proof of Theorem 5.10, we obtain the conclusion by application of Vitali's Theorem. □

5.5 Can we use the Mountain Pass Theorem for $\lambda < \lambda^*$?

Before answering the question in the title of the present section, let us make some following formal considerations.

Let $F : \mathbb{R} \to \mathbb{R}$ the function "defined" by $F(t) := J(u + tv)$, where $u, v \in H_0^1(\Omega)$ and

$$J(w) := \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \lambda \int_\Omega e^w\,dx.$$

Then, by formal calculations we obtain

$$\begin{aligned} F'(t) &= \int_\Omega \nabla u \cdot \nabla v\,dx + t\int_\Omega |\nabla v|^2\,dx - \lambda\int_\Omega e^u e^{tv} v\,dx,\\ F''(t) &= \int_\Omega |\nabla v|^2\,dx - \lambda\int_\Omega e^u e^{tv}|v|^2\,dx, \end{aligned}$$

If $u = \underline{u}(\lambda)$ is the minimal solution of $(5.1)_\lambda$, we have seen that

$$\left.\begin{aligned} F'(0) &= \int_\Omega \nabla\underline{u}(\lambda)\cdot\nabla v\,dx - \lambda\int_\Omega e^{\underline{u}(\lambda)} v\,dx = 0,\\ F''(0) &= \int_\Omega |\nabla v|^2\,dx - \lambda\int_\Omega e^{\underline{u}(\lambda)}|v|^2\,dx > 0, \end{aligned}\right. \qquad \forall v \in H_0^1(\Omega).$$

So, $\underline{u}_\lambda$ is a local minimum of J. Since $J\big(t\underline{u}(\lambda)\big) \to -\infty$ as $t \to +\infty$, there should be another critical point different from $\underline{u}(\lambda)$.

The problem in this argument is that in dimension $N \geq 3$, we don't have in general $e^u \in L^1(\Omega)$ for $u \in H_0^1(\Omega)$, so that the function F is not well defined. Moreover, in dimension $N = 2$, it follows from the Trudinger-Moser inequality that if $u \in H_0^1(\Omega)$, there exists $k > 0$ such that $e^{ku^2} \in L^1(\Omega)$. So, in this case we can apply the mountain pass theorem to show that for every $\lambda < \lambda^*$, there exists $u(\lambda)$ solution of $(5.1)_\lambda$ with $u(\lambda) > \underline{u}(\lambda)$.

For $N \geq 3$ and assuming that Ω is star-shaped, we have the following result. This result was communicated to us by X. Cabré [6], but it was already proved in [25].

Theorem 5.13. *Let $N \geq 3$. If $\Omega \subset \mathbb{R}^N$ is star-shaped, there exists $\lambda_0 > 0$ such that for $0 \leq \lambda < \lambda_0$, the problem $(5.1)_\lambda$ does not have any other solution in $\mathcal{R}$ than $\underline{u}(\lambda)$.*

Proof. We can assume without loss of generality that Ω is star shaped with respect to the origin.

Suppose that we have another regular solution $u \in \mathcal{R}$ of $(5.1)_\lambda$, $\lambda < \lambda_0$. Then, $u > \underline{u}$, where for simplicity we denote $\underline{u} := \underline{u}(\lambda)$ and $w := u - \underline{u}$. So, we have

$$\begin{cases} -\Delta w = \lambda(\mathrm{e}^u - \mathrm{e}^{\underline{u}}) \ \text{ in } \Omega, \\ w = 0 \ \text{ on } \Gamma, \end{cases} \tag{5.16}$$

By multiplying both sides of equation (5.16) by $x \cdot \nabla w(x)$, we have for its left hand side (see the calculations for the Pohozaev identity),

$$-\int_\Omega \Delta w(x)\big(x \cdot \nabla w(x)\big)\,dx = \left(1 - \frac{N}{2}\right)\int_\Omega |\nabla w(x)|^2\,dx - \frac{1}{2}\int_\Gamma \big(\sigma \cdot \nu(\sigma)\big)|\nabla w(\sigma)|^2\,d\sigma.$$

Since we are assuming that Ω is star-shaped with respect to the origin, it follows that $\sigma \cdot \nu \geq 0$ for every $\sigma \in \Gamma$ and we obtain

$$-\int_\Omega \Delta w(x)\big(x \cdot w(x)\big)\,dx \leq \left(1 - \frac{N}{2}\right)\int_\Omega |\nabla w(x)|^2\,dx \tag{5.17}$$

Concerning the right hand side and by observing that $\nabla\big[\mathrm{e}^w - 1 - w\big] = \big[\mathrm{e}^w - 1\big]\nabla w$, we have

$$\begin{aligned}\int_\Omega [\mathrm{e}^{u(x)} - \mathrm{e}^{\underline{u}(x)}](x \cdot \nabla w(x))\,dx &= \int_\Omega \mathrm{e}^{\underline{u}(x)}\big[\mathrm{e}^{w(x)} - 1\big](x \cdot \nabla w(x))\,dx \\ &= \int_\Omega \mathrm{e}^{\underline{u}(x)}\,\nabla\big[\mathrm{e}^{w(x)} - 1 - w(x)\big] \cdot x\,dx.\end{aligned} \tag{5.18}$$

It is clear that

$$\operatorname{div}\Big(\mathrm{e}^{\underline{u}}\big[\mathrm{e}^w - 1 - w\big]x\Big) = \mathrm{e}^{\underline{u}}(\nabla\underline{u} \cdot x)[\mathrm{e}^w - 1 - w] + \mathrm{e}^{\underline{u}}\big(\nabla[\mathrm{e}^w - 1 - w] \cdot x\big) + N\mathrm{e}^{\underline{u}}[\mathrm{e}^w - 1 - w].$$

Since $\mathrm{e}^w - 1 - w = 0$ on Γ, it follows from Gauss Theorem that

$$\int_\Omega \operatorname{div}\Big(\mathrm{e}^{\underline{u}(x)}\big[\mathrm{e}^{w(x)} - 1 - w(x)\big]x\Big)dx = 0.$$

Therefore,

$$\begin{aligned}\int_\Omega e^{\underline{u}(x)} \left(\nabla \left[e^{w(x)} - 1 - w(x)\right] \cdot x\right) dx \\ = -\int_\Omega e^{\underline{u}(x)} \left(\nabla \underline{u}(x) \cdot x\right) \left[e^{w(x)} - 1 - w(x)\right] dx \\ - N \int_\Omega e^{\underline{u}(x)} \left[e^{w(x)} - 1 - w(x)\right] dx.\end{aligned}$$

As $|x \cdot \nabla \underline{u}| \le M$ for some $M > 0$ and $e^{\underline{u}} \left[e^w - 1 - w\right] \ge 0$, we have

$$\begin{aligned}\int_\Omega e^{\underline{u}(x)} \left(\nabla \left[e^{w(x)} - 1 - w(x)\right] \cdot x\right) dx \\ \ge -(N + M) \int_\Omega e^{\underline{u}(x)} \left[e^{w(x)} - 1 - w(x)\right] dx.\end{aligned} \tag{5.19}$$

From (5.17), (5.18) and (5.19), we have

$$\begin{cases} -\displaystyle\int_\Omega \Delta w (x \cdot \nabla w)\, dx \le \left(1 - \frac{N}{2}\right) \int_\Omega |\nabla w|^2\, dx \\ \lambda \displaystyle\int_\Omega \left[e^u - e^{\underline{u}}\right] (x \cdot \nabla w)\, dx \ge -\lambda (N + M) \int_\Omega e^{\underline{u}} \left[e^w - 1 - w\right] dx \end{cases}$$

from which we obtain

$$\frac{N-2}{2} \int_\Omega |\nabla w|^2\, dx \le \lambda (N + M) \int_\Omega e^{\underline{u}} \left[e^w - 1 - w\right] dx \tag{5.20}$$

On the other hand, we have from (5.16),

$$-\frac{N-2}{4} \int_\Omega |\nabla w|^2\, dx = -\lambda \frac{N-2}{4} \int_\Omega \left[e^u - e^{\underline{u}}\right] w\, dx \tag{5.21}$$

So, adding (5.20) and (5.21), we get

$$\frac{N-2}{4} \int_\Omega |\nabla w|^2\, dx + \lambda \frac{N-2}{4} \int_\Omega \left[e^u - e^{\underline{u}}\right] w \le \lambda (N + M) \int_\Omega e^{\underline{u}} \left[e^w - 1 - w\right] dx \tag{5.22}$$

Now, we consider

$$\int_\Omega e^{\underline{u}} \left[e^w - 1 - w\right] dx \le \frac{1}{k} \int_{\Omega \cap \{w \ge k\}} e^{\underline{u}} \left[e^w - 1 - w\right] w\, dx + \int_{\Omega \cap \{0 \le w \le k\}} e^{\underline{u}} \left[e^w - 1 - w\right] dx$$

For $k = (N-2)/4(N+M)$ and the fact that $\mathrm{e}^w - 1 - w \leq \mathrm{e}^k w^2/2$ if $0 \leq w \leq k$, we obtain from (5.22),

$$\begin{aligned} &\frac{N-2}{4}\int_\Omega |\nabla w|^2\,dx + \lambda\frac{N-2}{4}\int_\Omega \left[\mathrm{e}^u - \mathrm{e}^{\underline{u}}\right] w \\ &\leq \lambda\frac{N-2}{4}\int_\Omega \mathrm{e}^{\underline{u}}\left[\mathrm{e}^w - 1 - w\right] w\,dx + \lambda(N+M)\int_\Omega \frac{\mathrm{e}^{\underline{u}}\,\mathrm{e}^k}{2} w^2\,dx. \end{aligned} \tag{5.23}$$

Therefore,

$$\frac{N-2}{4}\int_\Omega |\nabla w|^2\,dx \leq \lambda(N+M)\frac{\mathrm{e}^k}{2}\int_\Omega \mathrm{e}^{\underline{u}} w^2\,dx \leq \lambda M' \int_\Omega w^2\,dx.$$

and we can choose λ_0 such that that $w = 0$ if $\lambda < \lambda_0$. With this contradiction the proof is finished. □

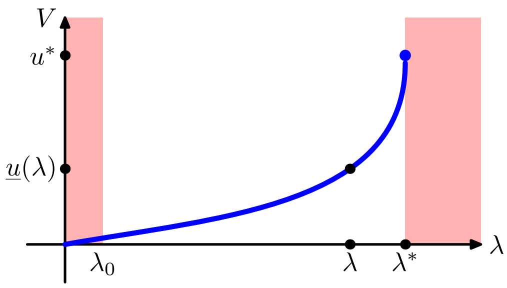

Figure 5.3. There is no solution in the right shadowed region and no regular solution different from $\underline{u}(\lambda)$ in the left shadowed region.

Remark 5.14. In particular, we can infer from the Theorem 5.13 that we cannot apply the mountain pass theorem for λ small enough. However, this result does not say that there is no other solution than $\underline{u}(\lambda)$ for λ small because this eventual other solution might be *not regular* (even more singular than $\mathcal{S}$)

5.6 Case $N \leq 9$. Solutions near (λ^*, u^*)

For $N \leq 9$ we know that $u^* \in \mathcal{R}$. We also know that for $\lambda > \lambda^*$ there is no solution. Therefore, we cannot apply the implicit function theorem at (λ^*, u^*) and this implies that the linearized operator at u^* is no longer coercive. So, there exists φ_1 such that

$$\begin{cases} -\Delta\varphi_1 = \lambda^* \mathrm{e}^{u^*}\varphi_1 & \text{in } \Omega, \\ \varphi_1 \in H_0^1(\Omega), \ \varphi_1 > 0, \end{cases}$$

i.e., φ_1 is the (unique if normalized) eigenfunction corresponding to this problem.

Let us look for solutions (λ, u) near (λ^*, u^*), with $\lambda < \lambda^*$, of the form

$$u = \varepsilon\varphi_1 + v + u^*, \quad v \perp \varphi_1.$$

By a straightforward calculation we have

$$-\Delta u - \lambda e^u = -\Delta v - \lambda^* e^{u^*} v - (\lambda - \lambda^*) e^{u^*+v+\varepsilon\varphi_1} - \lambda^* e^{u^*}\big[e^{v+\varepsilon\varphi_1} - \varepsilon\varphi_1 - v - 1\big].$$

This leads us to consider the function $F : \mathbb{R} \times [0, \lambda^*) \times V$ defined by

$$F(\varepsilon, \lambda, v) := -\Delta v - \lambda^* e^{u^*} v - (\lambda - \lambda^*) e^{u^*+v+\varepsilon\varphi_1} - \lambda^* e^{u^*}\big[e^{v+\varepsilon\varphi_1} - \varepsilon\varphi_1 - v - 1\big],$$

where $V := W^{2,p}(\Omega) \cap H_0^1(\Omega) \cap \varphi_1^\perp$. From Lemma 5.3 we know that F is C^1 if $p > N/2$ and it is clear that $F(0, \lambda^*, 0) = 0$. Let us consider the equation

$$D_{(\lambda,v)}F(0, \lambda^*, 0)[\mu, w] = \frac{\partial F}{\partial \lambda}(0, \lambda^*, 0)[\mu] + \frac{\partial F}{\partial v}(0, \lambda^*, 0)[w] = f \in L^p(\Omega),$$

which means that

$$-\Delta w - \lambda^* e^{u^*} w = f + \mu e^{u^*}. \tag{5.24}$$

In order to apply the Implicit Function Theorem we must show that (5.24) admits a unique solution (μ, w). But we know from the Fredholm alternative (see Theorem 2.26), that this equation has a solution if, and only if, $f + \mu e^{u^*}$ is orthogonal to φ_1, which is clear if we choose

$$\mu_* := \int_\Omega f(x)\varphi_1(x)\,dx \left(\int_\Omega e^{u^*(x)}\varphi_1(x)\right)^{-1} > 0$$

So, the problem

$$\begin{cases} -\Delta w - \lambda^* e^{u*} = f + \mu_* e^{u^*}, \\ \quad w \in H_0^1(\Omega), \quad w \perp \varphi_1, \end{cases}$$

has a unique solution w, and we can apply the Implicit Function Theorem, i.e., there exist $\varepsilon_0 > 0$, $\nu > 0$ and $r > 0$ such that the equation $F(\varepsilon, \lambda, v) = 0$ has solutions in $(-\varepsilon_0, \varepsilon_0) \times (\lambda^* - \nu, \lambda^* + \nu) \times B_r(0)$ and all solutions are the form $\lambda = \lambda(\varepsilon)$, $v = v(\varepsilon)$, where $\varepsilon \mapsto \big(\lambda(\varepsilon), v(\varepsilon)\big)$ is of class C^1 and $\lambda(0) = \lambda^*$, $v(0) = 0$.

In short, we show that for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ there exists a solution of $(5.1)_{\lambda(\varepsilon)}$ of the form

$$u(\varepsilon) = u^* + \varepsilon\varphi_1 + v(\varepsilon),$$

where $F\big(\varepsilon, \lambda(\varepsilon), v(\varepsilon)\big) = -\Delta u(\varepsilon) - \lambda(\varepsilon) e^{u^*} u(\varepsilon) = 0.$

Note that as $F\big(\varepsilon, \lambda(\varepsilon), v(\varepsilon)\big) = 0$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ we obtain by implicit differentiation at $\varepsilon = 0$,

$$-\frac{\partial \lambda}{\partial \varepsilon}(0)\mathrm{e}^{u^*} - \Delta \frac{\partial v}{\partial \varepsilon}(0) - \lambda^* \mathrm{e}^{u^*} \frac{\partial v}{\partial \varepsilon}(0) = 0$$

which implies that

$$\lambda_1 := \frac{\partial \lambda}{\partial \varepsilon}(0) = 0 \quad \text{and} \quad v_1 := \frac{\partial v}{\partial \varepsilon}(0) = 0.$$

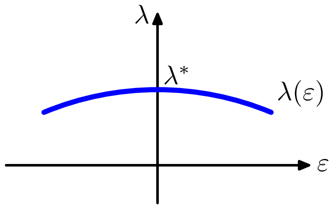

Figura 5.4. The graph of $\varepsilon \mapsto \lambda(\varepsilon)$ near the origin.

Since F is C^2 (if we assume that p is large enough), we can consider the development of $\lambda(\varepsilon)$ and $v(\varepsilon)$, i.e.,

$$\begin{cases} \lambda(\varepsilon) = \lambda^* + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + o(\varepsilon^2), \\ v(\varepsilon) = 0 + \varepsilon v_1 + \varepsilon^2 v_2 + o(\varepsilon^2), \end{cases}$$

where $\lambda_1 = 0$ and $v_1 = 0$. Hence $F\big(\varepsilon, \lambda(\varepsilon), v(\varepsilon)\big) = 0$ writes as

$$-\Delta v(\varepsilon) - \lambda^* \mathrm{e}^{u^*} v(\varepsilon) = (\lambda(\varepsilon) - \lambda^*)\mathrm{e}^{u^* + v(\varepsilon) + \varepsilon\varphi_1} + \lambda^* \mathrm{e}^{u^*}\big[\mathrm{e}^{v(\varepsilon) + \varepsilon\varphi_1} - v(\varepsilon) - \varepsilon\varphi_1 - 1\big].$$

From the asymptotic development, we have

$$(\lambda(\varepsilon) - \lambda^*)\mathrm{e}^{u^* + v(\varepsilon) + \varepsilon\varphi_1} = \varepsilon^2 \mathrm{e}^{u^*}\lambda_2 + o(\varepsilon^2),$$
$$\lambda^* \mathrm{e}^{u^*}\big[\mathrm{e}^{v(\varepsilon) + \varepsilon\varphi_1} - v(\varepsilon) - \varepsilon\varphi_1 - 1\big] = \varepsilon^2 \frac{\lambda^* \mathrm{e}^{u^*}}{2}\varphi_1^2 + o(\varepsilon^2),$$

So, discarding the terms of $o(\varepsilon^2)$, we obtain

$$-\Delta v_2 - \lambda^* \mathrm{e}^{u^*} v_2 = \mathrm{e}^{u^*}\lambda_2 + \frac{\lambda^* \mathrm{e}^{u^*}}{2}\varphi_1^2. \tag{5.25}$$

From the Fredholm Alternative, a necessary and sufficient condition for existence of a unique solution v_2 of (5.25) is that the right hand side term be orthogonal to φ_1, which is achieved by

$$\lambda_2 := -\frac{\lambda^*}{2} \int_\Omega \mathrm{e}^{u^*(x)}\varphi_1^3(x)\,dx \left(\int_\Omega \mathrm{e}^{u^*(x)}\varphi_1(x)\,dx\right)^{-1} < 0$$

Hence, for $\lambda < \lambda^*$ close to λ^*, we have two solutions of $(5.1)_\lambda$ corresponding to the mapping

$$\varepsilon \in (-\varepsilon_0, \varepsilon_0) \mapsto \lambda(\varepsilon) \in (\lambda^* - \nu, \lambda^*],$$

namely one for $\varepsilon > 0$ and one for $\varepsilon < 0$, more precisely,

$$\begin{aligned} u_1(\varepsilon) &:= u^* + \varepsilon\varphi_1 + v(\varepsilon), \quad \varepsilon \in (-\varepsilon_0, 0]; \\ u_2(\varepsilon) &:= u^* + \varepsilon\varphi_1 + v(\varepsilon), \quad \varepsilon \in [0, \varepsilon_0); \end{aligned}$$

Then,

$$\int_\Omega \big(u_1(\varepsilon)(x) - u^*(x)\big)\varphi_1(x)\,dx = \varepsilon \int_\Omega \varphi_1^2(x)\,dx < 0.$$

So, for $\varepsilon < 0$ we have $u_1(\varepsilon) = \underline{u}(\lambda(\varepsilon))$, i.e., it is the solution obtained previously. The same argument shows that if $\varepsilon > 0$ we have $u_2(\varepsilon) > \underline{u}(\lambda(\varepsilon))$, which corresponds to a new solution. This means that (λ^*, u^*) is a *turning point*, as shown in the figure.

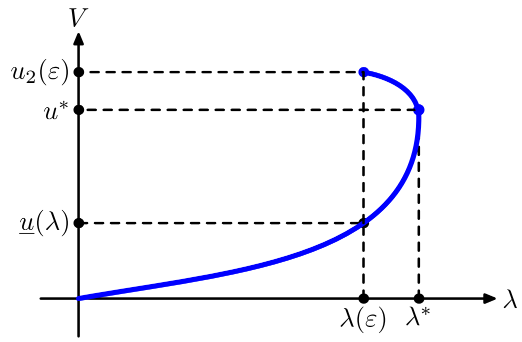

Figure 5.5. Picture of the turning point (λ^*, u^*).

Proposition 5.15. *The linearized operator at $u_2(\varepsilon)$ is not positive for every $\varepsilon > 0$ small enough.*

Proof. Let $\varepsilon \in (0, \varepsilon_0)$. We know that

$$\begin{cases} \lambda(\varepsilon) = \lambda^* + \varepsilon^2\lambda_2 + o(\varepsilon^2), \quad \lambda_2 < 0; \\ u_2(\varepsilon) = u^* + \varepsilon\varphi_1 + v(\varepsilon), \quad v(\varepsilon) \perp \varphi_1. \end{cases} \tag{5.26}$$

Let

$$I := \int_\Omega |\nabla\varphi_1(x)|^2\,dx - \lambda(\varepsilon) \int_\Omega e^{u_2(\varepsilon)(x)} \varphi_1^2(x)\,dx. \tag{5.27}$$

We also know that

$$\int_\Omega |\nabla\varphi_1(x)|^2\,dx - \lambda^* \int_\Omega e^{u^*(x)}\varphi_1^2(x)\,dx = 0. \tag{5.28}$$

So, subtracting (5.28) from (5.27) and considering (5.26), we obtain for $\varepsilon > 0$ small enough,

$$\begin{aligned} I &= \lambda^* \int_\Omega e^{u^*}\varphi_1^2\,dx - \lambda(\varepsilon)\int_\Omega e^{u_2(\varepsilon)}\varphi_1^2\,dx \\ &= \lambda^* \int_\Omega e^{u^*}\varphi_1^2\,dx - (\lambda^* + \varepsilon^2\lambda_2)\int_\Omega e^{u^*}\left[e^{\varepsilon\varphi_1 + v(\varepsilon)}\right]\varphi_1^2\,dx + o(\varepsilon^2) \\ &= \lambda^* \int_\Omega e^{u^*}\varphi_1^2\,dx - \lambda^*\int_\Omega e^{u^*}\left[1+\varepsilon\varphi_1\right]\varphi_1^2\,dx + o(\varepsilon^2) \\ &= -\varepsilon\lambda^*\int_\Omega \varphi_1^3\,dx + o(\varepsilon^2) < 0. \end{aligned}$$

This proves that $-\Delta - \lambda(\varepsilon)e^{u_2(\varepsilon)}$ is not a positive operator, as we wanted to prove. □

Remark 5.16. Since we know from Corollary 5.12 that $\mathcal{R}$ is relatively compact in $\mathcal{S}$, a natural question is to ask under what conditions the limit of regular solutions is regular or not. In the case where Ω is the unit ball $B_1(0)$, Gelfand showed for $3 \le N \le 9$ and for $\lambda^{**} = 2(N-2)$ that the problem $(5.1)_{\lambda^{**}}$ has a sequence of regular solutions $\{u_n\}_{n\in\mathbb{N}}$ with $u_n(0) \to +\infty$. So, from Corollary 5.12, this sequence has a limit u^{**}. As we will see below, $u^{**}(x) = -2\ln(|x|)$, which belongs to $\mathcal{S}\setminus\mathcal{R}$. Moreover, for $N \ge 10$, Joseph and Lundgren [14] showed that the curve of minimal solutions $\lambda \mapsto \underline{u}(\lambda)$ is unbounded in $L^\infty(\Omega)$. In fact in this case, $\lambda^{**} = \lambda^*$ and

$$\underline{u}(\lambda) \underset{\lambda\uparrow\lambda^*}{\rightharpoonup} \ln\left(\frac{1}{|x|^2}\right) \quad \text{weakly in } H_0^1(\Omega).$$

This shows that Corollary 5.12 can be considered as an optimal result.

5.7 Radial symmetric solutions in a ball

In this section we will consider $\Omega = B_R(0)$ for $R > 0$, i.e., the ball of $\mathbb{R}^N$ with radius R and center at the origin, where $N \ge 3$.

We know from Gidas, Ni and Nirenberg [12] that a regular solution is radial. Hence, each solution in $\mathcal{S}$ that is limit of regular solutions is also radial. A natural question is to know if every singular solution of (5.1) is radial. The answer is negative, as has been proved by Matano [17] and Rebai [22]. They have showed that in $\mathbb{R}^3$, for each x_0 near 0, there exists a solution of (5.1) with an isolated singularity at x_0. So, these solutions cannot be limit of regular solutions and, in particular, this shows that the result of Gidas, Ni and Nirenberg cannot be extended to singular solutions.

The main result of this section is the following.

Theorem 5.17. *Let* $\lambda > 0$ *and* $u \in L^1\big(B_R(0) \setminus \{0\}\big)$ *a radial function such that* $e^u \in L^1\big(B_R(0) \setminus \{0\}\big)$ *and*

$$\begin{cases} -\Delta u = \lambda e^u \quad \text{in } B_R(0) \setminus \{0\}, \\ u(x) = 0 \quad \text{for } |x| = R, \\ \lim\limits_{|x| \to 0} u(x) = +\infty. \end{cases} \tag{5.29}$$

Then,

$$\lambda = \frac{2(N-2)}{R^2} \quad \text{and} \quad u(x) = \ln\left(\frac{R^2}{|x|^2}\right).$$

Consequently, $u \in \mathcal{S}$ *and it is the unique singular radial solution of* (5.1) *in* $B_R(0)$.

Before proceeding with the proof, we need the following result.

Proposition 5.18. *Let* $\lambda > 0$ *and* $u \in L^1\big(B_R(0) \setminus \{0\}\big)$ *be a radial function such that* $e^u \in L^1\big(B_R(0) \setminus \{0\}\big)$ *and that it is bounded from below on a neighbourhood of the origin. If* u *satisfies*

$$-\Delta u = e^u \quad \text{in } B_R(0) \setminus \{0\},$$

then u *can be extended to a radial function (still denoted by* u*) in* $\mathbb{R}^N \setminus \{0\}$, $u \in C^\infty\big(\mathbb{R}^N \setminus \{0\}; \mathbb{R}\big)$ *satisfying*

$$-\Delta u = \lambda e^u \quad \text{in } \mathbb{R}^N \setminus \{0\} \quad \text{and} \quad \nabla u(x) \cdot x < 0, \ \forall x \neq 0.$$

Proof. Since u is radial, we have $u(x) = U(|x|)$, where $U :]0, R[\to \mathbb{R}$ with

$$-\frac{d}{dr}\left(r^{N-1}\frac{dU}{dr}\right) = \lambda r^{N-1} e^U \quad \text{in }]0, R[\ . \tag{5.30}$$

So,

$$\frac{dU}{dr} \in L^1_{\text{loc}}(]0, R[), \ \frac{d^2U}{dr^2} \in L^1_{\text{loc}}(]0, R[) \ \text{and} \ U \in C^0(]0, R[).$$

By iteration, it follows that $U \in C^\infty(]0,R[)$ and the mapping $r \mapsto r^{N-1}\frac{dU}{dr}$ is strictly decreasing.

Let us assume that there exists $r_0 \in (]0,R[)$ such that $\frac{dU}{dr}(r_0) \geq 0$. Then, for $r < r_0/2$, we have

$$r^{N-1}\frac{dU}{dr}(r) > \left(\frac{r_0}{2}\right)^{N-1}\frac{dU}{dr}(r_0/2) > r_0^{N-1}\frac{dU}{dr}(r_0) \geq 0.$$

Therefore, for $C := (r_0/2)\frac{dU}{dr}(r_0/2)$, we have

$$\frac{dU}{dr}(r) > \frac{C}{r^{N-1}}, \quad \forall\, r < \frac{r_0}{2},$$

from which we obtain that

$$\lim_{r\to 0} U(r) = -\infty$$

in contradiction with the hypothesis that U is bounded from below on a neighborhood of 0.

Hence,

$$\frac{dU}{dr}(r) < 0 \quad \forall\, r \in]0,R[\,.$$

As U is a solution of (5.30), it can be extended to a maximal interval $]a,\overline{R}]$ with $a \leq 0$ and $\overline{R} \geq R$. To complete the proof, we must show that $\overline{R} = +\infty$.

Let us assume that $\overline{R} < +\infty$. By the same arguments as before, we obtain

$$\frac{dU}{dr}(r) < 0 \quad \forall\, r \in [0,\overline{R}[\,.$$

Since U is decreasing, it is clear that

$$\lambda \int_{R/2}^{\overline{R}} \mathrm{e}^U(r) r^{N-1} dr < +\infty$$

and we have from (5.30) that the limit of $r^{N-1}\frac{dU}{dr}(r)$ as $r \to \overline{R}$ is finite. So, the same is true for U, which is in contradiction with the maximality if $\overline{R}$. $\square$

With the Proposition 5.18 in hands, we can prove the following result, from which Theorem 5.17 is an immediate consequence.

Theorem 5.19. *Let $\lambda > 0$ and $u \in L^1_{\mathrm{loc}}\big(\mathbb{R}^N \setminus \{0\}\big)$ a radial function such that $\mathrm{e}^u \in L^1_{\mathrm{loc}}\big(\mathbb{R}^N \setminus \{0\}\big)$ and*

$$\begin{cases} -\Delta u = \lambda \mathrm{e}^u & \text{in } \mathbb{R}^N \setminus \{0\}, \\ \lim\limits_{|x|\to 0} u(x) = +\infty. \end{cases} \tag{5.31}$$

Then,

$$u(x) = \ln\left(\frac{1}{|x|^2}\right) - \ln\left(\frac{\lambda}{2(N-2)}\right).$$

Consequently, we have

$$u \in H^1_{\text{loc}}(\mathbb{R}^N), \quad e^u \in L^1_{\text{loc}}(\mathbb{R}^N)$$

and u is the unique radial solution of

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to 0} u(x) = +\infty. \end{cases} \tag{5.32}$$

From the Proposition 5.18, the problem 5.31 consists to look for a function $U \in C^\infty(]0,+\infty[\,;\mathbb{R})$ such that

$$-\frac{d}{dr}\left(r^{N-1}\frac{dU}{dr}\right) = \lambda r^{N-1} e^U \text{ in }]0,+\infty[. \tag{5.33}$$

It is clear that a regular solution satisfies $\frac{dU}{dr}(0) = 0$, so that a singular one must satisfy $\lim_{r\to 0} U(r) = +\infty$.

The proof of Theorem 5.19 follows as consequence of several lemmas. In the first one we associated the solutions of (5.33) to a 2-dimensional dynamic system which we will analyse. This is an idea by L. Tartar [27], where one of the advantages is that the critical points have a finite distance.

Lemma 5.20. *Let U be a solution of* (5.33) *and associated to $V : \mathbb{R} \to \mathbb{R}$ defined as $V(s) := U(e^s)$, we consider the following functions*

$$v(s) := \frac{dV}{ds}(s) \quad \text{and} \quad w(s) = \frac{d^2V}{ds}(s) + (N-2)\frac{dV}{ds}(s).$$

Then, v and w satisfy the system

$$\begin{cases} \dfrac{dv}{ds} = w - (N-2)v, \\ \dfrac{dw}{ds} = (v+2)w, \end{cases} \quad \forall\, s \in \mathbb{R} \tag{5.34}$$

and

$$-w(s)e^{-2s}e^{-V(s)} = \lambda, \quad \forall\, s \in \mathbb{R}. \tag{5.35}$$

Conversely, if (v,w) is a trajectory of (5.34) *defined on $\mathbb{R}$ and*

$$U(r) := \int_0^{\ln(r)} V(s)\, ds + a, \quad (a = U(1)), \tag{5.36}$$

then $w(s)e^{-2s}e^{V(s)}$ is a constant λ (independent of s) and U is a solution of (5.33) *for this λ.*

Proof. The Eq. (5.33) can be written as

$$\frac{d^2U}{dr^2} + \frac{(N-1)}{r}\frac{dU}{dr} = \lambda \mathrm{e}^U. \tag{5.37}$$

Since $V(s) = U(\mathrm{e}^s)$, we have

$$\frac{dV}{ds} = \frac{dU}{dr}\mathrm{e}^s \quad \text{and} \quad \frac{d^2V}{ds^2} = \frac{d^2U}{dr^2}\mathrm{e}^{2s} + \frac{dU}{dr}\mathrm{e}^s.$$

So, by multiplying bough sides of (5.37) by e^{2s} we can write

$$\frac{d^2V}{ds^2} + (N-2)\frac{dV}{ds} = \lambda \mathrm{e}^{2s}\mathrm{e}^V. \tag{5.38}$$

The first differential equation of (5.34) follows directly from the definition of v and w. Moreover, from the definition of w and (5.38), we have

$$\lambda = w(s)\mathrm{e}^{-2s}\mathrm{e}^{-V(s)}, \quad \forall\, s \in \mathbb{R}. \tag{5.39}$$

Since λ is a constant, the second differential equation of (5.34) follows by differentiating (5.39)

The converse follows directly by differentiation of (5.36) and direct calculations. □

Remark 5.21. If U is a regular solution of (5.33), the corresponding trajectory (v, w) of (5.34) satisfies

$$\lim_{s\to-\infty} v(s) = 0 \quad \text{and} \quad \lim_{s\to-\infty} w(s) = 0. \tag{5.40}$$

As we will see in the sequel, (5.40) characterizes, among the trajectories of (5.34), those that correspond to regular solutions of (5.33).

Remark 5.22. From the fact that $\lambda > 0$ and (5.35), it is clear that $w(s) < 0$ for every $s \in \mathbb{R}$. Moreover, we know from Proposition 5.18 that $\frac{dU}{dr}(r) < 0$ for all $r \in]0, +\infty[$. So, we have $v(s) < 0\ \forall\, s \in \mathbb{R}$. Therefore, we will consider the trajectories of (5.34) restricted to the sector $\{v < 0, w < 0\}$.

It is evident that $O = (0, 0)$ and $A = \big(-2, -2(N-2)\big)$ are the only stationary points of (5.34). The following three lemmas give characterizations of these statonary points.

Lemma 5.23. *Let $N \geq 3$. Then O is a hyperbolic stationary point for (5.34). The v-axis is the stable manifold and the unstable manifold is tangent to $w - Nv = 0$.*

Proof. The linearized system in a neighborhood of O is associated to the matrix

$$L_O = \begin{pmatrix} -(N-2) & 1 \\ 0 & 2 \end{pmatrix}$$

whose eigenvalues are $\mu_0 = -(N-2)$ and $\mu_1 = 2$. Associated to μ_0 we have the eigenvector $(1,0)$ which is tangent to the stable manifold and the eigenvector associated to μ_1 is $(1,N)$ which is tangent to the unstable manifold. □

Lemma 5.24. *If $N \geq 10$, A is an attractive point. If $3 \leq N \leq 9$, A is a spiral attractive point.*

Proof. The linearized system in a neighborhood of A is associated to the matrix

$$L_A = \begin{pmatrix} -(N-2) & 1 \\ -2(N-2) & 0 \end{pmatrix}$$

If $N \geq 10$, L_A has the following two real and negative eigenvalues

$$\mu_{\pm} = \frac{1}{2}\left[(2-N) \pm \sqrt{(N-2)(N-10)}\right]$$

and we see that A is attractive. But, for $3 \leq N \leq 9$ the eigenvalue of L_A are

$$\nu_{\pm} = \frac{1}{2}\left[(2-N) \pm \mathbf{i}\sqrt{(N-2)(10-N)}\right].$$

Since the real part of $\nu_{\pm}$ are strictly negative, we see that A is spiral attractive. Therefore, in a neighborhood of A, the trajectories are spirals converging to A as $s \to +\infty$. □

Lemma 5.25. *The unstable manifold of $O = (0,0)$ is an heteroclinic orbit joining the points O and A.*

Proof. Let Σ be a trajectory (v,w) of (5.34) such that

$$\lim_{s\to-\infty} \big(v(s), w(s)\big) = (0,0).$$

We have to show that

$$\lim_{s\to+\infty} \big(v(s), w(s)\big) = \big(-2, -2(N-2)\big).$$

We claim that the trajectory Σ lies above the line of equation $w - Nv = 0$. Indeed, if we define $\phi(s) := w(s) - Nv(s)$, we have

$$\frac{d}{ds}\phi(s) = -(N-2)\phi(s) + v(s)w(s) \geq -(N-2)\phi(s),$$

from which we get

$$\frac{d}{ds}\big(\mathrm{e}^{(N-2)s}\phi(s)\big) \geq 0.$$

But we know that

$$\lim_{s\to-\infty} \phi(s) = 0.$$

Therefore,

$$\mathrm{e}^{(N-2)s}\phi(s) \geq 0, \quad \forall\, s \in \mathbb{R}$$

and we conclude that $\phi(s) \geq 0$ for all $s \in \mathbb{R}$.
(1) Let us assume $N \geq 10$. We consider the line passing at A and with slope

$$d := \frac{1}{2}\Big[(N-2) + \sqrt{(N-2)(N-10)}\Big] < N-2,$$

i.e., $w+2(N-2) = d(v+2)$. This line crosses the line of equation $w-Nv=0$ at the point B and we have a triangle OAB in the region $v<0$ and $w<0$.

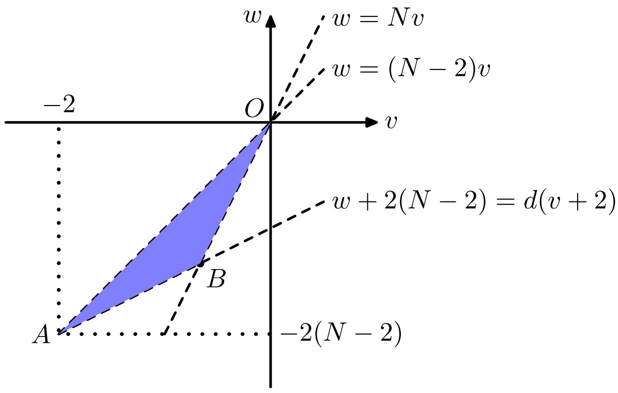

Figure 5.6. The triangle which contains the trajectory Σ.

From the above claim, the trajectory Σ near O belongs to the interior of this triangle and cannot leave it by the segment $\overline{OB}$, because it is part of the line $w = Nv$. On the other hand, since

$$\overline{OA} = \big\{(v,w)\,;\, w = (N-2)v,\ -2 \leq v \leq 0\big\},$$

if $P = (v_P, w_P) \in \overline{OA}$, we have from (5.34)

$$\frac{dv}{ds}(P) = w_P - (N-2)v_P = 0 \quad \text{and} \quad \frac{dw}{ds}(P) = v_P(v_P+2)(N-2) < 0.$$

Hence, the trajectory Σ does not leave the triangle through the segment $\overline{OA}$.

Now, let us show that in fact Σ cannot leave the triangle. First of all, if $P = (v_P, w_P) \in \overline{AB}$ we see that

$$\frac{dv}{ds}(P) = w_P - (N-2)v_P < 0 \quad \text{and} \quad \frac{dw}{ds}(P) = (v_p + 2)w_P < 0$$

because $\overline{AB}$ is below $\overline{OA}$, $w_p < 0$ and $-2 < v_P < 0$. Therefore, if Σ would leave the triangle through $\overline{AB}$ we should have

$$\frac{dw}{ds}(P)/\frac{dv}{ds}(P) = \frac{(v_p+2)w_P}{w_P - (N-2)v_P} \geq d. \tag{5.41}$$

Since we know that $w_P - (N-2)v_P < 0$, (5.41) is equivalent to

$$(v_p + 2)w_P \leq d\big[w_P - (N-2)v_P\big]. \tag{5.42}$$

But $w_P-(N-2)v_P = d(v_P+2)-2(N-2)-(N-2)v_P = (v_P+2)\big[d-(N-2)\big]$. So, we have from (5.42) and de fact that $v_P + 2 > 0$,

$$(v_P+2)w_P \leq d(v_p+2)\big[d-(N-2)\big] \quad \Rightarrow \quad w_P \leq d\big[d-(N-2)\big].$$

From he definition of d, we have

$$d\big[d-(N-2)\big] = \frac{1}{4}\big[(N-2)(N-10) - (N-2)^2\big] = -2(N-2)$$

so that $w_P \leq -2(N-2)$, which is impossible if $P \in \overline{AB}$ and $P \neq A$. □

Consequently, Σ does not leave the triangle and the limit as $s \to +\infty$ is the point A because it is attractive, i.e.,

$$\lim_{s\to+\infty}\big(v(s), w(s)\big) = \big(-2, -2(N-2)\big).$$

(2) Now we assume $3 \leq N \leq 9$. In this case we will prove that the trajectory Σ is a spiral around A which converges to A. The local behavior of Σ can be described by the linearized system, so we can choose a closed neighborhood $\mathcal{V}$ of A such that all trajectory that meets $\mathcal{V}$ converges to A as $s \to +\infty$ following a spiral.

We know that Σ stays above the line $w - Nv = 0$. Let $C = (-2, -2N)$ and $D = \big(-2n/(N-2), -2N\big)$. The orientation of the vector field

$$\mathbf{f}(v,w) := \big(w - (N-2)v, (v+2)w\big)$$

on the segments $\overline{CD}$, $\overline{DE}$ and $\overline{EO}$ shows that the trajectory Σ cannot leave the polygonal region $OCDE$. On the other hand, Σ cannot meet $\overline{EO}$ or converge to O as $s \to +\infty$ because the v axis is the unique trajectory that converges to O as $s \to +\infty$.

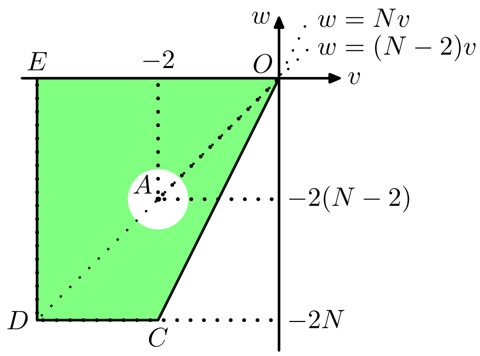

Figure 5.7. The poligonal region containing the heteroclinic orbit spinning in a spiral around A.

Let us suppose that Σ does not touch $\mathcal{V}$. If it is the case, it stay in a compact set contained in the polygonal region $OCDE$ where the only critical point is O. Since it does not converge to O, it should converge to a periodical trajectory around a critical point, which is necessarily A (cf. [13]). In order to prove that there is no such an orbit, we follow the argument of Bendixson's criteria. Let γ be a periodic trajectory contained in the polygonal region $OCDE$ surrounding the point A. Then, the vector field $\mathbf{f}$ is tangent to γ and if $\mathbf{n}$ is the normal to γ, we have from Green's Theorem

$$0 = \int_\gamma \mathbf{f} \cdot \mathbf{n} \, d\gamma = \int_\omega \operatorname{div} \mathbf{f}(v, w) \, dv dw.$$

where

$$\omega = \big\{(v, w)\,;\, w_- \le w \le w_+,\ v_-(w) \le v \le v_+(w)\big\}$$

and

$$v_-(w) < -2 < v_+(w) < 0 \ \forall w \in]w_-, w_+[.$$

But it is clear that $\operatorname{div} \mathbf{f}(v, w) = v - (N-4) = (v+1) - (N-3) \le v + 1$, from which we get a contradiction, because

$$\begin{aligned}\int_\omega \operatorname{div} \mathbf{f}(v, w) \, dv dw &\le \int_{w_-}^{w_+} \left(\int_{v_-(w)}^{v_+(w)} (v+1) \, dv \right) dw \\ &= \frac{1}{2} \int_{w_-}^{w_+} \big[v_+(w) - v_-(w)\big] \big[v_+(w) + v_-(w) + 2\big] \, dw < 0.\end{aligned}$$

Therefore, it does not exist such a periodic trajectory γ and Σ must go into $\mathcal{V}$ and so converges to A as $s \to +\infty$ following a spiral.

With the following lemma we have as immediate consequence the proof of Theorem 5.19.

Lemma 5.26. *The only trajectory of* (5.34) *such that the associated function V satisfies*

$$\lim_{s \to -\infty} V(s) = +\infty \tag{5.43}$$

is the trajectory reduced to the point $A = \big(-2, -2(N-2)\big)$.

Proof. Let Σ_0 be a trajectory associated to V satisfying (5.43) and assume that

$$\lim_{s \to -\infty} v(s) = -2. \tag{5.44}$$

Then, for each $n \in \mathbb{N}$, there exists a s_n, $-n-1 < s_n < -n$ such that

$$v(-n-1) - v(-n) = \frac{dv}{ds}(s_n) = w(s_n) - (N-2)v(s_n).$$

So,

$$w(s_n) = v(-n-1) - v(-n) + (N-2)v(s_n) \underset{n \to +\infty}{\longrightarrow} -2(N-2).$$

and we have

$$\lim_{n \to \infty} \big(v(s_n), w(s_n)\big) = \big(-2, -2(N-2)\big) = A.$$

Since $s_n \to -\infty$ and A is attractive, the only possibility is that Σ_0 is the trajectory reduced to the single point A.

In order to prove that there does not exist another trajectory associated to V satisfying (5.43), it is sufficient to prove that (5.44) is the unique possibility. So, let as assume that $\widetilde{\Sigma}$ is another trajectory different from Σ_0 and satisfying (5.43). We have to consider four steps to eliminate all possibilities.
Step 1: *The trajectory $\widetilde{\Sigma}$ cannot satisfy the condition*

$$\lim_{s \to -\infty} v(s) = v_0, \quad -2 < v_0 \le 0.$$

Indeed, if such a condition is possible, there exist $\alpha > 0$ and S_0 such that for all $s \le S_0$, $v(s) \ge -2 + \alpha$. Hence, for all $s \le S_0$,

$$\frac{dw}{ds}(s) = \big(v(s) + 2\big)w(s) \le \alpha w(s),$$

which implies that

$$w(S_0)e^{\alpha(s-S_0)} \le w(s) < 0, \quad \forall s \le S_0.$$

and
$$\lim_{s\to-\infty} w(s) = 0 \quad \text{and} \quad \lim_{s\to-\infty} \frac{dv}{ds}(s) = -(N-2)v_0.$$
But we have necessarily $v_0 = 0$, because otherwise $\lim_{s\to-\infty} v(s) = -\infty$. So,
$$\lim_{s\to-\infty} \big(v(s), w(s)\big) = (0,0) = O$$

This means that this trajectory converges to the stationary point O and so, it will be tangential to the unstable manifold of O, i.e., to the line $w - Nv = 0$. Therefore, for each $\varepsilon > 0$ there exist $S_1 \in \mathbb{R}$ such that for every $s \le S_1$, $0 \le w(s) - Nv(s) < \varepsilon v(s)$. So, for every $s < S_1$,
$$\frac{d}{ds}\big(e^{(-2+\varepsilon)s} v(s)\big) \le 0$$
and we conclude that
$$\frac{dV}{ds}(s) = v(s) \ge v(S_1) e^{(-2+\varepsilon)S_1} e^{(2-\varepsilon)s},$$
This means that the function $V(s) = U(e^s)$, which is a decreasing function, is bounded from below as $s \to -\infty$ and so $U(t)$ has a limit as $t \to 0$, which says that u is not singular.

Step 2: *The trajectory $\widetilde{\Sigma}$ cannot satisfy the condition*
$$\lim_{s\to-\infty} v(s) = v_0, \quad v_0 < -2.$$

Indeed, if such a condition is possible, there exist $\alpha > 0$ and S_2 such that for all $s \le S_2$, $v(s) < -2 - \alpha$. Hence, for all $s \le S_2$,
$$\frac{dw}{ds}(s) = \big(v(s) + 2\big) w(s) > -\alpha w(s),$$
which implies that
$$w(s) < w(S_2) e^{\alpha(S_2 - s)}, \quad \forall s \le S_2.$$
and
$$\lim_{s\to-\infty} w(s) = -\infty \quad \text{and} \quad \lim_{s\to-\infty} \frac{dv}{ds}(s) = -\infty.$$
From the Mean Value Theorem, for each $n \in \mathbb{N}$, there exists $-n < s_n < -n + 1$ such that
$$-v(-n) < v(-n+1) - v(-n) = \frac{dv}{ds}(s_n) \underset{n\to+\infty}{\longrightarrow} -\infty$$

and we have a contradiction.

Step 3: *The trajectory* $\widetilde{\Sigma}$ *cannot satisfy the condition*

$$\lim_{s\to-\infty} v(s) = -\infty. \tag{5.45}$$

If we assume that $\lim_{s\to-\infty} v(s) = -\infty$, the application of the Mean Value Theorem as before gives us

$$\lim_{s\to-\infty} w(s) = -\infty.$$

We claim that

$$\lim_{s\to-\infty} \frac{w(s)}{v(s)+2} = +\infty.$$

Indeed, since

$$\frac{dw}{ds} = (v+2)w,$$

we have by integrating on $[s, S_3]$

$$w(s) = w(S_3)\mathrm{e}^{-\int_s^{S_3}(v(\sigma)+2)\,d\sigma},$$

and so

$$\frac{w(s)}{v(s)+2} = w(S_3)\frac{\mathrm{e}^{-\int_s^{S_3}(v(\sigma)+2)\,d\sigma}}{\big(v(s)+2\big)} = \frac{w(S_3)}{\big(v(s)+2\big)\mathrm{e}^{\int_s^{S_3}(v(\sigma)+2)\,d\sigma}}.$$

Let us denote

$$\delta(s) := \big(v(s)+2\big) \quad \text{and} \quad \psi(s) := \delta(s)\mathrm{e}^{\int_s^{S_3}\delta(\rho)\,d\rho}.$$

Then we can write

$$\frac{w(s)}{v(s)+2} = \frac{w(S_3)}{\psi(s)}$$

and it suffices to show that $\psi(s) \to 0^-$ as $s \to -\infty$ to prove the claim.

First of all, we remark that

$$\psi(s) = \delta(s)\mathrm{e}^{\int_s^{S_3}\delta(\rho)\,d\rho} = -\frac{d}{ds}\left(\mathrm{e}^{\int_s^{S_3}\delta(\rho)\,d\rho}\right).$$

Hence,

$$\int_s^{S_3} \psi(\tau)\,d\tau = -1 + \mathrm{e}^{\int_s^{S_3}\delta(\rho)\,d\rho}.$$

This implies that ψ is integrable in $]-\infty, S_3]$ because

$$\lim_{\rho\to-\infty}\delta(\rho)=-\infty \Rightarrow \lim_{s\to-\infty}\int_s^{S_3}\delta(\rho)\,d\rho=-\infty \Rightarrow \lim_{s\to-\infty}\int_s^{S_3}\psi(\tau)\,d\tau=-1$$

Since $\delta(s)<0$ for s small enough, this implies $\psi(s)<0$, we have necessarily

$$\lim_{s\to-\infty}\psi(s)=0^-$$

and the claim is proved. Therefore,

$$\lim_{s\to-\infty}\frac{dv}{ds}(s)=\lim_{s\to-\infty}\big[w(s)-(N-2)v(s)\big]=-\infty.$$

So, $v(s)$ is decreasing for s small enough, which is incompatible with the hypothesis (5.45).

Step 4: The trajectory $\widetilde{\Sigma}$ cannot satisfy the condition

$$\liminf_{s\to-\infty}v(s)<\limsup_{s\to-\infty}v(s). \tag{5.46}$$

Let us suppose that (5.46) holds. Since v is analytic and it is not monotone near $-\infty$, the zeros of $\frac{dv}{ds}$ form a decreasing sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n\to-\infty$ as $n\to+\infty$. So, by defining $M(s_n):=\big(v(s_n),w(s_n)\big)$, we can say that, for every $n\in\mathbb{N}$, $M(s_n)$ belongs to the line $w-(N-2)v=0$, which we will call Δ, i.e.,

$$M(s_n)\in\Delta,\quad \forall n\in\mathbb{N}.$$

We claim that the zeros of $\frac{dw}{ds}$ form a sequence $\{t_n\}_{n\in\mathbb{N}}$ satisfying

$$s_{n+1}<t_n<s_n \quad\text{and}\quad v(t_n)=-2.$$

Indeed, if $\eta(s):=\big(w(s)-(N-2)v(s)\big)\mathrm{e}^{(N-2)s}$, then $\eta(s_{n+1})-\eta(s_n)=0$ and from Rolle's Theorem, there exists $t_n\in]s_{n+1},s_n[$ such that $\frac{d\eta}{ds}(t_n)=0$. But,

$$\begin{aligned}\frac{d\eta}{ds}(s)&=\mathrm{e}^{(N-2)s}\Big[\frac{dw}{ds}(s)-(N-2)\frac{dv}{ds}(s)+(N-2)w(s)-(N-2)^2v(s)\Big]\\&=\mathrm{e}^{(N-2)s}\Big[v(s)+2\Big]w(s).\end{aligned}$$

It is clear that $w(t_n)\neq 0$ because otherwise the trajectory would belong entirely to the v-axis. Therefore, $v(t_n)=-2$. Let us suppose now that there exist $t_n,t'_n\in]s_{n+1},s_n[$, $t_n\neq t'_n$ such that $\frac{d\eta}{ds}(t_n)=\frac{d\eta}{ds}(t'_n)$. Then $v(t_n)=v(t'_n)=-2$ ad from Rolle's Theorem there exist $t''_n\in]s_(n+1),s_n[$ with

$\frac{dv}{ds}(t''_n) = 0$, which is impossible because we are assuming that $\{s_n\}_{n\in\mathbb{N}}$ is the sequence of zeros of $\frac{dv}{ds}$.

In the sequel, we denote by $M(s)$ the point of the trajectory $\widetilde{\Sigma}$, i.e., $M(s) := \big(v(s), w(s)\big)$ and we analyze separately two cases: $N \geq 10$ and $3 \leq N \leq 0$.

Case $N \geq 10$. Let us consider the following closed curve

$$\Gamma_1 := \big\{M(s)\,;\, s \in [s_3, s_1]\big\} \cup \big[M(s_3), M(s_1)\big],$$

where $\big[M(s_3), M(s_1)\big]$ denotes the segment of the line $w - Nv = 0$ with extremities at $M(s_1)$ and $M(s_3)$. It is clear that A is located in the interior of Γ and O is in its exterior. Therefore Γ intersect the heteroclinic orbit joining O to A at some point P which P does not belong to the segment $\big[M(s_3), M(s_1)\big]$ because we have seen that the heteroclinic orbit does not touch the line $w - Nv = 0$. So, $P \in \big\{M(s)\,;\, s \in [s_3, s_1]\big\}$ which is a regular trajectory. But this is impossible and we conclude that, if $N \geq 10$, $\widetilde{\Sigma}$ does not exist.

Case $3 \leq N \leq 9$. The heteroclinic orbit Σ joining O and A comes from O (for $s = -\infty$) and, as s grows, it meets the line $\{v = -2\}$ for the first time at Q_0 (with $w < -2(N-2)$) and then the line Δ at P_0 (with $v < -2$) and then again the line $\{v = -2\}$ at Q_1 (with $w > -2(N-2)$) and so on, spinning in a spiral around A. This iterative process gives $\{P_n\}_{n\in\mathbb{N}}$ such that, for $k \in \mathbb{N}$,

$$P_{2k} \in \Delta \cap \{v < -2\} \quad \text{and} \quad P_{2k+1} \in \Delta \cap \{-2 < v < 0\} =]OA[.$$

On the other hand we know that $M(s_n) \in \Delta$ and that $\{v(s_n)\}_{n\in\mathbb{N}}$ oscillates around -2. So, we can suppose that $M(s_1) \in]OA[$ and then

$$M(s_{2k+1}) \in]OA[\quad \text{and} \quad M(s_{2k}) \in \Delta \cap \{v < -2\}.$$

Statement: *There exists $m \geq 1$ such that for each $k \geq m$, $M(s_{2k+1}) \in]OP_1[$.*

Before proceeding to prove this statement, we remark that it allows us to conclude the proof Lemma 5.26 using the argument as in the case $N \geq 10$. Indeed, let us consider the bounded curve

$$\Gamma_k := \big\{M(s)\,;\, s \in [s_{2k+3}, s_{2k+1}]\big\} \cup \big[M(s_{2k+3})M(s_{2k+1})\big].$$

This curve contains Σ in its interior and O in its exterior. So, it meets the heteroclinic orbit Σ at a point that does not belong to $\big[M(s_{2k+3})M(s_{2k+1})\big]$ because $\big[M(s_{2k+3})M(s_{2k+1})\big] \subset]OP_1[$ and the conclusion follows as in the case $N \geq 3$.

Now, we have to prove the statement. Since Σ converges to A following a spiral, we know that

$$P_{2k+3} \in]AP_{2k+1}[\quad \text{and} \quad P_{2k+1} \underset{n\to+\infty}{\longrightarrow} A.$$

Since $M(s_1) \in]OA[$, there exists k_0 such that $M(s_1) \in]P_{2k_0+3}P_{2k_0+1}[$. Then, as $\{s_n\}_{n\in\mathbb{N}}$ is decreasing, $M(s_3) \in]P_{2k_0+1}P_{2k_0-1}[$ and more generally,

$$M(s_{2n-1}) \in]P_{2k+3}P_{2k+1}[\;\Rightarrow\; M(s_{2n+1}) \in]P_{2k+1}P_{2k-1}[.$$

Indeed, since the singular trajectory can not meet Σ, the points $M(s)$, with $s_{2n+1} < s < s_{2n-1}$ are guided by the following arcs of Σ

$$]P_{2k+3}P_{2k+1}[\quad \text{and} \quad]P_{2k+1}P_{2k-1}[$$

As a consequence, we have

$$\forall\, k \geq k_0, \quad M(s_{2k+3}) \in]OP_1[$$

and the statement is proved.

Finally, from (1), (2), (3) and (4) we see that for a singular trajectory, we have

$$\lim_{s\to-\infty} v(s) = -2,$$

and we have the proofs of Lemma 5.26 and Theorem 5.19. □

We remark that the trajectories of (5.34) that correspond to radial solutions of (5.1) in a ball of radius 1 are carried by the heteroclinic orbit which joins the points O and A. The singular solution is reduced to the point A. The regular solutions start (when $s = 0$) from a point in the orbit (if it exists) such that $w(0) = \lambda$ and follow the heteroclinic orbit to converge towards O when $s \to -\infty$.

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