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ROBUST MODELING OF MULTIVARIATE FINANCIAL DATA

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Abstract

The bottom line in many statistical analysis in finance is the basic issue of modeling a set of multivariate data. Financial data are characterized by their fat tails containing some proportion of extreme observations. We propose a simple model able to capture these main characteristics, and to provide a good fit for the bulk of the data as well as for the atypical observations. Basically, we use a robust covariance estimator to define the center and orientations of the data, and the classical sample covariance to estimate how inflated could this distribution be by the effect of extreme observations. Estimation of the model is done either empirically or by maximum likelihood based on elliptical distributions. Simulation experiments verified the adequacy of the model to real data. We provide illustrations of the usefulness of the proposed procedure, in particular when constructing efficient frontiers. We show that robust portfolios may yield higher cumulative returns and have more stable weights compositions.

Classification code: C51

Key Words: Robust Estimation; Multivariate Financial Data; Outliers; Breakdown Point; Mean-Variance Optimal Portfolios; Simulations.

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1 Introduction

The bottom line in many statistical analysis in finance is the basic issue of modeling a set of multivariate data. Examples include the replication of the efficient frontier to construct confidence intervals for the corresponding portfolios weights (Michaud, 1998); or the generation of multivariate data to investigate the accuracy of risk measures estimates; or the simulation of multivariate data to assess the chances of occurring specific adverse scenarios; and so on. The lack of a good fit for the multivariate data often drives practitioners to replicate the data using simple bootstrap techniques. However, this approach also possesses its limitations, the greater concern being how to deal with time dependency in the data. Bootstrap techniques for time series data do exist, but require large data sets, which may have the drawback of covering periods of different economies or regulatory regimes.

Several models in finance rely on simplified assumptions. The Mean-Variance (MV) model of Markowitz (1959) assumes the multivariate normal distribution for a collection of assets. In this and other contexts (such as least squares estimation of linear regression models), the assumption of the multivariate normal distribution is primarily due to its mathematical tractability and statistical interpretations. However, it is now well known (Bekaert and Harvey (1997), among others) that financial returns distributions are heavy tailed containing some proportion of extreme observations.

Extreme observations are even more common in emerging markets. They may or may not be considered outliers (this is a frequent discussion topic), but certainly they seem to be related to a data generating process different from the one generating the vast majority of the observations. Clearly, the multivariate normal is not a reasonable assumption for neither emerging nor developed markets data. It also seems obvious that classical estimation methods, characterized by assigning equal weights to each data point, will not succeed in such environment. We will provide illustrations on how classical estimates fail.

The problem of fitting multivariate distributions to financial data has been investigated by other authors, including Jobson and Korkie (1981), Embrechts, McNeil and Strauman (1999). The challenge we face in this paper is that of obtaining a good representation and a good fit for the bulk of the data as well as for the extreme observations. Using our proposal, one does not have to worry about which and how many observations are outliers. The proposed procedure does it automatically. Once one has a good model, he can simulate the data, perform scenario analysis, obtain replications for the efficient frontier, construct robust confidence intervals for the portfolios weights, and so on. We will provide examples of such applications.

Before getting into more technical details, we give more motivation exploring the classical estimation of multivariate data, in particular its effects on the estimation of the covariance structure. We recall that the multivariate location and the covariance structure are sufficient to characterize an elliptical distribution. In addition, they are the only inputs in several statistical tools used in finance, for example, in the MV optimization procedure, where the widely used estimators are the classical sample covariance matrix and sample mean. As maximum likelihood estimators under normality, these estimators possess desirable statistical properties under the true model. However, their asymptotic breakdown point (definition given in the next section) is equal to zero (Maronna, 1976), which means that they are badly affected by extreme observations.

The effects of atypical points on the ellipsoid (Johnson and Wichern, 1990) associated to an estimate of the covariance structure are at least two (Rousseeuw and van Zomeren, 1990): (1) they may inflate its volume; (2) they may tilt its orientation. The first effect is related to inflated scale estimates. The second is the worst one, and may show up as switching the correlations signs.

To illustrate both effects we show in Figure 1 a dramatic example using the MSCI-EAFE and the American T-Bill returns expressed in Brazilian *reais* because this country has experienced a major currency devaluation recently. The 72 data points are monthly returns from January, 1995 to December, 2000. This figure shows the set of points with same statistical distance to the center, the ellipsoids (see (1)) associated to the classical sample covariance matrix computed with (dotted line) and without (solid line) the atypical points. We observe that the classical estimates provide inflated ellipsoids. More important, the outliers (the two most extreme correspond to December, 1998 and February, 1999 — the Brazilian devaluation) rotate the axes of the ellipsoid computed from the classical estimates, and mask the (correct) orientation given by the robust one.

<<Insert Figure 1 here>>

At such data configuration, robust and classical estimates will yield completely different pairwise covariances or correlations estimates. For the two variables in Figure 1, the classical and the robust estimates of the correlation coefficient are respectively 0.88 and 0.17. Note that for dimensions higher than 2 the multivariate outliers are harder to spot and we cannot rely on the graphical inspection anymore.

The remaining of this paper is organized as follows. In Section 2 we propose a statistical model and a robust estimation procedure for multivariate data, and illustrate using real data from emerging markets. In Section 3 we carry out three simulation experiments to verify the goodness of fit of the new method. The new estimates are compared to the classical ones in terms of bias and relative efficiency. In Section 4 we investigate the effect of the data modeling on asset allocation. Using real data we compare the performances of the robust and classical MV optimal portfolios. It is shown that the robust portfolios may yield higher cumulative returns and seem to possess more stable weight structures. In Section 5 we summarize the results and offer our conclusions.

2 Model and Estimation

We first fix the notation that will be used and give the definitions of some concepts that will be needed. Let $\Re^p$ denote the p-dimensional Euclidean space and $X = (X_1 \cdots X_p)'$ a random vector on $\Re^p$ with some distribution F. A random sample of size n of F is $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$, where $\mathbf{x}_i = (x_{i1} \cdots x_{ip})'$, for $i = 1, ..., n$. If F is absolutely continuous, the collection $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$, $n \geq p+1$, is in general position (all points must be different) with probability one.

The determinant of a $p \times p$ matrix $\mathbf{A}$ is denoted by $|\mathbf{A}|$. The eigenvalues of $\mathbf{A}$ are $\lambda_1(\mathbf{A}) \geq \cdots \geq \lambda_p(\mathbf{A})$, and for $i = 1, ..., p$, the $\mathbf{e}_i$'s are the eigenvectors, $\mathbf{e}_i \mathbf{e}_i' = 1$,

$\mathbf{e}_i\mathbf{e}_j' = 0$. Let $PDS(p)$ be the class of all $p \times p$ positive definite and symmetric matrices. Every element $\mathbf{A}$ in $PDS(p)$ has a root $\mathbf{R}$, a matrix such that $\mathbf{A} = \mathbf{R}\mathbf{R}'$. A matrix $\mathbf{A}$ is positive definite if and only if $\lambda_1(\mathbf{A}), ..., \lambda_p(\mathbf{A}) > 0$.

The Mahalanobis distance between two vectors $\mathbf{v}$ and $\mathbf{u}$ in $\Re^p$ with respect to a matrix $\mathbf{C}$ in $PDS(p)$ is defined as

$$d(\mathbf{v}, \mathbf{u}, \mathbf{C}) = \sqrt{(\mathbf{v}-\mathbf{u})'\mathbf{C}^{-1}(\mathbf{v}-\mathbf{u})} \ . \tag{1}$$

When $\mathbf{C} = \mathbf{I}$, where $\mathbf{I}$ is the diagonal matrix $diag(1 \cdots 1)$, (1) is just the Euclidean distance between $\mathbf{v}$ and $\mathbf{u}$, and it is denoted by $\| \cdot \|$.

An ellipsoid $E(\mathbf{u}, \mathbf{C}, r)$ in $\Re^p$, with center $\mathbf{u}$ and covariance structure $\mathbf{C}$, is

$$E(\mathbf{u}, \mathbf{C}, r) = \{\mathbf{x} \in \Re^p : (\mathbf{x}-\mathbf{u})'\mathbf{C}^{-1}(\mathbf{x}-\mathbf{u}) \leq r^2\} \ , \tag{2}$$

where r is a positive real number. The magnitude of the ellipsoid is determined by r and $\mathbf{C}$, and when $\mathbf{C} = \mathbf{I}$ it reduces to a ball with center $\mathbf{u}$ and radius r. The volume of any ellipsoid $E(\mathbf{u}, \mathbf{C}, r)$ is equal to $\sqrt{|\mathbf{C}|}\left(\frac{(\pi r^2)^{p/2}}{\Gamma(p/2+1)}\right)$, and the lengths of the p axes are equal to $2r\sqrt{\lambda_j(\mathbf{C})}$, $j = 1, ..., p$.

A location estimator of the multivariate center μ of F, based on a collection of n points, is a vector valued function of $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ and will be denoted by $\widehat{\mu}(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n)$ or simply $\widehat{\mu}$. A covariance estimator of the covariance matrix $\mathbf{\Sigma}$ of F, based on a collection of n points, is a function of $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ taking on values in $PDS(p)$, and will be denoted $\widehat{\mathbf{\Sigma}}(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n)$ or simply $\widehat{\mathbf{\Sigma}}$.

The robustness of an estimator may be investigated with respect to its local behavior under small pertubations on the data, and also with respect to its global behavior under large pertubations of a given situation. Here we are more interested in the breakdown point of an estimator, which is a global robustness property proposed first by Hampel (1968,1971).

This concept is related to the amount of extreme values which can "break down" the estimator. It is a measure which tells us up to what fraction of extreme values in the sample (or up to what distance from the assumed distribution F) the estimator still gives reliable information about F.

A location estimator breaks down if contamination drives the estimator to the boundaries of the parameter space. The finite sample replacement breakdown point (Donoho and Huber, 1983) of a location estimator $\widehat{\mu}$ at a collection $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n)$ is defined as the smallest fraction m/n of outliers that can take the estimator over all bounds:

$$\epsilon^*(\widehat{\mu}, \mathbf{X}) = \min_{1 \leq m \leq n} \left\{ \frac{m}{n} : \sup_{\mathbf{Y}_m} \| \widehat{\mu}(\mathbf{X}) - \widehat{\mu}(\mathbf{Y}_m) \| = \infty \right\} \tag{3}$$

where the supremum in (3) is taken over all possible corrupted collections $\mathbf{Y}_m = (\mathbf{y}_1, ..., \mathbf{y}_m, \mathbf{x}_{i_m+1}, ..., \mathbf{x}_{in})$ that can be obtained from $\mathbf{X}$ by replacing any m points $\mathbf{x}_{i_1}, ..., \mathbf{x}_{i_m}$ of $\mathbf{X}$ by arbitrary values $\mathbf{y}_1, ..., \mathbf{y}_m$.

For example, the breakdown point of the sample mean is $1/n$, the smallest possible value, measuring its sensitivity to extreme values. The best possible breakdown point among all translation equivariant estimators of multivariate location is $\lfloor \frac{n+1}{2} \rfloor / n$, where $\lfloor x \rfloor$ means the largest integer smaller or equal to x. Two factors related to location estimators are relevant to this work: (1) most estimators have a breakdown point that does not depend on the data; (2) location estimators with a high breakdown point necessarily must sacrifice tail performance (Lopuhaä and Rousseeuw, 1991).

A scale estimator breaks down if contamination drives the estimator to either zero or $+\infty$, situations named by Huber (1981) as "implosion" or "explosion". In the multivariate case this idea can be expressed using the eigenvalues of $\widehat{\Sigma}(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n)$. The finite sample replacement breakdown point of a covariance estimator $\widehat{\Sigma}$ at a collection $\mathbf{X}$ is defined as the smallest fraction m/n of outliers that can take either the largest eigenvalue $\lambda_1(\widehat{\Sigma})$ over all bounds, or take the smallest eigenvalue $\lambda_p(\widehat{\Sigma})$ arbitrarily close to zero:

$$\epsilon^*(\widehat{\Sigma}, \mathbf{X}) = \min_{1 \leq m \leq n} \left\{ \frac{m}{n} : \sup_{\mathbf{Y}_m} M(\widehat{\Sigma}(\mathbf{X}), \widehat{\Sigma}(\mathbf{Y}_m)) = \infty \right\} \tag{4}$$

where the supremum is taken over all possible corrupted collections $\mathbf{Y}_m$ as in (3), where $M(\widehat{\Sigma}(\mathbf{X}), \widehat{\Sigma}(\mathbf{Y}_m)) = \max\{||\lambda_1(\widehat{\Sigma}(\mathbf{X})) - \lambda_1(\widehat{\Sigma}(\mathbf{Y}_m))||, ||\lambda_p(\widehat{\Sigma}(\mathbf{X}))^{-1} - \lambda_p(\widehat{\Sigma}(\mathbf{Y}_m))^{-1}||\}$, and where $||x||$ means the absolute value of x.

For example, the breakdown point of the univariate sample variance is the smallest possible value $1/n$, illustrating its sensitivity to outliers. The best possible breakdown point among all affine equivariant covariance estimators at collections $\mathbf{X}$ in general position is $\lfloor\frac{n-p+1}{2}\rfloor/n$ (again, $\lfloor x\rfloor$ means the largest integer smaller or equal to x). It is important to note that it depends on p.

A covariance affine equivariant estimator which attains the maximum possible breakdown point is the Minimum Covariance Determinant (MCD) estimator of Rousseeuw (1985). The MCD location estimator $\widehat{\mu}_h(\mathbf{X})$ is defined as *the mean of the h points of* $\mathbf{X}=\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_n$ *for which the determinant of the sample covariance is minimal*. We will denote the classical sample covariance by S.

The (raw) MCD covariance estimator $\widehat{\mathbf{\Sigma}}_{raw}$ is the sample covariance of those h points. By taking $h=\lfloor\frac{n+p+1}{2}\rfloor$, the MCD attains the breakdown point of $\lfloor\frac{n-p+1}{2}\rfloor/n$ at any $\mathbf{X}$ in general position. In the univariate case the MCD estimator reduces to the sample mean and sample variance of the h points with the smallest sample variance. This estimator is asymptotically normal and converges at rate $\sqrt{n}$.

A concern in robustness is consistency of an estimator at the normal model. To obtain consistency, the "raw" covariance estimate based on the h points is multiplied by a factor, yielding the $\widehat{\mathbf{\Sigma}}_h$. This factor is median of the n squared Mahalanobis distances (1), $d^2(\mathbf{x}_i,\widehat{\mu}_\mathbf{h},\widehat{\mathbf{\Sigma}}_{raw})$, divided by $\chi^2_{p,0.5}$. The notation $\chi^2_{p,q}$ means the (upper) $100\cdot q$ % quantile of a chisquare distribution with p degrees of freedom. The final MCD estimator (see Rouseeuw and Van Driessen, 1999) is a one-step weighted sample mean and sample covariance computed with weigths:

$$\begin{aligned} w_i &= 1 \quad if\ d(\mathbf{x}_i,\widehat{\mu}_\mathbf{h},\widehat{\mathbf{\Sigma}}_h)\leq\sqrt{\chi^2_{p,0.975}} \\ w_i &= 0 \quad otherwise\ . \end{aligned} \tag{5}$$

Let us denote by h^+ the number of points used to obtain the reweighted estimator with weights given in (5). The final MCD estimator, denoted by $\widehat{\mathbf{\Sigma}}_{h^+}$, possesses the same high breakdown point of the initial estimate and reasonable asymptotic properties. Note that $n-h^+$ extreme points were not used to compute the $\widehat{\mathbf{\Sigma}}_{h^+}$, and we will use this information later.

We can interpret the MCD estimator as an estimator able to measure the "outlyingness" of any point $\mathbf{x}_i$, relative to the center of the collection $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$. The weights given in (5) were assigned according to the degree of outlyingness. This is also the default procedure of SPLUS (SPLUS, 2000), to be used in this paper.

Finally, we give the definition of a family of elliptically contoured distributions. A family of distributions $\{\mathcal{E}_{\mu,\Sigma;g} : \mu \in \Re^p, \Sigma \in PDS(p); g \text{ a known function}\}$, is a family of elliptical distributions (Kelker (1970), Cambanis, Huang, and Simons (1981)), if each member has density given by

$$|\mathbf{R}|^{-1} g(\| \mathbf{R}^{-1}(\mathbf{x} - \mu) \|)$$

where $\mathbf{R}\mathbf{R}' = \mathbf{\Sigma}$. A special member of the family is the spherically symmetric distribution, $\mathcal{S}_g$, which has density given by

$$g(\| \mathbf{x} \|) \,.$$

Every spherical distribution $\mathcal{S}_g$ generates the whole family $\mathcal{E}_{\mu,\Sigma;g}$ by means of a affine transformation $\mathbf{x} \mapsto \mathbf{R}\mathbf{x} + \mu$. Examples of elliptical families are the multivariate normal, $\Phi_{\mu,\Sigma}$, obtained with $g(y) = (2\pi)^{-\frac{p}{2}} \exp\{-\frac{1}{2}y^2\}$, and the multivariate t-student distribution with ν degrees of freedom, $t^{\nu}_{\mu,\Sigma}$.

For the cases $p \geq 2$, Davies (1987) showed that at an elliptical distribution $\mathcal{E}_{\mu,\Sigma;g}$, the MCD estimators $\widehat{\mu}_h$ and $\widehat{\mathbf{\Sigma}}_{h^+}$ are consistent, respectively for μ and $\mathbf{\Sigma}$. This fact will guarantee, ahead in this paper, that we are estimating the correct quantity.

2.1 Statistical Model

To obtain a good representation for the p-dimensional data we propose, as in Huber (1981), the contaminated model

$$F^{\epsilon} = (1-\epsilon)F(\mu, \mathbf{\Sigma}) + \epsilon F^{*}(\mu, \mathbf{\Sigma}^{*}) \tag{6}$$

where ϵ is the contaminating proportion, and where the underlying distribution F and the contaminating distribution F^* are members of families of elliptical distributions $\{\mathcal{E}_{\mu,\Sigma;g}\}$, with same μ. The $(p \times p)$ covariance matrix $\mathbf{\Sigma}$ represents the (predominant) dependence structure of the usual business days, or, in other words, the covariance structure of the data cloud without the outliers. $\mathbf{\Sigma}^*$ is the covariance matrix of an extended data cloud containing also most of the atypical points. In model (6), the ellipsoids (see (2)) associated to $\mathbf{\Sigma}$ and $\mathbf{\Sigma}^*$, for fixed $\mathbf{x}$, have the same orientation but different volumes. We explain in the following paragraphs how these characteristics are derived from the choice of the eigenvectors and eigenvalues of $\mathbf{\Sigma}$ and $\mathbf{\Sigma}^*$.

In practice, because ϵ is small, the contaminating distribution in (6) typically produces spurious extreme observations seeming to follow an orientation structure different of that of the usual days, or $\mathbf{\Sigma}$. As illustrated in Figure 1, when using the classical sample covariance matrix S, these few points can tilt the orientation of the axes of its associated ellipsoid. To avoid this problem we propose to estimate the correct orientation using $\mathbf{\Sigma}_{h^+}$, the high breakdown point MCD estimate of covariance matrices. The volumes of $\mathbf{\Sigma}$ and $\mathbf{\Sigma}^*$ will be estimated respectively based on the volumes of the robust $\mathbf{\Sigma}_{h^+}$ and classical S covariance estimates. We estimate the proportion ϵ empirically or by maximum likelihood.

More formally, our proposal is based on the spectral decomposition of positive definite matrices, which is a property held by legitimate estimates of covariance matrices.

Given a $p \times p$ symmetric positive definite matrix $\mathbf{A}$, its spectral decomposition gives $\mathbf{A} = \lambda_1(\mathbf{A})\mathbf{e}_1\mathbf{e}_1' + ... + \lambda_p(\mathbf{A})\mathbf{e}_p\mathbf{e}_p'$. A compact representation is $\mathbf{A} = \mathbf{\Gamma\Delta\Gamma}'$, where $\mathbf{\Delta}$ is a diagonal matrix with diagonal equal to $(\lambda_1(\mathbf{A}), ..., \lambda_p(\mathbf{A}))$, and $\mathbf{\Gamma} = (\mathbf{e}_1 \cdots \mathbf{e}_p)$. The axes of the ellipsoid associated to $\mathbf{A}$ have orientations given by the eigenvectors of $\mathbf{A}$, and lenghts proportional to the square root of the eigenvalues of $\mathbf{A}$.

Let

$$\mathbf{\Sigma} = \mathbf{\Gamma\Delta\Gamma}'$$

be the spectral decomposition of the covariance matrix of the bulk of the p-variate data with no contamination. We propose the extended covariance matrix to be

$$\mathbf{\Sigma}^* = \mathbf{\Gamma}\mathbf{\Delta}^*\mathbf{\Gamma}',$$

where $\mathbf{\Delta}^*$ is the diagonal matrix $\mathbf{\Delta}^* = \mathbf{\Delta}/\delta$ where δ is a correction factor. For any $(p \times p)$ symmetric matrix $\mathbf{\Sigma}$ it is true that $tr(\mathbf{\Sigma}) = \sum_i \lambda_i(\mathbf{\Sigma})$, where $tr(\mathbf{\Sigma})$, the sum of the diagonal elements of $\mathbf{\Sigma}$, is the trace of the matrix $\mathbf{\Sigma}$ (Johnson and Wichern, 1990). Since the elements in the diagonal of $\mathbf{\Sigma}$ are the variances, it is easy to see the relation between the magnitudes of the variances and the sizes of the eigenvalues. Thus the correction factor δ will be based on the robust and classical variance estimates. We clarify all these details in the next subsection.

2.2 Estimation Procedure

Consider the $n \times p$ data set $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n)$. Our estimation procedure for model (6) is:

STEP 1. Compute the robust multivariate location and covariance matrix estimates $(\widehat{\mu}_h, \widehat{\mathbf{\Sigma}}_{h^+})$. We emphasize that the MCD was chosen also because it points out the $n - h^+$ extreme observations, and does not assume any particular distribution for the data.

STEP 2. Compute the spectral decomposition of $\widehat{\mathbf{\Sigma}}_{h^+}$, $\widehat{\mathbf{\Sigma}}_{h^+} = \widehat{\mathbf{\Gamma}}_{h^+}\widehat{\mathbf{\Delta}}_{h^+}\widehat{\mathbf{\Gamma}}'_{h^+}$.

STEP 3. Compute S, the (classical) sample covariance matrix based on the n points.

STEP 4. Compute the correction vector $\widehat{\delta}$ as the ratio between the robust scale estimates (square root of the diagonal of $\widehat{\mathbf{\Sigma}}_{h^+}$) and the classical standard deviations (square root of the diagonal of S). The elements of $\widehat{\delta}$ are typically less than 1. Define $\widehat{\mathbf{\Delta}}^* = \widehat{\mathbf{\Delta}}_{h^+}/\widehat{\delta}$. This is equivalent to multiply the robust eigenvalues by a correction factor which is greater than 1.

STEP 5. Compute an estimate for the extended covariance matrix $\mathbf{\Sigma}^*$ as $\widehat{\mathbf{\Sigma}}^* = \widehat{\mathbf{\Gamma}}_{h^+}\widehat{\mathbf{\Delta}}^*\widehat{\mathbf{\Gamma}}'_{h^+}$. In this way, the eigenvalues of the inflated covariance estimate are the eigenvalues of the robust one corrected by appropriate factors depending on the data. It is easy to see that $\widehat{\Sigma^*}$ is positive definite, since its eigenvalues are all positive.

STEP 6. Estimate the proportion ϵ of contamination. Here we have two options. As antecipated, one possibility is to estimate ϵ empirically, using the fraction of observed extreme points

$$\widehat{\epsilon} = \frac{n - h^+}{n} ,$$

thus letting the data speak for themselves. Another possibility is to estimate ϵ by maximum likelihood. In this case, to maximize the log-likelihood of (6) both F and F^* must be specified, and we suggest to choose members of elliptical distributions. In fact, this is a very reasonable assumption for the central part of the multivariate data. As a strong support for this suggestion we recall the famous Winsor's principle, quoted by John Tukey (1960, p.457): "all distributions are normal in the middle".

The final robust estimator is $(1 - \widehat{\epsilon})\widehat{\Sigma}_{h^+} + \widehat{\epsilon}\widehat{\Sigma}^*$. This follows since if $\mathbf{x}$ has distribution F^ϵ as in model (6) then

$$\mathbf{x} = \mathbf{x}_1 \mathbf{1}_{(U<1-\epsilon)} + \mathbf{x}_2 \mathbf{1}_{(U \geq 1-\epsilon)}$$

where $\mathbf{1}$ is the indicator function, U is a standard uniform random variable, $\mathbf{x}_1 \sim F$, and $\mathbf{x}_2 \sim F^*$. From $E[\mathbf{x}] = E[E[\mathbf{x}|U]]$, the expected value of $\mathbf{x}$ is $E[\mathbf{x}_1](1-\epsilon) + E[\mathbf{x}_2](\epsilon) = \mu$, and the covariance of $\mathbf{x}$ is $(1-\epsilon)\Sigma + (\epsilon)\Sigma^*$.

We denote the multivariate location robust estimator by $\widehat{\mu}_h$, and the two robust proposed estimators by $\widehat{\Sigma}_{emp}$, when ϵ is estimated empirically and no assumption is made about F and F^*, or by $\widehat{\Sigma}_{ML}$, when ϵ is estimated by maximum likelihood.

To illustrate, Figure 2 shows the new estimator for other two components of the data set used previously. The solid line corresponds to the empirical estimation, the dotted line to the maximum likelihood estimation, and the dash-dotted line to the classical estimation. Again we can see that the robust estimates seem to capture the right orientation and volume of the collection of points.

<<Insert Figure 2 here>>

In the next sections we continue in the investigation of practical financial applications of the proposed model and estimation procedure.

3 Modeling Financial Data

How good are the proposed estimators when estimating the center and the covariance structure of a multivariate data set? How do they compare to widely used sample estimates? Do they perform well when data are contaminated, or come from mixtures of ellipticall distributions, and when they actually come from an elliptical distribution? To answer these questions we now carry on a small simulation experiment.

To verify if our model is a good representation of financial data, we performed the following experiment. First, we assumed a (true) model which could represent some of the characteristics of multivariate financial returns. Then, we generated 500 data sets according to this (true) model. For each simulated data we computed the robust and classical covariance estimators. Summaries of the simulations are compared to the (true) parameters values. A weakness of this verification is that the only choice for generation of multivariate data available in most of the statistical softwares, including SPLUS, is the normal model. Therefore, at this moment, we are not able to simulate data from other elliptical distributions. The best we can do is to contaminate the data assuming mixtures of normals, or point mass contamination.

We chose the true model to be a 5-dimensional normal distribution, with some center μ_0 and covariance $\mathbf{C}_0$, contaminated with a given proportion of extreme observations. The outliers are obtained by adding to the randomly generated values a contaminating arbitrary value, in order to produce what is known by additive outliers (Huber, 1981). The additive outliers are a reasonable representation of the effect of major news and interventions, which do not change the data generating process, but cause large short-time effects.

To be more realistic, we chose the (true) covariance matrix $\mathbf{C}_0$ as the covariance computed from a real data set (to be used in Section 4). The choice of the true parameters is irrelevant, since we are interested in measuring the closeness (or lack of) of estimates to the true values, in average. They were thus chosen, without loss of generality, to be $\mu_0 = (0,0,0,0,0)$ and $\mathbf{C_0}$ given by

$$\mathbf{C_0} = \begin{bmatrix} 3.905 & 0.032 & 0.818 & 0.411 & 0.600 \\ 0.032 & 0.010 & 0.006 & 0.008 & 0.004 \\ 0.818 & 0.006 & 0.876 & 0.196 & 0.221 \\ 0.411 & 0.008 & 0.196 & 0.588 & 0.125 \\ 0.600 & 0.004 & 0.221 & 0.125 & 0.363 \end{bmatrix}. \qquad (7)$$

Also, the contaminating values were different for each marginal, and based on the atypical points observed in each column of the real data[3].

We performed three experiments. Experiment 1 puts contamination only in two marginals (mimicking a local turmoil). Experiment 2 contaminates all 5 variables, to represent periods of global crisis. Experiment 3 adds no contamination, to verify how efficient under the Gaussian model is the robust estimator.

For each (1300×5) simulated data set we computed the classical and the proposed robust estimates of the true 20 parameters $\mu_0 = (0,0,0,0,0)$, and, for $i,j = 1,...,5$, the variances and covariances σ_{ij} given in (7). To be fair, in these experiments we compared the sample estimates to the $\widehat{\Sigma}_{emp}$. The output summarizing the 500 results are the mean, the standard deviation, the square root of the mean squared error (RMSE), and the median of the absolute value of errors (MAE) of the estimates of the 20 parameters.

In the first experiment the robust procedure showed smaller bias, RMSE and MAE, in 4 out of 5 estimated means; in 3 out of the 5 estimated variances; and in 6 out 10 estimated covariances. We report in Table 1, for each estimator, the worst cases in each class of parameters (mean, variance, and covariance). The worst classical cases are in rows 1, 3, and 5. We observe the inflated classical RMSE values implied by the large bias of the point estimates.

[3]A less subjective procedure for defining the outlying values could have been used by applying the concept of robust distances of Rousseeuw and van Zomeren (1990).

Table 1: ***Results for 6 parameters (out of 20) from Experiment 1. Contamination proportion is 3% on two variables.***

PAR.	TRUE	ROBUST				CLASSICAL			
μ_i and σ_{ij}		MEAN	STDEV	RMSE	MAE	MEAN	STDEV	RMSE	MAE
μ_1	0.000	-0.003	0.061	0.004	0.198	-0.009	0.082	0.007	0.232
μ_5	0.000	0.000	0.017	0.000	0.107	0.000	0.016	0.000	0.103
σ_{11}	3.905	3.690	0.162	0.073	0.468	8.135	0.250	17.954	2.058
σ_{55}	0.363	0.356	0.015	0.001	0.126	0.363	0.013	0.000	0.092
σ_{12}	0.032	0.027	0.006	0.000	0.078	1.915	0.081	3.550	1.369
σ_{34}	0.196	0.185	0.022	0.001	0.129	0.197	0.020	0.000	0.118

In the second experiment the robust procedure showed better RMSE and MAE performances for all 20 parameters. Some classical estimates showed very large biases, which is exactly the effect of the zero breakdown point. Table 2 shows the results for the same parameters in previous table. For example, note the classical point estimates of σ_{11}, σ_{55}, and σ_{12}.

Table 2: ***Results for 6 parameters (out of 20) from Experiment 2. Contamination proportion is 3% on all five variables.***

PAR.	TRUE	ROBUST				CLASSICAL			
μ_i and σ_{ij}		MEAN	STDEV	RMSE	MAE	MEAN	STDEV	RMSE	MAE
μ_1	0.000	0.001	0.055	0.003	0.199	-0.041	0.076	0.007	0.248
μ_5	0.000	-0.003	0.016	0.000	0.104	-0.016	0.026	0.001	0.144
σ_{11}	3.905	3.649	0.155	0.090	0.507	8.419	0.290	20.460	2.125
σ_{55}	0.363	0.340	0.015	0.001	0.152	1.005	0.042	0.414	0.799
σ_{12}	0.032	0.026	0.006	0.000	0.082	2.022	0.104	3.967	1.406
σ_{34}	0.196	0.180	0.022	0.001	0.143	0.487	0.027	0.085	0.540

In the third experiment the two methods provided surprisingly close values. In many cases the difference between the average values of the two estimators was in the fourth decimal place. The worst case was when estimating σ_{11}, when the robust average estimate was 3.660 and the classical was 3.893. In this case the RMSE were, respectively, 0.084 and 0.021. The relative efficiency, computed as the ratio between the squares of the STDEV (standard deviation) of both estimates and for all 20 parameters ranged between 1.041 and 1.253, a very satisfactory result.

We emphasize that finding the best fit for multivariate data is not an end, but a means of providing statistical inferences. For example, computations of risk measures, simulations of data to study and measure the chances of adverse scenarios, etc. We did some verifications on the accuracy of such applications, which we do not report since this would make the paper too long.

4 Assessing Robust Portfolios Performances

As already commented, financial analysts are typically interested in the applications that follow a (good) fit of a multivariate data set. For instance, the estimates $(\widehat{\mu_h}, \widehat{\Sigma}_{emp})$ may be used as inputs for asset allocation. The MV model is probably the most used model for efficiently allocate capital among risky asset classes. Estimation of the efficient frontier is almost always done via the classical sample mean $\bar{\mathbf{x}}$ and sample covariance S. However, different statistical estimates define different efficient frontiers. One of the most important causes of limitation of MV optimization in practice is the lack of optimality presented by classical estimates. For the problem of assessing the effect of classical estimates on the estimation of MV efficient frontiers, see Klein and Bawa (1976), Jobson and Korkie (1981), Best and Grauer (1991), and Britte and Jones (1999).

The literature suggests alternatives for the estimation of the inputs in the MV model. Examples include shrinkage estimators or Bayesian procedures (see West and Harrison, 1998). These suggestions can be found in Michaud (1998), but Fletcher and Hillier (2001) find little difference between the performance of portfolios with weights estimated according to Markowitz (1959) and Michaud's (1998) alternatives. Also, Handa and Tiwari (2000) find that incorporating parameter uncertainty through a Bayesian approach does not improve portfolio performance.

Portfolios constructed based on high breakdown point estimates are meant to be used for long term objectives, since they capture the dynamics of the majority of the business days, the "normal" days. On the other hand, the efficient frontier resulting from the use of classical estimates may reflect neither the usual nor the atypical days, as illustrated in the Introduction. In what follows we apply the $\widehat{\Sigma}_{emp}$ and the $\widehat{\Sigma}_{MV}$, together with $\widehat{\mu_h}$, to a real data set, and construct robust portfolios that should reflect the behavior of both the usual and the extreme days.

We chose emerging markets data due to their high frequency of atypical observations. The benefits of international diversification have been examined in the litterature and, from a US investor point of view, Li et al. (2001) found them to be small using emergent markets data. We take a different approach in one empirical exercise and address the asset allocation problem from the point of view of an investor in an emerging market considering more volatile asset classes than those that would be considered by an US investor. Thus, while there may be little gain from international diversification and alternative approaches to estimate the inputs of MV optimization for an US investor, this may not be true for other investors.

The five assets composing the portfolio are: 1. The Brazilian index IBOVESPA; 2. A Brazilian fixed income (CDI), benchmark for money market yields; 3. The American index S&P500; 4. The MSCI EAFE index to represent the rest of the world; 5. The J. P. Morgan Latin American EMBI to represent US dollar emerging market bonds (Brady Bonds); all denominated in dollars. We use 1413 daily percentual returns from January, 2, 1996 to May, 31, 2001. The data possess several extreme points observed during local and global crisis periods.

We will perform out-of-sample analysis of several aspects of the robust and classical optimal portfolios and investigate, in particular which one could yield cumulative returns. To this end, we split the data in two parts. The first part of the data, the estimating period, is used to compute the robust and classical inputs for the MV optimization procedure. The second part, the testing period, is used in the comparisons.

4.1 Cumulative Returns

The first aspect analyzed was the trajectory of the portfolios' cumulative returns over the testing period. Three portfolios in the efficient frontier were used in the comparisons: (*a*) the portfolio possessing some fixed target daily percentual return v, say, $v = 0.08\%$; (*b*) the minimum risk and (*c*) the maximum return portfolios. There is no particular reason for the choice of the daily target return of $v = 0.08\%$. This was just a portfolio return value existing in both frontiers. The portfolios' performances were assessed by implementing the portfolios' allocations (given in Table 3) computed at the baseline $t = 1013$, the end of the estimating period, at which the estimates were obtained, up to $t = 1413$, the end of the testing period. The weights were kept fixed during the testing period.

Table 3: *Portfolios compositions at the baseline* $t = 1013$, *based on the robust and classical inputs.*

	% Daily Return	% Risk (St.Dev.)	WEIGHTS IBOVESPA	CDI	S&P500	EAFE	EMBI
(a) Fixed Target Return Portfolios							
Robust $\widehat{\Sigma}_{emp}$	0.080	0.253	0.045	0.677	0.000	0.000	0.278
Robust $\widehat{\Sigma}_{ML}$	0.080	0.302	0.063	0.693	0.000	0.000	0.243
Classical	0.080	0.944	0.000	0.067	0.824	0.034	0.075
(b) Minimum Risk Portfolios							
Robust $\widehat{\Sigma}_{emp}$	0.060	0.063	0.000	0.999	0.000	0.000	0.001
Robust $\widehat{\Sigma}_{ML}$	0.060	0.075	0.000	1.000	0.000	0.000	0.000
Classical	0.043	0.573	0.000	0.4552	0.1495	0.2708	0.125
(c) Maximum Return Portfolios							
Robust $\widehat{\Sigma}_{emp}$	0.165	1.986	1.000	0.000	0.000	0.000	0.000
Robust $\widehat{\Sigma}_{ML}$	0.165	2.354	1.000	0.000	0.000	0.000	0.000
Classical	0.088	1.082	0.000	0.000	1.000	0.000	0.000

Figure 3 shows the cumulative returns of the portfolios (*a*), and Figure 4 shows the results for portfolios (*b*) and (*c*). As a benchmark and just for the sake of comparisons, we also plot the trajectory of the equally weighted (EW) portfolio. The figures display returns cumulated over the 400-days period. In all three scenarios analysed the portfolios constructed using $\widehat{\Sigma}_{emp}$ (the black line) dominate the classical ones (gray line). The middle (dotted) line in Figure 2 corresponds to the EW. The two robust portfolios performances were so close that we plotted only the empirical. Also the corresponding weights were very close.

<<Insert Figure 3 here>>

The out-of-sample performance of the portfolios depend upon their intrinsic characteristics, but also whether or not the testing and the estimating periods are compatible. In other words, for the comparisons to be meaningful, the inputs computed with and without the observations in the testing period should be close. The poor performance of the maximum return portfolios of Figure 4 at the end of the testing period of 400 days may be due to the fact that the estimating sample and the end of the testing sample represent quite different market behaviors. To verify this concern, and to assess the variability of the returns accumulated over the testing period, we carry out the following analysis.

<<Insert Figure 4 here>>

We again split the data in a estimating sample of size 1013 and a testing period of size 400. Using the baseline estimates we compute three portfolios: the minimum risk (Mi), the maximum return (Ma), and a "central" portfolio (Me), whose return is, given an efficient frontier, the average between the returns of its portfolios of minimum risk and maximum return[4]. The cumulative return over a 100-days period is computed for each one. Then, the following 10 observations ($t = 1014$ to $t = 1023$) are added to the estimating sample. All computations are repeated, robust and classical portfolios of the three types (Mi, Ma, Me) are obtained at the baseline

[4]Typically, the robust and the classical efficient frontiers occupy different regions in the Risk × Return space. Therefore, it is quite difficult to compare robust and classical portfolios. By choosing the "central" portfolio, we aim to characterize a portfolio designed for investors possessing the same degree of risk aversion, half way between the minimum and maximum risks for any given efficient frontier.

$t = 1023$, and estimates for the final value of the 100-days accumulated returns of all portfolios are saved. We repeat this process until we have 1313 observations in the sample, thus obtaining 31 representations of the returns of the (6) portfolios at the baselines and at the end of the 100-days periods. Only the the Σ_{emp} was applied.

The objective was to characterize the distributions of the returns and risks of the robust and classical portfolios at two points of the time: At the baselines ($t = 1013, 1023, ..., 1313$), and also at the end of the 100-days periods ($t = 1113, 1123, ..., 1413$).

We first characterize the distribution of the portfolios constructed at the baselines. Figure 5 shows the distribution of the three robust and classical baseline portfolios. In this figure, the notations RMi, RMe, and RMa (CMi, CMe, CMa) stand for the robust (classical) portfolios of the three types. The plot at left shows the returns. We observe that the robust portfolios are more stable, with a distribution located at higher values and possessing smaller variability. For example, for the minimum risk portfolios, the smaller observed robust value was greater than the highest observed classical one. We also carried out a formal paired t-test to test equality of the means of the returns. For all three types of portfolios the p-value was zero against the alternative hypothesis of the robust mean return being greater than the classical one.

<<Insert Figure 5 here>>

The risks associated to the 31 portfolios are box-plotted at the right hand side of Figure 5. We also note a smaller variability of the (also smaller) robust quantities. The baseline effect on the portfolios may be noticed from the high variability of the returns computed for each type of portfolio. However, the robust ones showed more stability through time.

Next, we investigated the distribution of the cumulative returns by examining the 100-days accumulated values associated to the 31 baseline portfolios. Table 4 gives summaries of the results. We observe that, for all three types, the distributions of the accumulated returns of the robust portfolios are located at the right of the classical ones. For example, the median of the central robust portfolio is 0.407, whereas the classical central portfolio distribution is located at -1.495.

Table 4: *Quantiles of the distribution of the 100-days cumulative returns for the three types of portfolios.*

	Probabilities				
	0.05	0.25	0.50	0.75	0.95
Minimum Risk Portfolios (Mi)					
Robust	-3.7247	0.2054	1.4816	4.0401	10.8487
Classical	-8.5662	-5.7592	-1.7407	1.6105	6.9451
Central Portfolios (Me)					
Robust	-8.7699	-3.3460	0.4071	4.3120	11.1023
Classical	-8.0000	-3.8494	-1.4955	2.1490	6.2649
Maximum Return Portfolios (Ma)					
Robust	-21.7927	-15.3349	-10.0274	-1.8232	13.5189
Classical	-26.9746	-18.6547	-12.0028	-3.2452	11.8448

4.2 Weights Stability

We also investigated the stability of the weights associated to the robust and classical portfolios. This is an important issue since the portfolios' compositions are usually kept fixed during some period, here during 100-days. To check this assumption, we again split the data in two parts. The first part contains 1313 observations and it is used to estimate the robust and classical MV inputs. The idea is to observe, for a given portfolio, how the weights change as long as new data points are incorporated in the analysis. Thus, successive days were incorporated into the analyzes (and the sample sizes increased to $1313 + i$, $i = 1, ..., 100$). In this way we could assess how stable the portfolios weights are during a 100-days period. The minimum risk, the maximum return, and the "central" portfolio are used. The compositions of the portfolios at the baseline day $t = 1313$ are given in Table 5.

Table 5: *Portfolios' compositions at baseline* $t = 1313$ *under robust and classical estimation.*

	Return	Risk	IBOVESPA	CDI	S&P500	EAFE	EMBI
Minimum Risk Portfolios							
Robust $\widehat{\Sigma}_{emp}$	0.058	0.099	0.000	0.988	0.000	0.000	0.012
Classical	0.039	0.560	0.000	0.486	0.099	0.228	0.187
Maximum Return Portfolios							
Robust $\widehat{\Sigma}_{emp}$	0.120	2.093	1.000	0.000	0.000	0.000	0.000
Classical	0.088	2.722	1.000	0.000	0.000	0.000	0.000
Central Return Portfolio							
Robust $\widehat{\Sigma}_{emp}$	0.089	0.409	0.000	0.389	0.000	0.000	0.611
Classical	0.063	0.992	0.071	0.000	0.567	0.000	0.362

The results are that the weights are very stable for the two extreme portfolios, under both robust and classical procedures, which usually puts weight 1 to some variable. However, the weights associated with the robust and classical central portfolios are different. This can be seen in Figure 6 where we boxplot the weights associated to the 5 components of the central portfolio under the robust (left) and classical (right) approaches. The robust weights presented less variability for all 5 components. Thus the robust portfolios seem to have more stable weights, thus reducing portfolio rebalancing costs.

<<Insert Figure 6 here>>

5 Conclusions

In this paper we proposed a statistical model and estimation procedure for heavy tailed multivariate elliptical data containing some proportion of atypical observations. These are characteristics typically found in financial data, in particular in emerging markets data.

The rationale behind the proposal is that the true correlations among the variables are those observed in the vast majority of the business days. Extreme observations may show up locally or globally, and whenever they occur this may result in spurious correlations if classical estimates are used. This is mainly due to the fact that these atypical observations may tilt the orientation of the axes of the ellipsoid associated to the covariance matrix estimate.

Our robust model and estimation procedure is expected to reflect the pattern of usual days, via the high breakdown point correlation structure, and also the effect of atypical days, via classical estimates of variances.

We performed three simulation experiments to check the performance of the proposal when compared to the widely used sample estimates. We generated data from a model representing some of the characteristics of financial returns, and contaminated it with some proportion of additive outliers. We found that the robust estimates presented smaller biases and smaller root mean squared error. Under no contamination they presented very reasonable relative efficiency with respect to the maximum likelihood estimates.

Then we looked to one of the most important statistical applications in finance that is particularly sensitive to input estimates errors: the MV efficient frontiers. Several aspects of the out-of-sample performance of the robust and classical portfolios were investigated. We found that robust portfolios typically yield higher accumulated returns. Also, for any given type of portfolio (minimum risk, maximum return, central portfolio), the robust portfolios showed a more concentrated distribution with higher expected returns. We also concluded that the baseline choice has a stronger effect on classical portfolios than on the robust ones. In other words, due to their definition and statistical properties, the robust estimates were able to reduce the instability of the optimization process. Finally, we found that this stability property carried over to the weights associated to the robust portfolios.

Endnotes

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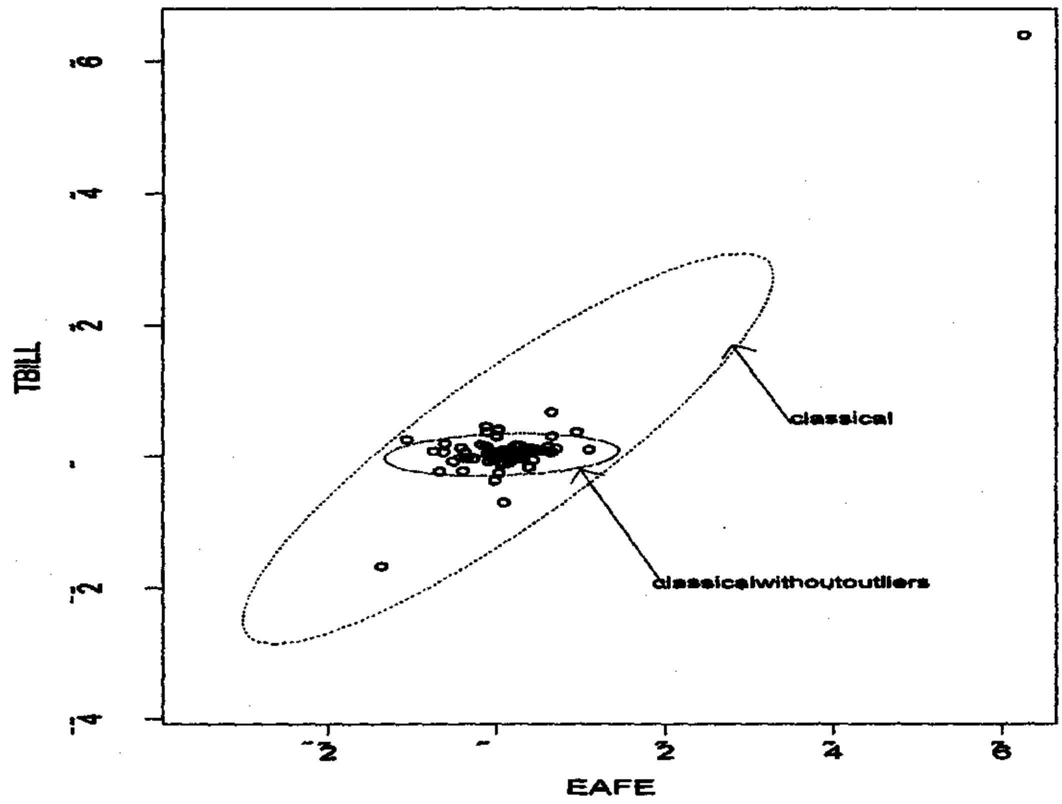

Figure 1: *Ellipsoids of constant probability equal to 0.999 for the monthly returns of the EAFE and T.BILL. Points on the curves are at the same statistical distance from the corresponding centers, thus possessing the same likelihood of occurrence.*

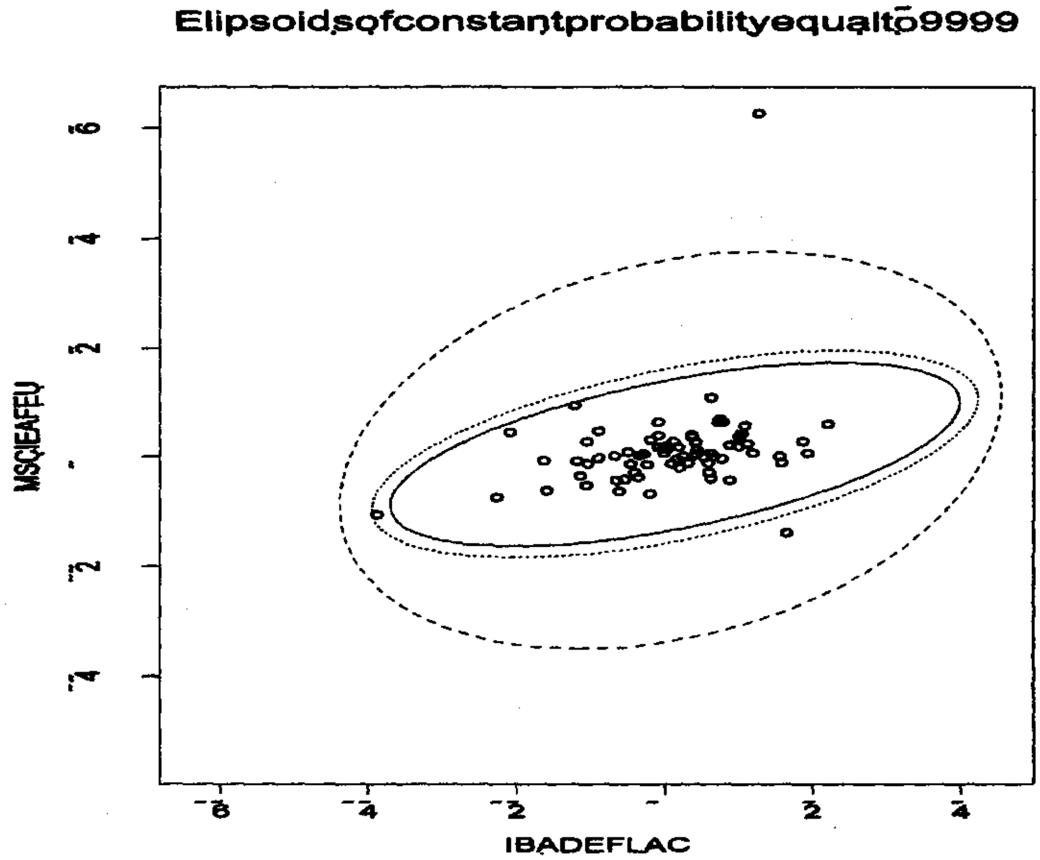

Figure 2: *Ellipsoids of constant probability equal to 0.999 for the monthly returns of the IBA and EAFE. The solid line corresponds to the empirical estimation, the dotted line to the maximum likelihood estimation, and the dash-dotted line to the classical estimators.*

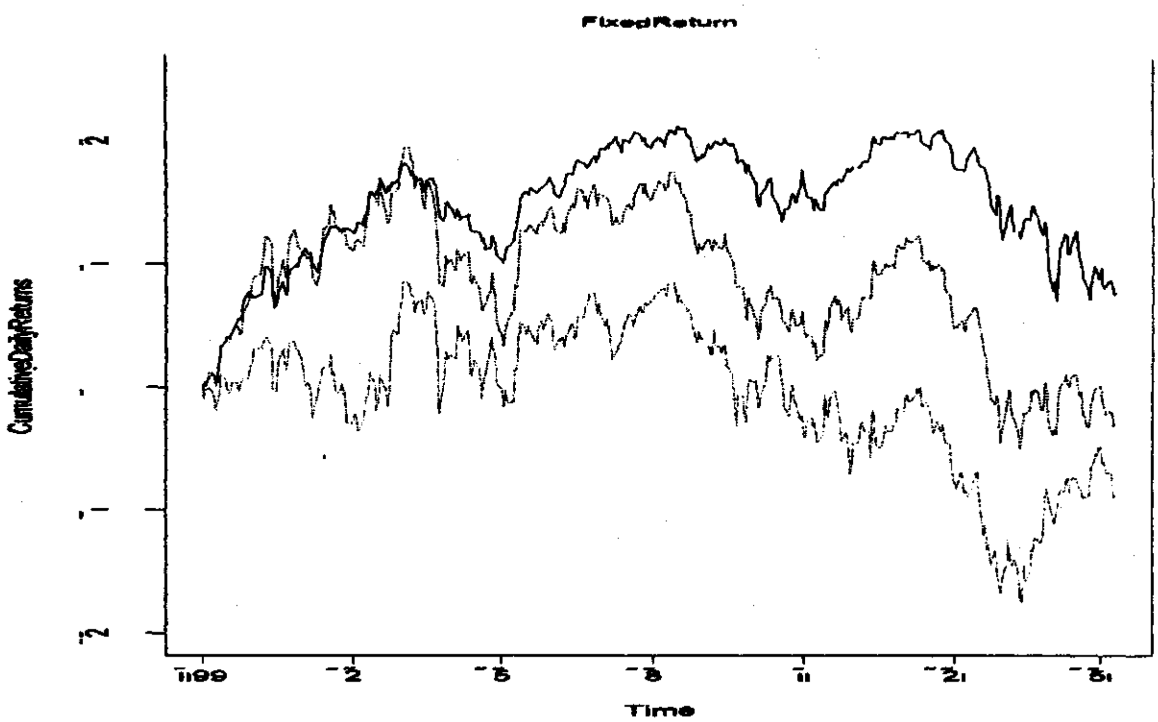

Figure 3: *Cumulative (%) daily returns of portfolios with target daily return equal to 0.08%. The black line corresponds to the robust portfolio. The gray line to the classical portfolio. The dotted line corresponds to the equally weighted portfolio.*

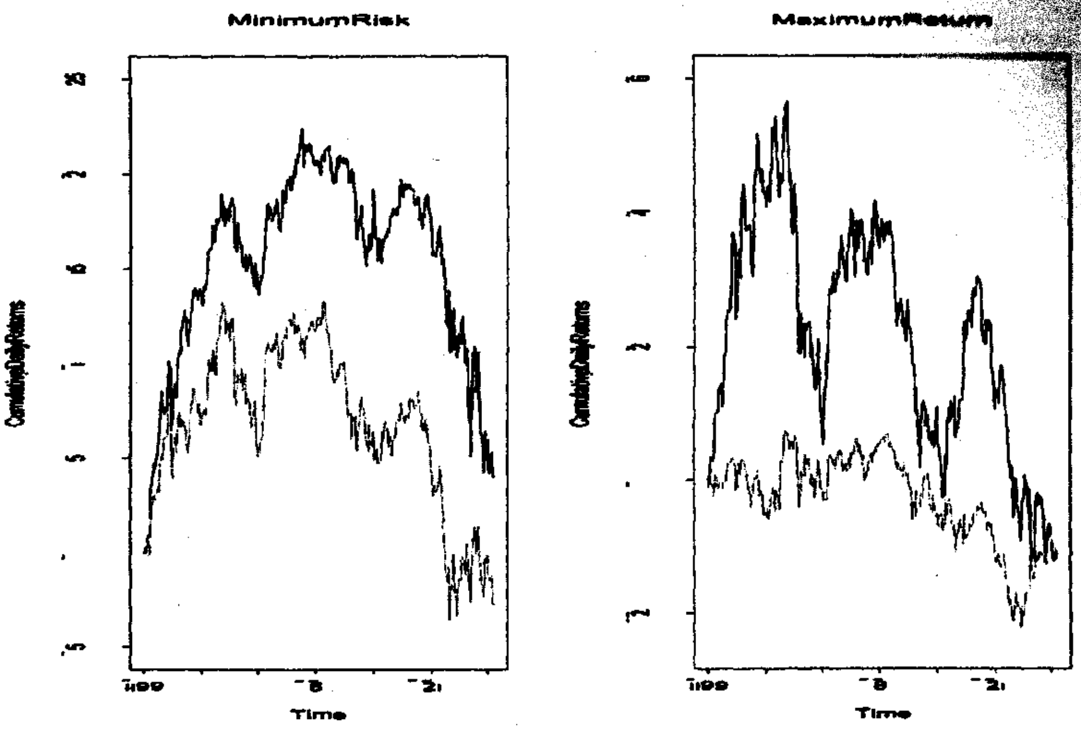

Figure 4: *Cumulative (%) daily returns for the robust (black line) and the classical (gray line) portfolios. At left: portfolios with minimum risk. At right: portfolios yielding maximum returns.*

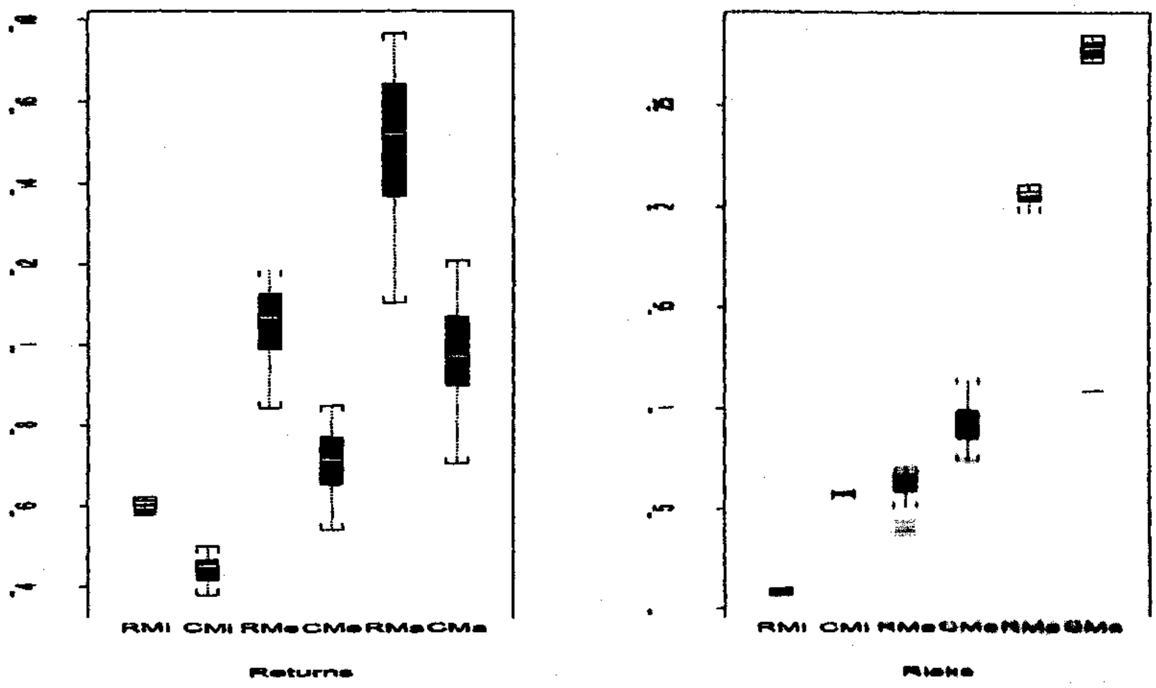

Figure 5: *Distribution of the returns and risks of the baseline portfolios (t = 1013, 1023, ...) under the robust and classical approaches.*

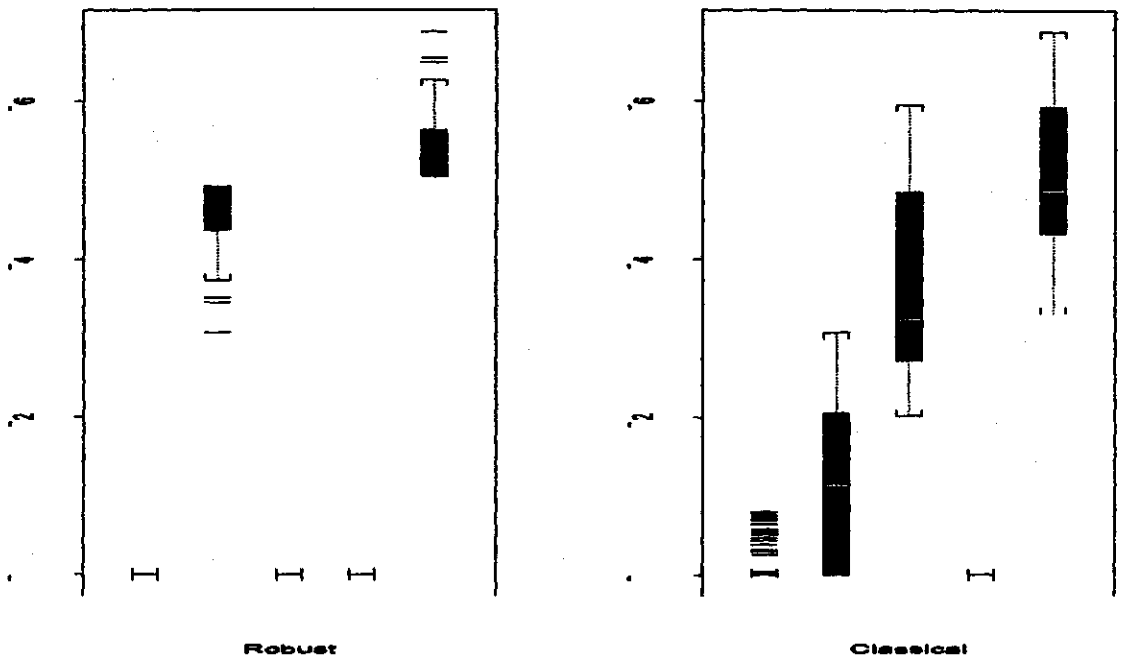

Figure 6: *Boxplots of robust and classical weigths associated to the five components of the central portfolio.*