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Topics in Quantum Field Theory: On-Shell Methods and Vacuum Polarization near a Boundary

Gabriel Almeida Zarpelon

Dissertação de Mestrado apresentada ao Programa de Pós-Graduação em Física do Instituto de Física da Universidade Federal do Rio de Janeiro - UFRJ, como parte dos requisitos necessários para a obtenção do título de Mestre em Ciências (Física).

Orientador: Henrique Boschi Filho

Coorientador: Thales Azevedo

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Resumo

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Nesta dissertação, apresentamos tópicos atuais relacionados à descrição quântica dos campos, pilar fundamental para a física de partículas e de interações fundamentais. Discutimos os métodos tradicionais de cálculos de elementos de matriz e de amplitudes de espalhamento, os métodos *on-shell* recentes (como a descrição do espalhamento em termos de autovetores do operador helicidade, a recursão de Britto-Cachazo-Feng-Witten (BCFW) e os cortes utilizados para a reconstrução de amplitudes com *loops*) e aplicações deste ferramental ao cálculo de processos na Eletrodinâmica Quântica e na Relatividade Geral de Einstein-Hilbert, mostrando como os métodos *on-shell* podem ser econômicos e eficientes, quando comparados aos métodos tradicionais. Além disso, discutimos efeitos inesperados que podem surgir quando condições de contorno não triviais originam classes de flutuações do vácuo dependentes da fronteira.

Palavras-chave: Teoria Quântica de Campos, Eletrodinâmica Quântica, Relatividade Geral, Métodos *On-Shell*, Recursão BCFW, Unitariedade Generalizada, Polarização do Vácuo, Correção de Uehling, Método das Imagens.

Abstract

Topics in Quantum Field Theory: On-Shell Methods and Vacuum Polarization near a Boundary

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Abstract da Dissertação de Mestrado apresentada ao Programa de Pós-Graduação em Física do Instituto de Física da Universidade Federal do Rio de Janeiro - UFRJ, como parte dos requisitos necessários para a obtenção do título de Mestre em Ciências (Física).

In this dissertation, we discuss current topics related to the quantum description of fields, a fundamental tool in the description of elementary particles and fundamental interactions. We will discuss the traditional methods for calculation of matrix elements and scattering amplitudes, and the newer on-shell paradigm, specifically, the Britto-Cachazo-Feng-Witten (BCFW) recursion and the discontinuity cuts featured in the generalized unitarity method. We will then, apply these tools in the calculation of QED processes, and in the context of the Einstein-Hilbert theory, showing how on-shell methods can be economic and efficient. We also discuss nontrivial phenomena which can occur when the presence of nontrivial boundary conditions induces classes of boundary dependent vacuum fluctuations.

Keywords: Quantum Field Theory, Quantum Electrodynamics, General Relativity, On-Shell Methods, BCFW recursion, Generalized Unitarity, Vacuum Polarization, Uehling Correction, Method of Images.

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Chapter 1

Introduction

Field theory has been a pillar for the physical description of various types of systems, ranging from the waves in a medium to the gravitational field, both as described by Newton and Einstein. The reason for such success is perhaps mysterious. After all, naively, most of the phenomena in our day to day lives are a consequence of the atoms and elementary particles, and–in light of the successes of non-relativistic quantum mechanics– the description of those constituents in terms of fields, constructs which are defined over *all* of space and/or time, might seem inadequate or unnecessary. Nevertheless, the description of physical fields has long been used to explain the dynamics of systems. However, it was only in the twentieth century that the idea of *quantum fields* came into existence.

While the point-like, discrete description of quantum particles has had a myriad of successes in explaining some of the phenomena which classical physics cannot describe, the attempts of formulating a relativistic theory of quantum mechanics have been effectively hijacked by the field description. This description has proven useful when the number of particles in a given system can vary and it is precisely the combination of the quantum uncertainty principle and the relativistic energy-mass equivalence which brings about the possibility of *particle creation*, even if the original system had only one particle to begin with. For this reason, instead of a Hilbert space description, one uses a Fock space description where the number of particles in each state is directly altered by use of

annihilation-creation operators.

This description was first attempted for the electromagnetic field and led to initial disappointment when it was seen that many of the processes the theory could describe led to divergent quantities. As infinities are usually ill-defined, those divergences led to disenchantment from the leading figures of the field, such as Dirac, who is famously credited with the first relativistic description of the electron and the formalism of quantum mechanics [1, 2, 3]. It was not until much later that the famous work of Hans Bethe, showed that the supposed infinite values arising from a quantum description of electromagnetism, once cut at a given energy scale, could fit well enough with the *Lamb shift* [4], a major mystery of spectroscopy at the time. This led to renewed interest in how to formally make sense of the divergences and, more importantly, extract physically sensible results, as Bethe had done. To this end, the famous works of Feynman, Schwinger and Tomonaga were crucial in developing the *quantum theory of electrodynamics* (QED) and making predictions such as the anomalous magnetic moment of the electron.[1]

From this point, much of the second half of the twentieth century saw a great development in the understanding of general quantum field theories, not only for electromagnetism, but also for other fields. This, ultimately, lead to the development of the *Standard Model* of particle physics, largely based on non-abelian gauge theories which were discussed by Yang, Mills and Shaw.

From this brief outline, it is clear that quantum theories of fields have carved a fundamental place in the way physicists describe the fundamental constituents of the world. However, the story is not finished (thankfully!), as not all has been well adjusted to this formalism.

Now a famous mantra, the original[2] theory of General Relativity (GR) is *non-renormalizable*. This implies that the systematic way to treat the divergent expressions which worked for

[1] For a comparative discussion of the approaches see the classic work [5].

[2] By "original" we mean the theory stemming from the Einstein-Hilbert Lagrangian.

other theories seems to break down for the highly nonlinear Einstein-Hilbert theory. To remedy this, an abundance of theories have been developed, many finding much more richness than simply as quantum theories of gravitation,[3] but to this day, no experimental verification has been able to ultimately rule out the candidates that have appeared.

While the full ultraviolet (UV) completion of gravity has not been understood, the advancements in experimental *and* theoretical findings have led to a renewed interest in calculations which, in some cases, were performed half a century ago. This is precisely the case with GR, whose quantum description as an *effective field theory* (EFT) has found applications in the groundbreaking measurements of gravitational waves [6, 7, 8].

A similar story also appears in the slightly better understood context of *quantum vacuum boundary effects*. The first calculations which provided evidence for nontrivial effects in the quantum vacuum from the mere presence of nontrivial boundary conditions[4] where done in the twentieth century, but nowadays have had renewed interest from the experimental verification of these phenomena. Such is the case with the famous *Casimir Effect* which predicts a small attractive force between two neutral, perfectly conducting plates in the vacuum of QED [9, 10, 11, 12] . However, many other interesting phenomena also appear as a direct byproduct of the vacuum fluctuations of the fields in contact with the boundary structures.[5]

In this thesis, we will discuss some of the applications of Quantum Field Theory (QFT) which are related to the above examples. We will first give a brief outline of the machinery which is used in the lengthy calculations of traditional QFT and also introduce in more detail the current on-shell approaches advocated by the so-called "amplitudes community." Then, in chapter 3, we will discuss the methods in the familiar case of QED. In chapter 4, we undertake a small detour from the on-shell methods and discuss a novel

[3]Such as string theory, which has served of inspiration to many tools now used for the demanding calculations in standard QFTs.

[4]By "nontrivial" we mean boundary conditions other than the usual asymptotic vanishing of the fields. This includes the (still fairly trivial) perfect conducting plates which are a staple of the Casimir effect.

[5]Another famous example is the Purcell effect [13].

phenomenon in the perturbative expansion of QED, which is related to the presence of a boundary. Finally, in chapter 5, we will apply the on-shell tools to General Relativity as a theory of gravitons and calculate some tree processes. Even at tree level, these become a difficult task with the standard Feynman diagram technique, and will serve as a good demonstration of the on-shell based approach.

Chapter 2

Quantum Field Theory Toolkit

In this chapter, we briefly introduce some of the basic building blocks of quantum field theory, with exemplifications in the context of quantum electrodynamics (both scalar and spinorial). These developments are mostly standard in multiple classic references such as [14, 15, 16, 17, 18, 19, 20], but nevertheless are important for the development to follow. Of course, the familiarized reader can skip directly to the spinor helicity (SHF) section, where we review recent developments in the area of amplitude calculations which have shed new light on the complicated perturbative calculations of particle physics. These will also serve an important role in the developments of chapter 4, when we discuss recent applications to general relativity.

2.1 Spinor and Scalar QED

To begin, let us work in quantum electrodynamics (QED). This corresponds to a quantum field theory of an abelian (massless) gauge field of spin 1, which is commonly called a photon. Denoting this field by $A_\mu(x)$, with $\mu = 0, 1, 2, 3$, the Lagrangian which describes this field (up to gauge fixing which we mention later) is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \,, \tag{2.1}$$

where $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. This is simply the Lagrangian for Maxwell's theory of electrodynamics with no sources and a four potential A_μ. The quantum properties of the field appear when one imposes the canonical formalism to A_μ, reinterpreting it as an operator which acts on vacuum states $|0\rangle$. Explicitly, one writes

$$A_\mu(x) = \sum_{s=1,2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left[\epsilon_\mu^{s\,*}(k) a^\dagger(k) e^{-ik\cdot x} + \epsilon_\mu^s(k) a(k) e^{ik\cdot x}\right] , \tag{2.2}$$

and interprets $a^\dagger, a$ as operators which create photons with momentum k and polarization $\epsilon_\mu^s(k)$. The s index is used to label each polarization vector, of which two are needed to fully describe the field. Also, $k \cdot x = k^0 x^0 - \mathbf{k}\cdot\mathbf{x} = E_k t - \mathbf{k}\cdot x$, such that the exponential factors can be understood as plane wave components.

The free theory can be developed from these basic ingredients and we refer the reader to the vast literature on the subject. Of course, we are interested in a theory with interactions coming from the interplay between other fields reacting to the A_μ photon field and vice versa.

First, let us discuss the theory with a scalar field φ coupled to the photon field, which slightly simplifies some of the theory since no spinors are introduced. In this case, let us consider that we have a complex scalar field φ of mass m whose Lagrangian is

$$\mathcal{L}_\phi = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* ,$$

Then, we can consider how the coupling to the photon can be constructed in a variety of ways. One of them is to consider a *local symmetry* parametrized by a space-time function $\alpha(x)$:

$$\phi \to \phi' = e^{-i\alpha(x)} \phi . \tag{2.3}$$

Clearly, the "mass" term in the Lagrangian is invariant under this. However, the kinetic term is not. In fact, it transforms as

$$\partial_\mu \phi \partial^\mu \phi^* \to (\partial_\mu - i\partial_\mu \alpha)\, \phi\, (\partial^\mu + i\partial^\mu \alpha)\, \phi^* ,$$

Now, remember the Lagrangian for the photon field: $-F^2/4$. This term also came with a specific gauge invariance, namely, all transformations parametrized by a change in the photon field such as

$$A_\mu \to A_\mu + \partial_\mu \alpha \,.$$

In fact, we can put any constant alongside the second term and so, for the sake of perturbation theory, we will add an $1/e$. Then, operating the derivatives, we see that

$$(\partial_\mu + ieA_\mu)\phi \to (\partial_\mu + ieA_\mu + i\partial_\mu\alpha)e^{-i\alpha}\phi = e^{-i\alpha}(\partial_\mu + ieA_\mu)\phi \,.$$

Thus, the product of this expression and its complex conjugate lead to a local gauge invariance. We denote this new *gauge covariant* derivative as

$$D_\mu\phi \equiv (\partial_\mu + ieA_\mu)\phi \,, \tag{2.4}$$

and add it to the scalar QED ($\mathcal{L}_{\phi\text{QED}}$) Lagrangian:

$$\mathcal{L}_{\phi\text{QED}} = \mathcal{L}_{QED} + \mathcal{L}_\phi + \mathcal{L}_{int} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)(D^\mu\phi)^* - m^2\phi\phi^* \,. \tag{2.5}$$

Before moving on, we need to remind the reader that this introductory setup skips over many important points. One of them is the concept of gauge fixing, which is of fundamental importance for the perturbative analysis of any QFT. Once again, we refer to the basic literature [14, 15, 16] that thoroughly discusses this issue. It will suffice to say that the actual Lagrangian for the gauge boson must include a gauge-fixing term which is conventionally taken to be

$$\mathcal{L}_{gauge\text{-}fix} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \,. \tag{2.6}$$

Then, the full photon Lagrangian is

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \,, \tag{2.7}$$

and this should be used to write $\mathcal{L}_{\phi\text{QED}}$, for example.

Expanding the covariant derivative, we have the following Lagrangian

$$\mathcal{L}_{\phi QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \phi\left(\partial^2 - m^2\right)\phi^* + e^2 A_\mu \phi A^\mu \phi^* + ieA_\mu(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi)\,. \quad (2.8)$$

Thus, apart from the straightforward propagators for the photon and the complex scalar field, we have three types of interaction vertices. Two of them are similar, and arise from the last two terms of the expanded Lagrangian above. Each of them comes with a factor of $-e$, the field A_μ and the scalar field and its conjugate. To understand the derivative term, consider the expansion of the scalar field and its derivative:

$$\begin{aligned} \phi &= \int \frac{d^3\mathbf{p}}{(2\pi)^3\sqrt{2\omega_p}}\left(a(\mathbf{p})e^{-ip\cdot x} + b^\dagger(\mathbf{p})e^{ip\cdot x}\right)\,, \\ \implies \partial^\mu\phi &= \int \frac{d^3\mathbf{p}}{(2\pi)^3\sqrt{2\omega_p}}\left(-ip^\mu a(\mathbf{p})e^{-ip\cdot x} + ip^\mu b^\dagger(\mathbf{p})e^{ip\cdot x}\right)\,. \end{aligned}$$

Then, it is natural to see that the presence of the derivatives will add a factor corresponding to the momentum of the associated scalar field, either incoming or outgoing. In passing, we also note that because of the commutation relations between ϕ and its conjugate momentum $\pi = \partial_0\phi$, the creation-annihilation operators a, b inherit similar commutation relations, namely:

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,, \quad (2.9)$$

$$[b(\mathbf{p}), b^\dagger(\mathbf{p}')] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,. \quad (2.10)$$

For the external states, it is useful to write the expressions for the annihilation-creation operators in terms of the complex field. Explicitly, given the conjugate momentum $\pi = \partial_0\phi$ one can write the well-known Fourier expansions:

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a(\mathbf{p})e^{i(\omega_p t - \mathbf{p}\cdot\mathbf{x})} + b^\dagger(\mathbf{p})e^{-i(\omega_p t - \mathbf{p}\cdot\mathbf{x})}\right)\,, \quad (2.11)$$

$$\pi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{i\omega_p}{\sqrt{2\omega_p}}\left(a(\mathbf{p})e^{i(\omega_p t - \mathbf{p}\cdot\mathbf{x})} - b^\dagger(\mathbf{p})e^{-i(\omega_p t - \mathbf{p}\cdot\mathbf{x})}\right) \quad (2.12)$$

which can be inverted, yielding

$$a(\mathbf{p}) = \int d^3\mathbf{x}\sqrt{\frac{\omega_p}{2}}\left(\phi(x)+\frac{1}{i\omega_p}\pi(x)\right)e^{-i(\omega_p t-\mathbf{p}\cdot\mathbf{x})}\,, \tag{2.13}$$

$$b^\dagger(\mathbf{p}) = \int d^3\mathbf{x}\sqrt{\frac{\omega_p}{2}}\left(\phi(x)-\frac{1}{i\omega_p}\pi(x)\right)e^{i(\omega_p t-\mathbf{p}\cdot\mathbf{x})}\,. \tag{2.14}$$

Now, because the matter particles are scalars, the external states do not introduce any factors in the S-matrix elements (these particles do not have any internal degrees of freedom). For this reason, one only needs to worry about the external factors for photons, which correspond to the polarization vectors ϵ^s_μ. Furthermore, from the form of the Lagrangian (2.8), we see that there are two types of interaction vertices. The first, which also appears in QED, is the vertex where two scalars couple to a photon, which is given by the fourth term in (2.8). Due to the derivative, one gets a momentum factor for one of the particles and an extra factor of $-ie$. For the second kind of vertex, one has the contribution arising from the third term of (2.8), which corresponds to two scalars interacting with two photons with a single intersection vertex.

The case of spinor QED is only different from the scalar theory by the appearance of the spinors that must be used to represent the non-integer spin particles, such as the electron. Thus, instead of adding the kinetic term of a complex scalar field, we must consider the Dirac fermion ψ, such that

$$\mathcal{L}_\psi = \overline{\psi}(i\partial\!\!\!/ - m)\psi\,. \tag{2.15}$$

A brief overview of the quantization of such spinor fields is presented in the standard literature [14, 20, 18, 17], and we reproduce the basics in (A.3).

Now, we must add a coupling between the photon field A_μ and the fermion field ψ. Using the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$, the full Lagrangian is then

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(iD\!\!\!/ - m)\psi \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ - m)\psi - e\overline{\psi}\gamma^\mu A_\mu\psi\,.\end{aligned} \tag{2.16}$$

Thus, we see that there is only one vertex made out of the intersection of three lines, two fermions and a photon, contributing a factor of $-ie\gamma_\mu$. In this sense, the vertex of spinor QED is relatively easier to calculate than the vertices of scalar QED. However, the main difference is the appearance of the spinors which come with the external legs associated with fermions. To see exactly where they come from, we can also look at the derivation of the LSZ formula for the specific case of QED.

2.1.1 LSZ reduction

Let us discuss how the physical S-matrix elements can be constructed in QED. We begin with spinors and then quickly outline the analogous case for the photon field. The exposition is heavily inspired by the one found in [15].

Recall that the Dirac spinor has the following Fourier decomposition (mostly negative signature)

$$\Psi(x) = \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(u_s(\mathbf{p}) b_s(\mathbf{p}) e^{-ip\cdot x} + v_s(\mathbf{p}) d_s^\dagger(\mathbf{p}) e^{ip\cdot x} \right) ,$$

The above can be inverted to give the creation-annihilation operators in terms of Ψ (A.45), (A.47):

$$b_s(\mathbf{p}) = \frac{1}{\sqrt{2\omega_\mathbf{p}}} \int d^3\mathbf{x}\, \overline{u}_s(\mathbf{p}) \gamma^0 \Psi(x) e^{ix\cdot p} , \tag{2.17}$$

$$b_s^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2\omega_\mathbf{p}}} \int d^3\mathbf{x}\, \overline{\Psi}(x) \gamma^0 u_s(\mathbf{p}) e^{-ix\cdot p} , \tag{2.18}$$

$$d_s(\mathbf{p}) = \frac{1}{\sqrt{2\omega_\mathbf{p}}} \int d^3\mathbf{x}\, \overline{\Psi}(x) \gamma^0 v_s(\mathbf{p}) e^{ix\cdot p} , \tag{2.19}$$

$$d_s^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2\omega_\mathbf{p}}} \int d^3\mathbf{x}\, \overline{v}_s(\mathbf{p}) \gamma^0 \Psi(x) e^{-ix\cdot p} . \tag{2.20}$$

Note that, considering the anti-commutation rules for the Dirac field and its conjugate momentum, the above operators satisfy anti-commutation relations of the form

$$\{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} = (2\pi)^3 \delta_{ss'} \delta^3(\mathbf{p} - \mathbf{p}') , \tag{2.21}$$

$$\{d_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} = (2\pi)^3 \delta_{ss'} \delta^3(\mathbf{p} - \mathbf{p}') , \tag{2.22}$$

while the anti-commutations between b_s, d_s vanish. For more details see eqs.(A.49),(A.48) of the appendix.

Let us now consider the construction of transition amplitudes in this theory. We can create n_1 quanta associated with $d_s^\dagger$ and n_2 quanta associated with the creation operator $b_s^\dagger$:

$$|i\rangle = \mathcal{C} \prod_{n=1}^{n_1} \prod_{m=1}^{n_2} d_{s_n}^\dagger(\mathbf{p}_n) b_{s_m}^\dagger(\mathbf{p}_m) \, |0\rangle \ , \tag{2.23}$$

where $C = \sqrt{\prod_{n=1}^{n_1} \prod_{m=1}^{n_2} 2\omega_{\mathbf{p_n}} 2\omega_{\mathbf{p_k}}}$ is the normalization factor associated with the choice of the Lorentz invariant measure. The question to be posed is how to generalize the above for the case of the interacting theory. For simplicity, let us consider only two particles coming and interacting. To isolate the scattering process, we consider that initially the two particles are far away enough as to allow us to consider that the system is not interacting. Then, we will allow the particles to interact and assume that after the scattering, they evolve far apart enough so that the system becomes free again. Note that the locality of the interaction is *necessary* for this construction, since otherwise, we would not be able to assume that the free theory states are asymptotically achievable[14, 15]. Thus, we consider an initial state $|\Phi_{in}\rangle$ constructed as

$$|\Phi_{in}\rangle = \lim_{t \to -\infty} d_{s_1}^\dagger(t, \mathbf{p}_1) b_{s_2}^\dagger(t, \mathbf{p}_2) \, |0\rangle \ .$$

We can rewrite each operator in terms of the fields by using equations (2.17), (2.18), (2.19), (2.20):

$$\begin{aligned}
d_s^\dagger(-\infty, \mathbf{p}) - d_s^\dagger(\infty, \mathbf{p}) &= -\int_{-\infty}^{\infty} dt \, \partial_0 d_s^\dagger(t, \mathbf{p}) \, , \\
&= -\int_{-\infty}^{\infty} dt \, \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \int d^3\mathbf{x} \, \overline{v}_s(\mathbf{p}) \gamma^0 \partial_0 \left(\Psi(x) e^{-ix\cdot p} \right) \, , \\
&= -\frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \int d^4x \, \overline{v}_s(\mathbf{p}) \left(\gamma^0 \partial_0 - i\gamma^0 p^0 \right) \Psi(x) e^{-ix\cdot p} \, , \\
&= -\frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \int d^4x \, \overline{v}_s(\mathbf{p}) \left(\gamma^0 \partial_0 - i(-\gamma^k p_k - m) \right) \Psi(x) e^{-ix\cdot p} \, , \\
&= -\frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \int d^4x \, \overline{v}_s(\mathbf{p}) \left(\gamma^0 \partial_0 + i\gamma^k p_k + im \right) \Psi(x) e^{-ix\cdot p} \, ,
\end{aligned}$$

where we used $\overline{v}_s(\mathbf{p})(\not{p}+m)=0 \implies \overline{v}_s(\mathbf{p})\gamma^0 p_0 = (-m-\gamma^k p_k)\overline{v}_s(\mathbf{p})$. Now, we use a trick to reinterpret p_k as the eigenvalue of the momentum operator, such that $p^k \to -i\partial^k$. We can do this by considering that $p_k \exp(-ix\cdot p) = i\partial_k \exp(-ix\cdot p)$. Thus, we can integrate by parts so as to make ∂_k act on Ψ and write

$$\begin{aligned} d_s^\dagger(-\infty,\mathbf{p}) - d_s^\dagger(\infty,\mathbf{p}) &= -\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\int d^4x\, \overline{v}_s(\mathbf{p}) e^{-ix\cdot p}\left(\gamma^0\partial_0 + \gamma^k\partial_k + im\right)\Psi(x)\,, \\ &= -i\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\int d^4x\, e^{-ix\cdot p}\,\overline{v}_s(\mathbf{p})\left(-i\not{\partial}+m\right)\Psi(x)\,. \end{aligned} \tag{2.24}$$

Similarly,

$$b_s^\dagger(-\infty,\mathbf{p}) - b_s^\dagger(\infty,\mathbf{p}) = i\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\int d^4x\,\overline{\Psi}(x)\left(i\overleftarrow{\not{\partial}}+m\right)u_s(\mathbf{p})e^{-ix\cdot p}\,. \tag{2.25}$$

from which we can also obtain expressions for the annihilation operators:

$$b_s(-\infty,\mathbf{p}) - b_s(\infty,\mathbf{p}) = i\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\int d^4x\, e^{ix\cdot p}\,\overline{u}_s(\mathbf{p})\left(-i\not{\partial}+m\right)\Psi(x)\,, \tag{2.26}$$

$$d_s(-\infty,\mathbf{p}) - d_s(\infty,\mathbf{p}) = -i\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\int d^4x\,\overline{\Psi}(x)\left(i\overleftarrow{\not{\partial}}+m\right)v_s(\mathbf{p})e^{ix\cdot p}\,. \tag{2.27}$$

Let us now go back to the transition amplitudes. Consider a process where particles of type "b" and type "d" come in and scatter outwards. The probability for this process is described by $\langle\Phi_{out}|\Phi_{in}\rangle$, where the asymptotic states are

$$|\Phi_{in}\rangle = \sqrt{2\omega_{\mathbf{p}_1}2\omega_{\mathbf{p}_2}}\,d_{s_1}^\dagger(-\infty,\mathbf{p}_1)b_{s_2}^\dagger(-\infty,\mathbf{p}_2)\,|0\rangle\ , \tag{2.28}$$

$$\langle\Phi_{out}| = \langle 0|\,\sqrt{2\omega_{\mathbf{p}_3}2\omega_{\mathbf{p}_4}}\,d_{s_3}(+\infty,\mathbf{p}_3)b_{s_4}(+\infty,\mathbf{p}_4)\,. \tag{2.29}$$

If we write the amplitude using equations (2.24),(2.25), (2.26), (2.27), we find

$$\begin{aligned} \langle\Phi_{out}|\Phi_{in}\rangle = i^4\int d^4x_1\int d^4x_2\int d^4x_3\int d^4x_4\, e^{-ix_1\cdot p_1 - ix_2\cdot p_2 + ix_3\cdot p_3 + ix_4\cdot p_4} \\ \times\overline{v}_{s_1}(\mathbf{p}_1)\left(-i\not{\partial}_{x_1}+m\right)\overline{u}_{s_4}(\mathbf{p}_4)\left(-i\not{\partial}_{x_4}+m\right) \\ \times\langle 0|\,T\left\{\Psi(x_1)\overline{\Psi}(x_2)\Psi(x_4)\overline{\Psi}(x_3)\right\}|0\rangle \\ \times\left(i\overleftarrow{\not{\partial}}_{x_3}+m\right)v_{s_3}(\mathbf{p}_3)(\mathbf{p}_2)\left(i\overleftarrow{\not{\partial}}_{x_2}+m\right)u_{s_2}(\mathbf{p}_2)\,. \end{aligned} \tag{2.30}$$

Here, T refers to the time ordering symbol, organizing the field operators such that they act with respect to decreasing times from left to right. Thus, all that is left is to calculate the matrix elements $\langle 0|\, T\left\{\Psi(x_1)\overline{\Psi}(x_2)\Psi(x_4)\overline{\Psi}(x_3)\right\}|0\rangle$. We are also able to extract the important factors coming from external fermions: $\overline{v}_s$ for incoming antiparticles, u_s for incoming particles, v_s for outgoing antiparticles and $\overline{u}_s$ for outgoing particles.

2.1.2 QED matrix elements

To calculate the matrix elements of the type discussed in the last section, one can use a number of different methods. The path integral method is based on the use of the functional generator of correlation functions Z. In QED, this is a functional of $J^\mu, \eta, \overline{\eta}$, currents associated with $\overline{\Psi}, \Psi, A_\mu$, and can be written as

$$Z[J^\mu,\eta,\overline{\eta}] \equiv \int [\mathcal{D}A][\mathcal{D}\Psi][\mathcal{D}\overline{\Psi}] \exp\left(i\int d^4x \left(\mathcal{L}_{QED} + J^\mu(x)A_\mu(x) + \overline{\Psi}(x)\eta(x) + \overline{\eta}(x)\Psi(x)\right)\right), \tag{2.31}$$

$$\mathcal{L}_{QED} = \overline{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}F_{\mu\nu}^2\,. \tag{2.32}$$

To write Z in a better form, we follow a standard procedure found in [15, 16, 14]. Let us consider each part of the Lagrangian that corresponds to free fields separately. For the fermion field action with the fermion sources, we can write the momentum space expression:

$$\begin{aligned} S_\Psi &= \int d^4x\, \overline{\Psi}(x)(i\not{\partial} - m)\Psi(x)\,, \\ &= \int d^4x \int \frac{d^4k}{(2\pi)^4}\int \frac{d^4k'}{(2\pi)^4}\Big\{\widetilde{\overline{\Psi}}(k')(\not{k} - m)\widetilde{\Psi}(k) + \widetilde{\overline{\eta}}(k')\widetilde{\Psi}(k) + \widetilde{\overline{\Psi}}(k')\widetilde{\eta}(k)\Big\}e^{-ix\cdot(k-k')}\,, \end{aligned}$$

and since

$$\int d^4x\, e^{-ix\cdot(k-k')} = (2\pi)^4\delta^4(k-k')\,,$$

we can write

$$S_\Psi = \int \frac{d^4k}{(2\pi)^4}\Big\{\widetilde{\overline{\Psi}}(k)(\not{k} - m)\widetilde{\Psi}(k) + \widetilde{\overline{\eta}}(k)\widetilde{\Psi}(k) + \widetilde{\overline{\Psi}}(k)\widetilde{\eta}(k)\Big\}\,. \tag{2.33}$$

To proceed, let us use a change of the field variable of the form

$$\widetilde{\Psi}(k) \to \widetilde{\Psi}'(k) = \widetilde{\Psi}(k) + \widetilde{\overline{\eta}}(k)(\not{k} - m)^{-1}\widetilde{\eta}(k)\,. \tag{2.34}$$

Using this in the integrand of the momentum-space action, we obtain

$$S_{\Psi'} = \int \frac{d^4k}{(2\pi)^4}\Big\{\widetilde{\overline{\Psi}}'(k)(\not{k} - m)\widetilde{\Psi}'(k) - \widetilde{\overline{\eta}}(k)(\not{k} - m)^{-1}\widetilde{\eta}(k)\Big\}\,. \tag{2.35}$$

Since the field change is a translation, the Jacobian associated with the measure in the path integral is unchanged: $\mathcal{D}[\Psi']\mathcal{D}[\overline{\Psi}'] = \mathcal{D}[\Psi]\mathcal{D}[\overline{\Psi}]$. Thus, the Ψ independent term can be taken outside the path integral, and we find

$$\begin{aligned} Z[\eta,\overline{\eta}] &= \exp\left(-i\int \frac{d^4k}{(2\pi)^4}\widetilde{\overline{\eta}}(k)(\not{k} - m)^{-1}\widetilde{\eta}(k)\right)Z[0,0]\,, \\ &= Z[0,0]\exp(iW_F[\eta,\overline{\eta}])\,, \end{aligned} \tag{2.36}$$

where W_F can also be written in real space as

$$W_F[\eta,\overline{\eta}] = -\int d^4x \int d^4y\, \overline{\eta}(x)G_F^0(x-y)\eta(y)\,, \tag{2.37}$$

and

$$G_F^0(x-y) = \lim_{\epsilon\to 0}\int \frac{d^4k}{(2\pi)^4}\frac{1}{\not{k} - m + i\epsilon}e^{-ik\cdot(x-y)}\,, \tag{2.38}$$

is the free fermion propagator in momentum space, and the ϵ has been explicitly stated in order to respect the Feynman prescription. It is worth mentioning that G^0 is actually a matrix. Multiplying by $\not{k} + m$ on the numerator and denominator, we can write

$$[G_F^0(x-y)]_{a\dot{a}} = \lim_{\epsilon\to 0}\int \frac{d^4k}{(2\pi)^4}\frac{\gamma^\mu_{a\dot{a}}k_\mu + m\mathbf{1}_{a\dot{a}}}{k^2 - m^2 + i\epsilon}e^{-ik\cdot(x-y)}\,,$$

so we can see that G_F^0 has two spinor indices. For the reader unfamiliar with this spinorial notation, we refer to appendix A.2 and references therein. Nevertheless, here we simply recall that spinor representations of the Lorentz group can be expressed in terms of spin 1/2 objects with definite parity. This allows one to construct two sets of two component

vectors, which are indexed by the dotted $\dot{a}$ and undotted a labels. These correspond to the usual intuition of "up and down" spin. Since the Dirac field groups particles and anti-particles, it is a field of four components, grouping both chiral components. Then, the gamma matrix γ^μ is a four vector comprised of 2 by 2 matrices which also carry spinorial indices (justifying the notation $\gamma^\mu_{a\dot{a}}$).

A similar discussion can be carried out for the photon field A, under the appropriate use of the gauge fixing term. We will skip over since this is identical to what was just done. The result is that the functional generator Z *without* the interaction can be written as

$$Z[J,\eta,\overline{\eta}] = Z[0,0,0]\exp(iW_F[\eta,\overline{\eta}])\exp(iW_A[J])\,, \tag{2.39}$$

with

$$\begin{aligned} W_F[\eta,\overline{\eta}] &= -\int d^4x \int d^4y\, \overline{\eta}(x) G^0_F(x-y)\eta(y)\,, & (2.40)\\ W_A[J] &= \frac{1}{2}\int d^4x \int d^4y\, J^\mu(x)\Delta^0_{\mu\nu}(x-y)J^\nu(y)\,, & (2.41) \end{aligned}$$

where $\Delta^0_{\mu\nu}$ is the free photon propagator. From Z as written in (2.39), one can calculate the matrix elements by taking consecutive functional derivatives. As an example, consider the time-ordered correlation of field operators $\langle\Omega|\ T\hat{A}^{\mu_1}(x_1)\hat{A}^{\mu_2}(x_2)\ldots\hat{A}^{\mu_n}(x_n)\,|\Omega\rangle$. It is given by

$$\begin{aligned} \left\langle \hat{A}^{\mu_1}(x_1)\hat{A}^{\mu_2}(x_2)\ldots\hat{A}^{\mu_n}(x_n)\right\rangle &= \int [\mathcal{D}A][\mathcal{D}\Psi][\mathcal{D}\overline{\Psi}]\, A^{\mu_1}(x_1)A^{\mu_2}(x_2)\ldots A^{\mu_n}(x_n)\, e^{iS}\Big|_{J,\eta,\overline{\eta}=0}\,,\\ &= \left(\frac{1}{i}\right)^n \left\{\frac{\delta}{\delta J^{\mu_1}(x_1)}\cdots\frac{\delta}{\delta J^{\mu_n}(x_n)}\int [\mathcal{D}A][\mathcal{D}\Psi][\mathcal{D}\overline{\Psi}] e^{iS}\right\}\Big|_{J,\eta,\overline{\eta}=0}\\ &= \left(\frac{1}{i}\right)^n \left\{\frac{\delta^n Z[J,\eta,\overline{\eta}]}{\delta J^{\mu_1}(x_1)\delta J^{\mu_2}(x_2)\cdots\delta J^{\mu_n}(x_n)}\right\}\Big|_{J,\eta,\overline{\eta}=0}\,. \end{aligned} \tag{2.42}$$

The same carries over for fields containing fermion operators. However, care must be taken when doing so, since Ψ and $\overline{\Psi}$ are Grassmann valued fields, or more formally, are given by

$$\Psi(x) = \sum_n \theta_n \psi(x)\,, \tag{2.43}$$

where θ_n are Grassmann numbers. Thus, one must choose a sign convention for the functional differentiation. We will consider that

$$\frac{\delta}{\delta\eta(y)}\big(\overline{\Psi}(x)\eta(x)\big) \;=\; -\overline{\Psi}(x)\frac{\delta\eta(x)}{\delta\eta(y)} = -\overline{\Psi}(x)\delta(x-y)\,. \tag{2.44}$$

With this, we can calculate, for example

$$\big\langle \Psi(x_1)\overline{\Psi}(x_2)\big\rangle = \frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x_1)}i\frac{\delta}{\delta\eta(x_2)}Z[J,\eta,\overline{\eta}]\Big|_{J,\eta,\overline{\eta}=0}\,. \tag{2.45}$$

To allow for the interaction term, we simply reinterpret the field terms in the interaction as functional derivatives acting on the free functional generator. Thus, we write

$$\begin{aligned}
Z[J,\eta,\overline{\eta}] &= \int[\mathcal{D}A][\mathcal{D}\Psi][\mathcal{D}\overline{\Psi}]\exp\!\left(-i\int d^4x\, e\overline{\Psi}\not{A}\Psi\right)e^{iS_{free}}\,,\\
&= \exp\!\left(-i\int d^4x\, ei\frac{\delta}{\delta\eta(x)}\gamma^\mu\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x)}\right)\int[\mathcal{D}A][\mathcal{D}\Psi][\mathcal{D}\overline{\Psi}]e^{iS_{free}}\,,\\
&= Z[0,0,0]\exp\!\left(-i\int d^4x\, ei\frac{\delta}{\delta\eta(x)}\gamma^\mu\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x)}\right)e^{i(W_F[\eta,\overline{\eta}]+W_A[J])}\,,\\
&= Z[0,0,0]\exp\!\left(-ie\int d^4x\, i\frac{\delta}{\delta\eta(x)}\gamma^\mu\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\frac{1}{i}\frac{\delta}{\delta\overline{\eta}(x)}\right)e^{i(W_{QED}[J,\eta,\overline{\eta}]}\,.
\end{aligned} \tag{2.46}$$

Using equation (2.46) we can obtain any matrix element which corresponds to green's functions of the theory. If they are used in equation (2.30), we can calculate S-matrix elements which are translated to scattering amplitudes.

2.1.3 Intermezzo: Feynman Rules

As we discussed, with the LSZ formalism and the matrix elements obtained from the generating functional, one can calculate scattering amplitudes. As is by now standard, this very long procedure has been short-cut by the introduction of the famous Feynman rules, which are just the outcomes of the procedure we discussed above. For QED, these are found in any standard reference of QFT and we refer the reader to [14, 20, 21, 16].

Even though the formalism developed is functional, to some extent, it is actually quite cumbersome to do calculations in. The number of Feynman diagrams which need to be

considered grows way too fast for direct brute force calculations to be viable. Thus, new methods to obtain the desired results have been developed. We generally work in mostly plus signature, unless stated otherwise.

2.2 Spinor Helicity formalism

Consider a particle with spin, such as an electron, such that in order to define an appropriate quantum state, we must consider both spin and momentum. We define the *helicity* of this particle as the projection of the spin onto the 3-momentum: $h = \vec{\mathbf{p}} \cdot \vec{\mathbf{S}}/|\vec{\mathbf{p}}|$. Thus, for a massless particle, which moves with the speed of light, this quantity is Lorentz invariant, since no boost can take us to a frame which moves faster than the particle, flipping the direction of the 3-momentum and consequently flipping the helicity, implying that no matter the frame, the helicity will keep its value. Also, it has extreme values of $h = \pm s$, for a particle with spin s.

Let us note that the concept of helicity is in fact much older than spinors. When one discusses light polarization, circular polarization is introduced as a basis that possesses chirality or handedness. Circularly polarized light corresponds to an eigenstate of the helicity operator, where $h = +$ corresponds to right-handedness, and $h = -$ to left-handedness.

In this sense, the spinor-helicity formalism is endowed with studying scattering amplitudes directly in terms of the helicity eigenstates. However, there is a deeper richness to the subject, especially when one considers how some of the mathematics was first developed. We will begin with the straightforward introduction of the formalism for massless particles. Then, we will reconsider our approach to make the massive formalism more natural.

2.2.1 Massless Formalism

For a spin 1/2 particle, the Dirac equation in momentum space reads:

$$(\not{p} + m)u(p) = 0\,,$$
$$(\not{p} - m)v(p) = 0\,,$$

such that in the massless case, $u = v$. We define helicity states by projecting u, v:

$$u_\pm = \frac{1}{2}(1 \pm \gamma_5)u(p)\,; \qquad v_\mp = \frac{1}{2}(1 \pm \gamma_5)v(p)\,, \tag{2.47}$$

which implies that in the massless case, $u_\pm = v_\mp$. These observations can be carried over to higher spin particles.

The starting point of the formalism is to translate the four momentum p in terms of bi-spinors. This is achieved by a mapping from four vectors to matrices which is realized by contractions with the mixed objects $\sigma^\mu_{a\dot{b}}, \bar{\sigma}^{\mu\dot{a}b}$. Specifically, under $\eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$, we have

$$p^{\dot{\alpha}\alpha} = \bar{\sigma}^{\mu\dot{\alpha}\alpha}p_\mu = \begin{pmatrix} -p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & -p^0 + p^3 \end{pmatrix}, \qquad p_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}p_\mu = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix}, \tag{2.48}$$

which can also be inverted, giving, for example, $p^\mu = \frac{1}{2}\mathrm{sgn}(g^{00})\sigma^\mu_{\alpha\dot{\alpha}}p^{\dot{\alpha}\alpha} = \frac{1}{2}\mathrm{sgn}(g^{00})\overline{\sigma}^{\mu\dot{\alpha}\alpha}p_{\alpha\dot{\alpha}}$.

Now, we observe that the determinants of these bi-spinors are $\det\big(p^{\dot{\alpha}\alpha}\big) = p_0^2 - \vec{\mathbf{p}}^2 = m^2 = 0 = \det(p_{\alpha\dot{\alpha}})$. Since the determinant is zero, we conclude that these objects, while being two by two matrices, are only of rank one. Therefore, we can decompose this matrix into the product of two vectors, which we call *helicity spinors*:

$$p^{\alpha\dot{\alpha}} = \lambda^\alpha\widetilde{\lambda}^{\dot{\alpha}}\,. \tag{2.49}$$

Note that the tilde is used to explicitly differentiate the right-handed weyl spinor from the left-handed one. To see how phyiscal parameters are written in this formalism, consider the mandelstalm invariant s for a massless particle 1 and a massless particle 2:

$$s_{12} = -(p_1 + p_2)^2 = -2p_1 \cdot p_2 = -2\,p_{1\mu}p_2^\mu = -\frac{1}{2}\sigma_{\mu\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}}p_1^{\alpha\dot{\alpha}}p_2^{\beta\dot{\beta}} = -\frac{1}{2}\sigma_{\mu\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}}\lambda_1^\alpha\widetilde{\lambda}_1^{\dot{\alpha}}\lambda_2^\beta\widetilde{\lambda}_2^{\dot{\beta}}\,.$$

Now, we can re express the product of the sigma matrices. Since they are invariant, we suppose that they must be proportional to invariant objects carrying two sets of spinor indices. Naturally:

$$\sigma_{\mu\alpha\dot{\alpha}}\sigma^{\mu}_{\ \beta\dot{\beta}} = C\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\,.$$

Consider the component $\alpha = 1, \beta = 2$. Here, $\varepsilon_{12} = 1 = \varepsilon_{\dot{1}\dot{2}}$. On the left handed side, the resulting component will be the sum $-1 + 0 + 0 - 1 = -2$. Thus, $C = -2$, and we find $\sigma_{\mu\alpha\dot{\alpha}}\sigma^{\mu}_{\ \beta\dot{\beta}} = -2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}$.[1] With this, we can finally write

$$s_{12} = \overbrace{\left(\varepsilon_{\alpha\beta}\lambda_1^{\alpha}\lambda_2^{\beta}\right)}^{[12]}\overbrace{\left(\varepsilon_{\dot{\alpha}\dot{\beta}}\widetilde{\lambda}_1^{\dot{\alpha}}\widetilde{\lambda}_2^{\dot{\beta}}\right)}^{-\langle 12\rangle} = -[12]\,\langle 12\rangle = [12]\,\langle 21\rangle\ . \tag{2.50}$$

Here, we introduced the notation which has become the trademark of the SH formalism, but we will detail it later. For now, we simply note that this can be generalized for multiple particles. If we consider n particles:

$$s_{1\ldots n} = \sum_{(ij)\in C_2^n} [ij]\,\langle ji\rangle\ , \tag{2.51}$$

where C_2^n refers to the different pairings of the n different labels, without repetition. The aesthetic simplicity of formula (2.51) is only a taste of this formalism for scattering amplitudes of massless particles. It is also fun to note that many different expressions involving the sigma and gamma matrices can be rewritten in terms of these helicity spinors and momentum bi-spinors. As an example, we write $\not{p} = \gamma^{\mu}p_{\mu}$ as

$$\not{p} = \begin{pmatrix} 0 & \sigma^{\mu}_{\ \alpha\dot{\beta}} \\ \bar{\sigma}^{\mu\dot{\alpha}\beta} & 0 \end{pmatrix} p_{\mu} = \begin{pmatrix} 0 & p_{\alpha\dot{\beta}} \\ p^{\dot{\alpha}\beta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda_{\alpha}\widetilde{\lambda}_{\dot{\beta}} \\ \widetilde{\lambda}^{\dot{\alpha}}\lambda^{\beta} & 0 \end{pmatrix}\ . \tag{2.52}$$

While this might just seem like a needless effort, there is an interesting relation which can be derived from equation ((2.52)). Consider the Weyl equations for the two components

[1] Note that if we chose the opposite convention for the Minkowski metric, we would get $C = 2$.

of the massless Dirac field:

$$\begin{aligned} \not{p} v_s(p) = 0\,; \qquad & \bar{u}_s \not{p} = 0\,, \\ \implies v_+(p) = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}\,; \qquad & v_-(p) = \begin{pmatrix} 0 \\ \widetilde{\lambda}^{\dot{\alpha}} \end{pmatrix}\,, \\ \implies \bar{u}_+(p) = \begin{pmatrix} \lambda^\alpha & 0 \end{pmatrix}\,; \qquad & u_-(p) = \begin{pmatrix} 0 & \widetilde{\lambda}_{\dot{\alpha}} \end{pmatrix}\,. \end{aligned}$$

The last expressions show that each helicity spinor is indeed associated with each possible spin solution for the Dirac field. It is important to keep in mind when expressions like the ones written above are using the spinors as components and as actual vector arrays. In the above equations, $v_s(p), u_s(p)$ are both 4 component objects, which implies that the spinors $\lambda, \widetilde{\lambda}$ written inside the parentheses are 2 component spinors (and the 0's are actually column/row zero vectors). From the above, we can also obtain the states of definite helicity by using the chiral projectors P_L and P_R defined as

$$P_L = \frac{1-\gamma^5}{2}\,, \qquad P_R = \frac{1+\gamma^5}{2}\,. \tag{2.53}$$

Then, for Dirac spinors $u(p)$ and $v(p)$, we have

$$\begin{aligned} P_{R/L} u(p) = u_\pm(p)\,; \qquad & P_{R/L} v(p) = v_\mp(p)\,; \\ \overline{u}(p) P_{R/L} = \overline{u}_\mp(p)\,; \qquad & \overline{v}(p) P_{R/L} = \overline{v}_\pm\,. \end{aligned}$$

As anticipated in ((2.50)), the helicity spinors can also be written in a "bra-ket" way by introducing the following notation [22], [15]:

$$\lambda^\alpha \to [p|^\alpha\,; \qquad \widetilde{\lambda}_{\dot{\alpha}} \to \langle p|_{\dot{\alpha}}\,, \tag{2.54}$$

$$\lambda_\alpha \to |p]_\alpha\,; \qquad \widetilde{\lambda}^{\dot{\alpha}} \to |p\rangle^{\dot{\alpha}}\,. \tag{2.55}$$

This notation allows us to write multiple expressions involving these helicity spinors. It is also important to note that some authors such as [23] use a different convention, such that

$$\lambda^\alpha \to |p\rangle^\alpha\,; \qquad \widetilde{\lambda}_{\dot{\alpha}} \to |p]_{\dot{\alpha}}\,, \tag{2.56}$$

$$\lambda_\alpha \to \langle p|_\alpha\,; \qquad \widetilde{\lambda}^{\dot{\alpha}} \to [p|^{\dot{\alpha}}\,. \tag{2.57}$$

We will follow the former, such that

$$\overline{u}_+(p) \to \begin{pmatrix} [p| & 0 \end{pmatrix} , \tag{2.58}$$

$$\overline{u}_-(p) \to \begin{pmatrix} 0 & \langle p| \end{pmatrix} , \tag{2.59}$$

$$v_+(p) \to \begin{pmatrix} |p] \\ 0 \end{pmatrix} , \tag{2.60}$$

$$v_-(p) \to \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} , \tag{2.61}$$

and, similarly, the Dirac spinors can be built

$$\overline{u}(p) = \begin{pmatrix} [p| & \langle p| \end{pmatrix} , \qquad u(p) = \begin{pmatrix} |p\rangle \\ |p] \end{pmatrix} \tag{2.62}$$

$$\overline{v}(p) = \begin{pmatrix} |p] & |p\rangle \end{pmatrix} , \qquad v(p) = \begin{pmatrix} |p] \\ |p\rangle \end{pmatrix} . \tag{2.63}$$

Here, we note that incoming particles of + helicity are associated with $u_+ = v_- \to |p\rangle$, while incoming particles with − helicity are associated with $u_- = v_+ \to |p]$. Then, if we use conventions where all particles are *outgoing*, crossing symmetry implies that *outgoing* particles with negative helicity will be associated with $|p\rangle$, while *outgoing* particles with positive helicity will be associated with $|p]$. If one chose the other convention, naming the dotted spinors with square brackets and un-dotted with angled ones, the same mnemonic -positive helicity means squared brackets and negative helicity means angled ones- would still be valid as long as one chose the convention where all particles are *incoming*. Thus, in general, positive helicity is associated with square spinors and negative with angled ones. As will become clear later, this notational divergence is not all that relevant for scattering amplitudes, since the relevant physical quantities are given in terms of the products $[]$, $\langle\rangle$ or symmetric combinations of projector-like objects. Nevertheless, as long as one chooses either all outgoing or all incoming, working directly with the angled-square kets is fairly error proof (although it can be confusing if the particular convention is not explicitly stated and one attempts to interpret the helicities in terms of their physical counter-parts).

Let us now introduce multiple identities which are useful in the evaluation of amplitudes. For example, consider the completeness relation of the spinors for the massless particle. Using the above:

$$\sum_{s=\pm} u_s(p)\overline{u}_s(p) = -\not{p} = |p]\langle p| + |p\rangle [p|\,, \tag{2.64}$$

which is the same for $v_\pm, \overline{v}_\pm$. Another important property is the so-called *Schouten identity*. Consider four particles such that their helicity spinors are labeled by $|1\rangle, |2\rangle \ldots |4\rangle$ and so on. Since these are two dimensional, we ought to be able to express them in terms of one another, say $|1\rangle = c_2\,|2\rangle + c_3\,|3\rangle$. However, since $\langle ii\rangle = 0$, we can simply contract with each label to obtain the coefficients:

$$|1\rangle = \frac{\langle 31\rangle}{\langle 32\rangle}\,|2\rangle + \frac{\langle 21\rangle}{\langle 23\rangle}\,|3\rangle\ ,$$

which, upon contracting with the fourth label, gives

$$\langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 42\rangle + \langle 14\rangle\langle 23\rangle = 0\,. \tag{2.65}$$

Another set of useful identities are related to contracting the gamma matrices γ^μ with two helicity spinors q, p with spinor components χ, λ, respectively. By writting out the matrix form of the objects:

$$\begin{aligned}\langle q|\,\gamma^\mu|p] &= \begin{pmatrix}0 & \widetilde{\chi}_{\dot\alpha}\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu_{\alpha\dot\alpha}\\ \overline{\sigma}^{\mu\dot\alpha\alpha} & 0\end{pmatrix}\begin{pmatrix}\lambda_\alpha\\ 0\end{pmatrix} = \lambda_\alpha\widetilde{\chi}_{\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\alpha}\,,\\ &= \lambda^\beta\widetilde{\chi}^{\dot\beta}\sigma^\mu_{\beta\dot\beta} = [p|\gamma^\mu\,|q\rangle\ ,\end{aligned} \tag{2.66}$$

and, similarly,

$$\langle q|\,\gamma^\mu\,|p\rangle = \begin{pmatrix}0 & \widetilde{\chi}_{\dot\alpha}\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu_{\alpha\dot\alpha}\\ \overline{\sigma}^{\mu\dot\alpha\alpha} & 0\end{pmatrix}\begin{pmatrix}\widetilde{\lambda}^{\dot\alpha}\\ 0\end{pmatrix} = 0 = [p|\gamma^\mu|q]\,. \tag{2.67}$$

In fact, equation ((2.66)) can be identified with the *Gordon identity* of QED, by choosing $q = p$, such that

$$\langle p|\,\gamma^\mu|p] = \lambda^\alpha\widetilde{\lambda}^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha} = p^{\alpha\dot\alpha}\sigma^\mu_{\alpha\dot\alpha} = 2p^\mu\,. \tag{2.68}$$

For one last demonstration, let us derive the *Fierz identities* which will be useful when defining polarization vectors. Contracting (2.66) with γ_μ, we can write

$$\begin{aligned}\gamma_\mu \langle q|\,\gamma^\mu|p] &= \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\alpha} \\ \overline{\sigma}^{\mu\dot\alpha\alpha} & 0 \end{pmatrix} \begin{pmatrix} 0 & \widetilde{\chi}_{\dot\beta} \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu_{\beta\dot\beta} \\ \overline{\sigma}^{\mu\dot\beta\beta} & 0 \end{pmatrix} \begin{pmatrix} \lambda_\beta \\ 0 \end{pmatrix}, \\ &= \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu \widetilde{\chi}_{\dot\beta}\lambda_\beta \\ \overline{\sigma}^{\mu\dot\alpha\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu \widetilde{\chi}_{\dot\beta}\lambda_\beta & 0 \end{pmatrix}.\end{aligned}$$

However, from $\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\beta\dot\beta} = -2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}$, we can obtain the contractions of the sigma matrices:

$$\begin{aligned}\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu &= \varepsilon^{\beta\gamma}\varepsilon^{\dot\beta\dot\gamma}\sigma_{\mu\dot\gamma\gamma}\sigma^\mu_{\alpha\dot\alpha} = -2\varepsilon^{\beta\gamma}\varepsilon^{\dot\beta\dot\gamma}\varepsilon_{\alpha\gamma}\varepsilon_{\dot\alpha\dot\gamma} = -2\delta^\beta{}_\alpha\delta^{\dot\beta}{}_{\dot\alpha}\,, \\ \overline{\sigma}^{\mu\dot\alpha\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu &= -2\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\,.\end{aligned}$$

Thus, we can write

$$\begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu \widetilde{\chi}_{\dot\beta}\lambda_\beta \\ \overline{\sigma}^{\mu\dot\alpha\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu \widetilde{\chi}_{\dot\beta}\lambda_\beta & 0 \end{pmatrix} = -2\begin{pmatrix} 0 & \delta^\beta{}_\alpha\delta^{\dot\beta}{}_{\dot\alpha}\widetilde{\chi}_{\dot\beta}\lambda_\beta \\ \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\widetilde{\chi}_{\dot\beta}\lambda_\beta & 0 \end{pmatrix} = -2\begin{pmatrix} 0 & \widetilde{\chi}_{\dot\alpha}\lambda_\alpha \\ \widetilde{\chi}^{\dot\alpha}\lambda^\alpha & 0 \end{pmatrix}.$$

Now, we can either identify $\widetilde{\chi}_{\dot\alpha}\lambda_\alpha = |p]_\alpha \langle q|_{\dot\alpha}$ and $\widetilde{\chi}^{\dot\alpha}\lambda^\alpha = |q\rangle^{\dot\alpha}\,[p|^\alpha$, or simply compute the projectors $|p]\,\langle q|$,$|q\rangle\,[p|$. Either way, we finally arrive at the desired result

$$-\frac{1}{2}\gamma_\mu \langle q|\,\gamma^\mu|p] = |p]\,\langle q| + |q\rangle\,[p|\,. \tag{2.69}$$

Another useful way of expressing the above is by contracting with other helicity spinors $[r|, |s\rangle$ on both sides, yielding

$$[r|\gamma_\mu\,|s\rangle\,\langle q|\,\gamma^\mu|p] = 2[pr]\,\langle qs\rangle\,, \tag{2.70}$$

which is especially useful for 4 point scattering at tree level. Here, we note the implicit property that $[p|\,|r\rangle = 0$, that is, different representations cannot be contracted. Another very useful identity for n particle scattering is the momentum conservation. Considering all momenta outgoing, this implies $\sum_j p^\mu_j = 0$. Contracting with the gamma matrices, one finds

$$\sum_{j=1,2\ldots n} \not{p}_j = -\sum_{j=1,2\ldots n} (|j]\,\langle j| + |j\rangle\,[j|) = 0\,. \tag{2.71}$$

Thus, contracting with the positive helicity spinor $[i|$ and negative helicity spinor $|k\rangle$ and recalling that $[i|\,|j\rangle = 0 = [j|\,|k\rangle$, we have

$$\sum_{j=1,2...n} [ij]\,\langle jk\rangle = 0\,. \tag{2.72}$$

Note that one could also remove k and i from the label summation, since $[ii] = \langle kk\rangle = 0$. This expression is particularly useful for cancellations of common factors during a given amplitude calculation, as will be seen later.

From our discussion, any Lorentz index object can be mapped to a spinor indexed object using these spinors, as we did with the four momentum $p^\mu \to p^{a\dot{a}}$. This is effectively a mapping from $SO(1,3)$ (the Lorentz group) to $SL(2,\mathbb{C})$ (two dimensional matrices with complex entries). As such, we can also write any other tensor using this procedure (see the apendix (A.5) for examples of this with the generators of the Lorentz group). Of special use are polarization vectors ϵ^μ which are needed to describe external states for photons, for example. They can be embedded in spinor indices as $\epsilon^{a\dot{a}} = \epsilon^\mu \overline{\sigma}_\mu^{a\dot{a}}$. To construct them explicitly we follow [24].

First, recall that λ^a and $\widetilde{\lambda}^{\dot{a}}$ are members of vector spaces such that it is completely sensible to construct linear combinations of their products to construct objects like $\epsilon^{a\dot{a}}$. Since $\epsilon^{a\dot{a}}$ is a bi-spinor (a four dimensional object), we should expect that we need four spinors to complete the basis. Mathematically, we mean

$$\epsilon_\pm^{a\dot{a}} = a_\pm \lambda^a \widetilde{\lambda}^{\dot{a}} + b_\pm \xi^a \widetilde{\xi}^{\dot{a}} + c_\pm \lambda^a \widetilde{\xi}^{\dot{a}} + d_\pm \widetilde{\lambda}^{\dot{a}} \xi^a\,, \tag{2.73}$$

where $\xi, \widetilde{\xi}$ are the two commuting spinors needed to complete the basis. By using the expected properties of the polarization vector/bi-spinor, one can solve the above. The result is

$$\epsilon_+^{\dot{\alpha}\alpha}(\lambda;\xi) = \sqrt{2}\frac{\tilde{\lambda}^{\dot{\alpha}}\xi^\alpha}{[\lambda\xi]}\,, \tag{2.74}$$

$$\epsilon_-^{\alpha\dot{\alpha}}(\lambda;\xi) = \sqrt{2}\frac{\lambda^\alpha\tilde{\xi}^{\dot{a}}}{\langle\lambda\xi\rangle}\,, \tag{2.75}$$

up to an arbitrary phase choice (and normalization). With the above, one can construct the standard 4 vector representation of these vectors by writing $\epsilon_{\pm}^{\mu}(p) = \epsilon_{\pm}^{a\dot{a}}\sigma_{a\dot{a}}^{\mu}$. The results are

$$\epsilon_{+}^{\mu}(p) = \frac{\langle q|\,\gamma^{\mu}|p]}{\sqrt{2}\,\langle pq\rangle}\,, \tag{2.76}$$

$$\epsilon_{-}^{\mu}(p) = \frac{[q|\gamma^{\mu}\,|p\rangle}{\sqrt{2}[pq]}\,, \tag{2.77}$$

where q is a reference momentum which is arbitrary (as long as it is not parallel to p), and comes from the gauge freedom of the spin-1 particle. One can also eliminate the gamma matrices by using equation ((2.69)), such that

$$\not{\epsilon}_{+} = \frac{\sqrt{2}}{\langle qp\rangle}\left(|p]\,\langle q| + |q\rangle\,[p|\right)\,, \tag{2.78}$$

$$\not{\epsilon}_{-} = \frac{\sqrt{2}}{[qp]}\left(|p\rangle\,[q| + |q]\,\langle p|\right)\,. \tag{2.79}$$

As expected, the polarization vectors $\epsilon_{\pm}$ satisfy all the usual properties, which can be verified directly by using the above forms. For example, given a vector k^{μ}:

$$k\cdot\epsilon_{+}(p,q) = \frac{\langle q|\,\not{k}|p]}{\sqrt{2}\,\langle pq\rangle} = \frac{\langle kq\rangle\,[kp]}{\sqrt{2}\,\langle pq\rangle}\,, \tag{2.80}$$

$$k\cdot\epsilon_{-}(p,q) = \frac{[q|\not{k}\,|p\rangle}{\sqrt{2}[pq]} = \frac{[kq]\,\langle kp\rangle}{\sqrt{2}[pq]}\,, \tag{2.81}$$

which implies that

$$p\cdot\epsilon_{\pm}(p,q) = 0\,, \tag{2.82}$$

$$q\cdot\epsilon_{\pm}(p,q) = 0\,. \tag{2.83}$$

Similarly, the products of polarization vectors yield

$$\epsilon_{+}(k;q)\cdot\epsilon_{+}(k';q') = \frac{\langle q|\,\gamma^{\mu}|k]\,\langle q'|\,\gamma_{\mu}|k']}{2\,\langle kq\rangle\,\langle k'q'\rangle} = \frac{\langle qq'\rangle\,[kk']}{\langle kq\rangle\,\langle k'q'\rangle}\,, \tag{2.84}$$

$$\epsilon_{+}(k;q)\cdot\epsilon_{-}(k';q') = \frac{\langle q|\,\gamma^{\mu}|k][q'|\gamma_{\mu}\,|k'\rangle}{2\,\langle kq\rangle\,[k'q']} = \frac{\langle qk'\rangle\,[kq']}{\langle kq\rangle\,[k'q']}\,, \tag{2.85}$$

$$\epsilon_{-}(k;q)\cdot\epsilon_{-}(k';q') = \frac{[q|\gamma^{\mu}\,|k\rangle\,[q'|\gamma_{\mu}\,|k'\rangle}{2[kq][k'q']} = \frac{\langle kk'\rangle\,[qq']}{[kq][k'q']}\,. \tag{2.86}$$

From the expressions above, we immediately see that

$$\epsilon_{\pm}(p,q)\cdot\epsilon_{\pm}(p,q)=0\,, \tag{2.87}$$

$$\epsilon_{\pm}(p,q)\cdot\epsilon_{\mp}(p,q)=-1\,, \tag{2.88}$$

$$(\epsilon_{\pm})^{*}=\epsilon_{\mp}\,, \tag{2.89}$$

which are the standard properties satisfied by polarization vectors.

2.2.2 Massive Formalism

The discussion so far was completely restricted to the massless case. However, the formalism can be carried over to massive particles as well. To see how the extension naturally arises, we reproduce the beautiful discussion of what a particle is, given in [25], using the mostly negative signature in order to agree with their results.

Let us consider that single particle states are labelled with a momentum p and other possible degrees of freedom σ. Thus, in the abstract Hilbert space $\mathcal{H}$, we will denote single particle states with $|p,\sigma\rangle$. Now, since we can obtain any momentum by using an appropriate boost, if we want to relate say $|k,\sigma\rangle$ to $|p,\sigma\rangle$, we must find a representation $U(L)$ of the Lorentz transformation $L(p;k)$ which acts on the states in $\mathcal{H}$, given that $p^{\mu}=L^{\mu}_{\nu}k^{\nu}$. As we know, for physical observables, the representations must be either unitary or anti-unitary and since the Lorentz group is a continuous one, we are restricted to taking U unitary. Thus

$$|p,\sigma\rangle=U(L(p;k))\,|k,\sigma\rangle\ . \tag{2.90}$$

Since these are representations of the Lorentz group, for any two transformations Λ_1,Λ_2, we necessarily have $U(\Lambda_1\Lambda_2)=U(\Lambda_1)U(\Lambda_2)$, which means that we can write

$$U(\Lambda)\,|p,\sigma\rangle=U(\Lambda)U(L(p;k))\,|k,\sigma\rangle=U(\Lambda L(p;k))\,|k,\sigma\rangle\ . \tag{2.91}$$

However, $\Lambda L(p;k)=L(\Lambda p;k)$ (we are boosting the momentum after $k\to p$), such that

we can also write

$$\begin{aligned} U(\Lambda)\,|p,\sigma\rangle &= U\left(L(\Lambda p;k)\right)U\left(L^{-1}(\Lambda p;k)\Lambda L(p;k)\right)|k,\sigma\rangle \\ &= U(L(\Lambda,p;k))U\left(W(\Lambda,p,k)\right)|k,\sigma\rangle\ , \end{aligned} \tag{2.92}$$

where one defines $W(\Lambda.p,k) \equiv L^{-1}(\Lambda p;k)\Lambda L(p;k)$. Note that applying this Lorentz transformation to k gives $Wk = L^{-1}(\Lambda p;k)\Lambda L(p;k)k = k$, such that W leaves k invariant. In this context, one can form a subgroup of the Lorentz group ($SO(1,D-1)$) which leave the four momenta k invariant. This is the so-called *little group*. For a representation of this little group, say, $D_{\sigma\sigma'}(W)$ we have

$$U\left(W(\Lambda,p,k)\,|k,\sigma\rangle = D_{\sigma\sigma'}(W(\Lambda,p,k))\,|k,\sigma'\rangle\ , \tag{2.93}$$

which implies, once inserted in (2.92) that

$$\begin{aligned} U(\Lambda)\,|p,\sigma\rangle &= U\left(L(\Lambda p;k)\right)D_{\sigma\sigma'}(W)\,|k,\sigma'\rangle\ , \\ &= D_{\sigma\sigma'}(W)\,|\Lambda p,\sigma'\rangle\ . \end{aligned} \tag{2.94}$$

Equation (2.94) tells us that particle states $|p,\sigma\rangle$ transform under representations of the little group, $D_{\sigma\sigma'}(W)$. This innocuous property is of extreme importance, because the little group *is not* the entire Lorentz group and, thus, explains why the embedding of particles into fields (which are representations of the full Lorentz group) can cause redundancies which often lead to unnecessary complications in calculations. For context, in $D = 4$, the little group is $U(1)$ for massless particles (which correspond to phase changes for photons) while for massive particles, it is $SU(2)$.

For the massive particles, the description of momentum via the bispinor $p_{a\dot{a}}$ only has one small distinction, namely that $p^2 = m^2 \neq 0$. Since we still have $\det p_{a\dot{a}} = p^2$, we cannot decompose $p_{a\dot{a}}$ in terms of the spinors $\lambda_a\tilde{\lambda}_{\dot{a}}$. However, we can decompose it in terms of rank 2 spinors $\lambda_a{}^I$, $\tilde{\lambda}_{\dot{a}I}$ as

$$p_{a\dot{a}} = \lambda_a{}^I\tilde{\lambda}_{\dot{a}I}\,, \tag{2.95}$$

such that, now, $\det p^{a\dot{a}} = \det\lambda \times \det\tilde{\lambda} = m^2$. Thus, we can construct the massive helicity spinors/ twistors in much the same way as the massless ones. In terms of them, the Dirac equations in the Weyl representation become

$$p_{a\dot{a}}\tilde{\lambda}^{\dot{a}I} = m\lambda_a{}^I\,, \tag{2.96}$$

$$p_{a\dot{a}}\lambda^{aI} = -m\tilde{\lambda}_{\dot{a}}{}^I\,. \tag{2.97}$$

To differentiate the massive case from the massless, we use bold symbols to denote them. Thus, following the convention of ([23]), we write

$$\lambda_a{}^I = \left|p^I\right\rangle_a\,; \qquad \tilde{\lambda}^{\dot{a}I} = |p^I]^{\dot{a}}\,, \tag{2.98}$$

$$\lambda^{\,aI} = \left\langle p^I\right|^a\,; \qquad \tilde{\lambda}^I{}_{\dot{a}} = [p^I|\,{}_{\dot{a}}\,. \tag{2.99}$$

The $SU(2)$ indices I, J can be raised and lowered using the antisymmetric ϵ_{IJ}: $\lambda_{aI} = \lambda_a{}^J\epsilon_{IJ}$, $\lambda_a{}^I = \epsilon^{IJ}\lambda_{aJ}$, with the convention that $\epsilon_{12} = -1$ and $\epsilon^{12} = 1$. This means that the momentum bispinors are written as

$$p_{a\dot{a}} = \left|p^I\right\rangle_a [p_I|\,{}_{\dot{a}} = \left|p^I\right\rangle_a [p^J|\,{}_{\dot{a}}\epsilon_{JI} = -\left|p_I\right\rangle_a [p^I|\,{}_{\dot{a}}\,, \tag{2.100}$$

$$p^{\dot{a}a} = -|p^I]^{\dot{a}}\,\langle p_I|^a = |p^I]^{\dot{a}}\left\langle p^J\right|^a\epsilon_{JI} = |p_I]^{\dot{a}}\left\langle p^I\right|^a\,. \tag{2.101}$$

Similarly, one can compute the spinor contractions $\left\langle p^I p^J\right\rangle$ and $[p^I p^J]$ which, contrary to the massless case, are not zero:

$$[p^I p^J] = m\epsilon^{IJ}\,, \qquad [p_I p_J] = -m\epsilon_{IJ}\,, \qquad [p^I p_J] = -m\delta^I{}_J\,, \tag{2.102}$$

$$\left\langle p^I p^J\right\rangle = -m\epsilon^{IJ}\,, \qquad \langle p_I p_J\rangle = m\epsilon_{IJ}\,, \qquad \left\langle p^I p_J\right\rangle = m\delta^I{}_J\,, \tag{2.103}$$

By using the above, one can recover the Dirac equation in terms of the massive helicity spinors. Simply contract both sides of Eqs.(2.100), (2.101) with the corresponding helicity

spinors. This yields

$$p_{a\dot{a}}|p^I]^{\dot{a}} = m\left|p^I\right\rangle_a , \qquad \left\langle p^I\right|^a p_{a\dot{a}} = -m[p^I|_{\dot{a}} , \tag{2.104}$$

$$p^{\dot{a}a}\left|p^I\right\rangle_a = m|p^I]^{\dot{a}} , \qquad [p^I|_{\dot{a}}p^{\dot{a}a} = -m\left\langle p^I\right|^{\dot{a}} . \tag{2.105}$$

To construct Dirac spinors in terms of these helicity spinors, we can write

$$\not{p} = \begin{pmatrix} 0 & p_{\alpha\dot{\beta}} \\ p^{\dot{\alpha}\beta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & |p\rangle_\alpha{}^I\,[p|\ {}_{\dot{\beta}I} \\ |p]^{\dot{\alpha}I}\,\langle p|^\beta{}_I & 0 \end{pmatrix} . \tag{2.106}$$

Adding a $\mathbb{1}m$ and using the expression for the mass in terms of the helicity spinor contractions, we find

$$\not{p} + \mathbb{1}\, m = \begin{pmatrix} -\delta_\alpha{}^\beta\left\langle p^I p^J\right\rangle\epsilon_{JI} & \left|p^I\right\rangle_\alpha[p_I|\ {}_{\dot{\beta}} \\ |p^I]^{\dot{\alpha}}\,\langle p_I|^\beta & -\delta^{\dot{\alpha}}{}_{\dot{\beta}}[p_I p_J]\epsilon^{JI} \end{pmatrix} = u^I(p)\epsilon_{IJ}\overline{u}^J(p) , \tag{2.107}$$

which means that we can identify

$$u^I(p) = \begin{pmatrix} \left|p^I\right\rangle_a \\ |p^I]^{\dot{a}} \end{pmatrix} , \tag{2.108}$$

$$\overline{u}_I(p) = \begin{pmatrix} -\langle p_I|^a & [p_I|_{\dot{a}} \end{pmatrix} . \tag{2.109}$$

If one subtracted m from $\not{p}$, the analogous expressions can be found for $v, \overline{v}$:

$$v^I(p) = \begin{pmatrix} \left|p^I\right\rangle_a \\ -|p^I]^{\dot{a}} \end{pmatrix} , \tag{2.110}$$

$$\overline{v}_I(p) = \begin{pmatrix} \langle p_I|^a & [p_I|_{\dot{a}} \end{pmatrix} . \tag{2.111}$$

Now, as presented in [25], we can discuss the high energy behaviour of these helicity variables to see how they can be related to the massless ones. The basic idea is to expand the massive spinors λ^I_α, $\tilde{\lambda}^I_{\dot{\alpha}}$ in terms of massless helicity spinors $\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}, \eta_\alpha, \tilde{\eta}_{\dot{\alpha}}$ and $SU(2)$ vectors ζ^{-I}, ζ^{+I}:

$$\lambda^I_\alpha = \lambda_\alpha\zeta^{-I} + \eta_\alpha\zeta^{+I} ; \qquad \tilde{\lambda}^I_{\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}}\zeta^{+I} + \tilde{\eta}_{\dot{\alpha}}\zeta^{-I} . \tag{2.112}$$

Since $\left\langle \lambda^I\tilde{\lambda}_J\right\rangle = m\delta^I{}_J$, we also have that $\langle\lambda\eta\rangle = m = [\tilde{\lambda}\tilde{\eta}]$, with $\epsilon_{IJ}\zeta^{+I}\zeta^{-J} = 1$. Now, the null spinors can be taken to be the eigenstates the 1/2 spin operator $\hat{p}\cdot\mathbf{S}$ (the helicity), which, implies $\lambda_\alpha = \sqrt{E+p}\xi^+_\alpha$ and $\eta_\alpha = \sqrt{E-p}\xi^-_\alpha$, with

$$\xi^+ = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix} , \qquad \xi^- = \begin{pmatrix} -\sin(\theta/2)e^{-i\phi} \\ \cos(\theta/2) \end{pmatrix} . \tag{2.113}$$

Then, when the high energy limit is taken, we can expand the square roots to see that $\sqrt{E+p} \sim \sqrt{2E}$, and $\sqrt{E-p} \sim m/\sqrt{2E}$. Consequently, η_α goes with $m/\sqrt{2E}$, which vanishes against λ_α. Therefore, to analyze the high energy limit, we can expand a given amplitude in terms of these null spinors and find the limit $m \to 0$. To be more specific, a given massive amplitude for a particle with little group indices $I_1, \dots I_{2s}$, once expanded in terms of the null spinors, becomes

$$M^{\{I_1 \dots I_{2s}\}} = \sum_h \left((\xi^+)^{s+h}(\xi^-)^{s-h}\right)^{I_1 \dots I_{2s}} M_h(\lambda, \tilde{\lambda}, \eta, \tilde{\eta})\,, \tag{2.114}$$

where $M_h(\lambda, \tilde{\lambda}, \eta, \tilde{\eta})$ is a function of the null spinors, satisfying the usual little group scailling properties: $M_h(w^{-1}\lambda, w\tilde{\lambda}, w\eta, w^{-1}\tilde{\eta}) = w^{2h} M_h(\lambda, \tilde{\lambda}, \eta, \tilde{\eta})$. Then, the high energy limit is to be taken by

$$\lim_{m\to 0} M_h(\lambda, \tilde{\lambda}, \eta = m\hat{\eta}, \tilde{\eta} = m\hat{\tilde{\eta}})\,. \tag{2.115}$$

Let us carry out this explicit procedure for a massive particle of spin 1 with a given massive amplitude

$$M^{\{IJ\}}(1^-, 2^-, \mathbf{3}) = \frac{\langle \mathbf{3}1\rangle \langle \mathbf{3}2\rangle}{\langle 21\rangle} = \frac{\langle 3^{(I}1\rangle \langle 3^{J)}2\rangle}{\langle 21\rangle}\,. \tag{2.116}$$

The decomposition can then be written as $\lambda_3^{I\alpha} = \lambda_3^\alpha \zeta_-^I + \eta_3^\alpha \zeta_+^I$, which, upon insertion into the above, yields

$$\begin{aligned}\frac{1}{\langle 21\rangle}(\langle 31\rangle \zeta_-^I + \langle \eta_3 1\rangle \zeta_+^I)(\langle 32\rangle \zeta_-^J + \langle \eta_3 2\rangle \zeta_+^J) = \zeta_-^I\zeta_-^J \frac{\langle 31\rangle \langle 32\rangle}{\langle 21\rangle} + \zeta_-^I\zeta_+^J \frac{\langle 31\rangle \langle \eta_3 2\rangle}{\langle 21\rangle} \\ + \zeta_+^I\zeta_-^J \frac{\langle \eta_3 1\rangle \langle 32\rangle}{\langle 21\rangle} + \zeta_+^I\zeta_+^J \frac{\langle \eta_3 1\rangle \langle \eta_3 2\rangle}{\langle 21\rangle}\end{aligned}$$

We see from the above three terms, the first corresponding to 3^-, the second and third corresponding to 3^0 and the last corresponding to 3^+. In the high energy limit (HE), we

can simply read off, from the above, the corresponding amplitudes:

$$\text{HE}(-1) \quad : \quad \frac{\langle 31\rangle\,\langle 32\rangle}{\langle 21\rangle} \to \frac{\langle 31\rangle\,\langle 32\rangle}{\langle 21\rangle} \tag{2.117}$$

$$\text{HE}(0) \quad : \quad \frac{\langle 31\rangle\,\langle \eta_3 2\rangle + \langle \eta_3 1\rangle\,\langle 32\rangle}{\langle 21\rangle} \to 0 \tag{2.118}$$

$$\text{HE}(+1) \quad : \quad \frac{\langle \eta_3 1\rangle\,\langle \eta_3 2\rangle}{\langle 21\rangle} \to \frac{[31][32]}{[21]}\,. \tag{2.119}$$

The last equation above can be obtained by the following manipulations:

$$\begin{aligned}
\frac{\langle \eta_3 1\rangle\,\langle \eta_3 2\rangle}{\langle 21\rangle} &= \frac{\overbrace{[3\tilde{\eta}_3]\,\langle \eta_3 1\rangle}^{[3|p_2|1\rangle}\overbrace{[3\tilde{\eta}_3]\,\langle \eta_3 2\rangle}^{[3|p_1|2\rangle}}{\langle 21\rangle\, m^2} \\
&= \frac{[32]\,\langle 21\rangle\,[31]\,\langle 12\rangle}{\langle 21\rangle\, m^2} \\
&= \frac{[32][31]}{[21]}\frac{\overbrace{\langle 12\rangle\,[21]}^{2p_1\cdot p_2}}{m^2} \\
&= \frac{[32][31]}{[21]}\frac{\overbrace{(p_1+p_2)^2}^{=p_3^2=m^2}}{m^2} \\
&= \frac{[32][31]}{[21]}\,.
\end{aligned}$$

Thus, we see that in the high energy limit, the amplitudes naturally decompose into the expected helicity weights, while the longitudinal mode vanishes. As explained in [25], this careful expansion is only really needed in the more pathological cases and in general, one can obtain the HE limit by simply un-bolding the symmetric components of the spinor products.

2.2.3 Little Group Constraints

Now let us comment more on the properties of amplitudes under the little group, as pointed out in [26, 25]. As we mentioned, pure Feynman amplitudes are Lorentz invariant objects, as they are effectively expected values of quantum fields, objects which are representations of the Lorentz group. On the other hand, the actual scattering amplitudes,

which come with contractions in the incoming/outgoing particle polarizations/external line factors. These external factors are necessary in order to reintroduce the covariance with respect to the little-group.

Thus, a given amplitude A must also carry the respective group indices associated with the little group representations which the particles belong to. For massless particles whose little group is U(1), these are just the helicities and one writes $A_4^{h_1,h_2,h_3,h_4}$ (or $A_4(1^{h_1}2^{h_2}3^{h_3}4^{h_4})$) for a 4 point amplitude with massless particles. This way of calculating the amplitudes by assigning specific values for the helicities of the particles is a way of recasting the amplitude in terms of a covariant little-group object which is sometimes called a *helicity* amplitude.

For a massive particle, the little-group SU(2) slightly complicates the situation. To better understand this, consider the usual case of the angular momentum operators J_x, J_y, J_z. In this case, these operators don't commute and one cannot specify a base of eigenvectors of all these operators. Thus, one usually chooses a direction, say J_z, and defines $J_z\,|j,m_j\rangle = m_j\hbar\,|j,m_j\rangle$, $J^2\,|j,m_j\rangle = s(s+1)\hbar^2\,|j,m_j\rangle$, with j being an integer and m_j assuming all values $-j \leq m_j \leq j$. Of course this is problematic for the discussion of Lorentz invariants, since one is specifically choosing a preferred reference frame. For this reason, the approach advocated in [25] is to recast the irreducible representations in terms of symmetric SU(2) tensors. In this sense, it is natural to consider the amplitudes we desire in terms of symmetric SU(2) indices. For a massless spin 1 particle with helicity +1, a massive spin 2 particle, a massless spin 3/2 particle with helicity +3/2 and a massive spin 1/2 particle, the amplitude with the explicit little group indices transforms according to

$$A_4^{\{+1\},\{I_1,I_2,I_3,I_4\},\{+3/2\},\{J\}} \to W^{I_1}{}_{I_1'} W^{I_2}{}_{I_2'} \dots W^{I_4}{}_{I_4'} W^{J}{}_{J'} (w_1)^2 (w_3)^3 A_4^{\{+1\},\{I_1',I_2',I_3',I_4'\},\{+3/2\},\{J'\}}\,, \tag{2.120}$$

where W corresponds to the $SU(2)$ transformations for the spin 1/2 representation, i.e.:

$\left|p^I\right\rangle = W^I{}_J \left|p^J\right\rangle$, $\left\langle p^I\right| = (W^{-1})^I{}_J \left\langle p^J\right|$, and w_1, w_3 correspond to the $U(1)$ transformations, i.e.: $\lambda_{3\alpha} = w_3 \lambda_{3\alpha}$ (which is just a phase $w_3 = e^{i\theta}$). Furthermore, analogous to what happens with the group structure constants of a color ordered amplitude, one can recast the amplitude in terms of spinor indices, by stripping factors containing the massive little group:

$$\mathcal{A}^{I_1 \dots I_{2s}}(p) = \left|p^{I_1}\right\rangle_{\alpha_1} \left|p^{I_2}\right\rangle_{\alpha_2} \dots \left|p^{I_{2s}}\right\rangle_{\alpha_{2s}} A^{\{\alpha_1 \dots \alpha_{2s}\}} = [p^{I_1}|_{\dot{\alpha}_1} [p^{I_2}|_{\dot{\alpha}_2} \dots [p^{I_{2s}}|_{\dot{\alpha}_{2s}} \tilde{A}^{\{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}\}} \,. \tag{2.121}$$

Each reduced amplitude $A, \tilde{A}$ can be interchanged by contracting with $p_{\alpha\dot{\alpha}}/m$, as the Weyl equations mix the basis spinors (see eq. (2.105)).

By using the proper form with which the amplitudes must transform, one can constrain the functional form of the amplitudes and characterize them as done in [25].

Massless Three Particle Amplitudes

To illustrate these selection rules, we first consider the full massless amplitudes. Then, considering that a generic n-point amplitude is a function of the spinors $|i\rangle, |i], h_i$ for particle i, the appropriate transformation properties when one transforms $|i\rangle \to w\,|i\rangle, |i] \to w^{-1}|i]$ are

$$A_n(\{|1\rangle, |1], h_1\} \dots \{w\,|i\rangle, w^{-1}|i], h_i\} \dots) = w^{2h_i} A_n(\{|1\rangle, |1], h_1\} \dots \{|i\rangle, |i], h_i\} \dots) \,, \tag{2.122}$$

that is, the amplitude is *homogeneous* function of the helicity spinors. This property is called *little group scailing* by some references [22, 24] and it is specially powerful for three point amplitudes.

For three point amplitudes, the conservation of momentum $p_1^\mu + p_2^\mu + p_3^\mu = 0$ implies that the mandelstalm invariants $s_{ij} = 2\langle ij\rangle [ji]$ all vanish. Thus, $\langle ij\rangle [ji] = 0$ and one would have either $[ji] = 0$ or $\langle ij\rangle = 0$. The problem is that, for real momenta, $[ji] = \langle ij\rangle^*$

and thus, both vanish. This requires all three particle massless amplitudes to vanish. However, let us consider for a moment that the momenta are complex. If that is the case, then the bispinors $p_{\alpha\dot{\alpha}}$ are no longer constrained to be hermitian and $[ji] \neq \langle ij\rangle^*$. This implies that if $s_{ij} = 0$ (by momentum conservation), one either has $[ij] = 0$ or $\langle ij\rangle = 0$, but not necessarily both. Therefore, a given three point amplitude could still be constructed by choosing either a representation with angle or square brackets. Let us suppose

$$A_3(1^{h_1}2^{h_2}3^{h_3}) = c\,\langle 12\rangle^{x_{12}}\,\langle 23\rangle^{x_{23}}\,\langle 31\rangle^{x_{31}}\ . \tag{2.123}$$

Then, by the little group scaling of this amplitude, we know that, for example, $\langle 12\rangle \to w_1^{-x_{12}} w_2^{-x_{12}}\,\langle 12\rangle$, so that scaling every particle we find

$$\begin{aligned} A_3(1^{h_1}2^{h_2}3^{h_3}) &= c w_1^{-x_{12}-x_{31}} w_2^{-x_{12}-x_{23}} w_3^{-x_{23}-x_{31}}\,\langle 12\rangle^{x_{12}}\,\langle 23\rangle^{x_{23}}\,\langle 31\rangle^{x_{31}}\ , \\ &= c w_1^{2h_1} w_2^{2h_3} w_3^{2h_3}\,\langle 12\rangle^{x_{12}}\,\langle 23\rangle^{x_{23}}\,\langle 31\rangle^{x_{31}}\ . \end{aligned} \tag{2.124}$$

Thus, we have a system of three equations:

$$2h_1 = -x_{12} - x_{31}\,, \qquad 2h_2 = -x_{12} - x_{23}\,, \qquad 2h_3 = -x_{23} - x_{31}\,, \tag{2.125}$$

which yield

$$A_3(1^{h_1}2^{h_2}3^{h_3}) = c\,\langle 12\rangle^{h_3-h_1-h_2}\,\langle 23\rangle^{h_1-h_2-h_3}\,\langle 31\rangle^{h_2-h_1-h_3}\ . \tag{2.126}$$

If one went back and, instead of imposing that the square brackets vanish, did so with the angle brackets, the result would've been

$$A_3(1^{h_1}2^{h_2}3^{h_3}) = c[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_1+h_3-h_2}\,. \tag{2.127}$$

Then, a natural question would be which of these would be the "correct" amplitude. The answer is that it depends on the theory one is working in. As noted in [22], for spin 1 massless particles, the amplitude $A_3(1^-2^-3^+)$ becomes:

$$A_3(1^-2^-3^+) = c\frac{\langle 12\rangle^3}{\langle 23\rangle\,\langle 31\rangle}\,, \qquad \text{or} \qquad A_3(1^-2^-3^+) = c\frac{[23][31]}{[12]^3}\,. \tag{2.128}$$

Since the spinors have mass dimension 1/2, the square and angled brackets have mass dimension 1, which means that the first expression above has mass dimension 1, while the second, mass dimension -1 (disregarding the coupling constant c). The first, could arise from a $AA\partial A$ interaction in a given Lagrangian (as is the case in standard Yang-Mills), but the second, could not, and one would need to consider another theory where such amplitude could appear (one example, given in [22], is a theory with non-local interactions such as $AA\frac{\partial}{\Box}A$).

As another example, let us consider a three point amplitude for gravitons (spin 2). Then:

$$A_3^{grav}(1^-2^-3^+) = c\frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}\,. \tag{2.129}$$

Surprisingly, this amplitude, which has the correct mass dimension coming from a $hh\partial h$ term in the Einstein-Hilbert linearized Lagrangian, is the square of the spin 1 amplitude in Eq.(2.128). This is a characteristic of the double copy conjecture for gravity theories (see the review [27]).

Massive Three Particle Amplitudes

As we have discussed, the little group structure of the massless amplitudes provide very restrictive constraints, such that one can find expressions for these amplitudes without need for Feynman rules. A natural question is then if this is also the case for massive particles.

As portrayed in [25], it is possible to classify the three particle massive amplitudes, but a singular structure can only be found when one is considering a single massive external leg, for which the helicity constraints of the other two legs are enough to restrict the full amplitude. When more massive legs are added, the amplitude has different representations in terms of the helicity weights and the SU(2) group indices. However, in any of these cases, it is true that one can write the amplitude in terms of the massless spinor indices as in Eq.(2.121). In this case, one can choose which chirality spinors to strip. We will

follow [25] in stripping factors in the undotted basis. Thus, a three point amplitude with two massless particles 1 and 2, and a third massive with spin s, carrying SU(2) indices $I_1, \dots I_{2s}$, will be written as:

$$A_3^{h_1,h_2,\{I_1 \dots I_{2s}\}} = \lambda^{I_1}_{\alpha_1} \dots \lambda^{I_{2s}}_{\alpha_{2s}} A_3^{h_1,h_2,\{\alpha_1 \dots \alpha_{2s}\}} \,, \tag{2.130}$$

with the amplitude in the right hand side being a function of the massless spinors.

The general strategy for the classification of these structures will then be to expand this stripped amplitude in terms of some basis of the $SL(2,\mathbb{C})$ space. Let us first consider two massless legs h_1, h_2 and a massive leg. In general, naming the two spinors that will serve as the basis η_α, ξ_α, we can envisage a general structure of the form

$$A^{h_1,h_2,\{\alpha_1,\dots \alpha_{2s}\}} = \sum_{i=0}^{2s} C_i (\eta^i \xi^{2s-i})^{\{\alpha_1,\dots \alpha_{2s}\}} [12]^{x_{12}} \,, \tag{2.131}$$

where the powers of the spinors are symbolic in the sense that they are dressed with the SL(2, $\mathbb{C}$) indices in curly brackets outside the round brackets. Note that the last factor ([12]) could also be written as $\langle 12 \rangle$, if desired. Now, a natural choice for η, ξ would be the spinors of the two massless particles, i.e.: $\eta = \lambda_1, \xi = \lambda_2$. On the other hand, as we discussed, the amplitude must scale as $w_1^{2h_1}$ under $\lambda_1 \to w_1 \lambda_1, \tilde{\lambda}_1 \to w_1^{-1} \tilde{\lambda}_1$. In this case, the above expansion gets heavily simplified by the helicity scaling of the massless spinors. Consider, for clarification, a term with $i = 4$. Then, the structure of the expansion would have the form

$$(\lambda_1^{\alpha_1} \lambda_1^{\alpha_2} \lambda_1^{\alpha_3} \lambda_1^{\alpha_4} \lambda_2^{\alpha_5} \dots \lambda_2^{\alpha_{2s}}) [12]^{x_{12}} \,,$$

which clearly does not satisfy the helicity scaling, since particle 1 scales as $x_{12} - 4$ and particle 2 scales as $x_{12} - 2s + 4$. Let us first determine x_{12}. By scaling both particles, we find $2h_1 = x_{12} - i$ and $2h_2 = x_{12} - 2s + i$. Adding, we conclude that $x_{12} = s + h_2 + h_1$. Then, we have $i = x_{12} - 2h_1 = s + h_2 - h_1$, which implies that there is only one polynomial which gets picked out, namely

$$A^{h_1,h_2,\{\alpha_1,\dots \alpha_{2s}\}} = C_{s+h_2-h_1} (\lambda_1^{s+h_2-h_1} \lambda_2^{s+h_1-h_2})^{\{\alpha_1,\dots \alpha_{2s}\}} [12]^{s+h_2+h_1} \,. \tag{2.132}$$

To determine $C_{s+h_2-h_1}$, we can consider the mass dimensions of the stripped amplitude and the coupling constant g. The conclusion is that the three point, one massive leg amplitude is uniquely determined by

$$A^{h_1,h_2,\{\alpha_1,\dots\alpha_{2s}\}} = \frac{g}{m^{2s+h_1+h_2-1}}(\lambda_1^{s+h_2-h_1}\lambda_2^{s+h_1-h_2})^{\{\alpha_1,\dots\alpha_{2s}\}}[12]^{s+h_2+h_1}\,. \tag{2.133}$$

Interestingly enough, the above form implies multiple no go theorems for three point amplitudes (see [25]).

Let us move on to two massive legs. Now, we cannot use the two massless spinors to form a basis. If the particles do not have the same mass, one can choose the λ from the massless particle and take the second spinor as $\tilde{\lambda}$ contracted with one of the massive momenta, i.e.: $(\eta_\alpha, \xi_\alpha) = (\lambda_\alpha, (p_{1\alpha\dot{\beta}}/m_1)\tilde{\lambda}^{\dot{\beta}})$ (p_1 being the momentum of one of the massive legs). Then, considering that for two spins s_1, s_2, the total number of elements in the $SU(2)\otimes SU(2)$ direct group is $s_1+s_2-|s_1-s_2|$, the sum becomes

$$A^{h,\{\alpha_1,\dots\alpha_{2s_1}\},\{\beta_1,\dots\beta_{2s_2}\}} = \sum_{i=0}^{s_1+s_2-|s_1-s_2|} C_i(\lambda^i[(p_1/m_1)\tilde{\lambda}]^{s_1+s_2-i})^{\{\alpha_1,\dots\alpha_{2s_1}\},\{\beta_1,\dots\beta_{2s_2}\}}\,. \tag{2.134}$$

If we scale the massless particle, we find that the degree of the polynomials in the spinors must differ by $2h$, but apart from this, there is no restriction which singles out a single structure for the polynomials. Thus, for example, the above could also be written as

$$A^{h,\{\alpha_n\}|_1^{2s_1}\{\beta_m\}|_1^{2s_2}} = \sum_{i=0}^{s_1+s_2-|s_1-s_2|} C_i\left([\lambda]^{(s_1+s_2-|s_1-s_2|)/2-h}[(p_1/m_1)\tilde{\lambda}]^{(s_1+s_2-|s_1-s_2|)/2+h}\right)^{\{\alpha_n\}|_1^{2s_1}\{\beta_m\}|_1^{2s_2}}$$

or, as done in [25],

$$A^{h,\{\alpha_1,\dots\alpha_{2s_1}\},\{\beta_1,\dots\beta_{2s_2}\}} = \sum_{i=1}^{s_1+s_2-|s_1-s_2|+1} C_i\left([\lambda]^{s_1+s_2-h}[p_1\tilde{\lambda}]^{s_1+s_2+h}\right)^{\{\alpha_1,\dots\alpha_{2s_1}\},\{\beta_1,\dots\beta_{2s_2}\}}\,. \tag{2.135}$$

In conclusion, because of the two sets of SU(2) indices, we cannot isolate a single helicity structure in general. If the masses are equal, however, the situation is a bit different since now the choice $\eta_\alpha = \lambda_\alpha$ and $\xi_\alpha = p_{1\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}$ does not form a basis. This can be seen

from basic kinematics for the three particles, because (denoting the massless particle's momentum by p_3):

$$\eta^\alpha \xi_\alpha = \lambda_3^\alpha p_{1\alpha\dot\beta}\tilde\lambda_3^{\dot\beta} = \langle 3|p_1|3] = 2p_1\cdot p_3 = -2p_1\cdot(p_1+p_2) = -2m^2 - 2p_1\cdot p_2\,.$$

Since $p_3^2 = (p_1+p_2)^2 = 2m^2 + 2p_1\cdot p_2 = 0$, we see that the above vanishes. Thus, λ_3^α and $p_{1\alpha\dot\beta}\tilde\lambda_3^{\dot\beta}$ are not linearly independent.

In principle, we could attempt to form an expansion using some other object with appropriate spinor indices. A classic example of this would be the anti-symmetrical tensor $\epsilon_{\alpha\beta}$. The problem with this choice, is the absence of the helicity weight which is needed. However, as pointed out in [25], even though the usual choice of the spinors is no longer valid, their proportionality constant can be used to construct a proper basis. Specifically, we write

$$x\lambda_{3\alpha} = \frac{1}{m}p_{1\alpha\dot\alpha}\tilde\lambda_3^{\dot\alpha}\,; \qquad \tilde\lambda_3^{\dot\alpha} = x\frac{1}{m}p_1^{\dot\alpha\alpha}\lambda_{3\alpha}\,, \tag{2.136}$$

which could be contracted with a reference spinor r to yield

$$x = \frac{\langle r|p_1|3]}{m\langle 3r\rangle} = \frac{m[r3]}{[r|p_1|3\rangle}\,. \tag{2.137}$$

Under a scaling of the massless particle 3, the x factor scales as w^{+2}, just like a polarization. For this reason, [28] also uses the notation

$$x_+ := \frac{\langle r|p_1|3]}{m\langle 3r\rangle} =: \frac{1}{x_-} := \frac{m[r3]}{[r|p_1|3\rangle}\,, \tag{2.138}$$

from which x_+ scales as w^{+2} and x_- scales as w^{-2}. Since these objects carry helicty scalings, we can use them together with λ_3 and $\epsilon_{\alpha\beta}$ to write a proper basis. This leads us to the following form

$$A^{h\{\alpha_1\ldots\alpha_{2s_1}\}\{\beta_1\ldots\beta_{2s_2}\}} = \sum_{i=|s_1-s_2|}^{s_1+s_2} C_i x_+^{h+i}\left([\lambda_3]^{2i}[\epsilon]^{s_2+s_1-i}\right)^{\{\alpha_1\ldots\alpha_{2s_1}\}\{\beta_1\ldots\beta_{2s_2}\}}\,, \tag{2.139}$$

which correctly scales as w^{2h} under $\lambda\to w^{-1}\lambda$. Reference [25] also uses the alternative form

$$A^{h\{\alpha_1\ldots\alpha_{2s_1}\}\{\beta_1\ldots\beta_{2s_2}\}} = \sum_{i=|s_1-s_2|}^{s_1+s_2} C_i x_+^h\left([\lambda_3]^{2i}\left[\frac{p_1\tilde\lambda_3}{m}\right]^i[\epsilon]^{s_2+s_1-i}\right)^{\{\alpha_1\ldots\alpha_{2s_1}\}\{\beta_1\ldots\beta_{2s_2}\}}\,.$$

The x factor plays a dual role in the construction of the amplitudes. Firstly, it has a *spurious pole*, which can be seen by its denominator in Eq.(2.137), and is important when one is constructing higher point amplitudes from these lower ones, since this procedure requires the appropriate poles in each of the channels to ensure the consistent factorization of the expressions (more in [25]).

Secondly, this x factor can be used to directly identify what a minimal coupling amplitude is. For massless particles, the minimal coupling is understood as a coupling with the lowest mass dimensions possible. However, as the spinors are given in powers of the helicities, this implies that the minimal coupling is made of opposite helicity particles. To see this, we might simply look at the general little group three particle massless amplitudes given in expressions 2.126 and 2.127. Then, the minimal coupling for massive particles will be the term which reproduces these massless amplitudes in the high energy limit. Thus, for a photon, the minimal coupling to two spin s states of the same mass is given by the following structures

$$A^{+1}_{\text{min }\{\alpha_1\ldots\alpha_{2s}\}\{\beta_1\ldots\beta_{2s}\}} = x_+ m\left(\prod_{i=1}^{2s}\epsilon_{\alpha_i\beta_i}+\text{sym}(\alpha_i,\beta_i)\right), \tag{2.140}$$

$$A^{-1}_{\text{min }\{\dot\alpha_1\ldots\dot\alpha_{2s}\}\{\dot\beta_1\ldots\dot\beta_{2s}\}} = x_- m\left(\prod_{i=1}^{2s}\epsilon_{\dot\alpha_i\dot\beta_i}+\text{sym}(\dot\alpha_i,\dot\beta_i)\right). \tag{2.141}$$

As an example, consider two same mass fermions coupled to a positive helicity photon. By eq.(2.139), the supposed minimal coupling amplitude with mass dimension 1 would be given by

$$m x_+ \epsilon_{\alpha_1\beta_1}$$

such that, inserting the external massive SU(2) spinors, one would have

$$m x_+ \lambda^{I\alpha_1}\lambda^{J\beta_1}\epsilon_{\alpha_1\beta_1} = m x_+ \langle \mathbf{12}\rangle \tag{2.142}$$

In general, we consider that minimal coupled three point amplitudes, for two massive

spin s legs and a helicity h massless leg, are

$$A^{+h}_{min} = x_+^h \frac{\langle \mathbf{12} \rangle^{2s}}{m^{2s-1}}\,, \qquad A^{-h}_{min} = x_-^h \frac{[\mathbf{12}]^{2s}}{m^{2s-1}}\,. \tag{2.143}$$

Note that we are expressing each amplitude in terms of the simplest basis, but one could express A^+_{min} in terms of square spinors. For explicit expressions, see [26]. Lastly, we note that the other terms which appear in the expansions for the amplitudes can be directly interpreted in terms of higher multipole contributions [25, 29].

2.3 Factorization and On-Shell Recursion

In this section, we outline a very powerful procedure to build higher point amplitudes from lower point ones. This allows one to calculate n point gluon (or graviton) tree amplitudes "only" from the knowledge of the three point amplitudes (there are subtleties), The strategy for this procedure is based on complexifying momenta by appropriate shifts and the factorization properties of amplitudes, both of which we briefly discuss below.

2.3.1 Momentum Shifts and Factorization

At first, let us consider the general analytic properties of tree amplitudes. Since there are no loops, the possible divergences in these objects come entirely from propagators going on-shell and are given by poles in the complex plane.

To better understand this behaviour, let us introduce *complex momentum shifts*. Following [22], consider an n point amplitude with momenta p_i^μ. Now, let us label r_i^μ, $i = 1, \ldots n$, n complex vectors which satisfy the following properties:

- $\sum_{i=1}^n r_i^\mu = 0$,
- $r_i \cdot r_j = 0$,
- $r_i \cdot p_i = 0$ (no sum).

With the above, we define *shifted* momenta $\hat{p}_i$ such that

$$\hat{p}_i^\mu(z) := p_i^\mu + z r_i^\mu \,. \tag{2.144}$$

Because of the above properties, these shifted complex momenta satisfy the on-shell and momentum conservation properties:

$$\hat{p}_i(z)^2 = 0 \,, \qquad \sum_{i=1}^{n} \hat{p}_i^\mu(z) = 0 \,. \tag{2.145}$$

Furthermore, if a given amplitude $A_n(\{p_i\})$ is expressed in terms of the shifted momenta, it will be a complex function $\hat{A}_n(z)$.

Now, let us consider a subset of the particles $S \in I$ (with $I = \{1, 2 \ldots, n\}$). Then, define the *regional momentum* P_S as

$$P_s^\mu := \sum_{i \in S} p_i^\mu \implies \hat{P}_s^\mu := \sum_{i \in S} \hat{p}_i^\mu(z) \,. \tag{2.146}$$

Then, $\hat{P}_S^2$ is linear in the complex variable z:

$$\hat{P}_S^2 = \sum_{i,j \in S} p_i \cdot p_j + z \sum_{i,j \in S} (p_i \cdot r_j + p_j \cdot r_i) = P_S^2 + 2 z P_S \cdot R_S \,, \tag{2.147}$$

with $R_S^\mu = \sum_{i \in S} r_i^\mu$. Now, given the propagator dependency on the regional momentum is $1/\hat{P}_S(z)^2$, the amplitude $A_n(z)$ will have simple poles in the complex z plane at

$$z_S = -\frac{P_S^2}{2 P_S \cdot R_S} \,. \tag{2.148}$$

Since the subset prevents the regional momenta from satisfying $P_S^2 = 0$, $z_S \neq 0$ for arbitrary momenta, the poles of the amplitude are all away from the origin. Now, let us consider $\hat{A}_n(z)/z$. For a curve that encircles all of the poles, including the one at the origin, the total residue of this expression is

$$\hat{A}_n(0) + \sum_{z_S} \operatorname*{Res}_{z \to z_S} \frac{\hat{A}_n(z)}{z} + B(\infty) = 0 \,, \tag{2.149}$$

where we can identify $\hat{A}_n(0)$ with the unshifted amplitude $A_n(\{p_i\}_{i=1,\ldots n})$ and $B(\infty)$ is the residue at $z = \infty$. The above can be used to make explicit an important property of

amplitudes: *factorization*. Suppose we write $\hat{A}_n(z) = \hat{A}_L(z)(\hat{P}_S)^{-2}\hat{A}_R(z)$, something completely valid for an arbitrary tree amplitude. Then, the left and right pieces $\hat{A}_R(z)$,$\hat{A}_L(z)$ both are pole free at $\hat{P}_S^2 = 0$. In terms of z, the $\hat{P}_S^2$ pole can be conveniently expressed in terms of z_S as

$$\hat{P}_S^2 = -\frac{P_S^2}{z_S}(z - z_S)z\,. \tag{2.150}$$

Then, the residues for the regional momenta are easily calculated:

$$A_n = \sum_{z_S} \frac{\hat{A}_L(z_S)\hat{A}_R(z_S)}{P_S^2} - B(\infty)\,. \tag{2.151}$$

Now, *iff* $B(\infty) = 0$, one can completely determine the amplitude A_n in terms of the lower point amplitudes $\hat{A}_L$,$\hat{A}_R$. Let us note, however, that the vanishing of the boundary term $B(\infty)$ is not a trivial manner and, in fact, there are certain shifts for which this term cannot be neglected. Since the evaluation of this term is not trivial, it is generally considered that a shift which ensures $B(\infty) \to 0$ is better.

2.3.2 Britto-Cachazo-Feng-Witten Recursion

The Britto-Cachazo-Feng-Witten (BCFW) recursion is a recursion method which closely follows the general shift procedure discussed above. Specifically, consider shifting only two of the helicity spinors, say i, j, by a complex factor z such that

$$\hat{\lambda}_i^\alpha(z) = \lambda_i^\alpha - z\lambda_j^\alpha\,, \qquad \hat{\tilde{\lambda}}_i^{\dot{\alpha}} = \tilde{\lambda}_i^{\dot{\alpha}}\,, \tag{2.152}$$

$$\hat{\lambda}_j^\alpha = \lambda_j^\alpha\,, \qquad \hat{\tilde{\lambda}}_j^{\dot{\alpha}}(z) = \tilde{\lambda}_j^{\dot{\alpha}} + z\tilde{\lambda}_i^{\dot{\alpha}}\,. \tag{2.153}$$

Then, accordingly, the momenta will shift as

$$\hat{p}_i^{\dot{\alpha}\alpha}(z) = \tilde{\lambda}_i^{\dot{\alpha}}(\lambda_i^\alpha - z\lambda_j^\alpha)\,, \qquad \hat{p}_j^{\dot{\alpha}\alpha}(z) = (\tilde{\lambda}_j^{\dot{\alpha}} + z\tilde{\lambda}_i^{\dot{\alpha}})\lambda_j^\alpha\,. \tag{2.154}$$

This effective "mixing" of the spinor components preserves the physically sensible properties of the bispinors, namely momentum conservation $\hat{p}_i^{\dot{\alpha}\alpha}(z) + \hat{p}_j^{\dot{\alpha}\alpha}(z) = p_i^{\dot{\alpha}\alpha} + p_j^{\dot{\alpha}\alpha}$ and the

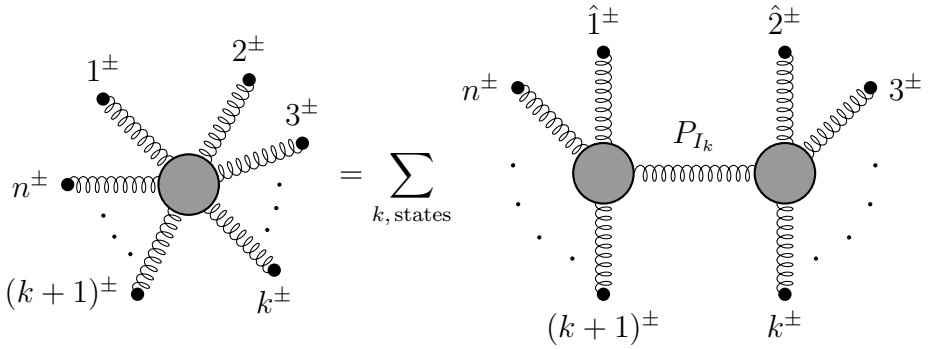


Figure 2.1: BCFW shift of legs $1, 2$ yielding a factorization of the tree amplitude with inner on-shell momentum P_{I_k}. Note that the inner states refer specifically to the associated helicities which must be flipped as the momenta crosses the inner line.

on-shell conditions $\hat{p}_i^{\dot{\alpha}\alpha}(z)\hat{p}_{i\alpha\dot{\alpha}}(z) = \hat{p}_i^2(z) = 0 = \hat{p}_j^2(z)$, just as we required of the general r^μ shifts above.

In terms of these new complexified momenta, a given n-point gluon amplitude will also be complexified $A_n \to A_n(z)$. Then, as with the general shifts, one can write the n point amplitude in terms of the lower point amplitudes:

$$A_n = \sum_I \hat{A}_L[1, \ldots \hat{i}, -\hat{P}_I] \frac{1}{P_I^2} \hat{A}_R[\hat{P}_I, \hat{j}, \ldots, n](z_I)\,, \tag{2.155}$$

where I corresponds to the regional momentum. It is worth mentioning that the $[j, i\rangle$ shift performed is such that the i and j particle are separated by an inner line propagator, otherwise, there would be no pole and the factorization would not work. Also, The summation over the I label implies summing over the amplitudes with the possible regional momenta excluding the shifted momenta i, j and also includes a summation over the helicities of the inner line momentum. For a generic picture which is conventional for this type of discussion see fig.2.1.

It is important to point out that not all choices of momenta pairs are acceptable in order to enforce the vanishing of the boundary term. In fact, for pure Yang-Mills theory, the precise scaling with z of the amplitudes are known[22, 23, 24]:

$$[+,+\rangle \sim \frac{1}{z}\,; \qquad [+,-\rangle \sim z^3\,; \qquad [-,-\rangle \sim \frac{1}{z}\,; \qquad [-,+\rangle \sim \frac{1}{z}\,. \tag{2.156}$$

One can use known expressions for amplitudes to explicitly check the scalings above.

As a first, simpler example, let us calculate the MHV four point gluon scattering amplitude $A_4(1^-2^-3^+4^+)$ using the BCFW shift. Then, there are three acceptable shifts corresponding to $[++\rangle, [-+\rangle, [--\rangle$. Let us consider a $[1,2\rangle$ shift, such that $|\hat{1}] = |1] + z|2]$ and $\left|\hat{2}\right\rangle = |2\rangle - z\,|1\rangle$. Then, the factorization implies

$$A_4[1^-, 2^-, 3^+, 4^+] = \sum_{h=\pm} A_3[\hat{1}^-, \hat{P}_{13}^h, 3^+] \frac{1}{P_{13}^2} A_3[-\hat{P}_{13}^{-h}, \hat{2}^-, 4^+]\,. \tag{2.157}$$

Note that there is no summation on the regional momentum because the color ordered amplitude must respect the particle orderings. Now, we recall the little group constraints on three particle kinematics, which give us the expressions for the amplitudes:

$$A_3[\hat{1}^-, \hat{P}_{13}^+, 3^+] = g\frac{[\hat{P}_{13}3]^3}{[3\hat{1}][\hat{1}\hat{P}_{13}]}\,; \qquad A_3[\hat{1}^-, \hat{P}_{13}^-, 3^+] = g\frac{\langle\hat{1}\hat{P}_{13}\rangle^3}{\langle\hat{P}_{13}3\rangle\left\langle 3\hat{1}\right\rangle}\,, \tag{2.158}$$

$$A_3[-\hat{P}_{13}^+, \hat{2}^-, 4^+] = g\frac{[4(-\hat{P}_{13})]^3}{[-\hat{P}_{13}\hat{2}][\hat{2}4]}\,; \qquad A_3[-\hat{P}_{13}^-, \hat{2}^-, 4^+] = g\frac{\langle-\hat{P}_{13}\hat{2}\rangle^3}{\left\langle\hat{2}4\right\rangle\langle 4(-\hat{P}_{13})\rangle}\,. \tag{2.159}$$

Here, we must choose a convention for the $|-p], |-p\rangle$ helicity spinors. This is an effective analytic continuation and to some extent, is arbitrary as long as $|-p]\langle -p| = -p^{\dot{\alpha}\alpha}$. Depending on the metric signature, some authors([24]) choose $|-p] = i|p], |-p\rangle = -i\,|p\rangle$, while others ([22]) choose $|-p\rangle = -\,|p\rangle\,, |-p] = |p]$. We follow the first. Furthermore, there is a simplification for one of the three particle amplitudes. Recall that the on-shell condition for the inner momentum implies:

$$\hat{P}_{13}^2 = 0 = 2\hat{p}_1 \cdot p_3 = \left\langle\hat{1}3\right\rangle[3\hat{1}] = \langle 13\rangle\,[3\hat{1}]\,. \tag{2.160}$$

Since $\langle 13\rangle \neq 0$, it is clear that the on-shell condition requires $[3\hat{1}] = 0$. This, in turn, implies that $[\hat{P}_{13}3] = 0$, after all:

$$\left|\hat{P}_{13}\right\rangle[\hat{P}_{13}3] = -(\hat{p}_1 + p_3)_{\alpha\dot{\alpha}}|3]^{\dot{\alpha}} = -\hat{p}_{1\dot{\alpha}\alpha}|3]^{\dot{\alpha}} = \left|\hat{1}\right\rangle[\hat{1}3] = |1\rangle\,[\hat{1}3] = 0\,. \tag{2.161}$$

Thus, the amplitude $A_3[\hat{1}^-, \hat{P}_{13}^+, 3^+] = 0$. Consequently, we only need to worry about two

terms, one of them being the exchange $(3 \leftrightarrow 4)$:

$$\begin{aligned} A_4[1^-,2^-,3^+,4^+] &= g^2 \frac{\langle \hat{1}\hat{P}_{13}\rangle^3}{\langle \hat{P}_{13}3\rangle \langle 3\hat{1}\rangle} \frac{1}{P_{13}^2} \frac{[4(-\hat{P}_{13})]^3}{[-\hat{P}_{13}\hat{2}][\hat{2}4]}\,, \\ &= g^2 \frac{\langle \hat{1}\hat{P}_{13}\rangle^3 [4\hat{P}_{13}]^3}{\langle \hat{P}_{13}3\rangle \langle 3\hat{1}\rangle [\hat{P}_{13}\hat{2}][\hat{2}4]} \frac{1}{\langle 13\rangle [31]}\,, \\ &= g^2 \frac{\left(\langle 1|\,\hat{P}_{13}|4]\right)^3}{[2|\hat{P}_{13}\,|3\rangle \langle 31\rangle [24]} \frac{1}{\langle 13\rangle [31]}\,, \end{aligned} \tag{2.162}$$

where, in the last line, we used the shift properties (namely that $\left|\hat{1}\right\rangle = |1\rangle$ and $|\hat{2}] = |2]$) in order to un-hat the spinors. The above can be further simplified by use of

$$\langle 1|\,\hat{P}_{13}|4] = \langle 1|\,\hat{p}_1 + p_3|4] = \langle 13\rangle\,[43]\,, \qquad [2|\hat{P}_{13}\,|3\rangle = [2|\hat{p}_1\,|3\rangle = -[2\hat{1}]\left\langle \hat{1}3\right\rangle\,, \tag{2.163}$$

and since, $\left\langle \hat{1}\right| = \langle 1|$ and $[2\hat{1}] = [21]$, we have

$$A_4[1^-,2^-,3^+,4^+] = g^2 \frac{\left(\langle 13\rangle\,[43]\right)^3}{[12]\,\langle 13\rangle\,\langle 31\rangle\,[24]} \frac{1}{\langle 13\rangle\,[31]} = g^2 \frac{[34]^3}{[12][24][31]}\,. \tag{2.164}$$

In principle, the above is the desired answer. However, we can rewrite it, by using $\hat{P}_{13} = -\hat{P}_{24}$ in the development. Then, one has

$$A_4[1^-,2^-,3^+,4^+] = g^2 \frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle}\,. \tag{2.165}$$

The above is the Park-Taylor MHV amplitude for the four point case. In fact, we have proven it for the 3 point and 4 point. Thus, if we can show it for the n-th case, we will have inductively proven it. This is not difficult to do. Consider the MHV amplitude $A_n[1^-2^-3^+\ldots n^+]$ under the same $[1,2\rangle$ shift above. Then, we have

$$A_n[1^-,2^-,3^+,\ldots,n^+] = \sum_{k=3}^{n-1} \sum_{h=\pm} A_{n-k+3}[\hat{1}^-,\hat{P}_I^h,k^+,\ldots,n^+] \frac{1}{P_I^2} A_{k-1}[-\hat{P}_I^h,\hat{2}^-,3^+\ldots(k-1)^+]\,. \tag{2.166}$$

This general expression can be simplified. First, we note that all amplitudes with a single flipped helicity vanish, as well as the same helicity amplitudes, since the reference momenta in the gluon polarizations can be chosen in such a way that the contractions

yield zero (see [30]). Thus, the above sum in k corresponding to all partitions of the momentum across the on-shell line can be reduced to only two diagrams:

$$\begin{aligned} A_n[1^-,2^-,3^+,\ldots,n^+] &= A_3[\hat{1}^-,\hat{P}_{1n}^+,n^+]\frac{1}{P_{1n}^2}A_{n-1}[-\hat{P}_{1n}^-,\hat{2}^-,3^+\ldots(n-1)^+] \\ &\quad + A_{n-1}[\hat{1}^-,\hat{P}_{23}^-,4^+,\ldots,n^+]\frac{1}{P_{23}^2}A_3[-\hat{P}_{23}^+,\hat{2}^-,3^+]\,. \end{aligned} \tag{2.167}$$

Once again, however, there are further simplifications coming from the three point kinematics. As we discussed, $\hat{P}_{1n}^2 = 0$ implies $[\hat{P}_{1n}n] = 0$ and thus, the whole mostly positive helicity three point amplitude vanishes: $A_3[\hat{1}^-,\hat{P}_{1n}^+,n^+] = 0$. Thus, we are left with a single expression:

$$\begin{aligned} A_n[1^-,2^-,3^+,\ldots,n^+] &= A_{n-1}[\hat{1}^-,\hat{P}_{23}^-,4^+,\ldots,n^+]\frac{1}{P_{23}^2}A_3[-\hat{P}_{23}^+,\hat{2}^-,3^+] \\ &= \frac{\langle\hat{1}\hat{P}_{23}\rangle^4}{\langle\hat{1}\hat{P}_{23}\rangle\langle\hat{P}_{23}4\rangle\ldots\langle n\hat{1}\rangle}\frac{1}{\langle 23\rangle\,[32]}\frac{[3(-\hat{P}_{23})]^3}{[-\hat{P}_{23}\hat{2}][\hat{2}3]}\,. \end{aligned} \tag{2.168}$$

Now, we carry the same manipulations as before:

$$\langle\hat{P}_{23}4\rangle[\hat{P}_{23}\hat{2}] = \langle 4|\,\hat{P}_{23}|\hat{2}] = \langle 34\rangle\,[32]\,, \qquad \langle\hat{1}\hat{P}_{23}\rangle[\hat{P}_{23}3] = \langle 12\rangle\,[32]\,, \tag{2.169}$$

which leads to

$$A_n[1^-,2^-,3^+,\ldots,n^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\ldots\langle n1\rangle}\,. \tag{2.170}$$

2.3.3 Massive Factorization

So far, we discussed the recursive procedure for the massless case. However, the only required ingredient for the discussion of the recursive procedure was the factorization properties of the amplitudes when the inner momentum line goes on-shell. When we add massive particles, the main change to the basic argument is the position of the poles which are now shifted by the pole mass. Thus, if one has two massless particles, say $1, n$, a generic $[1,n\rangle$ shift with an inner massive particle would lead to [24, 31]:

$$A_n[1,\ldots,n] = \sum_I \sum_{s=-j}^{j} A_L[\hat{1},2,\ldots,\hat{P}_I^j]\frac{1}{P_I^2-m^2}A_R[-\hat{P}_I^{-j},\ldots,\hat{n}]\,. \tag{2.171}$$

While this procedure follows the BCFW construction more strictly, the general arguments of factorization and their use to glue together lower point amplitudes to construct higher point ones can be directly explored without necessarily relying on the complex shifts we have discussed. This is the procedure carried out in [25] to construct higher point massive amplitudes with any number of massless and massive external legs. In this case, the idea is to match the numerators corresponding to the expected analytic structure of the tree-level amplitudes in each of the channels present in the scattering. This method is even faster than BCFW, but the cost is that it often produces spurious poles which are not physical and are related to the contact terms which are not captured by the analytic properties. We will explicitly use this procedure to construct the four point amplitude for graviton-scalar scattering later.

2.4 Unitarity and Cut-Reconstruction

Now, let us discuss how the exploitation of the unitarity of the S-matrix can be used to simplify calculations and produce non trivial results independent of the specific contents of the theories at hand.

2.4.1 Optical Theorem

As a first application of the unitarity properties of the S matrix, we review the optical theorem. Recall that, if the theory is unitary, we must have that the probabilities of measurements associated with a given state $|\Psi;t\rangle$ at any t must add to unity. By the norm of this state, we must have that the Hamiltonian is Hermitian, $H = H^\dagger$. Then, this automatically implies that the S-matrix $S = T\exp\left(-i\int H\,dt\right)$ will satisfy $S^\dagger S = 1$.

Now, let us apply these properties to the calculation of amplitudes. The S-matrix is related to the transfer T matrix by $S = 1 + iT$, and the scattering amplitudes are related to T by

$$\langle f|\,T\,|i\rangle = (2\pi)^4\delta\left(\Sigma_i p_i\right)\mathcal{A}(i\to f)\,, \tag{2.172}$$

where $|i\rangle, |f\rangle$ are the initial and final states, respectively. Given the properties of the S-matrix, the transfer matrix satisfies

$$(1 - iT^\dagger)(1 + iT) = 1 \implies T^\dagger T = i(T^\dagger - T)\,. \tag{2.173}$$

Thus,

$$\langle f|\,(T^\dagger - T)\,|i\rangle \;=\; (2\pi)^4\delta\left(\Sigma_f p_f - \Sigma_i p_i\right)\left(\mathcal{A}^*(i \to f) - \mathcal{A}(i \to f)\right)\,,$$

and, using (2.173):

$$\langle f|\,T^\dagger T\,|i\rangle = i(2\pi)^4\delta\left(\Sigma_i p_i\right)\left(\mathcal{A}^*(i \to f) - \mathcal{A}(i \to f)\right)\,. \tag{2.174}$$

Now, let us insert a complete set of multi particle states $|X\rangle$ of the Hilbert space $\mathcal{H}$ inside the above. For clarity, X is a generic label to include all possible multi particle states, i.e.: $|p_1, \ldots, p_n\rangle, |p_1\rangle$ or mixtures of fermionic and bosonic particles. In this context, the phase space measure to be included is the Lorentz invariant one, such that the completeness relation reads:

$$\mathbb{1} = \sum_X \int \prod_{i\in X} \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i}\,|X\rangle\,\langle X| = \sum_X \int d\Pi_X\,|X\rangle\,\langle X|\,. \tag{2.175}$$

Insertion of the above into the LHS of Eq.(2.174) yields

$$\begin{aligned}\sum_X \int d\Pi_X\,\langle f|\,T^\dagger\,|X\rangle\,\langle X|\,T\,|i\rangle \;&=\; \sum_X \int d\Pi_X\,(2\pi)^8\delta\left(\Sigma_f p_f - \Sigma_X p_X\right)\delta\left(\Sigma_X p_X - \Sigma_i p_i\right)\\ &\quad\times\mathcal{A}^*(f \to X)\mathcal{A}(i \to X)\,,\\ &=\; \sum_X (2\pi)^8\delta\left(\Sigma_f p_f - \Sigma_i p_i\right)\int d\Pi_X\,\delta\left(\Sigma_X p_X - \Sigma_i p_i\right)\\ &\quad\times\mathcal{A}^*(f \to X)\mathcal{A}(i \to X)\,,\end{aligned}$$

and, thus,

$$\mathcal{A}(i \to f) - \mathcal{A}^*(i \to f) = i(2\pi)^4\sum_X \int d\Pi_X\,\delta\left(\Sigma_X p_X - \Sigma_i p_i\right)\mathcal{A}^*(f \to X)\mathcal{A}(i \to X)\,. \tag{2.176}$$

Considering that the LHS above is just the imaginary part of the amplitude, one can write

$$2i\mathrm{Im}\mathcal{A}(i \to f) = i \sum_X \int d\Pi_X \, (2\pi)^4 \delta \left(\Sigma_X p_X - \Sigma_i p_i\right) \mathcal{A}^*(f \to X)\mathcal{A}(i \to X) \, . \quad (2.177)$$

Equation (2.177) is known as the *generalized optical theorem*. By inspection, one can see a surprising property pertaining to tree amplitudes. If the $\mathcal{A}(i \to f)$ represents a tree amplitude, then, since the RHS has a product of two amplitudes, the equation would turn inconsistent, that is, the LHS would be first order in the coupling while the RHS would be second order. Thus, in order for the above equation to be order by order consistent, the amplitude appearing in the LHS must be of a higher order than the ones appearing in the RHS.

While this might seem like a restriction, it is actually a valuable property. It allows the construction of (at least the imaginary parts of) higher order amplitudes using smaller order ones. Specifically, via the above, one can hope to construct a 1 loop amplitude by calculating tree level amplitudes and, hopefully, avoiding many of the difficulties associated with loop calculations. This has been developed in the context of *generalized unitarity cuts* ([24],[27],[22]).

Before moving on, consider the case for which the final state is equal to the initial, $|i\rangle = |f\rangle$. For a massive one-particle state $|i\rangle$, the *decay rate* $\Gamma(i \to X)$ is written as

$$\Gamma(i \to X) = \frac{1}{2m_i} \int d\Pi_X \, (2\pi)^4 \delta^4(\Sigma_X p_X - \Sigma_i p_i) |\mathcal{A}(i \to X)|^2 \, , \quad (2.178)$$

which implies that

$$\mathrm{Im}\mathcal{A}(i \to i) = \sum_X m_i \Gamma(i \to X) = m_i \Gamma_{\mathbf{total}} \, , \quad (2.179)$$

that is, the decay rate is directly related to the imaginary part of the amplitude. Similarly, considering a two particle state $|I\rangle$, the scattering cross section in the CM frame $\sigma(I \to I)$ is

$$\sigma(I \to X) = \frac{1}{4E_{CM}|\vec{p}_i|} \int d\Pi_X \, (2\pi)^4 \delta^4(\Sigma_X p_X - \Sigma_i p_i) |\mathcal{A}(I \to X)|^2 \, , \quad (2.180)$$

which implies that

$$\mathrm{Im}\mathcal{A}(I \to I) = 2E_{CM}|\vec{p}_i| \sum_X \sigma(I \to X)\,. \tag{2.181}$$

Equation (2.181) is often known as the standard optical theorem for historical reasons. It relates the imaginary part of an amplitude to the scattering cross section.

2.4.2 Cutting Amplitudes

Since imaginary parts of amplitudes seem to play a prominent role in calculations, let us consider the imaginary part of a simple scalar propagator:

$$\mathrm{Im}\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{2i}\left(\frac{1}{p^2 - m^2 + i\epsilon} - \frac{1}{p^2 - m^2 - i\epsilon}\right) = \frac{-\epsilon}{(p^2 - m^2)^2 + \epsilon^2}\,. \tag{2.182}$$

For $\epsilon \to 0$, this clearly vanishes except for $p^2 = m^2$, the mass pole of the propagator. To find out what happens at the pole, recall the known formula (Sokhotski-Plemelj theorem)

$$\lim_{\epsilon\to 0} \frac{1}{p^2 - m^2 \pm i\epsilon} = \mathcal{P}\frac{1}{p^2 - m^2} \mp i\pi\delta(p^2 - m^2)\,, \tag{2.183}$$

where $\mathcal{P}$ is Cauchy principal value. From the above, we find that

$$\mathrm{Im}\lim_{\epsilon\to 0} \frac{1}{p^2 - m^2 + i\epsilon} = -\pi\delta(p^2 - m^2)\,. \tag{2.184}$$

In words, the propagator is real whenever it is *off-shell*, but acquires an imaginary part when it goes *on-shell* (unless one absorbs an i factor into the propagator, which flips the statement).

Now, let us discuss how the procedure of extracting the imaginary part of a diagram is directly related to putting intermediate particles on shell (here, we closely follow the exposition in [17]). At first, let us consider scalar particles. Their Feynman propagators in momentum space can be conveniently written as

$$D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} = \frac{i}{2\omega_p}\left(\frac{1}{p^0 - \omega_p + i\epsilon} - \frac{1}{p^0 + \omega_p - i\epsilon}\right) \tag{2.185}$$

with $\omega_p = \sqrt{\vec{\mathbf{p}}^2 + m^2}$ and $(p^0)^2 - \omega_p^2 = p^2 - m^2$. However, from (2.183), we can also write

$$\frac{1}{p^0 - \omega_p + i\epsilon} - \frac{1}{p^0 - \omega_p - i\epsilon} = -2\pi i \delta(p^0 - \omega_p)\,. \tag{2.186}$$

Thus:

$$\begin{aligned} D_F(p) &= \frac{i}{2\omega_p}\left(\frac{1}{p^0 - \omega_p + i\epsilon} - \frac{1}{p^0 - \omega_p - i\epsilon} + \frac{1}{p^0 - \omega_p - i\epsilon} - \frac{1}{p^0 + \omega_p - i\epsilon}\right), \\ &= \frac{i}{2\omega_p}\left(-2\pi i\delta(p^0 - \omega_p) + \frac{1}{p^0 - \omega_p - i\epsilon} - \frac{1}{p^0 + \omega_p - i\epsilon}\right), \\ &= \frac{\pi}{\omega_p}\delta(p^0 - \omega_p) + D_{Ret}(p)\,, \end{aligned} \tag{2.187}$$

with $D_{Ret}(p)$ being the retarded propagator

$$D_{Ret}(p) := \frac{i}{2\omega_p}\left(\frac{1}{p^0 - \omega_p - i\epsilon} - \frac{1}{p^0 + \omega_p - i\epsilon}\right), \tag{2.188}$$

whose poles all lie in the upper p^0 complex plane. Now, for simplicity consider a 1 loop amplitude in a ϕ^3 theory. Then, the general expression will be

$$\begin{aligned} i\mathcal{A}_{\phi^3} &= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} D_F(k) D_F(p-k)\,, \\ &= -g^2 \int \frac{d^4k}{(2\pi)^4}\left[\frac{\pi}{\omega_k}\delta(k^0 - \omega_k) + D_{Ret}(k)\right]\left[\frac{\pi}{\omega_{k-p}}\delta(p^0 - k^0 - \omega_{p-k}) + D_{Ret}(k-p)\right]. \end{aligned}$$

The term containing the two retarded propagators can be integrated using a contour which avoids all of the poles. Similarly, the term containing both deltas vanishes, afterall, in the frame where $\vec{\mathbf{p}} = 0, p^0 = m$, both deltas being satisfied would require $k^0 = -\omega_k = \omega_k$, which is impossible. From these arguments, the amplitude would become

$$\begin{aligned} \mathcal{A}_{\phi^3} &= ig^2 \int \frac{d^4k}{(2\pi)^4}\left[\frac{\pi}{\omega_k}\delta(k^0 - \omega_k) D_{Ret}(k-p) + D_{Ret}(k)\frac{\pi}{\omega_{k-p}}\delta(p^0 - k^0 - \omega_{p-k})\right], \\ &= ig^2 \int \frac{d^4k}{(2\pi)^4}\left[\frac{\pi}{\omega_k}\delta(k^0 - \omega_k) D_F(k-p) + D_F(k)\frac{\pi}{\omega_{k-p}}\delta(p^0 - k^0 - \omega_{p-k})\right]. \end{aligned}$$

The last passage is justified because, upon using relation (2.187) again, the terms with double deltas will vanish once again. Now, if we want the imaginary part of the above

amplitude, the only terms with an imaginary part in the integrand come from the propagators. Then, multiplying the i and using Eq.(2.184), we write

$$\mathrm{Im}(\mathcal{A}_{\phi^3}) = g^2 \int \frac{d^4k}{(2\pi)^4} \left[\frac{\pi^2}{\omega_k} \delta(k^0 - \omega_k)\delta((k-p)^2 - m^2) + \frac{\pi^2}{\omega_{k-p}} \delta(k^2 - m^2)\delta(p^0 - k^0 - \omega_{p-k}) \right] .$$

In the p^μ rest frame, $p^0 = m$ and the second term has deltas which require $k^0 = m + \omega_k = \omega_k$ (which is impossible, once again). Thus, the second term vanishes. The first, however, can be rewritten considering that

$$\delta(k^2 - m^2) = \frac{\delta(k^0 + \omega_k)}{2\omega_k} + \frac{\delta(k^0 - \omega_k)}{2\omega_k} , \tag{2.189}$$

and since the integration with a term like $\delta(k^0 + m)$ vanishes, the result is that

$$2\mathrm{Im}(\mathcal{A}_{\phi^3}) = -g^2 \int \frac{d^4k}{(2\pi)^4} (-2\pi i)\delta(k^2 - m^2)(-2\pi i)\delta((k-p)^2 - m^2) . \tag{2.190}$$

This implies that the extraction of the imaginary part can be achieved by putting both particles inside the loop on shell, which implies making the substitutions $D_F(k) \to -2\pi i \delta(k^2 - m^2)\theta(k^0)$, where the $\theta(k^0)$ enforces positive energy states. This procedure is actually a general result due to Cutkosky [32], and consists of the so-called *cutting rules*:

1. Draw a line which intercepts each propagator of a given diagram.

2. For each cut propagator, put it on shell, i.e. replace the propagator with $-2\pi i \delta^+(D)$, where, D is the momentum polynomial appearing in the propagator's denominator and $\delta^+(D) = \delta(D)\theta(p^0)$.

3. Sum over all possible cuts and intermediate particle states.

4. Identify the result as $2i\mathrm{Im}\mathcal{A}$.

Often, the Cutkosky rules are rephrased in terms of the discontinuity Disc of an amplitude, defined as

$$\text{Disc}\mathcal{A}(p) = \mathcal{A}(p+i\epsilon) - \mathcal{A}(p-i\epsilon) = 2i\text{Im}\mathcal{A}(p) = i\text{Disc}\; i\mathcal{A}(p)\,. \tag{2.191}$$

To summarize, if we have an amplitude $\mathcal{A}$ with n vertices V_i, m propagators with momenta p_i, after performing an on-shell cut, the diagram will be divided in two parts. Label all the momenta to the left of the cut as $p_1, p_2, \ldots p_{m_1}$ and the momenta to the right as $p_{m_1+1}, \ldots p_{m_2}$, with the momenta intercepting the cut labeled as $p_{m_2+1}, \ldots p_m$ (the way in which this partition in made is not restrictive). Then, its discontinuity can be written in generic form as

$$\text{Disc}_{\mathcal{A}} = \int \prod_{i=1}^{m} \frac{d^d p_i}{(2\pi)^d} \prod_{j=1}^{n} V_j \prod_{k_1=1}^{m_1} \frac{1}{p_{k_1}^2 - m_{k_1}^2} \times \left(\prod_{l=m_2+1}^{m} (-2\pi i)^4 \delta^+(p_l^2 - m_l^2) \right) \times \prod_{k_2=m_1+1}^{m_2} \frac{1}{p_{k_2}^2 - m_{k_2}^2}\,. \tag{2.192}$$

The expression above can also be written using the Lorentz invariant phase space measure which appears after one integrates the on-shell delta. Then, one can write

$$\text{Disc}_{\mathcal{A}} \sim \int d\text{LIPS}\,(\ell_1, \ldots \ell_n) \mathcal{A}_{\text{left}}(\ell_1 \ldots, \ell_n, p_1 \ldots, p_{m_1}) \times \mathcal{A}_{\text{right}}(-\ell_1 \ldots, -\ell_n, p_{m_1+1} \ldots, p_{m_2})\,, \tag{2.193}$$

with

$$d\text{LIPS} = \prod_i \frac{d^{d-1} p_i}{(2\pi)^{d-1} 2E_i} \delta(\sum_j p_j)\,. \tag{2.194}$$

In Eq. 2.193, $\mathcal{A}_{\text{left}}$ corresponds to the diagram resulting from the left of the cut and $\mathcal{A}_{\text{right}}$ to the right side one. In figure 2.2 we give an example for a 2-loop correction to the photon propagator (2.2).

The use of the cutting rules to reconstruct loop-amplitudes is a fashionable way to reduce the work needed in calculating some loop integrals. Initially, the method was developed with the aid of dispersion relations, but now, it has been revived with the *general unitarity method*. This, however, requires a further study of loop amplitudes and the way in which they can be reduced to the so-called *master integrals*.

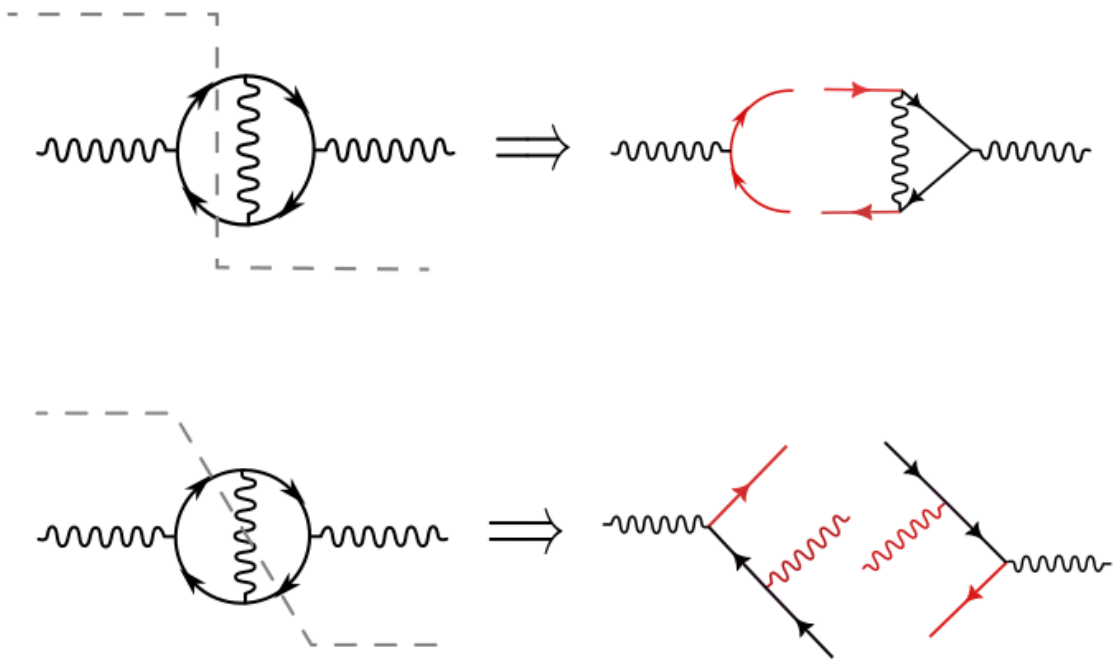


Figure 2.2: Exemplification of expression 2.193 for a two loop correction to the photon propagator. The diagrams on the right correspond to the lower order diagrams one multiplies, where the red propagator lines correspond to the on-shell pieces which are integrated over the Lorentz invariant measure, with their respective intermediate states summed over.

2.5 Generalized Unitarity Methods

The *Generalized Unitarity Method* [33, 23, 24, 22, 34, 35] consists of an extension of the unitarity cuts described above. Specifically, it depends on two main ingredients: the obtaining of the discontinuities of a given loop amplitude by the use of on-shell cuts and the ability to reexpress these in terms of a suitable basis of integrals which are, in principle, known. This method is specially useful at 1-loop order, where it has been well known that a generic 1-loop amplitude can be reduced to a linear combination of scalar integrals, that is, Feynman integrals appearing in a scalar theory (up to rational functions which pose ambiguities to be constrained [35, 36]).

2.5.1 Master Integrals

As discussed before, an important property of the calculation of loop amplitudes lies in the fact that they can be expressed in terms of a basis of scalar integrals. In fact, at 1-loop order one only needs to consider four primitive diagrams: the tadpole, bubble, triangle and box diagrams. Each of these 1PI loop diagrams are associated with corresponding Feynman integrals which can be evaluated using dimensional regularization, as we have done explicitly for the bubble diagram (which corresponds to the vacuum polarization phenomena).

Let us introduce these integrals to make the discussion more specific. In general, we denote an n point 1 loop scalar Feynman integral I_n as

$$I_n := \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\prod_{i=0}^{n-1}[(P_i+\ell)^2 - m_i^2]}\,, \tag{2.195}$$

where $P_i = \sum_{i=0}^{n-1} p_i$ and $p_0 = 0$. Note that some authors [37, 38] prefer to define the above with a pre-factor $\mathcal{N}_d$, which is related to the dimensional regularization scale parameter μ^d. Also note that we are considering all outgoing momenta.

Then, the specific set of integrals we are interested in, are the first $n = 1, 2, 3, 4$ cases of the above, which correspond to the tadpole, bubble, triangle and box diagrams, respectively. The explicit computations for the massless and massive box can be found in, for example, [37]. We quote here, the results for these three integrals in dimensional regularization with $d = 4 - 2\epsilon$([39]):

$$I_2(x) = \frac{c_\Gamma}{\epsilon(1-2\epsilon)}(-x)^{-\epsilon} = c_\Gamma\left\{\frac{1}{\epsilon} - \ln(-x) - 2\right\}, \tag{2.196}$$

$$I_3(x) = \frac{c_\Gamma}{\epsilon^2}(-x)^{-1-\epsilon} = -\frac{c_\Gamma}{x}\left\{\frac{1}{\epsilon^2} - \frac{\ln(-x)}{\epsilon} + \frac{\ln^2(-x)}{2}\right\}, \tag{2.197}$$

$$I_4(x,y) = \frac{c_\Gamma}{xy}\left\{-\frac{2}{\epsilon^2}\left[(-x)^{-\epsilon} + (-y)^{-\epsilon}\right] + \ln^2\left(\frac{x}{y}\right) + \pi^2\right\}, \tag{2.198}$$

$$c_\Gamma = \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{2.199}$$

2.5.2 Veltman-Passarino Tensor Reduction

Given a generic one loop amplitude, the resulting Feynman integral will in general not be a scalar integral. Instead, it will have powers of the momenta in the numerator, as

$$I_n[N(\ell)] := \int \frac{d^d\ell}{(2\pi)^d} \frac{N(\ell)}{\prod_{i=0}^{n-1}[(P_i+\ell)^2 - m_i^2]}\,, \tag{2.200}$$

where $N(\ell)$ is some polynomial of the loop momenta and $I_n[1]$ reduces to the scalar integrals discussed above. The tensor reduction method which we quote from works of Veltman and Passarino consists of a way to rewrite the above integral as a linear combination of the scalar integrals. This allows one to re-express higher point 1 loop integrals in terms of the cited I_1, I_2, I_3, I_4 scalar integrals discussed above.

The method for obtaining the coefficients of the resulting linear combination of the scalar integrals exploits Lorentz invariance in order to constrain the form of the final expression. Let us consider some simple cases to exemplify the general algorithm.

First, consider the integral $I_n[\ell^\mu]$. Then, after integration, this can only depend on an linear combination of the independent $n-1$ external momenta (since the conservation of momentum poses a constraint for one of them, say p_n). Thus, we write

$$I_n[\ell^\mu] = \sum_{i=1}^{n-1} C_i p_i^\mu = \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu}{\prod_{i=0}^{n-1}[(P_i+\ell)^2 - m_i^2]}\,. \tag{2.201}$$

Now, contracting with an arbitrary external p_j^μ, one has

$$I_n[\ell\cdot p_j] = \sum_{i=1}^{n-1} C_i (p_i\cdot p_j) = \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell\cdot p_j}{\prod_{i=0}^{n-1}[(P_i+\ell)^2 - m_i^2]}\,. \tag{2.202}$$

Now, let us note that $p_j = P_j - P_{j-1}$. Then, the following holds:

$$p_j\cdot\ell = \frac{1}{2}\left\{\left[(\ell+P_j)^2 - m_j^2\right] - \left[(\ell+P_{j-1})^2 - m_{j-1}^2\right] - P_j^2 + P_{j-1}^2 + m_j^2 - m_{j-1}^2\right\} \tag{2.203}$$

By inserting this expression into the numerator of the $I_n[p_j\cdot\ell]$ integral, one can decompose it into a sum of each of the terms above, none of which contain the ℓ^μ momentum (and

thus, are all scalar integrals). However, the first two terms, containing the propagator terms, will cancel the factors in the denominator. Thus, for example:

$$I_n\left[\left((\ell+P_j)^2-m_j^2\right)\right]=\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{\prod_{\substack{i=0\\i\neq j}}^{n-1}[(P_i+\ell)^2-m_i^2]}=:I_{n-1}^{(j)}[1]\,. \tag{2.204}$$

Here, we defined the $n-1$ point scalar integral with the j-th propagator pinched out and labelled it $I_{n-1}^{(j)}[1]$. Thus, when one substitutes Eq.2.203 into the numerator of the integral $I_n[p_j\cdot\ell]$, one finds

$$\sum_{i=1}^{n-1}C_i(p_i\cdot p_j)=\frac{1}{2}\left\{I_{n-1}^{(j)}[1]-I_{n-1}^{(j-1)}[1]+(P_{j-1}^2-P_j^2+m_j^2-m_{j-1}^2)I_n[1]\right\}\,. \tag{2.205}$$

Now, let us interpret the LHS of the above as a matrix equation with column vectors C_i and a matrix $G_{ij}:=p_i\cdot p_j$ called the *Gram matrix*. If the Gram matrix is invertible ($\det(\Delta)\neq 0$), the linear equations above can be solved and one has

$$C_i=\frac{1}{2}\sum_j G_{ij}^{-1}\left\{I_{n-1}^{(j)}[1]-I_{n-1}^{(j-1)}[1]+(P_{j-1}^2-P_j^2+m_j^2-m_{j-1}^2)I_n[1]\right\}\,. \tag{2.206}$$

Thus, we have, in principle, successfully reduced the $I_n[\ell^\mu]$ integral to a linear combination of scalar $I_n[1], I_{n-1}^{(j)}[1], I_{n-1}^{(j-1)}[1]$ integrals. This procedure can be followed for more powers of the momenta in the numerator, yielding expansions for $I_n[\ell^{\mu_1}\ell^{\mu_2}\ldots\ell^{\mu_i}]$, but this is at the expense of considerably lengthier expressions. Now, let us briefly comment on G_{ij}. Its inverse is formally given by

$$G^{-1}=\frac{1}{\det(G)}\,\mathrm{adj}\,G\,, \tag{2.207}$$

such that, for a situation with three external momenta, one has $\det(G)=p_1^2p_2^2-(p_1\cdot p_2)^2$. If this determinant is zero, one has $p_1^2p_2^2=(p_1\cdot p_2)^2$ and there is no inverse. This defines a singular region for which the above algorithm does not work. There are, however, ways to work around this apparent impediment, such as expanding around another momentum $\tilde{p}$ (see [24, 38]).

On the other hand, this same procedure can be used to construct expressions for an n point scalar integral $I_n[1]$ in terms of a linear combination of lesser point scalar integrals.

This means that any 1 loop integral $I_n[N(\ell)]$ can, in principle, be written as a linear combination of scalar, lesser point integrals. The reduction can be carried out until there are only the basic integrals I_2, I_3, I_4, up to a rational part, leading to the generic expression

$$I_n[N] = \sum_j a_j I_2^{(j)}[1] + \sum_j b_j I_3^{(j)}[1] + \sum_j c_j I_4^{(j)}[1] + R_n \,. \tag{2.208}$$

It is worth mentioning that in supersymmetric Yang-Mills theories, it has been shown that it is possible to reduce *all* 1-loop amplitudes to a single box integral I_4. This property is not enjoyed by other theories and is just one case where the addition of supersymmetry provides interesting outcomes.

2.5.3 Cut Reconstruction

Now, let us discuss the generalized unitary method as outlined in [33, 24, 40, 41, 22, 23, 38]. This method is heavily inspired by the standard Cutcosky discontinuity method discussed above, but with an added caveat: more cuts. Instead of simply performing a double cut, one could go and cut the resulting sub-amplitudes, yielding more cuts. The reason that this is useful comes from the fact that by cutting the amplitude, one can read off the coefficients associated with a suitable integral basis. To see this more exactly, recall that, as we discussed before, applying a reduction method (such as the Veltman-Passarino algorithm) allows one to reduce an arbitrary n point 1 loop amplitude to a linear combination of the scalar box, triangle and bubble integrals plus rational terms. However, these master integrals are only sensitive to *some* cuts, since they have a different number of propagators in the loop (bubbles have two, triangles three, boxes, four). Thus, each of these integrals allow one to constrain the form of the complete amplitude by matching the multiple cuts that can be performed.

Take, for example, an n point 1 loop amplitude $A_n^{(1)}$. As we discussed before, using the bubble-triangle-box basis, one can write

$$A_n^{1-\text{loop}} = \sum_k a_k I_2^{(k)}[1] + \sum_k b_k I_3^{(k)}[1] + \sum_k c_k I_4^{(k)}[1] + R_n \,, \tag{2.209}$$

with R_n referring to rational terms which are insensitive to any cut. Then, if one performs a standard *double-cut* in the i, j channel, one is effectively calculating the discontinuity of the diagrams:

$$\text{Disc}_{\{ij\}} A_n^{1-\text{loop}} = \sum_k a_k \text{Disc}_{\{ij\}} I_2^{(k)} + \sum_k b_k \text{Disc}_{\{ij\}} I_3^{(k)} + \sum_k c_k \text{Disc}_{\{ij\}} I_4^{(k)} \,, \quad (2.210)$$

where all integrals are sensitive to such cuts. Notice that the rational term naturally drops out, which means that, if this term exists, it is unreachable by unitarity cuts (more on this later). On the other hand, one could proceed with more cuts, that is, putting more loop propagators on-shell. However, the box integral is the only sensitive to a quadruple cut: this is the *maximum cut* possible for this basis. Then, for a given cut, one has:

$$A_n^{1-\text{loop}}\Big|_{4(\text{cut})} = \sum_k c_k I_4^{(k)}\Big|_{4(\text{cut})} \,. \quad (2.211)$$

However, this can be further simplified by noting that the given cut will reduce the left hand side to a product of tree amplitudes summed over the inner particle intermediate states, while the right hand side only has one box integral corresponding to a given cut. It is then found that the coefficient can be directly given as ([34, 23]):

$$\begin{aligned} c_{i,j,k,l} &= \frac{1}{|S|} \sum_S A_n^{\text{tree}}(-p_1, i, \dots j-1, p_2) A_n^{\text{tree}}(-p_2, j, \dots, k-1, p_3) \\ &\quad \times A_n^{\text{tree}}(-p_3, k, \dots, l-1, p_4) A_n^{\text{tree}}(-p_4, l, \dots, i-1, p_1) \,, \end{aligned} \quad (2.212)$$

where the summation is carried out over the solution set of the on-shell conditions for the cut propagators.

As we discussed, to fully compute the amplitude, one would need to go over multiple cuts in order to match the coefficients and this is not always easy. A triple cut, for example, will yield contributions from the triangle and boxes, leading to more involved relations for the coefficients. However, by using a top-down approach, the use of previously obtained coefficients by higher cuts allows one to, in principle, obtain the full expansion coefficients even for the lower double cuts.

So far we have deliberately ignored the rational terms R_n which are unreachable by the standard cuts we discussed. These terms are often terms which come from integrations which are not UV divergent and can be evaluated with dimensional regularization, yielding finite functions. Therefore, any cut calculation has ambiguities related to the presence of these cut-undetectable terms.

On the other hand, there are theories for which such rational terms do not even exist. The most famous case is, once again, $\mathcal{N} = 4$ SYM. In fact, this particular case is much more interesting because not only are there no rational terms, but there are no UV divergences: the theory has no running couplings and, thus, no bubbles and triangle diagrams. For this reason, all 1 loop amplitudes can actually be expressed in terms of box functions and the cut reconstruction is particularly simple ([33, 22, 23]).

Nevertheless, there are situations where these ambiguities arising from the rational part must be taken into account. In order to find these terms, one has to use other tools apart from the unitary cuts discussed. An example of this is the use of collinear factorization properties to constrain these rational terms (see [41]). Another approach, is to perform the cuts in D dimensions, instead of 4. This implies obtaining the tree level amplitudes in D dimensions by using a suitable redefinition of spinor helicity variables. Specifically, one writes the loop momentum ℓ as a four dimensional piece k and a $D-4$ dimensional piece μ, $\ell = k+\mu$. The reason this works is that in dimensional regularization, one takes $D = 4 - 2\epsilon$, with $\epsilon \to 0$ implied by the end. This limit explicitly leaves the rational terms required to completely determine the amplitude. Therefore, by proceeding with the cuts in D dimensions and expanding the loop integration in terms of these, one can re obtain these terms [39, 41, 42].

Chapter 3

Applications I: Quantum Electrodynamics

Here, we discuss the first applications of the formalism we discussed before. At first we compute objects using the standard methods and then, using the newer on-shell approach, re-obtain many of the results. This will hopefully provide a clear case for the new amplitude techniques that have been developed.

3.1 Tree Level Amplitudes

First, let us discuss tree-level amplitudes. These calculations are in themselves important because of the prominent role many of their predictions have played in the development of quantum theory and the modern comprehension of the Standard Model. We will focus on Compton scattering which is arguably the most calculation intensive tree process in QED.

3.1.1 Standard Approach

Compton scattering consists of a process where a photon scatters off of an electron. We can associate it with the shorthand $\gamma f \to \gamma f$ (f stands for fermion, while γ for a photon), and the following feynman diagrams at tree level (Fig. 3.1).

As discussed before, the main difference between *spinor*-QED and *scalar*-QED is the presence of spinors. For comparison we will show both expressions for the amplitudes at

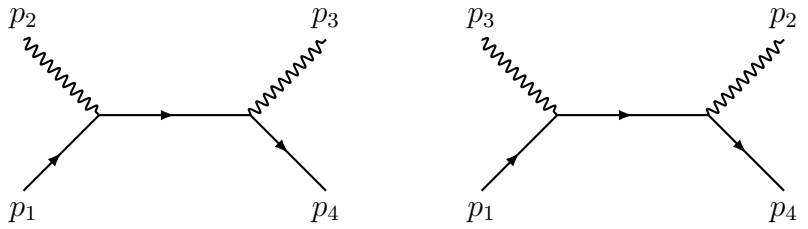


Figure 3.1: Diagrams for the Compton effect. The left one corresponds to the s-channel, while the right, to the t-channel.

tree level.

Scalar QED:

By applying the Feynman rules to the left diagram above, we would find

$$i\mathcal{M}_s = (-ie)\epsilon_\mu(p_2)(p_1^\mu + q^\mu)\frac{i}{q^2 - m^2 + i\epsilon}(-ie)(q^\nu + p_4^\nu)\epsilon_\nu^*(p_3)\,.$$

where $q = p_1 + p_2$. For the right diagram, we would write

$$i\mathcal{M}_t = (-ie)\epsilon_\mu^*(p_3)(p_1^\mu + q^\mu)\frac{i}{q^2 - m^2 + i\epsilon}(-ie)(q^\nu + p_4^\nu)\epsilon_\nu(p_2)\,,$$

where now, $q = p_1 - p_3$. The complete contribution is the sum:

$$i\mathcal{M} = -ie^2\epsilon_\mu(p_2)\epsilon_\nu^*(p_3)\left[\frac{(2p_1^\mu + p_2^\mu)(p_1^\nu + p_2^\nu + p_4^\nu)}{(p_1 + p_2)^2 - m^2} + \frac{(2p_1^\mu - p_3^\mu)(p_1^\nu - p_3^\nu + p_4^\nu)}{(p_1 - p_3)^2 - m^2}\right]\,. \quad (3.1)$$

Spinor QED:

For fermions, we need to include the spinors. In this case, the s-channel diagram would read

$$i\mathcal{M}_s = \overline{u}(p_4)(-ie\gamma^\nu)\epsilon_\nu^*(p_3)\frac{i(\not{p}_1 + \not{p}_2 + m)}{(p_1 + p_2)^2 - m^2 + i\epsilon}\epsilon_\mu(p_2)(-ie\gamma^\mu)u(p_1)\,. \quad (3.2)$$

While the t-channel diagram would become

$$i\mathcal{M}_t = \overline{u}(p_4)(-ie\gamma^\nu)\epsilon_\nu(p_2)\frac{i(\not{p}_1 - \not{p}_3 + m)}{(p_1 - p_3)^2 - m^2 + i\epsilon}\epsilon_\mu^*(p_3)(-ie\gamma^\mu)u(p_1)\,. \quad (3.3)$$

Thus, the total contribution would be the sum of the two expressions above. To actually calculate cross sections, we must take the square of the above expressions, which would

lead to a proliferation of terms which is standard in these perturbative calculations. A naive estimation would give 36 terms without evaluating the gamma matrices, which usually amounts to even more momentum factors. Most of these extra terms are obtained by relabeling other terms, but nevertheless, even a textbook calculation becomes a tedious task.

3.1.2 Modern Approaches

Let us now see how the newer methods based on spinor-helicity (SH) variables are used to compute the basic amplitudes discussed before.

Compton Scattering

$e^-(p_1)\gamma(p_2) \to e^-(p_4)\gamma(p_3)$. Once again, since we are considering all out-going particles, this process is equivalent to $0 \to e^+(p_1)\gamma(p_2)e^-(p_4)\gamma(p_3)$ and the physical momentum has the opposite sign for incoming particles, as do their physical helicities. As noted before, for Compton scattering, we have two diagrams which contribute (3.1).

Let us first consider the massless case. The general helicity amplitudes will be $A_4(\bar{f}^{h_1}\gamma^{h_2}\gamma^{h_3}f^{h_4})$. As we know, the vertex will contain the contractions of the helicity spinors and the gamma matrices. However, the external photon states will also cause contractions with the photon polarization vectors. Let us recall that in the SHF language, we have

$$\begin{aligned} \not{\epsilon}_+(p;q) &= \frac{\sqrt{2}}{\langle qp\rangle}(|p]\langle q| + |q\rangle[p|)\,, \\ \not{\epsilon}_-(p;q) &= \frac{\sqrt{2}}{[qp]}(|p\rangle\,[q| + |q]\,\langle p|)\,, \\ -\not{p} &= |p]\,\langle p| + |p\rangle\,[p|\,. \end{aligned}$$

Thus, the three terms coming from the polarizations and the $\not{p}$ factor from the fermion propagator will yield something which can still be written as the "projector" with the outside fermion helicity spinors sandwiching this inner part. Thus, if both fermions have the

same helicity, this will vanish. Consequently, one only needs to consider the amplitudes of the form $A_4(\bar{f}^{\pm}\gamma^{h_2}\gamma^{h_3}f^{\mp})$. For $A_4(\bar{f}^{+}\gamma^{+}\gamma^{-}f^{-})$ (1,2,3,4 represent the physically incoming fermion, photon and outgoing photon and fermion, respectively), we can use the SH formalism to express equations (3.2) and (3.3) as:

$$\begin{aligned} iA_4(1^+2^+3^-4^-) &= (ie)^2 \langle 4| \not{\epsilon}_+(p_2;q_2)\frac{-i(\not{p}_1+\not{p}_2)}{(p_1+p_2)^2}\not{\epsilon}_-(p_3;q_3)|1] \\ &\quad +(ie)^2 \langle 4| \not{\epsilon}_-(p_3;q_3)\frac{-i(\not{p}_1+\not{p}_3)}{(p_1+p_3)^2}\not{\epsilon}_+(p_2;q_2)|1]\,, \end{aligned} \tag{3.4}$$

where the first term corresponds to the left diagram of fig. 3.1) and the second corresponds to the right diagram. Using the representations in terms of the helicity spinors, one writes

$$\begin{aligned} A_4(1^+2^+3^-4^-) &= 2e^2 \langle 4| \frac{(|2]\langle q_2| + |q_2\rangle [2|)}{\langle q_2 2\rangle}\left(\frac{|1]\langle 1| + |1\rangle [1| + |2]\langle 2| + |2\rangle [2|}{s_{12}}\right) \\ &\quad \times \frac{(|3\rangle [q_3| + |q_3] \langle 3|)}{[q_3 3]}|1] \\ &\quad +2e^2 \langle 4| \frac{(|3\rangle [q_3| + |q_3] \langle 3|)}{[q_3 3]}\left(\frac{|1]\langle 1| + |1\rangle [1| + |3]\langle 3| + |3\rangle [3|}{s_{13}}\right) \\ &\quad \times \frac{(|2]\langle q_2| + |q_2\rangle [2|)}{\langle q_2 2\rangle}|1]\,, \\ &= 2e^2\frac{\langle 4q_2\rangle}{\langle q_2 2\rangle}\frac{[21]\langle 13\rangle}{s_{12}}\frac{[q_3 1]}{[q_3 3]} + 2e^2\frac{\langle 43\rangle}{[q_3 3]}\left(\frac{[q_3 1]\langle 1q_2\rangle + [q_3 3]\langle 3q_2\rangle}{s_{13}}\right)\frac{[21]}{\langle q_2 2\rangle}\,. \end{aligned}$$

Now, recall that q_2, q_3 are reference momenta which are directly related to the gauge freedom of the photon. Thus, whether we decide to keep them until the end of calculations or directly choose values for them which simplify the calculations, the final result should be independent (and indeed, it is). Then, we could take $q_3 = q_2 = 1$ and obtain directly

$$\begin{aligned} A_4(1^+2^+3^-4^-) &= -2e^2\frac{\langle 34\rangle\,[13]\,\langle 31\rangle\,[21]}{s_{13}[13]\,\langle 12\rangle} \\ &= 2e^2\frac{\langle 34\rangle\,[13]\,\langle 31\rangle\,[21]}{\langle 13\rangle\,[13][13]\,\langle 12\rangle} \\ &= -2e^2\frac{\langle 34\rangle\,[12]}{[13]\,\langle 12\rangle}\,. \end{aligned}$$

Note that, using momentum conservation after multiplying by $1 = \langle 34\rangle\,/\,\langle 34\rangle$, we can also write

$$-2e^2\frac{\langle 34\rangle\,[12]\,\langle 34\rangle}{[13]\,\langle 12\rangle\,\langle 34\rangle} = -2e^2\frac{\langle 34\rangle\,[12]\,\langle 34\rangle}{-[12]\,\langle 12\rangle\,\langle 24\rangle} = 2e^2\frac{\langle 34\rangle^2}{\langle 12\rangle\,\langle 24\rangle}\,. \tag{3.5}$$

Thus, we find that

$$A_4(1^+2^+3^-4^-) = 2e^2\frac{\langle 34\rangle^2}{\langle 12\rangle\,\langle 24\rangle}\,. \tag{3.6}$$

From the parity properties of the amplitude, we automatically obtain $A_4(1^-2^-3^+4^+) = A_4(1^+2^+3^-4^-)^*$:

$$A_4(1^-2^-3^+4^+) = 2e^2\frac{[34]^2}{[12][24]}\,. \tag{3.7}$$

Before considering the other amplitude configurations, let us mention that the calculation of the above amplitudes is straightforward: we pick an adequate shift of the external massless particles –for example, $[1,2\rangle$ for the second and $[3,4\rangle$ for the first– and multiply the lower three point amplitudes at each side of the inner on-shell propagator. Then, after adding the different contributions coming from the different helicity assignments of this inner on-shell particle, we must also remember to add the different permutations of the un-shifted particles.

Now, let us consider more possibilities for the helicity configurations of the amplitudes. Knowing that the fermions must have opposite helicities, left us with $2\times1\times2\times2 = 8$ possibilities. Above, we obtained 2 of them and the other 6 are $A_4(1^+2^-3^-4^-)$, $A_4(1^+2^+3^-4^+)$, $A_4(1^+2^-3^-4^+)$ and their conjugates. However, there is an argument which reduces these 6 amplitudes to only 2: photons with the same helicity also give vanishing contributions. To see this, consider the generic $A_4(\bar{f}^+\gamma^{h_2}\gamma^{h_3}f^-)$. Recycling expression (3.4), we have

$$\begin{aligned} iA_4(1^+\gamma^{h_2}\gamma^{h_3}4^-) &= (ie)^2 \langle 4| \not{\epsilon}_{h_2}(p_2;q_2)\frac{-i(\not{p}_1+\not{p}_2)}{(p_1+p_2)^2}\not{\epsilon}_{h_3}(p_3;q_3)|1] \\ &\quad +(ie)^2 \langle 4| \not{\epsilon}_{h_3}(p_3;q_3)\frac{-i(\not{p}_1+\not{p}_3)}{(p_1+p_3)^2}\not{\epsilon}_{h_2}(p_2;q_2)|1]\,. \end{aligned} \tag{3.8}$$

On the other hand, recall that

$$\langle q|\not{\epsilon}_+(p;q) = 0 = \not{\epsilon}_+(p;q)\,|q\rangle\ , \tag{3.9}$$

$$\not{\epsilon}_-(p;q)|q] = 0 = [q|\not{\epsilon}_-(p;q)\,. \tag{3.10}$$

Therefore, in expression (3.8), we see that choosing the reference momenta appropriately the whole amplitude vanishes. For positive helicities, we can take $q_2 = 4 = q_3$ and for negative ones, we can take $q_3 = 1 = q_2$. Also, note that this is solely dependent on the external fermions sandwiching the polarizations. For this reason, it is in general true that an amplitude will vanish if the helicities of the vector bosons are the same (be careful when interpreting in terms of the *physical* helicities).

This leaves us with only one more possibility, the amplitude $A_4(1^+2^-3^+4^-)$ and its conjugate. We have

$$\begin{aligned} A_4(1^+2^-3^+4^-) &= e^2 \langle 4| \not{\epsilon}_+(p_3;q_3)\frac{(\not{p}_1+\not{p}_2)}{(p_1+p_2)^2}\not{\epsilon}_-(p_2;q_2)|1] \\ &\quad +e^2 \langle 4| \not{\epsilon}_-(p_2;q_2)\frac{(\not{p}_1+\not{p}_3)}{(p_1+p_3)^2}\not{\epsilon}_+(p_3;q_3)|1]\,. \end{aligned} \tag{3.11}$$

Taking $q_2 = 1$, the first contribution vanishes and we have

$$\begin{aligned} A_4(1^+2^-3^+4^-) &= e^2 \langle 4| \not{\epsilon}_-(p_2;p_1)\frac{(\not{p}_1+\not{p}_3)}{(p_1+p_3)^2}\not{\epsilon}_+(p_3;q_3)|1]\,, \\ &= 2e^2\frac{[13]^2}{[12][24]}\,, \end{aligned} \tag{3.12}$$

which is the same as the amplitude $A_4(1^-2^-3^+4^+)$ with $4 \leftrightarrow 1$. The remaining amplitude, namely $A_4(1^-2^+3^-4^+)$ is obtained by conjugating the above.

At last, the total scattering amplitude $\mathcal{A}_4(0 \to \bar{f}f\gamma\gamma)$ will be given by a contribution of each of the helicity amplitudes we calculated. Note, however, that $|\mathcal{A}_4(0 \to \bar{f}f\gamma\gamma)|^2$ is formally a transition probability. This means that if we are interested in the transition probability for the process, we can sum the squares of the helicity amplitudes we calculated, after all, they represent the transition probabilities for each set of helicity prescriptions. Therefore, we write

$$\begin{aligned} |\mathcal{A}_4(0 \to \bar{f}f\gamma\gamma)|^2 &= \sum_{\{h_i\}} |A_4(\bar{f}^{h_1}\gamma^{h_2}f^{h_3}\gamma^{h_4})|^2 \,, \\ &= |A_4(1^+2^+3^-4^-)|^2 + |A_4(1^+2^-3^+4^-)|^2 + \cdots \\ &= 2\left\{|A_4(1^+2^-3^+4^-)|^2 + |A_4(1^+2^+3^-4^-)|^2\right\} \,. \end{aligned} \tag{3.13}$$

Using the results obtained above, we find

$$\begin{aligned} |\mathcal{A}_4(0 \to \bar{f}f\gamma\gamma)|^2 &= 8e^4\left(\frac{\langle 13\rangle^2\,[13]^2}{\langle 12\rangle\,\langle 24\rangle\,[12][24]} + \frac{\langle 34\rangle^2\,[34]^2}{\langle 12\rangle\,\langle 24\rangle\,[12][24]}\right) \,, \\ &= 8e^4\left(\frac{\langle 34\rangle\,[34]}{\langle 13\rangle\,[13]} + \frac{\langle 13\rangle\,[13]}{\langle 34\rangle\,[34]}\right) \,, \\ &= 8e^4\left(\frac{p_3\cdot p_4}{p_1\cdot p_3} + \frac{p_1\cdot p_3}{p_3\cdot p_4}\right) \,. \end{aligned} \tag{3.14}$$

This completes the calculation. Note that it agrees with the massive calculation for negligible masses $m_i \approx 0$. In terms of the Mandelstalm invariants ($2p_1\cdot p_3 = t$, $2p_1\cdot p_2 = s$) we have

$$|\mathcal{A}|^2 = 8e^4\left(\frac{t}{s} + \frac{s}{t}\right) \,. \tag{3.15}$$

Let us now consider the case of massive Compton amplitudes and obtain their high energy limit. First, let us recast the usual form of these amplitudes using the massive spinors introduced before. As we discussed for Compton scattering, there are two channels to worry about. If we take particles 1 and 3 to be massive, the general form of the

amplitudes will be $A_4(1^I, 2^{h_2}, 3^{h_3}, 4^J)$. By the standard Feynman rules, these are

$$\begin{aligned} A_4(1^I, 2^{h_2}, 3^{h_3}, 4^J) &= \epsilon^{\mu}_{2,h_2}\epsilon^{\nu}_{3,h_3}\overline{u}^I(p_1)\left(\frac{\gamma^\mu(\not{P}_{12}+m)\gamma^\nu}{s_{12}-m^2}+\frac{\gamma^\nu(\not{P}_{13}+m)\gamma^\mu}{s_{13}-m^2}\right)v^J(p_4)\,, \\ &= \overline{u}^{\{I}(p_1)\left(\frac{n_{12}}{s_{12}-m^2}+\frac{n_{13}}{s_{13}-m^2}\right)v^{J\}}(p_4)\,. \end{aligned} \tag{3.16}$$

Let us focus on the inner numerator factors n_{12} and n_{13}. By using the explicit matrix form of the factors inside, we can obtain the relevant expressions. Specifically, we recall that

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\beta} \\ \bar\sigma^{\mu\dot\alpha\beta} & 0 \end{pmatrix}, \qquad \not{p}+m = \begin{pmatrix} m\delta_\alpha{}^\beta & p_{\alpha\dot\beta} \\ p^{\dot\alpha\beta} & m\delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}, \tag{3.17}$$

which implies that

$$\begin{aligned} n_{12} &= \begin{pmatrix} 0 & \epsilon_{2,h_2,\alpha\dot\beta} \\ \epsilon^{\dot\alpha\beta}_{2,h_2} & 0 \end{pmatrix}\begin{pmatrix} m\delta_\beta{}^\sigma & (P_{12})_{\beta\dot\sigma} \\ (P_{12})^{\dot\beta\sigma} & m\delta^{\dot\beta}{}_{\dot\sigma} \end{pmatrix}\begin{pmatrix} 0 & \epsilon_{3,h_3,\sigma\dot\rho} \\ \epsilon^{\dot\sigma\rho}_{3,h_3} & 0 \end{pmatrix} \\ &= \begin{pmatrix} m\epsilon_{2,h_2,\alpha\dot\sigma}\epsilon^{\dot\sigma\rho}_{3,h_3} & \epsilon_{2,h_2,\alpha\dot\beta}(P_{12})^{\dot\beta\sigma}\epsilon_{3,h_3\sigma\dot\rho} \\ \epsilon^{\dot\alpha\beta}_{2,h_2}(P_{12})_{\beta\dot\sigma}\epsilon^{\dot\sigma\rho}_{3,h_3} & m\epsilon^{\dot\alpha\beta}_{2,h_2}\epsilon_{3,h_3,\beta\dot\rho} \end{pmatrix}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} n_{13} &= \begin{pmatrix} 0 & \epsilon_{3,h_3,\alpha\dot\beta} \\ \epsilon^{\dot\alpha\beta}_{3,h_3} & 0 \end{pmatrix}\begin{pmatrix} m\delta_\beta{}^\sigma & (P_{13})_{\beta\dot\sigma} \\ (P_{13})^{\dot\beta\sigma} & m\delta^{\dot\beta}{}_{\dot\sigma} \end{pmatrix}\begin{pmatrix} 0 & \epsilon_{2,h_2,\sigma\dot\rho} \\ \epsilon^{\dot\sigma\rho}_{2,h_2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} m\epsilon_{3,h_3,\alpha\dot\sigma}\epsilon^{\dot\sigma\rho}_{2,h_2} & \epsilon_{3,h_3,\alpha\dot\beta}(P_{13})^{\dot\beta\sigma}\epsilon_{2,h_2\sigma\dot\rho} \\ \epsilon^{\dot\alpha\beta}_{3,h_3}(P_{13})_{\beta\dot\sigma}\epsilon^{\dot\sigma\rho}_{2,h_2} & m\epsilon^{\dot\alpha\beta}_{3,h_3}\epsilon_{2,h_2,\beta\dot\rho} \end{pmatrix}. \end{aligned} \tag{3.19}$$

Now, note that the resulting matrices contain factors corresponding to different representations (dotted and un-dotted basis). To express the above in a single representation, one can use the massive Weyl equations to exchange representations. This amounts to using the following change of basis:

$$\begin{pmatrix} p^{\dot\mu\alpha}/m & \delta^{\dot\mu}{}_{\dot\alpha} \end{pmatrix}\begin{pmatrix} o_\alpha{}^\beta & o_{\alpha\dot\beta} \\ o^{\dot\alpha\beta} & o^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}\begin{pmatrix} \delta^\mu{}_\beta \\ p^{\dot\beta\mu}/m \end{pmatrix} = \mathcal{O}^{\dot\mu\mu}\,, \tag{3.20}$$

where the momentum p corresponds to a massive particle. Before changing the representations to sensible spinor indices, let us specify the helicities of the amplitudes, allowing us to use the known expressions for the polarizations. For the $A(1^I, 2^-, 3^+, 4^J)$ amplitude,

we get

$$n_{12}^{-+} = \frac{1}{[2r_2]\langle 3r_3\rangle}\begin{pmatrix} m\lambda_{2\alpha}[r_2 3]r_3^\rho & \lambda_{2\alpha}[r_2|P_{12}|r_3\rangle\,\tilde\lambda_{3\dot\rho} \\ r_2^{\dot\alpha}\langle 2|P_{12}|3]r_3^\rho & mr_2^{\dot\alpha}\langle 2r_3\rangle\,\tilde\lambda_{3\dot\rho}\end{pmatrix} \tag{3.21}$$

$$n_{13}^{-+} = \frac{1}{[3r_3]\langle 2r_2\rangle}\begin{pmatrix} mr_{3\alpha}[3r_2]\lambda_2^\rho & r_{3\alpha}[3|P_{13}|2\rangle\,r_{2\dot\rho} \\ \tilde\lambda_3^{\dot\alpha}\langle r_3|P_{13}|r_2]\lambda_2^\rho & m\tilde\lambda_3^{\dot\alpha}\langle r_3 2\rangle\,r_{2\dot\rho}\end{pmatrix}. \tag{3.22}$$

Note that, since $P_{12} = p_1^{\dot\alpha\alpha} + p_2^{\dot\alpha\alpha} = (|\mathbf{1}\rangle[\mathbf{1}| + |\mathbf{1}]\langle\mathbf{1}|) + (|2\rangle[2| + |2]\langle 2|)$, it is clear that many of the above factors simplify if we choose the reference momenta as $r_2 = p_3$ and $r_3 = p_2$. Then, for example, $[r_2|P_{12}|r_3\rangle = [3|p_1|2\rangle$, and the above expressions simplify to

$$n_{12}^{-+} = \frac{\langle 2|p_1|3]}{\langle 23\rangle[32]}\begin{pmatrix} 0 & \lambda_{2\alpha}\tilde\lambda_{3\dot\rho} \\ \tilde\lambda_3^{\dot\alpha}\lambda_2^\rho & 0\end{pmatrix} \tag{3.23}$$

$$n_{13}^{-+} = \frac{\langle 2|p_1|3]}{\langle 32\rangle[23]}\begin{pmatrix} 0 & \lambda_{2\alpha}\tilde\lambda_{3\dot\rho} \\ \tilde\lambda_3^{\dot\alpha}\lambda_2^\rho & 0\end{pmatrix}. \tag{3.24}$$

Thus, $n_{12}^{-+} = n_{13}^{-+}$. Now, applying the representation shift according to Eq.(3.20), we have

$$\begin{aligned} n_{12}^{-+} = n_{13}^{-+} &= \frac{\langle 2|p_1|3]}{\langle 23\rangle[32]}\begin{pmatrix}\frac{1}{m}p_4^{\dot\mu\alpha} & \delta^{\dot\mu}{}_{\dot\alpha}\end{pmatrix}\begin{pmatrix} 0 & \lambda_{2\alpha}\tilde\lambda_{3\dot\rho} \\ \tilde\lambda_3^{\dot\alpha}\lambda_2^\rho & 0\end{pmatrix}\begin{pmatrix}\delta^\mu{}_\rho \\ \frac{1}{m}p_1^{\dot\rho\mu}\end{pmatrix} \\ &= \frac{\langle 2|p_1|3]}{\langle 23\rangle[32]}\left(\tilde\lambda_3^{\dot\mu}\lambda_2^\mu + \frac{1}{m^2}p_4^{\dot\mu\alpha}\lambda_{2\alpha}\tilde\lambda_{3\dot\rho}\,p_1^{\dot\rho\mu}\right) \end{aligned} \tag{3.25}$$

It is noteworthy that the above expression has other possible representations, especially when it comes to the specific way one chooses the massive momenta to multiply the mixed representation expressions (3.23),(3.24). This specific form agrees with that of [25], with their particular assignment of particle labels. With the above, the result reads

$$\begin{aligned} A(1^I 2^- 3^+ 4^J) &= \overline{u}_1^{\{I}\frac{\langle 2|p_1|3]}{\langle 23\rangle[32]}\left(\tilde\lambda_3^{\dot\mu}\lambda_2^\mu + \frac{1}{m^2}p_4^{\dot\mu\alpha}\lambda_{2\alpha}\tilde\lambda_{3\dot\rho}\,p_1^{\dot\rho\mu}\right)\left(\frac{1}{s_{12}-m^2} + \frac{1}{s_{13}-m^2}\right)v_4^{J\}} \\ &= \overline{u}_1^{\{I}\frac{\langle 2|p_1|3]}{(s_{12}-m^2)(s_{13}-m^2)}\left(\tilde\lambda_3^{\dot\mu}\lambda_2^\mu + \frac{1}{m^2}p_4^{\dot\mu\alpha}\lambda_{2\alpha}\tilde\lambda_{3\dot\rho}\,p_1^{\dot\rho\mu}\right)\left(\frac{2p_1\cdot(p_2+p_3)}{\langle 23\rangle[32]}\right)v_4^{J\}}. \end{aligned}$$

Alternatively, one could use the form of Dirac spinors in Eq.(3.25) before applying the change of basis. Either way, the spinor contraction leads to (with the bold notation representing the symmetrization)

$$A(\mathbf{1}^-3^+\mathbf{4}) = \frac{2p_1\cdot(p_2+p_3)}{(s_{12}-m^2)(s_{13}-m^2)}\frac{\langle 2|p_1|3]}{\langle 23\rangle[32]}(\langle\mathbf{12}\rangle[\mathbf{34}] + [\mathbf{13}]\langle\mathbf{24}\rangle). \tag{3.26}$$

The above can be further simplified by recalling that $\langle 23\rangle\,[32] = 2p_2\cdot p_3 = (p_2+p_3)^2$ and thus:

$$\frac{2p_1\cdot(p_2+p_3)}{\langle 23\rangle\,[32]} = \frac{2p_1\cdot(p_2+p_3)}{(p_2+p_3)^2} = -\frac{2p_1\cdot(p_1+p_4)}{(p_1+p_4)^2} = -\frac{2m^2+2p_1\cdot p_4}{2m^2+2p_1\cdot p_4} = -1\,.$$

We conclude that the Compton amplitude is

$$A(\mathbf{1}2^-3^+\mathbf{4}) \;=\; \frac{\langle 2|\,p_1|3]}{(s_{12}-m^2)(s_{13}-m^2)}(\langle \mathbf{1}2\rangle\,[3\mathbf{4}] + [\mathbf{1}3]\,\langle 2\mathbf{4}\rangle)\,. \tag{3.27}$$

As before, the opposite helicity choice can be computed by parity:

$$A(\mathbf{1}2^+3^-\mathbf{4}) \;=\; \frac{\langle 3|\,p_1|2]}{(s_{12}-m^2)(s_{13}-m^2)}(\langle \mathbf{4}3\rangle\,[\mathbf{2}1] + [\mathbf{4}2]\,\langle \mathbf{3}1\rangle)\,. \tag{3.28}$$

From the two equations above, we can read off the high energy limit of the amplitudes. For Eq.(3.28), we un-bold the symmetric indices and take $m\to 0$ to find

$$\frac{\langle 31\rangle[12]}{\langle 12\rangle\,[21]\,\langle 13\rangle\,[31]}(\langle 43\rangle\,[21] + [42]\,\langle 31\rangle)\,, \tag{3.29}$$

from which we can see that the amplitude decomposes into two terms: one for 1^+4^- and another for 1^-4^+. Therefore, the amplitude $A_4(1^+2^+3^-4^-)$ is

$$\begin{aligned} A_4(1^+2^+3^-4^-) &= \frac{\langle 31\rangle[12]}{\langle 12\rangle\,[21]\,\langle 13\rangle\,[31]}\,\langle 43\rangle\,[21] \\ &= \frac{\langle 43\rangle\,[21]}{\langle 12\rangle\,[31]} \\ &= \frac{\langle 43\rangle^2}{\langle 12\rangle\,\langle 24\rangle}\underbrace{\frac{\langle 24\rangle\,[21]}{\langle 43\rangle\,[31]}}_{-\langle 42\rangle[21]} \\ &= \frac{\langle 34\rangle^2}{\langle 12\rangle\,\langle 24\rangle}\,, \end{aligned} \tag{3.30}$$

which agrees with Eq.(3.6), while the second term of Eq.(3.29) leads to the amplitude

$A_4(1^-2^+3^-4^+)$:

$$\begin{aligned} A_4(1^-2^+3^-4^+) &= \frac{\langle 31\rangle[12]}{\langle 12\rangle[21]\langle 13\rangle[31]}[42]\langle 31\rangle \\ &= \frac{\langle 13\rangle[24]}{\langle 12\rangle[31]} \\ &= \frac{\langle 13\rangle^2}{\langle 12\rangle\langle 24\rangle}\frac{\overbrace{\langle 24\rangle[24]}^{(p_2+p_4)^2=(p_1+p_3)^2=\langle 13\rangle[31]}}{\langle 13\rangle[31]} \\ &= \frac{\langle 13\rangle^2}{\langle 12\rangle\langle 24\rangle}, \end{aligned} \tag{3.31}$$

which also agrees with the result for the same configuration from the massless formalism. In this way, one can proceed the decompositions obtaining two massless amplitudes for each massive amplitude computed.

For the all positive helicity configurations, one has

$$n_{12}^{++} = \frac{1}{\langle 2\xi_2\rangle\langle 3\xi_3\rangle}\begin{pmatrix} m[23]\xi_{2\alpha}\xi_3^\rho & [2|P_{12}|\xi_3\rangle\,\xi_{2\alpha}\tilde{\lambda}_{3\dot\rho} \\ \langle\xi_2|\,P_{12}|3]\tilde{\lambda}_2^{\dot\alpha}\xi_3^\rho & m\langle\xi_2\xi_3\rangle\,\tilde{\lambda}_2^{\dot\alpha}\tilde{\lambda}_{3\dot\rho}\end{pmatrix}, \tag{3.32}$$

$$n_{13}^{++} = \frac{1}{\langle 2\xi_2\rangle\langle 3\xi_3\rangle}\begin{pmatrix} m[32]\xi_{3\alpha}\xi_2^\rho & [3|P_{13}|\xi_2\rangle\,\xi_{3\alpha}\tilde{\lambda}_{2\dot\rho} \\ \langle\xi_3|\,P_{13}|2]\tilde{\lambda}_3^{\dot\alpha}\xi_2^\rho & m\langle\xi_3\xi_2\rangle\,\tilde{\lambda}_3^{\dot\alpha}\tilde{\lambda}_{2\dot\rho}\end{pmatrix}. \tag{3.33}$$

Which becomes

$$\begin{aligned} A(\mathbf{1}2^+3^+\mathbf{4}) =& \frac{1}{s_{12}-m^2}\frac{1}{\langle 2\xi_2\rangle\langle 3\xi_3\rangle}\left(\langle\xi_3\mathbf{4}\rangle[\mathbf{1}2]\langle\xi_2|\,p_{12}|3] - \langle\mathbf{1}\xi_2\rangle\langle\xi_3\mathbf{4}\rangle[23]m\right. \\ &\left. + \langle\mathbf{1}\xi_2\rangle[\mathbf{3}4][2|p_{12}|\xi_3\rangle - [\mathbf{1}2][\mathbf{3}4]\langle\xi_2\xi_3\rangle\,m\right) \\ &+ \frac{1}{s_{13}-m^2}\frac{1}{\langle 2\xi_2\rangle\langle 3\xi_3\rangle}\left(\langle\xi_2\mathbf{4}\rangle[\mathbf{1}3]\langle\xi_3|\,p_{13}|2] - \langle\mathbf{1}\xi_3\rangle\langle\xi_2\mathbf{4}\rangle[32]m\right. \\ &\left. + \langle\mathbf{1}\xi_3\rangle[\mathbf{2}4][3|p_{13}|\xi_2\rangle - [\mathbf{1}3][\mathbf{2}4]\langle\xi_3\xi_2\rangle\,m\right) \end{aligned} \tag{3.34}$$

We see that the simplifications which follow from an appropriate choice of the reference momenta ξ_2, ξ_3 in the massless theory do not follow in the massive one. In this sense, it is no longer true that this same helicity amplitude is zero.

Now, the above calculations were done by effectively rewriting the expressions one would find from Feynman rules in terms of the SH formalism. It is then natural to wonder how much simpler the on-shell approach can get by exploiting the factorization properties

and building the four particle Compton amplitude from three particle amplitudes. While the BCFW method is perfectly applicable, let us show how the calculation can be further simplified by directly matching the poles in each channel.

For three massless particles $1, 2, 3$ interacting with spins s, 1, $-s$ respectively, the correctly weighted amplitudes with the coupling constant omitted read $A_3(1^s 2^+ 3^{-s}) = [12]^{2s+1}[23]^{1-2s}[31]^{-1}$, with the opposite helicity amplitude obtained by exchanging square brackets for angled ones. Since in Compton scattering one is not concerned with the annihilation-creation channel, we only need to worry about two channels, which in our assignments are the s and t. For the s-channel, the residue in the $A_4(1^{-s}2^+3^-4^s)$ amplitude would be

$$\mathrm{Res}A_4(1^{-s}2^+3^-4^s)\Big|_{s=0} = e\frac{[2k]^{2s+1}[12]^{1-2s}}{[1k]} \times \frac{\langle -k\,3\rangle^{2s+1}\langle 3\,4\rangle^{1-2s}}{\langle -4k\rangle} = e\frac{(\langle 13\rangle[24])^{2s}}{2^{2-2s}t}\langle 3|1-4|2]^{2-2s} \tag{3.35}$$

where we have used momentum conservation in the last factor to write $2\langle 3|\,p_1|2] = \langle 3|\,p_1 - p_4|2]$. For the t channel (obtained by interchanging $2 \leftrightarrow 3$), the residue would be

$$\mathrm{Res}A_4(1^{-s}2^+3^-4^s)\Big|_{t=0} = e\frac{(\langle 13\rangle[24])^{2s}}{s}\frac{1}{2^{2-2s}}\langle 3|1-4|2]^{2-2s}\,. \tag{3.36}$$

With both results, the correct factorizing amplitude would be simply

$$A_4(1^{-s}2^+3^-4^s) = e^2\frac{(\langle 13\rangle[24])^{2s}}{st}\frac{1}{2^{2-2s}}\langle 3|1-4|2]^{2-2s} \tag{3.37}$$

The above amplitude develops so called *spurious poles* for spin $s \geq 3/2$, once the momenta of particle 1 and 4 go collinear ($p_1 = p_4$). Since this spurious pole would imply unphysical propagation, it is clear that elementary particles of spin higher than 1 cannot couple in this way to photons. Furthermore, for the case of massless $1/2$ fermions, we find

$$A_4(1^-2^+3^-4^+) = e^2\frac{\langle 13\rangle[24]}{2st}\langle 3|1-4|2]\,, \quad A_4(1^+2^-3^+4^-) = e^2\frac{[13]\langle 24\rangle}{2st}\langle 2|1-4|3] \tag{3.38}$$

For the remaining amplitudes, namely $A_4(1^{+s}2^+3^-4^{-s})$ and its conjugate, one only needs to alter the spinor factors to correctly account for the helicity weights:

$$A_4(1^{+s}2^+3^-4^{-s}) = e^2\frac{(\langle 34\rangle[12])^{2s}}{2^{2-2s}st}\langle 3|1-4|2]^{2-2s}\,, \tag{3.39}$$

such that for QED:

$$A_4(1^+2^+3^-4^-) = e^2\frac{\langle 34\rangle[12]}{2st}\langle 3|p_1-p_4|2]\,, \quad A_4(1^-2^-3^+4^+) = e^2\frac{[34]\langle 12\rangle}{2st}\langle 2|p_1-p_4|3]\,. \tag{3.40}$$

Then, the full amplitude would be obtained by multiplying and adding each of the conjugates above. The result agrees with the one using the standard Feynman rules as a starting point, i.e. Eq. (3.15).

This approach of directly using factorization can be carried out for massive particles as well, but this is not without difficulties, though, since massive amplitudes have a well-known problem of spurious poles (specifically for gravitational Compton amplitudes, see [43]). A deeper discussion of this point will not be pursued here as we are merely giving an introduction to the modern formalism (see [25] for a lot more).

Bhabba Scattering

Consider now the simpler case of fermion-fermion scattering. Specifically, we consider the annihilation process $e^-e^+ \to e^-e^+$ which taking all outgoing particles turns into $0 \to e^-e^+e^+e^-$. Let us label the particles in order by $1,2,3,4$ (1 and 4 are fermions, while 2 and 3 are anti-fermions). Then, the expression for the scattering according to the Feynman rules is

$$A_4(f^{s_1}\bar{f}^{s_2}f^{s_3}\bar{f}^{s_4}) \;=\; -e^2\bar{u}_{s_4}(p_4)\gamma^\mu v_{s_3}(p_3)\frac{i\eta_{\mu\nu}}{(p_1+p_2)^2}\bar{u}_{s_2}(p_2)\gamma^\nu v_{s_1}(p_1) \tag{3.41}$$

Note that there is only one channel for this process since we are starting with two distinguishable particles and ending with other two distinguishable particles (an electron and a positron). The same would not be true for Coulomb scattering, for example. From this, we see that the calculation will be a lot simpler than the Compton case. For example, we can observe that amplitudes such as $A_4(1^\pm 2^\pm 3^\pm 4^\pm)$ all vanish, since they are proportional to $[p|\gamma^\mu|q]$ or $\langle p|\,\gamma^\mu\,|q\rangle$. The only four amplitudes which contribute to the annihilation

process are

$$A(1^-2^+3^+4^-) = -e^2\frac{[2|\gamma^\mu|1\rangle\,[3|\gamma_\mu|4\rangle}{s_{12}} = -2e^2\frac{[23]\langle 41\rangle}{\langle 12\rangle[21]} = 2e^2\frac{\langle 14\rangle^2}{\langle 12\rangle\langle 34\rangle}\,, \quad (3.42)$$

$$A(1^-2^+3^-4^+) = -e^2\frac{[2|\gamma^\mu|1\rangle\,[4|\gamma_\mu|3\rangle}{s_{12}} = -2e^2\frac{[24]\langle 31\rangle}{\langle 12\rangle[21]} = 2e^2\frac{\langle 13\rangle^2}{\langle 12\rangle\langle 34\rangle}\,, \quad (3.43)$$

$$A(1^+2^-3^+4^-) = 2e^2\frac{[13]^2}{[12][34]}\,, \quad (3.44)$$

$$A(1^+2^-3^-4^+) = 2e^2\frac{[14]^2}{[12][34]}\,. \quad (3.45)$$

Consequently, for pure annihilation, the transition amplitude for unpolarized scattering averaged over the initial helicity states (4, two for the electrons and two for the positrons) is

$$|\mathcal{A}|^2 = 2e^4\left(\frac{u^2+t^2}{s^2}\right) \quad (3.46)$$

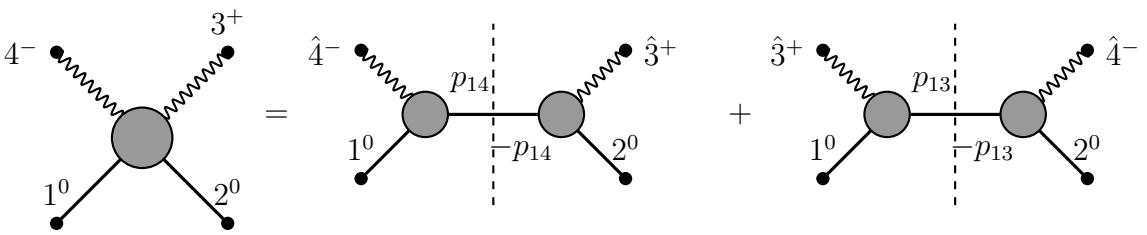
with $u = (p_1+p_4)^2$, $t = (p_1+p_3)^2$ and $s = (p_1+p_2)^2$. Note that this agrees perfectly with the massive calculation (once $m = 0$ is taken) and can be done in a few minutes.

Scalar QED

Let us obtain the four point amplitude in scalar QED for the process $A_4(\phi\gamma \to \phi\gamma)$ using the BCFW method. In this case, the relevant three point amplitudes are

$$A_3(1^0 2^+ p^0) = e\frac{[12][2p]}{[p1]}\,, \quad A_3(1^0 2^- p^0) = e\frac{\langle 12\rangle\langle 2p\rangle}{\langle 1p\rangle}\,. \quad (3.47)$$

The BCFW shifts can be carried out for each helicity configuration. For $A_4(1^0 2^0 3^+ 4^-)$, we use a $[4,3\rangle$ shift:



(3.48)

For the first diagram, we have

$$e^2\frac{\langle 1\hat{4}\rangle\langle\hat{4}\hat{p}_{14}\rangle}{\langle\hat{p}_{14}1\rangle}\frac{1}{p_{14}^2}\frac{[\hat{p}_{14}\hat{3}][\hat{3}2]}{[2\hat{p}_{14}]}=\frac{e^2}{p_{14}^2}\frac{\langle 14\rangle\langle 41\rangle([43]+[13])[32]}{\langle 14\rangle([12]+[42]+z[32])}=e^2\frac{[13]^2[32]}{[12][34][41]} \tag{3.49}$$

where we used the standard properties of the shift i.e.: $|\hat{4}]=|4]+z|3]\,,\ \left|\hat{3}\right\rangle=|3\rangle-z\,|4\rangle$ with $\left|\hat{4}\right\rangle=|4\rangle$, $|\hat{3}]=|3]$, and the Schouten identity $[41][32]=[31][42]+[21][43]$. For the second, the same manipulations are valid:

$$e^2\frac{[\hat{3}1][\hat{p}_{13}\hat{3}]}{[1\hat{p}_{13}]}\frac{1}{p_{13}^2}\frac{\langle\hat{4}\hat{p}_{24}\rangle\langle 2\hat{4}\rangle}{\langle\hat{p}_{24}2\rangle}=e^2\frac{\langle 14\rangle^2\langle 24\rangle}{\langle 12\rangle\langle 13\rangle\langle 43\rangle} \tag{3.50}$$

The summation of both is easier if we convert the first piece to angled brackets:

$$e^2\frac{[13]^2[32]}{[12][34][41]}=e^2\frac{\langle 14\rangle\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 43\rangle} \tag{3.51}$$

upon which summation gives

$$A_4(1^0 2^0 3^+ 4^-)=e^2\frac{\langle 14\rangle\langle 24\rangle}{\langle 13\rangle\langle 23\rangle} \tag{3.52}$$

The opposite parity is immediate by exchanging the angled for square brackets.

3.2 Loop Amplitudes

The perturbative expansion produces results at tree level which are fairly well behaved. This fortunate property is not shared by loop calculations, which are often divergent if naively computed. The divergences can be avoided by redefining the physical constants of the theory. In this regard, it is not true that this process is purely mathematical, since the redefinition of the constants implies *measurable* effects.

The importance of loop calculations can be understood diagrammatically. Consider, for example, the exchange of a photon by two fermions. The interaction vertex is given by the Lagrangian term $-e\overline{\psi}\gamma_\mu A^\mu\psi$, which necessitates two fermion lines to connect to a photon line. This basic interaction vertex leads to the possibility that any photon line in a diagram can be interrupted by a loop which, physically, can be described as a photon

creating a positron-electron pair which annihilates back into a photon. Diagrammatically, this implies that the simple exchange of a photon can be represented as an infinite sum of diagrams:

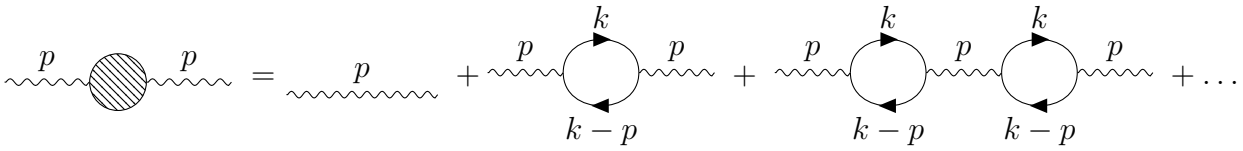


Figure 3.2: The loop corrections to the photon propagator. Note that we are only considering the very first terms and omitting possible corrections to the fermion loop in the form of electron/positron self energy effects.

In more general terms, any propagator without any corrections is often denoted a *bare* propagator, while a propagator with all of its corrections is denoted a *dressed* or *exact* propagator. It is important to note that these effects are *not* effects which appear in non-relativistic quantum mechanics. At this point, it is useful to define a *one particle irreducible* diagram (1PI for short) as any diagram that we cannot describe in terms of other allowed diagrams by "cutting" through some inner line. In this sense, these are diagrams which can be used to construct other diagrams by "gluing" them together, which mathematically implies multiplying the total amplitude by these specific partial amplitude factors. Thus, if we want, we can describe the dressed propagator by a sum of 1PI diagrams:

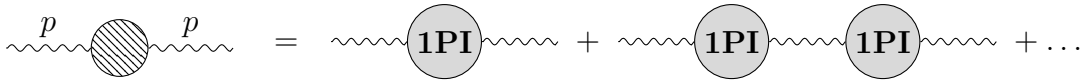


Figure 3.3: General expansion of the dressed propagator in terms of 1PI diagrams.

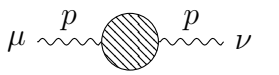
In other words, we can consider loop diagrams as possible corrections appearing in the exact propagator.

Let us consider a way of representing the 1PI expansion written before. For this, consider that the first 1PI blob is equal to a correction term which we will generically

denote $\Pi^{\mu\nu}$. The indices are needed in the case of QED, where the photon propagator has two Lorentz indices. Diagrammatically:

$$= \quad i\Pi^{\mu\nu}(p)\,.$$

This association means that, whenever we have the photon propagator, we will have the following series of corrections:



$$= \Delta^{(0)\mu\nu} + \Delta^{(0)\mu\rho} i\Pi_{\rho\sigma} \Delta^{(0)\sigma\nu} + \dots$$

In principle, $\Pi^{\mu\nu}$ should be Lorentz covariant, after all it is a propagator in Minkowski space. This implies that its dependence on momentum is restricted to be either $\eta^{\mu\nu} f_1(p^2)$ or $p^\mu p^\nu f_2(p^2)$, for some functions f_1, f_2. However, the Ward-Takahashi identity also restricts the particular dependence on these two terms since $p_\mu \Pi^{\mu\nu} = 0$. This, in turn, means that we can determine the form of $\Pi^{\mu\nu}$, namely $\Pi^{\mu\nu} = (\eta^{\mu\nu} p^2 - p^\mu p^\nu)\Pi(p^2)$, which allows us to focus on the last factor. To see how the full self energy chain expansion can be brought into a closed mathematical form, we note that the photon propagator can be written as

$$\Delta^{(0)}_{\mu\nu} = \frac{-i}{p^2 + i\epsilon} P_{\mu\nu} + \frac{-i}{(p^2 + i\epsilon)^2} p_\mu p_\nu \xi\,,$$

where $P_{\mu\nu} = \eta_{\mu\nu} - p_\mu p_\nu / p^2$ corresponds to a projector, satisfying $P_{\mu\nu} P^{\nu\alpha} = P_\mu{}^\alpha$. Then, noting that $i\Pi^{\mu\nu} = ip^2 \Pi(p^2) P^{\mu\nu}$, we can write the n-th iteration of the blob diagram as

$$\Delta^{(0)}_{\mu\alpha_1} i\Pi^{\alpha_1\beta_1} \Delta^{(0)}_{\beta_1\alpha_2} i\Pi^{\alpha_2\beta_2} \Delta^{(0)}_{\beta_2\alpha_3} i\Pi^{\alpha_3\beta_3} \Delta^{(0)}_{\beta_3\alpha_4} \dots i\Pi^{\alpha_n\beta_n} \Delta^{(0)}_{\beta_n\nu} = \frac{1}{p^2 + i\epsilon} \left(-i\Pi(p^2)\right)^n P_{\mu\nu}\,. \tag{3.53}$$

Then, the full 1PI diagrams are given by the series

$$\Delta^{(1)}_{\mu\nu} \;=\; \Delta^{(0)}_{\mu\nu} + \frac{1}{p^2 + i\epsilon} \sum_{n=1}^{\infty} (-i\Pi)^n P_{\mu\nu} = \frac{-i}{p^2 + i\epsilon} P_{\mu\nu} \sum_{n=0}^{\infty} (-i)^{n-1} \Pi^n + \frac{-i}{(p^2 + i\epsilon)^2} p_\mu p_\nu \xi\,. \tag{3.54}$$

Thus, the multiple corrections can be incorporated by using the form above. The last piece is explicitly dependent on the gauge parameter ξ and vanishes in Lorentz/Landau gauge with $\xi = 0$. Also, we note that this gauge dependent term, for S-matrix calculations, must yield no contribution. With this in mind, the dressed propagator is usually written in the literature as the summation of the formally divergent geometric series above, i.e.

$$\Delta_{\mu\nu}(p) \;=\; \frac{-i\eta_{\mu\nu}}{p^2(1-\Pi(p^2))}\,. \tag{3.55}$$

Note that, since the interaction vertex for a general theory is proportional to the bare coupling constant e, every time $\Delta_{\mu\nu}$ appears in a diagram, it will generally be accompanied by an e^2. Thus, if we want to simply account for the loop corrections, we could *redefine* the coupling constant of the vertex interaction. The physical significance of such a procedure will be discussed below. Nevertheless, the calculation of the loop corrected propagator is dependent on the calculation of the so-called self-energy $\Pi(p^2)$.

3.3 Vacuum Polarization Loop diagrams

3.3.1 Spinor QED

Now, in spinor QED, we have a fermion field ψ coupled to the photon field A_μ by the interaction term $-e\overline{\psi}\gamma^\mu A_\mu\psi$. In this sense, similar to ϕ^3 theory, the photon line can only join two fermion lines to create a loop. This gives rise to the following diagram.

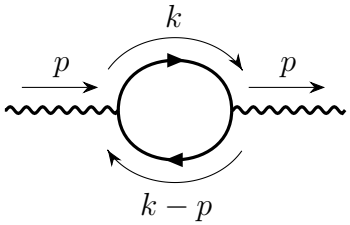


Figure 3.4: Vacuum polarization diagram. This is the one loop order correction to the photon propagator.

The Feynman rules lead to evaluation of the following amplitude,

$$\mathcal{A}_{loop} = \epsilon^*_\mu \epsilon_\nu \Pi^{\mu\nu} \,,$$
$$i\Pi^{\mu\nu}(p) = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \,\mathrm{Tr}\left[\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\gamma^\nu \frac{i(\not{k}-\not{p})+m)}{(k^\alpha-p^\alpha)^2-m^2+i\epsilon}\right]. \qquad (3.56)$$

Here the trace is taken with respect to the spinor indices and we have factored out the photon polarization vectors. The numerator can be simplified using the following identities

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4\eta^{\mu\nu} \,,$$
$$\mathrm{Tr}\big[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\big] = 4(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\alpha\nu}) \,. \qquad (3.57)$$

With these, and the fact that the trace of an odd number of gamma matrices is zero, the numerator can be calculated:

$$\begin{aligned}\mathrm{Tr}\big[\gamma^\mu(\not{k}+m)\gamma^\nu([\not{k}-\not{p}]+m)\big] &= k_\alpha(k_\beta-p_\beta)\,\mathrm{Tr}\big[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\big] + m^2\,\mathrm{Tr}[\gamma^\mu\gamma^\nu] \,, \\ &= 4\left(k^\mu(k^\nu-p^\nu) - k\cdot(k-p)\eta^{\mu\nu} + k^\nu(k^\mu-p^\mu) + m^2\eta^{\mu\nu}\right) .\end{aligned}$$

The integral becomes

$$-4e^2\int\frac{d^4k}{(2\pi)^4}\frac{k^\mu(k^\nu-p^\nu)+k^\nu(k^\mu-p^\mu)-\eta^{\mu\nu}(k\cdot(k-p)-m^2)}{(k^2-m^2+i\epsilon)((k-p)^2-m^2+i\epsilon)} \,.$$

Using Feynman parameters and setting $B = k^2-m^2+i\epsilon$, $A = (k-p)^2-m^2+i\epsilon$, we rewrite the integral as

$$-4e^2\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{k^\mu(k^\nu-p^\nu)+k^\nu(k^\mu-p^\mu)-\eta^{\mu\nu}(k\cdot(k-p)-m^2)}{[(k+p(x-1))^2-p^2x(x-1)-m^2+i\epsilon]^2} \,.$$

Now we make the change of variables $q^\mu \to k^\mu + p^\mu(x-1)$, such that the integral turns into

$$-4e^2\int_0^1 dx\int\frac{d^4q}{(2\pi)^4}\frac{K^{\mu\nu}(q,p)}{[q^2-p^2x(x-1)-m^2+i\epsilon]^2} \,,$$

where

$$\begin{aligned}K^{\mu\nu}(q,p) &= 2q^\mu q^\nu + 2[q^\mu p^\nu(1-x) + (\mu\leftrightarrow\nu)] + 2p^\mu p^\nu(1-x)^2 \\ &\quad -[q^\nu p^\nu + (\mu\leftrightarrow\nu)] - 2p^\mu p^\nu(1-x) + \eta^{\mu\nu}[m^2 - q^2 - 2q\cdot p(1-x) \\ &\quad -p^2(1-x)^2 + p\cdot q + p^2(1-x)] \,.\end{aligned}$$

Note that the integration over q is symmetric, so any term linear in q^μ integrates to zero. Similarly, $q^\mu q^\nu$ will integrate to zero, unless $\mu = \nu$, which implies we could substitute $q^\mu q^\nu$ for $c\eta^{\mu\nu}q^2$. The coefficient c can be determined by contracting both sides with $\eta_{\mu\nu}$, which leads to $c = 1/\eta^\mu_{\ \mu} = 1/d$, where d is the number of dimensions. With this in mind, we will drop all terms linear in q^μ and the integral can be written as

$$-4e^2 \int_0^1 \int \frac{d^4q}{(2\pi)^4} \frac{\eta^{\mu\nu}q^2\left(\frac{2}{d}-1\right) - 2p^\mu p^\nu x(1-x) + \eta^{\mu\nu}(m^2 + p^2x(1-x))}{q^2 - \Delta + i\epsilon}\,.$$

Here, we defined $\Delta = m^2 + p^2x(x-1)$, as usual. Now, to evaluate the integral above, we use dimensional regularization. For this procedure, we calculate the integral in d dimensions, and, as in the scalar case, write $d = 4 - \epsilon$ and expand around $\epsilon = 0$. Using formula (B.8) of the appendix, we can write

$$\begin{aligned}
\int \frac{d^dq}{(2\pi)^d} \frac{q^2}{(q^2 - \Delta + i\epsilon)^2} &= -\frac{i}{(4\pi)^{d/2}} \frac{d}{2}\Gamma\left(1 - \frac{d}{2}\right)\Delta^{\frac{d}{2}-1}\,, \\
\int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2 - \Delta + i\epsilon)^2} &= \frac{i}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right)\Delta^{\frac{d}{2}-2}\,,
\end{aligned}$$

where undoing the Wick rotation yields the negative sign in the first equation ($q^2 = -q_E^2$). The entire integral can be written in terms of these two. The integral proportional to q^2 gives

$$\begin{aligned}
-4e^2 g^{\mu\nu} \int \frac{d^dq}{(2\pi)^d} \frac{\left(\frac{2}{d}-1\right)q^2}{(q^2 - \Delta + i\epsilon)^2} &= \frac{4ie^2\eta^{\mu\nu}}{(4\pi)^{d/2}}\left(\frac{2}{d}-1\right)\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right)\Delta^{\frac{d}{2}-1}\,, \\
&= \frac{4ie^2\eta^{\mu\nu}}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\Delta^{\frac{d}{2}-1}\,, \\
&= \frac{4ie^2\eta^{\mu\nu}}{(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\Delta^{1-\frac{\epsilon}{2}}\,. \qquad (3.58)
\end{aligned}$$

Here, we wrote $d = 4 - \epsilon$ to simplify the expansion as $d \to 4 \implies \epsilon \to 0$. The remaining part of the numerator yields

$$-4e^2 A \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2 - \Delta + i\epsilon)^2} = \frac{-4ie^2}{(4\pi)^{2-\epsilon/2}}\Gamma\left(\frac{\epsilon}{2}\right)\Delta^{-\epsilon/2}A\,,$$

where $A = -2p^\mu p^\nu x(1-x) + \eta^{\mu\nu}(m^2 + p^2x(1-x))$. Putting all together, we have

$$i\Pi^{\mu\nu}(p) = -\frac{8ie^2}{(4\pi)^{2-\epsilon/2}}\left(\eta^{\mu\nu}p^2 - p^\mu p^\nu\right)\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\,\Delta^{-\epsilon/2}x(1-x)\,.$$

Inserting $\Delta = m^2 + p^2 x(x-1)$ and considering $\epsilon \to 0$, we expand and discard up to linear order:

$$\begin{aligned} i\Pi^{\mu\nu}(p) = & -\frac{8ie^2}{16\pi^2}\left(\eta^{\mu\nu}p^2 - p^\mu p^\nu\right)\left[1+\frac{\epsilon}{2}\ln 4\pi + O(\epsilon^2)\right]\left[\frac{2}{\epsilon} - \gamma_E + O(\epsilon)\right] \\ & \times \int_0^1 dx\left[\left(1 - \frac{\epsilon}{2}\ln\Delta + O(\epsilon^2)\right)\right], \\ = & -\frac{ie^2}{2\pi^2}\left(\eta^{\mu\nu}p^2 - p^\mu p^\nu\right)\int_0^1 dx\, x(1-x)\left\{\left(\frac{2}{\epsilon} + \ln\left[\frac{4\pi e^{-\gamma_E}}{m^2 + p^2x(x-1)}\right]\right)\right\}. \end{aligned}$$

Usually, a constant $\mu^2 = 4\pi e^{-\gamma_E}$ is defined and we obtain

$$\Pi^{\mu\nu}(p) = -\frac{e^2}{2\pi^2}\left(\eta^{\mu\nu}p^2 - p^\mu p^\nu\right)\int_0^1 dx\, x(1-x)\left\{\left(\frac{2}{\epsilon} + \ln\left[\frac{\mu^2}{m^2 + p^2x(x-1)}\right]\right)\right\}. \quad (3.59)$$

Note that the important part which appears in the denominator of the dressed propagator is the integral part of $\Pi^{\mu\nu}$, which is written as $\Pi(p^2)$, such that $\Pi^{\mu\nu}(p) = (p^2\eta^{\mu\nu} - p^\mu p^\nu)\Pi(p^2)$ with

$$\Pi(p^2) = -\frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left\{\left(\frac{2}{\epsilon} + \ln\left[\frac{\mu^2}{m^2 + p^2x(x-1)}\right]\right)\right\}. \quad (3.60)$$

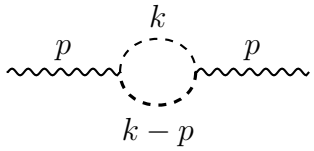
From the above result we see that the divergent term has been isolated and is contained in $\Pi(0)$. Thus, if we could subtract this term, the loop contribution would become finite. This is the idea of renormalization: redefine the Lagrangian so that we can subtract this divergent part from the perturbative expansion. In QED, the possible parameters are the electron mass m and the electron charge e, but often we will also need to rescale the fields themselves.

3.3.2 Scalar QED

In scalar QED, we consider scalar particles described by a field ϕ interacting with the photon field A_μ. The Lagrangian is given by

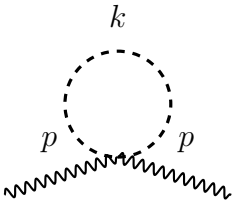
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \phi^*(\partial^2 - m^2)\phi + e^2A_\mu\phi A^\mu\phi^* - ieA_\mu(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi). \quad (3.61)$$

From the above, the interaction part allows two different corrections to the two-point photon propagator, one arising from the inner scalar loop with two vertexes:



$$= (-ie)^2 \int \frac{d^4p}{(2\pi)^4} \frac{i(2k^\mu - p^\mu)}{(k-p)^2 - m^2 + i\epsilon} \frac{i(2k^\nu - p^\nu)}{k^2 - m^2 + i\epsilon} \tag{3.62}$$

and



$$= 2ie^2\eta^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \,. \tag{3.63}$$

Thus, the correction to the two-point propagator is given by the sum of both contributions, yielding

$$i\Pi^{\mu\mu} := e^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2k^\mu - p^\mu)(2k^\nu - p^\nu) - 2\eta^{\mu\nu}[(k-p)^2 - m^2]}{[(k-p)^2 - m^2 + i\epsilon][k^2 - m^2 + i\epsilon]} \,. \tag{3.64}$$

Now one must proceed with the isolation of the divergences. Before this, however, there are further simplifications which can be used in the above integral. Firstly, note that the propagator correction will be of the form

$$D^{(1)}_{\mu\nu}(p) = D^{(0)}_{\mu\nu}(p) + \Pi_{\mu\nu}(p) \,.$$

As before, from Lorentz covariance, $\Pi_{\mu\nu}(p)$ must be a linear combination of the momentum $p_\mu p_\nu$ and the metric $\eta_{\mu\nu}$ (with coefficients depending on p^2), which allows us to write

$$\Pi_{\mu\nu}(p) = \eta_{\mu\nu} A(p^2) + p_\mu p_\nu B(p^2) \,. \tag{3.65}$$

However, once the photon propagator at tree level $D^{(0)}_{\mu\nu}$ is considered, the first correction becomes

$$\begin{aligned} D^{(1)}_{\mu\nu}(p) &= \frac{-i}{p^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right) + \eta_{\mu\nu}A(p^2) + p_\mu p_\nu B(p^2) \\ &= \frac{-i}{p^2}\left([1+A']\eta_{\mu\nu} - (1-\xi+B')\frac{p_\mu p_\nu}{p^2}\right) . \end{aligned} \tag{3.66}$$

The above shows that the parameter B can (upon a suitable redefinition) be absorbed into the gauge parameter ξ. Thus, in principle, the corresponding part of the tensor $\Pi_{\mu\nu}$ should not contribute to whatever physical process one is attempting to calculate using the corrected propagator. For this reason, we write $\Pi_{\mu\nu}(p) = \eta_{\mu\nu}A(p^2)$ and search for the component of $\Pi_{\mu\nu}$ proportional to $\eta_{\mu\nu}$. In Eq. (3.64), this amounts to neglecting terms proportional to p_μ, p_ν or the product $p_\mu p_\nu$. Thus, we redefine

$$i\Pi^{\mu\nu}(p) := 2e^2 \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu - \eta^{\mu\nu}[(k-p)^2 - m^2]}{[(k-p)^2 - m^2 + i\epsilon][k^2 - m^2 + i\epsilon]} \,. \tag{3.67}$$

As standard, we use Feynman parameters with $A = (k-p)^2 - m^2 + i\epsilon$ and $B = k^2 - m^2 + i\epsilon$ to rewrite the above integral as

$$\begin{aligned} i\Pi^{\mu\nu}(p) &:= 2e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu - \eta^{\mu\nu}[(k-p)^2 - m^2]}{(x[(k-p)^2 - m^2 + i\epsilon] + (1-x)[k^2 - m^2 + i\epsilon])^2} \\ &= 2e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu - \eta^{\mu\nu}[(k-p)^2 - m^2]}{[(k-xp)^2 + x(1-x)p^2 - m^2 + i\epsilon]^2} \,. \end{aligned} \tag{3.68}$$

Now, we substitute $k^\mu \to k^\mu - xp^\mu$, which allows us to write

$$i\Pi^{\mu\nu}(p) := 2e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{2(k^\mu - xp^\mu)(k^\nu - xp^\nu) - \eta^{\mu\nu}[(k-xp)^2 - m^2]}{[k^2 + x(1-x)p^2 - m^2 + i\epsilon]^2} \,. \tag{3.69}$$

Note that the symmetric integration of the momenta implies that any piece containing a single k_μ will yield an odd integrand, which evaluates to zero. By the same argument, the term containing $k^\mu k^\nu$ will only yield a non-zero contribution when $\mu = \nu$, which allows one to take $k^\mu k^\nu \propto \eta^{\mu\nu}k^2$. Contracting with $\eta^{\mu\nu}$, the proportionality coefficient is found to be the inverse of the trace of $\eta^\mu{}_\nu = \delta^\mu{}_\nu$, namely the number of dimensions d we are considering. Since we will work in dimensional regularization, where one takes $d = 4 - \epsilon$, we keep d as a variable and write

$$i\Pi^{\mu\nu}(p) := 2e^2\mu^{4-d}\eta^{\mu\nu} \int \frac{d^dk}{(2\pi)^d} \frac{\left(\frac{2}{d} - 1\right)k^2 - x^2p^2 + m^2}{[k^2 + x(1-x)p^2 - m^2 + i\epsilon]^2} \,, \tag{3.70}$$

where we recall that μ is an unphysical parameter added in order to preserve the dimensions of the physical variables during the regularization process. At last, we arrive

at

$$\Pi^{\mu\nu}(p) := 2ie^2\mu^{4-d}\eta^{\mu\nu}\int\frac{d^dk}{(2\pi)^d}\frac{\left(1-\frac{2}{d}\right)k^2+x^2p^2-m^2}{(k^2+\Delta+i\epsilon)^2}\,. \tag{3.71}$$

Here $\Delta := x(1-x)p^2 - m^2$, for convenience. To proceed, we use the identity (B.8) to write

$$\begin{aligned}\int\frac{d^dq}{(2\pi)^d}\frac{q^2}{(q^2-\Delta+i\epsilon)^2} &= -\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right)\Delta^{\frac{d}{2}-1}\,,\\ \int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2-\Delta+i\epsilon)^2} &= \frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\Delta^{\frac{d}{2}-2}\,,\end{aligned}$$

which implies that

$$\Pi^{\mu\nu}(p) = -\frac{2e^2}{(4\pi)^d}p^2\eta^{\mu\nu}\mu^{4-d}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\; x(2x-1)\Delta^{-(2-d/2)}\,. \tag{3.72}$$

Note that the only possibility consistent with the Ward identities $p_\mu\Pi^{\mu\nu}=0$ for the above is to add a term $-p^\mu p^\nu$, yielding the more general form

$$\Pi^{\mu\nu}(p) = -\frac{2e^2}{(4\pi)^d}(p^2\eta^{\mu\nu}-p^\mu p^\nu)\mu^{4-d}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\; x(2x-1)\Delta^{-(2-d/2)}\,. \tag{3.73}$$

Using the power expansions in terms of the parameter ϵ in $d = 4-\epsilon$, one gets

$$\Pi^{\mu\nu}(p) = -\frac{e^2}{8\pi^2}(p^2\eta^{\mu\nu}-p^\mu p^\nu)\int_0^1 dx\; x(2x-1)\left[\frac{2}{\epsilon}+\ln\left(\frac{\tilde{\mu}^2}{m^2-p^2x(1-x)}\right)+O(\epsilon)\right]\,, \tag{3.74}$$

where $\tilde{\mu} := 4\pi e^{-\gamma_E}\mu^2$ and γ_E is the Euler-Mascheroni constant. The above expression isolates the divergent term proportional to $1/\epsilon$ from the subsequent finite parts. Note the striking similarity to the spinor QED case.

3.3.3 Dispersion Relations

Before moving on, let us apply the cutting rules discussed in chapter 2 to obtain the vacuum polarization for QED. This topic is fairly old and extends to other areas of physics, but its exploitation of complexifying momenta (i.e. the variables serving as input) is yet

another case where this procedure lands itself helpful in the study of fairly complicated objects. In the more general case, it is related to the famous Kramers-Kronig relations which formally relate the imaginary and real parts of a given analytic function.

First, let us define the momentum space version of the 2 point function $\langle T\phi_A(x)\phi_B(y)\rangle$ generically as

$$i\Pi_{AB}(p^2) = \int d^4x \, \langle T\phi_A(x)\phi_B(y)\rangle \, e^{ip\cdot(x-y)} \, . \tag{3.75}$$

From the discussion of the spectral representation of the 2 point function, the analytical properties of $\Pi_{AB}(p^2)$ are clear: it is analytic almost everywhere in the complex p^2-plane (after the ϵ factor has been set to 0) but possesses a simple pole at $p^2 = m^2$ with residue equivalent to the mass renormalization constant (or $p^2 = 0$ with the residue associated to charge renormalization) and other singularities starting at $p^2 = 4m^2$. Thus, if we allow for complex momenta, we can write expressions which exploit these analytical properties.

First there is the so-called Schwartz Reflection Principle. For an analytic function $\Pi_{AB}(z)$ with $\Pi_{AB}(z) \in \mathbb{R}$ for $z \in I_{\mathbb{R}}$ (some interval of the real line), the reflection principle states that $\Pi_{AB}(z) = \Pi_{AB}(z^*)^*$. From this, we can equate real and imaginary parts such that

$$\mathrm{Re}\Pi_{AB}(z) = \mathrm{Re}\Pi_{AB}(z^*) \, , \qquad \mathrm{Im}\Pi_{AB}(z) = -\mathrm{Im}\Pi_{AB}(z^*) \, . \tag{3.76}$$

Thus, we find

$$\Pi_{AB}(z) - \Pi_{AB}(z^*) = 2i \, \mathrm{Im}\Pi_{AB}(z) =: \mathrm{Disc}\Pi_{AB}(z) \, . \tag{3.77}$$

Now, recalling the Cauchy formula, we can also write

$$\Pi_{AB}(p^2) = \frac{1}{2\pi i} \oint_C \frac{\Pi_{AB}(z)}{z - p^2} \, dz \, , \tag{3.78}$$

where C is a positively oriented curve which avoids all of the poles of Π_{AB} and encircles only p^2. From our knowledge of the spectral representation of Π, it is a known fact that this

function has a branch cut on the real axis at values bigger then the threshold $s_0 = 4m^2$, which is precisely the minimum energy needed to produce the particle-anti-particle pair. Thus, a suitable path is depicted in figure 3.5. Now, in order to further exploit the

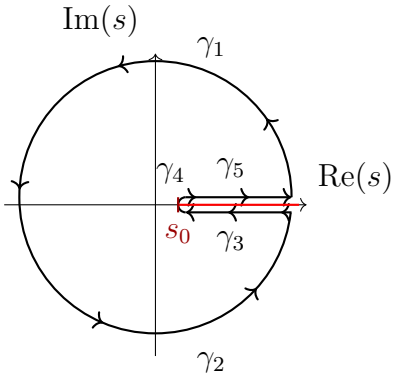


Figure 3.5: Key-hole contour for the dispersion integral. We call $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \gamma_5$, with the red line corresponding to the branch cut.

analytical properties of Π_{AB}, it would be useful if the above integral approached zero as the radius of the arc of circle $\gamma_1 \cup \gamma_2$ went to infinity, i.e. $R \to \infty$. Of course, this is not true since usually the 2 point function has at least UV divergences. However, as we will see explicitly in Section 3.4, the difference $\Pi_{AB}(p^2) - \Pi_{AB}(p_0^2)$ is finite. For the photon, specifically, it is also logarithmically dependent on the input momentum and since $\ln(x)/(1-x) \overset{x\to\infty}{\to} 0$, we should expect the above integral to vanish for $R \to \infty$. Thus, considering a fixed arbitrary momentum p_0, we subtract the corresponding expressions,

$$\begin{aligned} \Pi_{AB}(p^2) - \Pi_{AB}(p_0^2) &= \frac{1}{2\pi i} \oint_\Gamma \left(\frac{\Pi_{AB}(z)}{z - p^2} - \frac{\Pi_{AB}(z)}{z - p_0^2} \right) dz \\ &= \frac{1}{2\pi i} \oint_\Gamma \Pi_{AB}(z) \frac{p^2 - p_0^2}{(z - p^2)(z - p_0^2)}\, dz \ , \end{aligned} \tag{3.79}$$

where we have redefined the curve Γ such that it also encompasses p_0^2. For simplicity, taking $p_0 = 0$, we find

$$\Pi_{AB}(p^2) - \Pi_{AB}(0) = \frac{1}{2\pi i} \oint_\Gamma \frac{p^2\, \Pi_{AB}(z)}{z(z - p^2)}\, dz \ . \tag{3.80}$$

Now, since $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \gamma_5$ (see Fig. 3.5) and the integral along γ_1, γ_2 vanishes for $R \to \infty$, we must only worry about the remaining three contours. Furthermore, when

we take the limit $r \to 0$ for the small radius, the integral also vanishes, allowing us to neglect contributions from γ_4. The remaining curves are the ones parallel to the positive real axis and can be parametrized by $z = s + ir$, with the limit $r \to 0$ to be taken further on. Consequently,

$$\begin{aligned}
\oint_\Gamma \frac{p^2\,\Pi_{AB}(z)}{z(z-p^2)}\,dz &= p^2 \lim_{r\to 0^+}\left\{\int_{s_0}^{\infty} \frac{\Pi_{AB}(s+ir)}{(s+ir)(s+ir-p^2)}\,ds \right. \\
&\quad \left. + \int_{\infty}^{s_0} \frac{\Pi_{AB}(s-ir)}{(s-ir)(s-ir-p^2)}\,ds\right\} \\
&= p^2 \int_{s_0}^{\infty} \frac{\Pi_{AB}(s+ir) - \Pi_{AB}(s-ir)}{s(s-p^2)}\,ds \\
&= 2ip^2 \int_{s_0}^{\infty} \frac{\mathrm{Im}\Pi_{AB}(s)}{s(s-p^2)}\,ds\ , \qquad (3.81)
\end{aligned}$$

where one used the reflection principle in the last step. Thus, we find

$$\Pi_{AB}(p^2) - \Pi_{AB}(0) = \frac{p^2}{\pi}\int_{s_0}^{\infty} \frac{\mathrm{Im}\Pi_{AB}(s)}{s(s-p^2)}\,ds\ . \qquad (3.82)$$

Also, recall that if the renormalized 2-point function is used, $\Pi^R_{AB}(p^2)$, one can write

$$\Pi^R_{AB}(p^2) = \frac{p^2}{\pi}\int_{s_0}^{\infty} \frac{\mathrm{Im}\Pi_{AB}(s)}{s(s-p^2)}\,ds = \frac{p^2}{2\pi i}\int_{s_0}^{\infty} \frac{\mathrm{Disc}\Pi_{AB}(s)}{s(s-p^2)}\,ds\ . \qquad (3.83)$$

Whichever the form chosen, the main feature of these relations is the ability to reconstruct the entire Π_{AB} by knowledge of its imaginary part. Furthermore, as we saw before, obtaining the imaginary part of a given amplitude (or more precisely, its discontinuity) is particularly straightforward by applying the Cutkosky rules as shown in figure 3.6.

The corresponding expression for the cut is:

$$\begin{aligned}
2i\mathrm{Im}\Pi^{\mu\nu}(p^2) &= -(2\pi)^2 e^2 \int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\delta^+(p_1^2-m^2)\delta^+(p_2^2-m^2)\delta^4(p-p_1-p_2) \\
&\quad \times \mathrm{Tr}\Big[(\not{p}_1 + m)\gamma^\mu(-\not{p}_2 + m)\gamma^\nu\Big]\,. \qquad (3.84)
\end{aligned}$$

For simplicity, we define the trace (polarization sum) above as

$$\begin{aligned}
N^{\mu\nu} &:= e^2\,\mathrm{Tr}\Big[(\not{p}_1 + m)\gamma^\mu(-\not{p}_2 + m)\gamma^\nu\Big] \\
&= -4\left(p_1^{(\mu}p_2^{\nu)} - \eta^{\mu\nu}(m^2 + p_1\cdot p_2)\right)\,. \qquad (3.85)
\end{aligned}$$

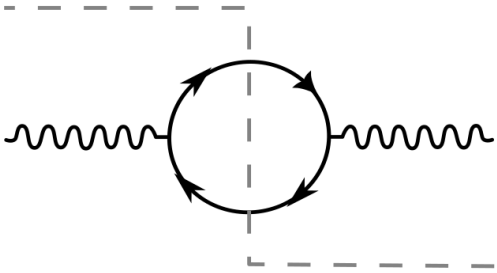


Figure 3.6: Vacuum Polarization cut yielding the discontinuity of the corresponding full amplitude.

Note that, considering momentum conservation, $(p_1 + p_2)^2 = 2m^2 + 2p_1 \cdot p_2 = p^2 = s$, such that the above can also be written as

$$N^{\mu\nu} = -4e^2(p_1^{(\mu}p_2^{\nu)} - \frac{1}{2}\eta^{\mu\nu}s)\,. \tag{3.86}$$

Now, if we use the identity

$$\delta^+(p_i^2 - m^2) = \frac{1}{2\omega_{p_i}}\left[\delta(p_i^0 - \omega_{p_i}) + \delta(p_i^0 + \omega_{p_i})\right]\theta(p_i^0) = \frac{\delta(p_i^0 - \omega_{p_i})}{2\omega_{p_i}} \tag{3.87}$$

and integrate the p_1^0, p_2^0 components, we find

$$\int \frac{d^3\vec{p}_1}{(2\pi)^3}\int \frac{d^3\vec{p}_2}{(2\pi)^3}\frac{\delta^3(\vec{p} - \vec{p}_1 - \vec{p}_2)\delta(E_p - \omega_{p_1} - \omega_{p_2})}{4\omega_{p_1}\omega_{p_2}}N^{\mu\nu}(\vec{p}_1, \vec{p}_2) = \int d\text{LIPS}\,(p)\,N^{\mu\nu}(p_1, p_2)\,, \tag{3.88}$$

Thus, in expression (3.84), the momenta are on shell, i.e. $p_1^0 = \omega_{p_1}, p_2^0 = \omega_{p_2}$. Note also that the integration recovers the Lorentz invariant phase space measure dLIPS (Eq. 2.194). Instead of directly evaluating the integral, the easiest way of proceeding is obtaining its value from restrictions on its possible form. By now this is standard: we use Lorentz covariance, the fact that the only momentum dependence can be on $\vec{p}$, and, if we want, even the Ward identities. First, let us define

$$I^{\mu\nu} := \int d\text{LIPS}(p)\,N^{\mu\nu}(p_1, p_2)\,, \tag{3.89}$$

where p_1^μ, p_2^μ are on shell and, thus, only depend on the associated three-vectors. The resulting integral must be dependent only on p. From Lorentz covariance, we have an

ansatz: $I^{\mu\nu} = A\eta^{\mu\nu} + Bp^\mu p^\nu$. To obtain A and B, we contract the form above with $g_{\mu\nu}$ and $p_\mu p_\nu$. This yields two equations,

$$I^\mu_{\ \mu} = 4A + p^2 B\,, \tag{3.90}$$

$$I^{\mu\nu} p_\mu p_\nu = 4Ap^2 + Bp^4\,. \tag{3.91}$$

However, since p is on-shell and assigned to a photon, $p^2 = 0$ and the right-hand side of Eq. (3.91) vanishes. This is simply the Ward identity, $I^{\mu\nu} p_\mu p_\nu = I^{\mu\nu} p_\mu = 0$. Thus, the correct ansatz we should have made is $I^{\mu\nu} = \widetilde{A}(sg^{\mu\nu} - p^\mu p^\nu)$, where s is the Mandelstam invariant $s = p^2 = (p_1 + p_2)^2$. Hence we can write

$$\begin{aligned}
\widetilde{A} &= \frac{1}{4s} I^\mu_{\ \mu} \\
&= \frac{1}{4s} \int d\mathrm{LIPS}\,(p) N^\mu_{\ \mu}(p_1, p_2) \\
&= -\frac{e^2}{s} \int d\mathrm{LIPS}\,(p) \left(2(p_1 \cdot p_2) - \frac{1}{2}\eta^\mu_{\ \mu} s \right) \\
&= 4e^2 \left(1 + \frac{2m^2}{s} \right) \int d\mathrm{LIPS}\,(p)\,,
\end{aligned} \tag{3.92}$$

where in the last step we used the facts that $2p_1 \cdot p_2 = (p_1 + p_2)^2 - p_1^2 - p_2^2 = s - 2m^2$ and $\eta^\mu_{\ \mu} = 4$. Thus, we actually only need to calculate the Lorentz invariant phase space factor above. This integral is directly calculated in many references [15, 17, 16]. We outline the procedure for the decay into two particles of different masses m_1, m_2, and then specify for $m_1 = m_2$. One has:

$$\begin{aligned}
\int d\mathrm{LIPS}\,(p) &= \int \frac{d^3\vec{p}_1}{(2\pi)^3} \int \frac{d^3\vec{p}_2}{(2\pi)^3} \frac{(2\pi)^4 \delta^4(p - p_1 - p_2)}{4E_{p_1} E_{p_2}} \\
&= \frac{1}{(2\pi)^2} \int d^3\vec{p}_1 \int d^3\vec{p}_2\, \delta^3(\vec{p} - \vec{p}_1 - \vec{p}_2) \frac{\delta(E_p - E_{p_1} - E_{p_2})}{4E_{p_1} E_{p_2}} \\
&= \frac{1}{(2\pi)^2} \int d^3\vec{p}_1\, \frac{\delta(E_p - E_{p_1} - E'_{p_1})}{4E_{p_1} E'_{p_1}}\,,
\end{aligned} \tag{3.93}$$

where $E_{p_1} = \sqrt{(\vec{p}_1)^2 + m_1^2}$ and $E'_{p_1} = \sqrt{(\vec{p} - \vec{p}_1)^2 + m_2^2}$. Note that if we switch to the Center of Mass frame in order to write $\vec{p} + \vec{p}_1 + \vec{p}_2 = 0$ (which implies $\vec{p} = \vec{0}$, and thus is

the rest frame of the momentum $\vec{p}$), we can simply take $E'_{p_1} = \sqrt{(\vec{p}_1)^2 + m_2^2}$. The above delta can be rewritten as

$$\delta(f(|\vec{p}_1|)) = \frac{\delta(|\vec{p}_1| - |\vec{k}|)}{|f'(x)|_{x\to|\vec{k}|}}\,, \tag{3.94}$$

where $\vec{k}$ is to be understood as the vector for which the function $f(|\vec{k}|) = 0$. This can be found after some algebra:

$$E_p - \sqrt{\vec{k}^2 + m_1^2} - \sqrt{\vec{k}^2 + m_2^2} = 0 \implies |\vec{k}| = \frac{1}{2E_p}\sqrt{E_p^4 - 2(m_1^2 + m_2^2)E_p^2 + (m_1^2 - m_2^2)}\,. \tag{3.95}$$

For our case, in which $m_1 = m_2$, one finds

$$|\vec{k}| = \frac{E_p}{2}\sqrt{1 - \frac{4m^2}{E_p^2}} = \frac{\sqrt{s}}{2}\sqrt{1 - \frac{4m^2}{s}}\,. \tag{3.96}$$

With this, the delta function is re-expressed as

$$\delta(E_p - E_{p_1} - E'_{p_1}) = \frac{\delta(|\vec{p}_1| - |\vec{k}|)}{\left|-\left(\frac{|\vec{p}_1|}{E_{p_1}} + \frac{|\vec{p}_1|}{E_{p_1}}\right)\right|} = \frac{E_{p_1}E'_{p_1}}{|\vec{p}_1|\sqrt{s}}\delta(|\vec{p}_1| - |\vec{k}|)\,, \tag{3.97}$$

and the integral can be readily calculated

$$\begin{aligned}
\int d\mathrm{LIPS}\,(p) &= \frac{1}{(2\pi)^2}\int d^3\vec{p}_1\,\frac{E_{p_1}E'_{p_1}}{|\vec{p}_1|\sqrt{s}}\frac{\delta(|\vec{p}_1| - |\vec{k}|)}{4E_{p_1}E'_{p_1}} \\
&= \frac{1}{(2\pi)^2 4\sqrt{s}}\int_0^{4\pi} d\Omega \int_0^{\infty} d|\vec{p}_1|\,|\vec{p}_1|\delta(|\vec{p}_1| - |\vec{k}|) \\
&= \frac{|\vec{k}|}{4\pi\sqrt{s}} = \frac{1}{8\pi}\sqrt{1 - \frac{4m^2}{s}}\,.
\end{aligned} \tag{3.98}$$

With this, we conclude that

$$\widetilde{A} = \frac{e^2}{2\pi}\left(1 + \frac{2m^2}{s}\right)\sqrt{1 - \frac{4m^2}{s}} \tag{3.99}$$

and

$$I^{\mu\nu} = (s\eta^{\mu\nu} - p^\mu p^\nu)\frac{e^2}{2\pi}\left(1 + \frac{2m^2}{s}\right)\sqrt{1 - \frac{4m^2}{s}}\,. \tag{3.100}$$

Thus, the imaginary part of $\Pi^{\mu\nu}$ is given by

$$\operatorname{Im}\Pi^{\mu\nu}(s) = \alpha(s\eta^{\mu\nu} - p^\mu p^\nu)\left(1 + \frac{2m^2}{s}\right)\sqrt{1 - \frac{4m^2}{s}}\,. \tag{3.101}$$

From the above, the use of the subtracted dispersion relation allows one to completely reconstruct the polarization tensor $\Pi^{\mu\nu}$,

$$\Pi_{\mu\nu}(p^2) = \frac{p^2}{\pi}\int_{s_0}^{\infty}\frac{\operatorname{Im}\Pi_{\mu\nu}(s)}{s(s-p^2)}\,ds\ . \tag{3.102}$$

The explicit calculation can be performed and Ref. [44] presents it. The same reference also discusses the application of dispersion relations to other scenarios, including QCD and vertex functions.

While the vacuum polarization is particularly simple to reconstruct in this manner, the procedure for more complicated scenarios quickly renders this dispersion-based approach unwieldy (see e.g. [24, 37]). For this reason, other methods have been devised, such as the generalized unitarity cuts.

3.4 Physical Effects From Vacuum Polarization

The corrections to the propagator given by loop diagrams have physically significant effects. Here we will consider the corrections to the Coulomb potential from the vacuum polarization loop calculated in the last sections. Before, let us make some important remarks on the physical significance of these loop corrections in the specific case of QED.

In general, given two fermions, the most general scattering process will be a sum of an infinite number of diagrams, to higher and higher orders of the coupling constant (see Fig. 3.7). Since these orders diminish drastically the observable effect of such processes, we are generally only interested in the first orders.

However, let us recall that, since the propagator is always attached to an interaction vertex, the effect of the loops can be absorbed into a proper redefinition of the coupling charge e. This is possible because the exact photon propagator always has a pole at

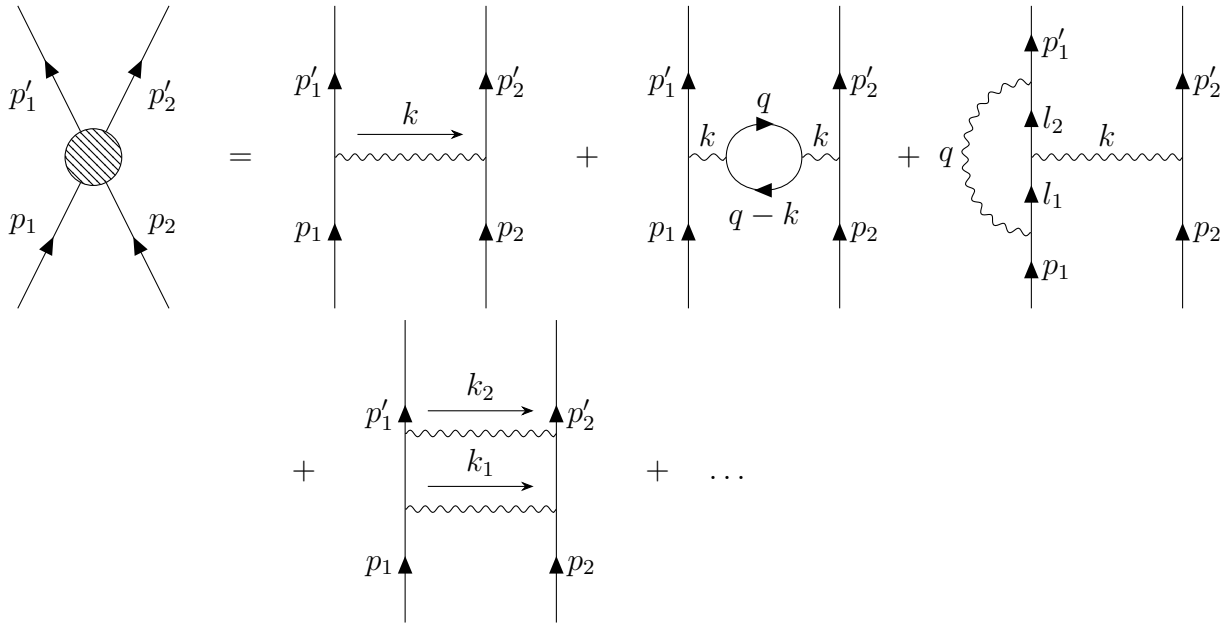


Figure 3.7: The diagrammatic expansion for the Coulomb interaction. The first is tree level, the second is a vacuum polarization type, the third corresponds to the electron self energy and the fourth is a box loop. Note that these diagrams are of order e^2 and e^4, while higher orders have been omitted.

momentum $p^2 = 0$, corresponding to a zero mass as can be seen by its (Lehmann-Källén) spectral representation (see Refs. [14, 21]). Thus, no matter the order of perturbation theory, the photon mass shall remain zero and the only other physical constant specific to pure QED which can be redefined is the electric charge e. If the photon is on-shell, we must have $p^2 = 0$, and thus the residue of the propagator $\Delta_{\mu\nu}$ will be rescaled as

$$\mathrm{Res}_{q^2\to 0}\,\Delta_{\mu\nu} = \frac{-i\eta_{\mu\nu}}{1-\Pi(0)} \equiv -i\eta_{\mu\nu}Z_3\,. \tag{3.103}$$

Therefore, a given scattering amplitude will acquire a correction by the above factor Z_3. Specifically, we will redefine the coupling $e \to e_R = e\sqrt{Z_3}$, so that every internal propagator gets the correct contribution $e^2 Z_3$. The new charge e_R is the renormalized electron charge and can be defined as the actual physical charge measured in experiments [16, 17, 18].

From the above discussion, given a scattering process, the amplitude will acquire a

factor

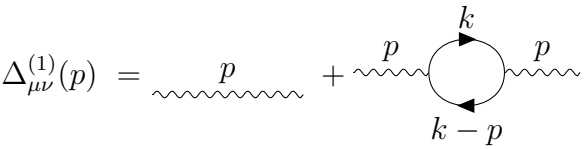
$$\begin{aligned}\frac{e^2}{1-\Pi(p^2)} &= \frac{e_R^2}{Z_3(1-\Pi(p^2))} \\ &= \frac{e_R^2}{(1-\Pi(0))^{-1}(1-\Pi(p^2))} \\ &= \frac{e_R^2}{(1+\Pi(0)+\Pi(0)^2+\dots)(1-\Pi(p^2))} \\ &= \frac{e_R^2}{1+\Pi(0)-\Pi(p^2)+\dots}\,, \end{aligned} \tag{3.104}$$

where the omitted terms in the denominator contain higher powers of the self energy and the coupling constant. At one loop order, we write $\Pi(0)-\Pi(p^2) \approx \Pi_1(0)-\Pi_1(p^2)$ and, using the result (3.59), we can write

$$\Pi^{(1)}(p^2)-\Pi^{(1)}(0) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left[1+\frac{p^2}{m^2}x(x-1)\right] \equiv \Pi_R^{(1)}(p^2)\,. \tag{3.105}$$

Note that the divergent $1/\epsilon$ term has vanished from this subtraction.

As we discussed before, given the simple photon propagator, the possible loop corrections lead to a difference between the free field propagator and the exact propagator in the interacting theory. Let us focus strictly on the propagator line for now. As stated, each correction to the free propagator can be counted by the number of loops we allow. Thus, the leading correction $\Delta_{\mu\nu}^{(1)}$ would be given by considering a single loop contribution, which we can obtain explicitly by using the result of the 1-loop calculation (3.59),

$\Delta_{\mu\nu}^{(1)}(p)$ = 

$$\begin{aligned} &= \Delta_{\mu\nu}^{(0)} + i\Delta_{\mu\rho}^{(0)}\Pi^{(1)\rho\sigma}\Delta_{\sigma\nu}^{(0)} \\ &= \frac{-i\eta_{\mu\nu}}{p^2}\left(1+\Pi^{(1)}(p)\right)\,, \end{aligned} \tag{3.106}$$

where we have unceremoniously dropped the terms proportional to $p^\mu p^\nu$ since, by the Ward identity, they must vanish when calculating physical amplitudes.

3.4.1 Uehling Potential

Spinor QED

Now, considering the one loop correction, let us investigate what effects can be derived for the standard physical objects in Maxwell's theory. To begin, recall that the Fourier transform of the Coulomb potential is related to the photon propagator. From Maxwell's equations, we know that

$$\partial^2 A^\nu - \partial_\mu \partial^\nu A^\mu = j^\nu(x)\,, \tag{3.107}$$

which, once inverted, gives

$$A^\mu(x) = \int d^4x'\, G^{\mu\nu}(x-x') j_\nu(x')\,. \tag{3.108}$$

Here, $G^{\mu\nu}(x-x')$ is a Green's function which solves the inhomogeneous equation with source term j_ν. As we know, this is not well defined because of the Gauge freedom, but for our purposes this can be glossed over by choosing Lorenz gauge, which implies that the second term in the LHS of equation (3.107) is zero and we can simply invert the D'alembertian operator. Now, we want the photon field to be quantized, such that the contributions from the loops can be accounted for. This can be achieved by interchanging

$$G^{\mu\nu}(x-x') \to i\,\langle 0|\,T\left\{\hat{A}^\mu(x)\hat{A}^\nu(x')\right\}|0\rangle = \Delta^{\mu\nu}(x-x')\,, \tag{3.109}$$

where $\Delta^{\mu\nu}(x-x')$ is the Feynman propagator for the photon. In this regard, the electrostatic potential of the interaction can be found by the zeroth component $A^0(x) = \phi(t,\mathbf{x})$,

$$\begin{aligned}
\phi(t,\mathbf{x}) &= -i\int d^4x'\,\Delta^{0\nu}(x-x')j_\nu(x') \\
&= -i\int \frac{d^4p}{(2\pi)^4}\Delta^{0\nu}(p)e^{-ip\cdot x}\int d^4x'\, j_\nu(x')e^{ip\cdot x'} \\
&= -i\int \frac{d^4p}{(2\pi)^4}\Delta^{0\nu}(p)\widetilde{j}_\nu(p)e^{-ip\cdot x}\,,
\end{aligned} \tag{3.110}$$

where $\widetilde{j}_\nu(p)$ is the Fourier transform of the source current. For a static point charge Q placed at $\mathbf{r}_s$, the current can be modeled by $j_\nu(x) = Q\delta^3(\mathbf{x}-\mathbf{r}_s)\delta_{\nu 0}$, which has a Fourier

transform given by

$$\widetilde{j}_\nu(p) = \int d^4x'\, j_\nu(x') e^{ip\cdot x'} = Q\delta_{\nu 0} e^{-i\mathbf{p}\cdot\mathbf{r}_s} \int dx^0\, e^{ip^0 x^0} = 2\pi Q \delta_{\nu 0} e^{-i\mathbf{p}\cdot\mathbf{r}_s}\delta(p^0)\,.$$

Inserting the above into equation (3.110), we get

$$\phi(t,\mathbf{x}) = -iQ \int \frac{d^3\mathbf{p}}{(2\pi)^3} \Delta^{00}(0,\mathbf{p}) e^{-i\mathbf{p}\cdot(\mathbf{r}_s-\mathbf{x})}\,. \tag{3.111}$$

Thus, as claimed, for a static classical point source, the potential produced is directly related to the Fourier transform of the 00 component of the photon propagator. Now, as we pointed out, the quantized photon field implies that the propagator can have loop corrections and, thus, be dressed. If we simply plug in the naked propagator, the result from equation (3.111) will be the standard Coulomb potential. On the other hand, having calculated the first correction to the naked propagator, we can instead plug in $\Delta^{(1)\mu\nu}$ from equation (3.106) to find the first quantum correction to the Coulomb potential. In this case, note that

$$\Delta^{(1)00}(p) = -i\frac{1+\Pi_R(p^2)}{p^2} \implies \Delta^{(1)00}(\mathbf{p}) = i\frac{1+\Pi_R(-\mathbf{p}^2)}{\mathbf{p}^2}\,,$$

which implies that the potential is

$$\phi(\mathbf{x}) = Q\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1+\Pi_R(-\mathbf{p}^2)}{\mathbf{p}^2} e^{-i\mathbf{p}\cdot(\mathbf{r}_s-\mathbf{x})}\,.$$

In Fourier space, the potential will be given by

$$\tilde{\phi}(\mathbf{p}) = \frac{Q}{\mathbf{p}^2}\left\{1 + \frac{e_R^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left[1+\frac{\mathbf{p}^2}{m^2}x(1-x)\right]\right\} e^{-i\mathbf{p}\cdot\mathbf{r}_s}\,. \tag{3.112}$$

This result can be used to compute the limiting behavior for small or large momenta. However, let us note that there is actually a way to compute explicitly the integral above. We outline the procedure presented in Refs. [45, 21]. Considering $\alpha = e_R^2/4\pi$ and integrating by parts to remove the logarithm, the integral can be written as

$$\Pi_R = -\frac{2\alpha}{\pi}\int_0^1 dx\, x^2\left(\frac{1}{2}-\frac{1}{3}x\right)\frac{(\mathbf{p}/m)^2(1-2x)}{1+(\mathbf{p}/m)^2x(1-x)}\,,$$

which, upon a change of variables of the form $v = 2x - 1$, becomes

$$\Pi_R = \frac{\alpha}{\pi}\frac{\mathbf{p}^2}{4m^2}\int_{-1}^{1} dv\, \frac{v^2\left(1-\frac{1}{3}v^2\right)}{1+(\mathbf{p}/2m)^2[1-v^2]}\,. \tag{3.113}$$

By inserting the above in the expression for the Fourier transformed potential, we can exchange the order of integration so that

$$V(\mathbf{x}) = \frac{Q}{4\pi}\left\{\frac{1}{|\mathbf{x}-\mathbf{r}_s|} + \frac{\alpha}{m^2}\int_0^1 dv\, v^2\left(1-\frac{1}{3}v^2\right)\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{r}_s)}}{1+(\mathbf{p}/2m)^2[1-v^2]}\right\}. \tag{3.114}$$

Now, recall that

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\boldsymbol{a}}}{\mathbf{p}^2+b^2} = \frac{1}{4\pi|\boldsymbol{a}|}e^{-b|\boldsymbol{a}|}\,, \tag{3.115}$$

which implies that the momentum integral in equation (3.114) is equivalent to

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{r}_s)}}{1+(\mathbf{p}/2m)^2[1-v^2]} = \frac{4m^2}{1-v^2}\frac{1}{4\pi|\mathbf{x}-\mathbf{r}_s|}e^{-|\mathbf{x}-\mathbf{r}_s|\frac{2m}{\sqrt{1-v^2}}}\,,$$

and, thus, labeling $r \equiv |\mathbf{x}-\mathbf{r}_s|$, we obtain

$$V(\mathbf{x}) = \frac{e_R}{4\pi r}\left\{1+\frac{\alpha}{\pi}\int_0^1 dv\,\frac{v^2(1-\frac{1}{3}v^2)}{1-v^2}\exp\left(-\frac{2m}{\sqrt{1-v^2}}r\right)\right\}. \tag{3.116}$$

In the literature, it is conventional to express the above in terms of $u = (1-v^2)^{-1/2}$ which yields

$$V(\mathbf{x}) = \frac{e_R}{4\pi r}\left\{1+\frac{2\alpha}{3\pi}\int_1^\infty du\left(1+\frac{1}{2u^2}\right)\frac{\sqrt{u^2-1}}{u^2}e^{-2mur}\right\}. \tag{3.117}$$

Equation (3.117) is known as the *Uehling potential* [46] and gives the first loop correction to the Coulomb interaction. Usually, the asymptotic analysis of the $r \gg 1/m$ and $r \ll 1/m$ regimes is discussed with the claim that the integral cannot be analytically solved. However, it was shown in Ref. [47] that the integral can be rewritten in terms of modified Bessel functions K_0 and its iterations K_i, while the authors of Ref. [48] gave a parameterization in terms of modified Bessel functions and the more general G-Meijer functions.

Scalar QED

A very similar discussion can be carried out for scalar QED. In this case, let us recall the form of the polarization tensor:

$$\Pi^{\mu\nu}(p^2) = (p^2\eta^{\mu\nu} - p^\mu p^\nu)\Pi_S(p^2)\,, \tag{3.118}$$

$$\Pi_S(p^2) = -\frac{e^2}{8\pi^2}\int_0^1 dx\, x(2x-1)\left[\frac{2}{\epsilon} + \ln\left(\frac{\tilde{\mu}^2}{m^2 - x(1-x)p^2}\right)\right]\,. \tag{3.119}$$

Once again, the renormalized polarization tensor can be written as $\Pi_S^R(p^2) := \Pi_S(p^2) - \Pi_S(0)$, which automatically ensures that $\Pi_S^R(0) = 0$. Then, as we discussed, one ensures that the pole of the photon propagator for the theory is maintained at the point $p^2 = 0$. This implies that

$$\Pi_S^R(p^2) = \frac{e_R^2}{8\pi^2}\int_0^1 dx\, x(2x-1)\ln\left(1 - \frac{p^2}{m^2}x(1-x)\right)\,. \tag{3.120}$$

Now, let us consider the potential originated from the above correction. This is actually not different from the case of spinor QED since the photon (EM field) still satisfies Maxwell's equations:

$$(\delta^\mu{}_\nu\partial^2 - \partial_\nu\partial^\mu)A^\nu(x) = j^\mu(x)\,. \tag{3.121}$$

Thus, for a classical potential $A_\mu(x)$, one could write

$$A_\mu(x) = \int d^4y\, G_{\mu\nu}(x-y)j^\nu(y)\,, \tag{3.122}$$

where $G_{\mu\nu}$ is a suitable Green's function which solves

$$(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)G^{\mu\nu}(x-y) = \delta^4(x-y)\,. \tag{3.123}$$

As before, the above is ill-defined from the Gauge invariance of the photon field, which is directly related to the gauge dependence of the photon propagator. Also as before, one can either go through the formal procedure of adding a gauge-fixing term which allows the inversion of the above, or simply choose a given gauge and compute the potential

in that specific gauge. The bottom line is that the photon propagator $\Delta_{\mu\nu}$ is a Green's function of the above, such that one can write

$$A_\mu(x) = -i \int d^4y \, \Delta_{\mu\nu}(x-y) j^\nu(y) \,, \tag{3.124}$$

$$\Delta_{\mu\nu}(p) = \frac{-i}{p^2 + i\epsilon} \left[\eta_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} \right] . \tag{3.125}$$

For the loop correction, one substitutes $\Delta_{\mu\nu}$ for the 1 loop corrected photon propagator, which in scalar QED is (in momentum space)

$$\Delta^{(1)}_{\mu\nu}(p) = \Delta^{(0)}_{\mu\nu}(p) + \Delta^{(0)}_{\mu\alpha}(p) i(p^2 \eta^{\alpha\beta} - p^\alpha p^\beta) \Pi^R_S(p^2) \Delta^{(0)}_{\beta\nu}(p) \,. \tag{3.126}$$

For the static charge $j^\nu(y) = Q\delta^\nu{}_0 \delta^3(\mathbf{y} - \mathbf{x}_s)$, the terms proportional to $p_\mu p_\nu$ in the above vanish in the calculation of the potential, which allows one to write

$$\begin{aligned} A^{(1)}_\mu(x) &= -iQ \int d^4y \int \frac{d^4p}{(2\pi)^4} \Delta^{(0)}_{\mu\nu}(p) [1 + \Pi^R_S(p^2)] 2\pi \delta^\nu{}_0 \delta^3(\vec{\mathbf{y}} - \vec{\mathbf{x}}_s) e^{-ip\cdot(x-y)} \\ &= -iQ \int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3} \Delta^{(0)}_{\mu 0}(0, \vec{\mathbf{p}}) [1 + \Pi^R_S(-\vec{\mathbf{p}}^2)] e^{i\vec{\mathbf{p}}\cdot(\vec{\mathbf{x}} - \vec{\mathbf{x}}_s)} \,. \end{aligned} \tag{3.127}$$

Writting the explicit form of the propagator and the polarization tensor, we have

$$A^{(1)}_0(\vec{\mathbf{x}}) = -Q \int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3} \frac{1}{-\vec{\mathbf{p}}^2} \left\{ 1 + \frac{e^2_R}{8\pi^2} \int_0^1 dx \, x(2x-1) \ln \left(1 + \frac{\vec{\mathbf{p}}^2}{m^2} x(1-x) \right) \right\} e^{i\vec{\mathbf{p}}\cdot(\vec{\mathbf{x}} - \vec{\mathbf{x}}_s)} \,,$$

so that the Fourier transform of the potential A_0 is

$$\tilde{A}^{(1)}_0(\vec{\mathbf{p}}) = \frac{Q}{\vec{\mathbf{p}}^2} \left\{ 1 + \frac{e^2_R}{8\pi^2} \int_0^1 dx \, x(2x-1) \ln \left(1 + \frac{\vec{\mathbf{p}}^2}{m^2} x(1-x) \right) \right\} e^{-i\vec{\mathbf{p}}\cdot\vec{\mathbf{x}}_s} \,. \tag{3.128}$$

To see the difference between this potential and the Uehling potential of spinor QED, we follow the procedure outlined before. First, integrating by parts the correction term in x yields

$$-\frac{e^2_R}{8\pi^2} \int_0^1 dx \, x^2 \left(\frac{2}{3} x - \frac{1}{2} \right) \frac{(1-2x)\, \vec{\mathbf{p}}^2/m^2}{1 + x(1-x)\vec{\mathbf{p}}^2/m^2} \,,$$

which, setting $v = 2x - 1$, becomes

$$\frac{e^2_R}{8\pi^2} \frac{1}{2} \frac{\vec{\mathbf{p}}^2}{4m^2} \int_{-1}^1 dv \, \frac{\frac{v^4}{3} + \frac{v^3}{2} - \frac{v}{6}}{1 + (1-v^2)\vec{\mathbf{p}}^2/4m^2} = \frac{e^2_R}{24\pi^2} \frac{\vec{\mathbf{p}}^2}{4m^2} \int_0^1 dv \, \frac{v^4}{1 + (1-v^2)\vec{\mathbf{p}}^2/4m^2} \,.$$

Using this expression in the equation for $A_0(\vec{\mathbf{x}})$, one has

$$A_0^{(1)}(\vec{\mathbf{x}}) = A_0^{(0)}(\vec{\mathbf{x}}) + Q\frac{e_R^2}{24\pi^2}\frac{1}{4m^2}\int_0^1 dv\, v^4 \int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3}\frac{e^{i\vec{\mathbf{p}}\cdot(\vec{\mathbf{x}}-\vec{\mathbf{x}}_s)}}{1+(1-v^2)\vec{\mathbf{p}}^2/4m^2}\,.$$

Integration of the momentum part yields

$$A_0^{(1)}(\vec{\mathbf{x}}) = A_0^{(0)}(\vec{\mathbf{x}}) + Q\frac{e_R^2}{24\pi^2}\frac{1}{4\pi|\vec{\mathbf{x}}-\vec{\mathbf{x}}_e|}\int_0^1 dv\,\frac{v^4}{1-v^2}e^{-\frac{2m|\vec{\mathbf{x}}-\vec{\mathbf{x}}_s|}{\sqrt{1-v^2}}}\,.$$

At last, setting $|\vec{\mathbf{x}}-\vec{\mathbf{x}}_s| \equiv r$ and changing to the variable $u = (1-v^2)^{-1/2}$, the integral term in the above can be put into the form

$$Q\frac{e_R^2}{24\pi^2}\frac{1}{4\pi r}\int_1^\infty du\,\frac{(u^2-1)^{3/2}}{u^4}e^{-2mur}\,.$$

Summarizing, for scalar QED, the 1 loop corrected potential for a static point charge Q is given by

$$A_0^{(1)}(r) = \frac{Q}{4\pi r}\left\{1+\frac{\alpha}{6\pi}\int_1^\infty du\,\frac{\sqrt{u^2-1}}{u^2}\left(1-\frac{1}{u^2}\right)e^{-2mur}\right\}\,. \tag{3.129}$$

As expected, both potentials for spinor and scalar QED are relatively similar, both being infinite superpositions of Yukawa potentials. As can be seen in the plot of figure 3.8, the scalar potential is more localized than the spinor potential, a feature which could be attributed to the exclusion principle which would delocalize the fermion cloud and increase the shielding range. To see how the charge distribution of the vacuum is altered by this correction, one could resort to Maxwells theory and recall that in the absence of time varying charge distributions, the total charge density $\rho_{tot}(\vec{\mathbf{r}}) = \rho_c(\vec{\mathbf{r}}) + \rho_{pol}(\vec{\mathbf{r}})$ can be found by

$$\boldsymbol{\nabla}\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{1}{\epsilon_0}\rho(\vec{\mathbf{r}}) = -\nabla^2 V(\vec{\mathbf{r}})\,, \tag{3.130}$$

with ϵ_0 being the classical vacuum permitivity and $V(\vec{\mathbf{r}})$ the electrostatic potential associated. For the case of the vacuum, treated as a polarizable medium, the contribution of the polarization charge density ρ_{pol}, induced by vacuum polarization, leads to

$$\rho_{pol}(\vec{\mathbf{r}}) = -\nabla^2\delta A_0^{(1)}(\vec{\mathbf{r}})\,, \tag{3.131}$$

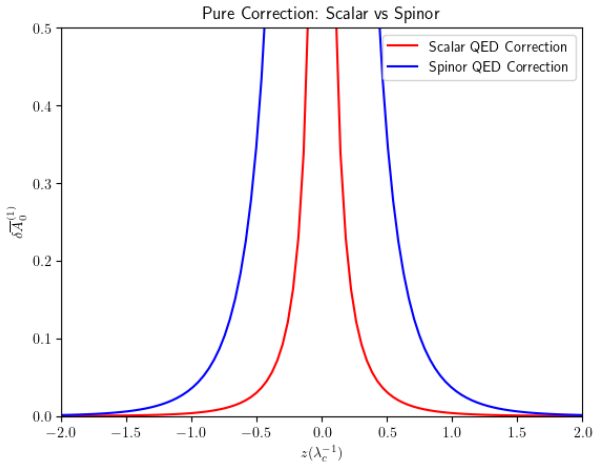


Figure 3.8: Plot of the static charge corrections found for spinor and scalar QED. Here, the terms have been normalized by $\alpha/6\pi^2$ and we set $m = 1, e_R = 1$ with $\alpha = 1/137$. This turns length dimensions into Compton wavelengths $r \sim \lambda_C/2\pi = \hbar/mc = 386$ fm.

with $\delta A_0^{(1)}(\vec{\mathbf{r}})$ referring to the pure correction term appearing in the Uehling potential. Substitution of the integral in Eq. (3.129) gives the distribution for scalar QED,

$$\rho^b_{pol}(r) \;=\; \frac{Q\alpha}{6\pi}\int_1^\infty du\left(1-\frac{1}{u^2}\right)\frac{\sqrt{u^2-1}}{u^2}\left\{\delta^3(\vec{\mathbf{r}})-\frac{m_b^2u^2}{\pi r}\right\}e^{-2m_bur}\,, \qquad (3.132)$$

Eq. (3.132) consists of a same sign charge superimposed at the position of the source and another charge distribution which is extended, and of opposite charge to the source. For comparison, in spinor QED, the same calculation yields a very similar result:

$$\rho^f_{pol}(r) = \frac{2Q\alpha}{3\pi}\int_1^\infty du\left(1+\frac{1}{2u^2}\right)\frac{\sqrt{u^2-1}}{u^2}\left\{\delta^3(\vec{\mathbf{r}})-\frac{m_f^2u^2}{\pi r}\right\}e^{-2m_fur}\,. \qquad (3.133)$$

To end this section, we make some remarks on the vacuum polarization correction described above. The Uehling potential still remains an active focus of research. Different representations and calculation schemes have been proposed [49, 48, 50, 51, 52], and higher-order corrections have been investigated [53, 54]. Generalizations of Uehling's result to modified QED theories have been analyzed [55, 56], as well as extensions to the hadronic case [57, 58] and to quantum gravity theories [59, 60, 61]. In addition, the Uehling potential has played a central role in the analysis of precise experimental tests for QED [62, 63] and in scattering phenomena [64, 65, 66, 67, 68]. This correction has

also been applied for a deep understanding of the spectra of many-electron atoms [69, 70, 71], ordinary molecules [72, 73], and muonic and pionic molecules [74, 75].

Until recently, there had been no studies concerning the influence of boundaries on the Uehling correction. However, it is well-known that the presence of objects can strongly affect the quantum vacuum. One of the most emblematic effects of this kind is the Casimir effect, which consists in its simplest form of a finite shift of the zero-point energy of the (quantized) electromagnetic field, giving rise to an attractive force between two parallel, uncharged, perfectly conducting plates [9, 10, 12, 76, 77].

Because of the possibility of new unexplored effects, we will take a small detour and undertake this investigation in the next chapter.

Chapter 4

Applications II: Vacuum Polarization near a Perfect Conductor

The topic of macroscopic structure effects is a fairly unexplored area of the perturbative expansion approach which we have discussed so far. By macroscopic structure, we mean an object which interacts with the quantum fields in such a way as to alter their propagators in some particular way. The simplest case of this, which we will discuss, is the presence of a perfectly conducting plate (a mirror) which imposes Dirichlet boundary conditions on the photon propagator. As we shall see, even this fairly simple scenario is already much more complicated than usual due to the nonlinear effects induced by the vacuum fluctuations [78]. This will allow us to discuss the influence of such simple boundary effects on the vacuum polarization correction to Coulomb's law.

Here, we will consider what happens to the Uehling potential near a perfect plate conductor which imposes Dirichlet boundary conditions on the differential equation governing the photon propagator. This is interesting because it will be comparable to an elementary calculation performed in classical Electromagnetism where one applies the so-called method of images to find the force felt by a static charge placed next to a perfect plate conductor. Before proceeding to the loop calculation, let us review the method of images in the context of the photon propagator.

4.1 Method of Images

The method of images is a well-known procedure to describe certain problems in classical electrodynamics in terms of simpler ones. The standard example is the study of a point charge placed near a perfectly conducting plane. In this case, two charges of opposite sign and placed at reflected positions with respect to the plane can be shown to yield the same electrostatic potential as the original configuration (in the semi-space where the real charge is).

In the study of Quantum Mechanics, the image method is still applicable in some situations. Specifically, it can be shown that the boundary conditions of a given problem can be used to write the desired propagator in terms of other propagators known for simpler situations. As an illustration, consider the standard case of the perfect plane positioned at $x^3 = 0$ (the $\mathcal{OXY}$ plane). Now, let us consider the photon propagator $\overline{\Delta}_{\mu\nu}(x;x')$. The Dirichlet boundary conditions imply that

$$\overline{\Delta}_{\mu\nu}(0;x') = 0\,, \tag{4.1}$$

that is, the photon is not allowed to penetrate the plane. The image method consists of rewritting this propagator in terms of the standard free photon propagators in the *absence of the plane*. Specifically, we consider

$$\overline{\Delta}_{\mu\nu}(x;x') = \Delta^{(0)}_{\mu\nu}(x-x') - \Delta^{(0)}_{\mu\nu}(x-\tilde{x}')\,, \tag{4.2}$$

where $\tilde{x}' := (x'^0, x'^1, x'^2, -x'^3)$. To see that the Dirichlet condition is indeed satisfied, we can write the explicit form of the propagators in position space. In Feynman gauge,

$$\begin{aligned}\Delta^{(0)}_{\mu\nu}(x-x') &= -i\eta_{\mu\nu}\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-x')}}{p^2+i\epsilon}\\ &= \frac{\eta_{\mu\nu}}{4\pi^2}\frac{1}{(x-x')^2}\,.\end{aligned} \tag{4.3}$$

Thus, the modified propagator under the Dirichlet boundary condition becomes

$$\overline{\Delta}_{\mu\nu}(x;x') = \eta_{\mu\nu}\frac{1}{4\pi^2}\left\{\frac{1}{(x^0-x'^0)^2-[(x^1-x'^1)^2+(x^2-x'^2)^2+(x^3-x'^3)^2]} - \frac{1}{(x^0-x'^0)^2-[(x^1-x'^1)^2+(x^2-x'^2)^2+(x^3+x'^3)^2]}\right\},$$

for which $\overline{\Delta}_{\mu\nu}(x^0,x^1,x^2,0;x'^0,x'^1,x'^2,x'^3)=0$, as required. It is noteworthy that we could also construct $\overline{\Delta}_{\mu\nu}(x;x')$ as

$$\overline{\Delta}_{\mu\nu}(x;x') = \Delta^{(0)}_{\mu\nu}(x-x') - \Delta^{(0)}_{\mu\nu}(\tilde{x}-x')\,, \tag{4.4}$$

such that now $\tilde{x}=(x^0,x^1,x^2,-x^3)$.

We can also explicitly describe these alterations by introducing the Fourier space decompositions

$$\Delta^{(0)}_{\mu\nu}(x-x') = \int\frac{d^4p}{(2\pi)^4}\Delta^{(0)}_{\mu\nu}(p)e^{-ip\cdot(x-x')}\,, \tag{4.5}$$

$$\Delta^{(0)}_{\mu\nu}(x-\tilde{x}') = \int\frac{d^4p}{(2\pi)^4}\Delta^{(0)}_{\mu\nu}(p)e^{-ip\cdot(x-\tilde{x}')}\,. \tag{4.6}$$

Since $p\cdot\tilde{x}' = p^0\tilde{x}'^0 - \mathbf{p}\cdot\tilde{\mathbf{x}}'$ and $\tilde{\mathbf{x}}' = (x',y',z') - (0,0,2z') = \mathbf{x}' - 2\mathbf{z}'$, we can write $p\cdot\tilde{x}' = p\cdot x' + 2p^3x'^3$, such that the $\Delta^{(0)}_{\mu\nu}(x-\tilde{x}')$ propagator becomes

$$\Delta^{(0)}_{\mu\nu}(x-\tilde{x}') = \int\frac{d^4p}{(2\pi)^4}\Delta^{(0)}_{\mu\nu}(p)e^{-ip\cdot(x-x')+2ip^3x'^3}\,.$$

For the second form, Eq. (4.4), we would have

$$\Delta^{(0)}_{\mu\nu}(\tilde{x}-x') = \int\frac{d^4p}{(2\pi)^4}\Delta^{(0)}_{\mu\nu}(p)e^{-ip\cdot(x-x')-2ip^3x^3}\,.$$

For the first prescription, Eq. (4.2), the total propagator $\overline{\Delta}_{\mu\nu}$ will be given by

$$\begin{aligned}\overline{\Delta}_{\mu\nu}(x;x') &= \Delta^{(0)}_{\mu\nu}(x-x') - \Delta^{(0)}_{\mu\nu}(x-\tilde{x}') \\ &= \int\frac{d^4p}{(2\pi)^4}\Delta^{(0)}_{\mu\nu}(p)\left(1-e^{2ip^3x'^3}\right)e^{-ip\cdot(x-x')} \\ &= \int\frac{d^4p}{(2\pi)^4}\Delta^{(0)}_{\mu\nu}(p)\chi(x'^3p^3)e^{-ip\cdot(x-x')}\,,\end{aligned} \tag{4.7}$$

where we have defined $\chi(\xi) := 1 - \exp(2i\xi)$. As for the second prescription, we simply make the change $x'^3 \to -x^3$ in the argument of χ,

$$\begin{aligned}
\overline{\Delta}_{\mu\nu}(x;x') &= \Delta^{(0)}_{\mu\nu}(x-x') - \Delta^{(0)}_{\mu\nu}(\tilde{x}-x') \\
&= \int \frac{d^4p}{(2\pi)^4}\Delta^{(0)}_{\mu\nu}(p)\left(1 - e^{-2ip^3x^3}\right)e^{-ip\cdot(x-x')} \\
&= \int \frac{d^4p}{(2\pi)^4}\Delta^{(0)}_{\mu\nu}(p)\chi^*(x^3p^3)e^{-ip\cdot(x-x')}\,. \qquad (4.8)
\end{aligned}$$

Thus, we see that the effect of the boundary conditions is effectively a change in the Fourier transformed $\overline{\Delta}_{\mu\nu}(p) \to \Delta^{(0)}_{\mu\nu}(p)\chi(x'^3p^3)$ or $\Delta^{(0)}_{\mu\nu}(p)\chi^*(x^3p^3)$. Furthermore, if we note that $p\cdot\widetilde{x} = \widetilde{p}\cdot x$, it is clear that

$$\int d^4p\,\Delta^{(0)}_{\mu\nu}(p)e^{-ip\cdot(x-\widetilde{x}')} = \int d^4p\Delta^{(0)}_{\mu\nu}(\widetilde{p})e^{-ip\cdot(\widetilde{x}-x')}\,. \qquad (4.9)$$

Thus, if $\Delta^{(0)}_{\mu\nu}(\widetilde{p}) = \Delta^{(0)}_{\mu\nu}(p)$, *both prescriptions are the same.* Since this is true in Feynman gauge and we will be working with this choice, we will choose the prescription that simplifies calculations.

To check the consistency of this treatment, let us re-obtain the basic result of classical electrodynamics for the electrostatic potential of the point charge near the plate. This is equivalent to considering the tree level diagram for the photon exchange of two particles, but taking the photon propagator to be modified by the presence of the plate. In this framework, we simply insert the propagator $\overline{\Delta}_{\mu\nu}(x;y)$ as the solution to Maxwell's equations [1]:

$$i\overline{A}_\mu(x) = \int d^4y\,\overline{\Delta}_{\mu\nu}(x;y)j^\nu(y) \qquad (4.10)$$

where $j^\nu(y)$ is the 4-current density. Working with the first form (4.7), we can write the above in terms of the Fourier transformed propagator:

$$\begin{aligned}
i\overline{A}_\mu(x) &= \int d^4y \int \frac{d^4p}{(2\pi)^4}\Delta^{(0)}_{\mu\nu}(p)\chi(y^3p^3)e^{-ip\cdot(x-y)}j^\nu(y) \\
&= \int \frac{d^4p}{(2\pi)^4}\Delta^{(0)}_{\mu\nu}(p)\tilde{J}^\nu(p)e^{-ip\cdot x}\,, \qquad (4.11)
\end{aligned}$$

[1] The i factor is needed for this specific form of the propagator, otherwise the potential would be imaginary.

where $\tilde{J}^\nu$ is the Fourier transform of the current density with the added $\chi(y^3p^3)$ factor. For a static point charge q placed at $\mathbf{x}_q$, we have $j^\nu(y) = q\delta^\nu_0\delta^3(\mathbf{y}-\mathbf{x}_q)$, so that

$$\widetilde{J}^\nu(p) = \int d^4y\, q\delta^\nu_0\delta^3(\mathbf{y}-\mathbf{x}_q)\chi(y^3p^3)e^{ip\cdot y} = 2\pi q\delta(p^0)\delta^\nu_0\chi(x^3_qp^3)e^{-i\mathbf{p}\cdot\mathbf{x}_q}\,. \tag{4.12}$$

Thus,

$$\overline{A}_\mu(x) \;=\; -iq\int\frac{d^3p}{(2\pi)^3}\Delta^{(0)}_{\mu0}(p^0=0,\mathbf{p})\chi(x^3_qp^3)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_q)}\,. \tag{4.13}$$

Inserting the explicit form of the propagator into the above, we find

$$\begin{aligned} A_\mu(x) &= q\int\frac{d^3p}{(2\pi)^3}\frac{\eta_{\mu0}}{\mathbf{p}^2}\chi(x^3_qp^3)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_q)} \\ &= q\eta_{\mu0}\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_q)}-e^{i\mathbf{p}\cdot(\mathbf{x}-\widetilde{\mathbf{x}}_q)}}{\mathbf{p}^2} \\ &= \eta_{\mu0}\frac{1}{4\pi}\left(\frac{q}{|\mathbf{x}-\mathbf{x}_q|}-\frac{q}{|\mathbf{x}-\widetilde{\mathbf{x}}_q|}\right), \end{aligned} \tag{4.14}$$

where, for $\mathbf{x}_q = (x_q, y_q, z_q)$ (with $z_q > 0$), we have $\widetilde{\mathbf{x}}_q = (x_q, y_q, -z_q)$. Since $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, we conclude that the zeroth component $A_0(x) = \phi(\mathbf{x})$ is the standard electrostatic potential. Thus, the prescription for the propagator is able to reproduce the classical result accurately. It is worth mentioning that the alternative prescription given by (4.4) gives the same result:

$$\begin{aligned} \overline{A}_\mu(x) &= -i\int\frac{d^4p}{(2\pi)^4}\Delta^{(0)}_{\mu\nu}(p)2\pi q\delta(p^0)\delta^\nu{}_0\chi^*(x^3p^3)e^{-i\mathbf{p}\cdot\mathbf{x}_q}e^{-ip\cdot x}\,, \\ &= -iq\int\frac{d^3\mathbf{p}}{(2\pi)^3}\Delta^{(0)}_{\mu0}(0,\mathbf{p})\chi^*(x^3p^3)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_q)}\,, \\ &= q\eta_{\mu0}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1-e^{-2ip^3x^3}}{\mathbf{p}^2}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_q)}\,, \\ &= \frac{q\eta_{\mu0}}{4\pi}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_q|}-\frac{1}{|\widetilde{\mathbf{x}}-\mathbf{x}_q|}\right), \end{aligned} \tag{4.15}$$

where $\widetilde{\mathbf{x}} = (x^1, x^2, -x^3)$. Because of the modulus, it is clear that $|\mathbf{x}-\widetilde{\mathbf{x}}_q| = |\widetilde{\mathbf{x}}-\mathbf{x}_q|$, which shows the validity of both prescriptions in this case.

4.2 Boundary Modified Loop Correction

Inspired by the results of last subsection, we can attempt to apply the image method to the loop corrected propagator. Pragmatically speaking, this could allow us to investigate the influence of a perfect plate conductor on the Uehling potential of QED.

According to the image method, the prescription for the Dirichlet boundary conditions would be given by equation (4.2). Since we are now considering the effect of the first loop correction, the corresponding diagrammatic representation will be (red lines are the modified photon propagator)

$$\overline{\Delta}^{(1)}_{\mu\nu}(p) \;=\; \mu \;\overset{p}{\sim}\; \nu \;+\; \mu \;\overset{p}{\sim}\; \overset{k}{\underset{k-p}{\circlearrowleft}} \;\overset{p}{\sim}\; \nu\,. \tag{4.16}$$

Considering the proper renormalization of the loop (by adding the counter term contribution which subtracts the divergent term), the above diagram can be most easily expressed as

$$\overline{\Delta}^{(1)}_{\mu\nu}(p) \;=\; \overline{\Delta}^{(0)}_{\mu\nu} + i\overline{\Delta}^{(0)}_{\mu\rho}\Pi^{\rho\sigma}_{R}\overline{\Delta}^{(0)}_{\sigma\nu}\,, \tag{4.17}$$

where $\Pi_R(p)$ is given by equation (3.105). For the case of the perfect plate positioned in the $\mathcal{OXY}$ plane, the modified propagator $\overline{\Delta}^{(0)}_{\mu\nu}(x;x')$ will still be given by (4.2). Thus, as before, the momentum space $\overline{\Delta}^{(0)}_{\mu\nu}(p)$ will obtain the factor $\chi(x'^3p^3) = 1 - \exp\big(2ip^3x'^3\big)$ and we can write each modified free propagator in terms of the free vacuum propagators as

$$\overline{\Delta}^{(0)}_{\mu\nu}(p) = \Delta^{(0)}_{\mu\nu}(p)\chi(x'^3p^3)\,. \tag{4.18}$$

The correction to the free propagator $\overline{\Delta}^{(0)}_{\mu\nu}$ must be treated with care. One way to analyze its structure is by going back to position space. Without the Dirichlet condition,

the matrix element which contributes to the loop correction is

$$\begin{aligned} i\left\langle A_\mu(x)A_\nu(y)\right\rangle^{(1)} &= ie^2\int d^4x_1\int d^4x_2\,\Delta^{(0)}_{\mu\alpha}(x-x_1)\gamma^\alpha G_F(x_1-x_2)\gamma^\beta G_F(x_2-x_1)\Delta^{(0)}_{\beta\nu}(x_2-y) \\ &= \int d^4x_1\int d^4x_2\,\Delta^{(0)}_{\mu\alpha}(x-x_1)i\Pi^{\alpha\beta}(x_1,x_2)\Delta^{(0)}_{\beta\nu}(x_2-y)\,, \end{aligned} \tag{4.19}$$

where G_F is the fermion propagator and $\Pi^{\alpha\beta}$ represents the whole loop part. If we impose that $\langle A_\mu(x)A_\nu(y)\rangle$ vanish at $x^3=0$, we would satisfy the Dirichlet boundary condition for the full propagator (the tree level term plus the loop correction). We can achieve this by simply replacing the first free photon propagator of expression (4.19) with its modified counterpart $\overline{\Delta}^{(0)}_{\mu\alpha}(x;x_1)$. This, however, would not make sense, since the other photon is also sensitive to the boundary conditions. To properly satisfy the Dirichlet condition for both endpoints ($x^3=0$ and $y^3=0$) we impose that both propagators appearing in equation (4.19) should be modified as depicted in (4.16).

Furthermore, we must restrict the integration domain to the region bounded by the plate, namely $\mathfrak{D}:=\mathbb{R}^{2,1}\times\mathbb{R}_+$, where $\mathbb{R}_+=[0,+\infty[$. In this case, the expression corresponding to the substitution of both photon propagators is

$$\begin{aligned} i\left\langle A_\mu(x)A_\nu(y)\right\rangle^{(1)} &= ie^2\int_{\mathfrak{D}} d^4x_1\int_{\mathfrak{D}} d^4x_2\,\overline{\Delta}^{(0)}_{\mu\alpha}(x;x_1)\gamma^\alpha G_F(x_1-x_2)\gamma^\beta G_F(x_2-x_1)\overline{\Delta}^{(0)}_{\beta\nu}(x_2;y) \\ &= \int_{\mathfrak{D}} d^4x_1\int_{\mathfrak{D}} d^4x_2\,\overline{\Delta}^{(0)}_{\mu\alpha}(x;x_1)i\Pi^{\alpha\beta}(x_1,x_2)\overline{\Delta}^{(0)}_{\beta\nu}(x_2;y)\,, \end{aligned} \tag{4.20}$$

and the relevant propagators in momentum space are

$$\overline{\Delta}^{(0)}_{\mu\alpha}(x;x_1) = \Delta^{(0)}_{\mu\alpha}(x-x_1)-\Delta^{(0)}_{\mu\alpha}(\tilde{x}-x_1)=\int\frac{d^4p}{(2\pi)^4}\Delta_{\mu\alpha}(p)e^{-ip\cdot(x-x_1)}\left[1-e^{-2ip^3x^3}\right] \tag{4.21}$$

$$\overline{\Delta}^{(0)}_{\beta\nu}(x_2;y) = \Delta^{(0)}_{\beta\nu}(x_2-y)-\Delta^{(0)}_{\beta\nu}(x_2-\tilde{y})=\int\frac{d^4p}{(2\pi)^4}\Delta^{(0)}_{\mu\alpha}(p)e^{-ip\cdot(x_2-y)}\left[1-e^{2ip^3y^3}\right] \tag{4.22}$$

where again $\tilde{x}^\mu=(x^0,x^1,x^2,-x^3)$ and $\tilde{y}^\mu=(y^0,y^1,y^2,-y^3)$. Note that we have used prescription (4.8) for the first propagator and (4.7) for the second. This is useful because

it allows us to factorize the boundary integration and the effect of the modification in the propagators as will be seen below.

Following the prescription above, the position space expression can be written in terms of the momentum space representations as usual:

$$\begin{aligned} i\left\langle A_\mu(x)A_\nu(y)\right\rangle^{(1)} &= \int\left(\prod_{i=1}^{4}\frac{d^4p_i}{(2\pi)^4}\right)\Delta^{(0)}_{\mu\alpha}(p_1)i\Pi^{\alpha\beta}(p_2,p_3)\Delta^{(0)}_{\beta\nu}(p_4) \\ &\quad\times e^{-ip_1\cdot x}\left[1-e^{-2ip_1^3x^3}\right]e^{ip_4\cdot y}\left[1-e^{2ip_4^3y^3}\right] \\ &\quad\times\int_{\mathfrak{D}}d^4x_1\int_{\mathfrak{D}}d^4x_2\,e^{ix_1\cdot(p_1-p_2+p_3)}e^{ix_2\cdot(p_2-p_3-p_4)}\,, \end{aligned} \tag{4.23}$$

The last two integrals can be decomposed into components perpendicular to the $\mathcal{OXY}$ plane and the parallel components:

$$\begin{aligned} \int_{\mathbb{R}^{2,1}\times\mathbb{R}^+}d^4x_2\,e^{ix_2\cdot(p_2-p_3-p_4)} &= \int_{-\infty}^{\infty}dx_2^0\,e^{ix_2^0(p_2^0-p_3^0-p_4^0)} \\ &\quad\times\int_{\mathbb{R}^2}d\mathbf{x}_{2\|}\,e^{i\mathbf{x}_{2\|}\cdot(\mathbf{p}_{4\|}+\mathbf{p}_{3\|}-\mathbf{p}_{2\|})}\int_0^{\infty}dx_2^3\,e^{ix_2^3(p_4^3+p_3^3-p_2^3)} \\ &= (2\pi)^3\delta(p_2^0-p_3^0-p_4^0)\delta^2(\mathbf{p}_{4\|}+\mathbf{p}_{3\|}-\mathbf{p}_{2\|}) \\ &\quad\times\left[\pi\delta(p_4^3+p_3^3-p_2^3)+i\mathcal{P}\left(\frac{1}{p_4^3+p_3^3-p_2^3}\right)\right]\,, \end{aligned} \tag{4.24}$$

where the Sokhotski-Plemejl theorem has been used (see appendix B.3.1) in the last equality and $\mathcal{P}$ denotes the Cauchy principal value. The x_1 integral follows an identical treatment and one would write

$$\begin{aligned} \int_{\mathbb{R}^{2,1}\times\mathbb{R}^+}d^4x_1\,d^4x_2\,e^{ix_1\cdot(p_1-p_2+p_3)}e^{ix_2\cdot(p_2-p_3-p_4)} &= (2\pi)^8\delta(p_1^0-p_2^0+p_3^0)\delta(p_2^0-p_3^0-p_4^0) \\ &\quad\times\delta^2(\mathbf{p}_{1\|}+\mathbf{p}_{3\|}-\mathbf{p}_{2\|})\delta^2(\mathbf{p}_{4\|}+\mathbf{p}_{3\|}-\mathbf{p}_{2\|}) \\ &\quad\times\left[\frac{1}{2}\delta(p_1^3+p_3^3-p_2^3)+\frac{i}{2\pi}\mathcal{P}\left(\frac{1}{p_2^3-p_3^3-p_1^3}\right)\right] \\ &\quad\times\left[\frac{1}{2}\delta(p_4^3+p_3^3-p_2^3)+\frac{i}{2\pi}\mathcal{P}\left(\frac{1}{p_4^3+p_3^3-p_2^3}\right)\right]\,. \end{aligned} \tag{4.25}$$

This contributes four terms to the integration, which we will name $a_{\mu\nu}, b_{\mu\nu}, c_{\mu\nu}, d_{\mu\nu}$. The first, $a_{\mu\nu}$, comprised of the first terms in each of the square brackets above, can be fully

integrated, since the product of deltas is just $\delta^4(p_1 - p_2 + p_3)\delta^4(p_2 - p_3 - p_4)$. Once integrated with the rest of the terms containing the propagators and phase factors, we recover a term which is identical to considering the full x_1^3, x_2^3 range of integrations, except by a factor of 1/4. This term, in fact, resembles the structure of the naive image method, thanks to the two χ factors with opposite phase signs.

The second term $b_{\mu\nu}$ is obtained by integrating after distributing the product in the square brackets above, and is symmetric by relabeling the particle indices with the third term $c_{\mu\nu}$. Each of them contains one of the Cauchy principal value parts. For $b_{\mu\nu}$, the momentum deltas containing $p_1 + p_3 - p_2$ can be assembled into a $\delta^4(p_1 + p_3 - p_2)$. By taking $p_2 = k$, $p_2 - p_3 = p$, one can write this contribution as

$$\begin{aligned}
b_{\mu\nu} &= \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \Delta^{(0)}_{\mu\alpha}(p_1)\chi^*_{p_1}(x^3) i\Pi^{\alpha\beta}(k, k-p_1) e^{-ip_1\cdot x} \\
&\quad \times \frac{i}{4\pi}\mathcal{P}\int d^4p_4\, \Delta^{(0)}_{\beta\nu}(p_4)\chi_{p_4}(y^3) e^{ip_4\cdot y} \frac{\delta^2(\mathbf{p}_{1\parallel} - \mathbf{p}_{4\parallel})\delta(p_1^0 - p_4^0)}{p_4^3 - p_1^3} \\
&= \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \Delta^{(0)}_{\mu\alpha}(p_1)\chi^*_{p_1}(x^3) i\Pi^{\alpha\beta}(k, k-p_1) e^{-ip_1\cdot x} \\
&\quad \times \frac{i}{4\pi}(-i\eta_{\beta\nu}) e^{i(p_1^0 y^0 - \mathbf{p}_{1\parallel}\cdot\mathbf{y}_\parallel)} \mathcal{P}\int_{-\infty}^{\infty} dp_4^3 \, \frac{e^{ip_4^3 y^3} - e^{-ip_4^3 y^3}}{\left[(p_4^3)^2 - \left((p_1^0)^2 - \mathbf{p}_{1\parallel}^2\right) - i\epsilon\right][p_4^3 - p_1^3]} \\
&= \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \Delta^{(0)}_{\mu\alpha}(p_1)\chi^*_{p_1}(x^3) i\Pi^{\alpha\beta}(k, k-p_1) e^{-ip_1\cdot x} \\
&\quad \times \frac{i}{2\pi}\eta_{\beta\nu} e^{i(p_1^0 y^0 - \mathbf{p}_{1\parallel}\cdot\mathbf{y}_\parallel)} \mathcal{P}\int_{-\infty}^{\infty} d\xi \, \frac{\sin(\xi y^3)}{(\xi^2 - \omega_{1\parallel}^2 - i\epsilon)(\xi - p_1^3)} \,,
\end{aligned} \tag{4.26}$$

where we have relabeled $\xi = p_4^3$ and defined

$$\omega_{1\parallel}^2 := \left((p_1^0)^2 - \mathbf{p}_{1\parallel}^2\right) . \tag{4.27}$$

The ξ integral above can be calculated (see appendix B.3.2) and the result is

$$\mathcal{P}\int_{-\infty}^{\infty} d\xi \, \frac{\sin(\xi y^3)}{(\xi^2 - \omega_{1\parallel}^2 - i\epsilon)(\xi - p_1^3)} = \frac{\pi}{\omega_{1\parallel}^2 - (p_1^3)^2}\left(e^{i\omega_{1\parallel} y^3} - \cos\left(p_1^3 y^3\right)\right) . \tag{4.28}$$

Inserting the above into equation (4.26), one finds

$$\begin{aligned} b_{\mu\nu} &= \int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4k}{(2\pi)^4}\Delta^{(0)}_{\mu\alpha}(p_1)\chi^*_{p_1}(x^3)i\Pi^{\alpha\beta}(k,k-p_1)e^{-ip_1\cdot x} \\ &\quad\times\frac{i}{2\pi}\eta_{\beta\nu}e^{i(p_1^0y^0-\mathbf{p}_{1\parallel}\cdot\mathbf{y}_\parallel)}\frac{\pi}{\omega_{1\parallel}^2-(p_1^3)^2}\left(e^{i\omega_{1\parallel}y^3}-\cos\!\left(p_1^3y^3\right)\right). \end{aligned} \tag{4.29}$$

Since the only term in the integrand that depends in k is the polarization tensor $\Pi^{\alpha\beta}$, one can interchange the order of integration and write

$$\begin{aligned} b_{\mu\nu} &= \int \frac{d^4p}{(2\pi)^4}\chi^*_p(x^3)\Delta^{(0)}_{\mu\alpha}(p)i\Pi^{\alpha\beta}(p)\frac{i\eta_{\beta\nu}}{2}\frac{1}{\omega_\parallel^2-(p^3)^2}\left(e^{i\omega_\parallel y^3}-\cos\!\left(p^3y^3\right)\right)e^{ip^3y^3}e^{-ip\cdot(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4}\chi^*_p(x^3)\Delta^{(0)}_{\mu\alpha}(p)i\Pi^{\alpha\beta}(p)B_{\beta\nu}(p;y^3)e^{-ip\cdot(x-y)}, \end{aligned} \tag{4.30}$$

where we have relabeled $p_1\to p$ and defined

$$B_{\beta\nu}(p;y^3) := \frac{i\eta_{\beta\nu}}{2}\frac{1}{\omega_\parallel^2-(p^3)^2}\left(e^{i\omega_\parallel y^3}-\cos\!\left(p^3y^3\right)\right)e^{ip^3y^3}. \tag{4.31}$$

In a completely analogous way, one can find for the third contribution $c_{\mu\nu}$:

$$\begin{aligned} c_{\mu\nu} &= \int \frac{d^4p_4}{(2\pi)^4}\int \frac{d^4k}{(2\pi)^4}\Delta^{(0)}_{\beta\nu}(p_4)\chi_{p_4}(y^3)i\Pi^{\alpha\beta}(k,k-p_4)e^{ip_4\cdot y} \\ &\quad\times\frac{i}{4\pi}\mathcal{P}\int d^4p_1\,\Delta^{(0)}_{\mu\alpha}(p_1)\chi^*_{p_1}(x^3)e^{-ip_1\cdot x}\frac{\delta^2(\mathbf{p}_{4\parallel}-\mathbf{p}_{1\parallel})\delta(p_4^0-p_1^0)}{p_4^3-p_1^3} \\ &= \int \frac{d^4p_4}{(2\pi)^4}\int \frac{d^4k}{(2\pi)^4}\Delta^{(0)}_{\beta\nu}(p_4)\chi_{p_4}(y^3)i\Pi^{\alpha\beta}(k,k-p_4)e^{ip_4\cdot y} \\ &\quad\times\frac{i}{2\pi}\eta_{\mu\alpha}e^{-i(p_4^0x^0-\mathbf{p}_{4\parallel}\cdot\mathbf{x}_\parallel)}\mathcal{P}\int_{-\infty}^{\infty}d\xi\,\frac{\sin(\xi x^3)}{(\xi^2-\omega_{4\parallel}^2-i\epsilon)\,(\xi-p_4^3)}. \end{aligned} \tag{4.32}$$

Using the result for the integral, $c_{\mu\nu}$ can be written as

$$\begin{aligned} c_{\mu\nu} &= \int \frac{d^4p}{(2\pi)^4}\chi_p(y^3)\Delta^{(0)}_{\beta\nu}(p)i\Pi^{\alpha\beta}(p)\frac{i\eta_{\mu\alpha}}{2}\frac{1}{\omega_\parallel^2-(p^3)^2}\left(e^{i\omega_\parallel x^3}-\cos\!\left(p^3x^3\right)\right)e^{-ip^3x^3}e^{-ip\cdot(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4}C_{\mu\alpha}(p;x^3)i\Pi^{\alpha\beta}(p)\Delta^{(0)}_{\beta\nu}(p)\chi_p(y)e^{-ip\cdot(x-y)}, \end{aligned} \tag{4.33}$$

with

$$C_{\mu\alpha}(p;x) := \frac{i\eta_{\mu\alpha}}{2}\frac{1}{\omega_\parallel^2-(p^3)^2}\left(e^{i\omega_\parallel x^3}-\cos\!\left(p^3x^3\right)\right)e^{-ix^3p^3}. \tag{4.34}$$

The fourth contribution $d_{\mu\nu}$ can be written in a similar way, but with the added complication that the polarization tensor $\Pi^{\alpha\beta}$ will appear nested inside the third integral. It is given by

$$\begin{aligned} d_{\mu\nu} &= \int \frac{d^4p_1}{(2\pi)^4}\chi^*_{p_1}(x^3)\Delta^{(0)}_{\mu\alpha}(p_1)e^{-i[p_1^0(x^0-y^0)-\mathbf{p}_1^{\parallel}\cdot(\mathbf{x}^{\parallel}-\mathbf{y}^{\parallel})]} \\ &\quad\times e^{ip_1^3x^3}\int_{-\infty}^{\infty}dp_4^3\,\chi_{p_4^3}(y^3)\Delta^{(0)}_{\beta\nu}(p_4^3,p_1^0,\mathbf{p}_1^{\parallel})e^{-ip_4^3y^3} \\ &\quad\times\frac{1}{4\pi^2}\int\frac{d^4k}{(2\pi)^4}\int_{-\infty}^{\infty}dp^3\,\frac{i\Pi^{\alpha\beta}(k;k^0-p_1^0,\mathbf{k}^{\parallel}-\mathbf{p}_1^{\parallel},k^3-p^3)}{(p^3-p_1^3)(p^3-p_4^3)}\,. \end{aligned} \tag{4.35}$$

Let us recall that the standard form of the polarization tensor (3.56) is obtained from $\Pi^{\alpha\beta}(k,k-p)$ as

$$i\Pi^{\alpha\beta}(p)=\int\frac{d^4k}{(2\pi)^4}i\Pi^{\alpha\beta}(k,k-p)=e^4\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^\alpha\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\gamma^\beta\frac{i[(\not{k}-\not{p})+m]}{(k-p)^2-m^2+i\epsilon}\right]. \tag{4.36}$$

Since the k integration is effectively absorbed into the definition of $\Pi^{\alpha\beta}(p)$, we can exchange the order of integration in the above and use the shorthand notation for $\Pi^{\alpha\beta}(p_1^0,\mathbf{p}_1^{\parallel},p^{\perp})$ to keep explicit the momentum variables which are being integrated. Performing these manipulations in $d_{\mu\nu}$, it can be written as

$$\begin{aligned} d_{\mu\nu} &= \int \frac{d^4p_1}{(2\pi)^4}\chi^*_{p_1}(x^3)\Delta^{(0)}_{\mu\alpha}(p_1)e^{-i[p_1^0(x^0-y^0)-\mathbf{p}_1^{\parallel}\cdot(\mathbf{x}^{\parallel}-\mathbf{y}^{\parallel})]} \\ &\quad\times e^{ip_1^3x^3}\int_{-\infty}^{\infty}dp_4^3\,\chi_{p_4^3}(y^3)\Delta^{(0)}_{\beta\nu}(p_4^3,p_1^0,\mathbf{p}_1^{\parallel})e^{-ip_4^3y^3} \\ &\quad\times\frac{1}{4\pi^2}\int_{-\infty}^{\infty}dp^3\,\frac{i\Pi^{\alpha\beta}(p_1^0,\mathbf{p}_1^{\parallel},p^3)}{(p^3-p_1^3)(p^3-p_4^3)}\,. \end{aligned} \tag{4.37}$$

To make contact with the calculations already executed, we can postpone the treatment of the polarization tensor and extract the dp_1^3 integration from the first measure. This

allows us to rewrite the above as

$$\begin{aligned} d_{\mu\nu} &= \int_{\mathbb{R}^{2,1}} \frac{dp_1^0\, d^2p_1^{\parallel}}{(2\pi)^6} e^{-i\overline{\mathbf{p}}_1\cdot(\overline{\mathbf{x}}-\overline{\mathbf{y}})} \int_{-\infty}^{\infty} dp^3\, i\Pi^{\alpha\beta}(\overline{\mathbf{p}}_1, p^3) \\ &\quad \times\mathcal{P}\int_{-\infty}^{\infty} dp_1^3\, \frac{\Delta^{(0)}_{\mu\alpha}(\overline{\mathbf{p}}_1, p_1^3)}{p^3 - p_1^3} \chi^*_{p_1}(x^3) e^{ip_1^3 x^3} \\ &\quad \times\mathcal{P}\int_{-\infty}^{\infty} dp_4^3\, \frac{\Delta^{(0)}_{\beta\nu}(\overline{\mathbf{p}}_1, p_4^3)}{p^3 - p_4^3} \chi_{p_4^3}(y^3) e^{-ip_4^3 y^3}\,. \end{aligned}$$

Here, $\overline{\mathbf{p}} := (p^0, \mathbf{p}^{\parallel})$ is the three vector with all of the components of the four vector p^μ except p^3, such that $\overline{\mathbf{p}}^2 := (p^0)^2 - (\mathbf{p}^{\parallel})^2$ and, thus, $|\overline{\mathbf{p}}| = \sqrt{(p^0)^2 - (\mathbf{p}^{\parallel})^2} = \omega_{p\parallel}$. The last integral is the same as the one appearing in $b_{\mu\nu}$:

$$\begin{aligned} \mathcal{P}\int_{-\infty}^{\infty} dp_4^3\, \frac{\Delta^{(0)}_{\beta\nu}(\overline{\mathbf{p}}_1, p_4^3)}{p^3 - p_4^3} \chi_{p_4}(y^3) e^{-ip_4^3 y^3} &= -2\eta_{\beta\nu}\mathcal{P}\int_{-\infty}^{\infty} d\xi\, \frac{\sin(\xi y^3)}{(\xi^2 - \overline{\mathbf{p}}_1^2 - i\epsilon)(\xi - p^3)}\,, \\ &= -\frac{2\pi\eta_{\beta\nu}}{\omega_{1\parallel}^2 - (p^3)^2}\left(e^{i\omega_{1\parallel} y^3} - \cos\big(p^3 y^3\big)\right)\,. \end{aligned} \tag{4.38}$$

The middle integral only acts on p_1^3 and will be explicitly given by:

$$\mathcal{P}\int_{-\infty}^{\infty} dp_1^3\, \frac{\Delta^{(0)}_{\mu\alpha}(\overline{\mathbf{p}}_1, p_1^3)}{p^3 - p_1^3} \chi^*_{p_1}(x^3) e^{ip_1^3 x^3} = 2\eta_{\mu\alpha}\mathcal{P}\int_{-\infty}^{\infty} d\xi\, \frac{\sin(\xi x^3)}{(\xi - p^3)(\xi^2 - \overline{\mathbf{p}}_1^2 - i\epsilon)}\,.$$

This integral is also the same one calculated for $c_{\mu\nu}$ and we find:

$$\mathcal{P}\int_{-\infty}^{\infty} dp_1^3\, \frac{\Delta^{(0)}_{\mu\alpha}(\overline{\mathbf{p}}_1, p_1^3)}{p^3 - p_1^3} \chi^*_{p_1}(x^3) e^{ip_1^3 x^3} = \frac{2\pi\eta_{\mu\alpha}}{\omega_{1\parallel}^2 - (p^3)^2}\left(e^{i\omega_{1\parallel} x^3} - \cos\big(p^3 x^3\big)\right)\,. \tag{4.39}$$

Using Eqs.(4.39,4.38) in Eq.(4.37), one obtains for $d_{\mu\nu}$:

$$d_{\mu\nu} = \int_{\mathbb{R}^{2,1}} \frac{dp_1^0\, d^2p_1^{\parallel}}{(2\pi)^3} \int_{\mathbb{R}} \frac{dp^3}{2\pi} e^{-i\overline{\mathbf{p}}_1\cdot(\overline{\mathbf{x}}-\overline{\mathbf{y}})} D_{\mu\alpha\beta\nu}(\overline{\mathbf{p}}_1, p^3; x^3, y^3) i\Pi^{\alpha\beta}(\overline{\mathbf{p}}_1, p^3) e^{ip^3(x^3 - y^3)}\,, \tag{4.40}$$

where

$$\begin{aligned} D_{\mu\alpha\beta\nu}(\overline{\mathbf{p}}_1, p^3; x^3, y^3) := -\frac{\eta_{\mu\alpha}\eta_{\beta\nu}}{(\omega_{1\parallel}^2 - (p^3)^2)^2} \Big\{ & e^{i\omega_{1\parallel}(x^3 + y^3)} + \cos\big(p^3 x^3\big)\cos\big(p^3 y^3\big) \\ & - e^{i\omega_{1\parallel} y^3}\cos\big(p^3 x^3\big) - e^{i\omega_{1\parallel} x^3}\cos\big(p^3 y^3\big)\Big\} e^{-ip^3(x^3 - y^3)} \end{aligned} \tag{4.41}$$

Relabeling $p_1^0, p_1^{\parallel}$ to $p^0, p^{\parallel}$, we can omit all of the labels referring to the momentum p_1 and simply write

$$d_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} D_{\mu\alpha\beta\nu}(p; x^3, y^3) i\Pi^{\alpha\beta}(p) e^{-ip\cdot(x-y)}\,. \tag{4.42}$$

With Eqs.(4.30,4.33,4.42), the first term with all of the deltas integrated (corresponding to $a_{\mu\nu}$) and the tree level modified propagator, we can write the Fourier decomposition of the 1 loop corrected two point function as

$$i\left\langle A_\mu(x)A_\nu(y)\right\rangle^{(1)} = \int \frac{d^4p}{(2\pi)^4}\overline{\Delta}^{(1)}_{\mu\nu}(p;x^3,y^3)e^{-ip\cdot(x-y)}\,, \tag{4.43}$$

with

$$\begin{aligned}\overline{\Delta}^{(1)}_{\mu\nu}(p;x^3,y^3) &= \overline{\Delta}^{(0)}_{\mu\nu}(p) + i\Pi^{\alpha\beta}(p)\left\{\frac{1}{4}\chi^*_p(x^3)\Delta^{(0)}_{\mu\alpha}(p)\Delta^{(0)}_{\beta\nu}(p)\chi_p(y^3) + D_{\mu\alpha\beta\nu}(p;x^3,y^3)\right.\\ &\left.+\chi^*_p(x^3)\Delta^{(0)}_{\mu\alpha}(p)B_{\beta\nu}(p;y) + C_{\mu\alpha}(p;x)\Delta^{(0)}_{\beta\nu}(p)\chi_p(y^3)\right\}. \end{aligned}\tag{4.44}$$

Note that, while we have explicitly written the above with respect to the z axis components x^3, y^3, one can easily generalize this for any plate positioning by simply exchanging the 3 superscript with a $\perp$ superscript to identify the perpendicular components to the plate.

Furthermore, since we have been able to maintain the structure of the polarization tensor $\Pi^{\alpha\beta}$, we make the assumption that the proper renormalization can still be carried out in the same way as the case without the plate. Pragmatically speaking, we substitute $e \to e_R$ and $\Pi^{\mu\nu} \to \Pi^{\mu\nu}_R(p) = (\eta^{\mu\nu}p^2 - p^\mu p^\nu)\Pi_R(p^2)$ where the renormalized scalar $\Pi_R(p^2)$ is still given by (3.105). With this in mind, the finite, renormalized version of 4.44 becomes

$$\begin{aligned}\overline{\Delta}^{(1)}_{\mu\nu}(p|x^\perp,y^\perp) &= \overline{\Delta}^{(0)}_{\mu\nu}(p) + i\Pi^{\alpha\beta}_R(p)\left\{\frac{1}{4}\chi^*_p(x^\perp)\Delta^{(0)}_{\mu\alpha}(p)\Delta^{(0)}_{\beta\nu}(p)\chi_p(y^\perp) + D_{\mu\alpha\beta\nu}(p;x^\perp,y^\perp)\right.\\ &\left.+\chi^*_p(x^\perp)\Delta^{(0)}_{\mu\alpha}(p)B_{\beta\nu}(p^\perp;y^\perp) + C_{\mu\alpha}(p^\perp;x^\perp)\Delta^{(0)}_{\beta\nu}(p)\chi_p(y^\perp)\right\}. \end{aligned}\tag{4.45}$$

4.3 Influence of the Perfect Conducting Plate on the Potential

Now, let us consider the contribution of the modified propagator given by (4.45) to the electrostatic potential generated by a static charge q. For a general current $j^\nu(y)$, one has

$$iA_\mu(x) = \int_{\mathbb{R}^{2,1}\times\mathbb{R}_+} d^4y\,\overline{\Delta}^{(1)}_{\mu\nu}(x;y)j^\nu(y)\,,$$

with $j^\nu(y) = q\delta^\nu{}_0\delta^3(\mathbf{y}-\mathbf{x}_s)$.

The first two terms in the expression for the potential will come from the respective first two terms of the modified propagator $\overline{\Delta}^{(1)}_{\mu\nu}$. However, the first of these is just the tree-level contribution from the image method and the second is exactly the same as the naive expression for the loop potential under the image method prescription. The new contributions come from the last three terms in equation (4.45), and are all dependent on the parameter $\omega_\parallel = \sqrt{(p^0)^2 - \mathbf{p}_\parallel^2}$. For the term proportional to $B_{\mu\alpha} = \eta_{\mu\alpha}B(p;y)$, one has, in Feynman gauge,

$$\eta_{\mu 0} q \int d^4y \, \frac{d^4p}{(2\pi)^4}\delta^3(\mathbf{y}-\mathbf{x}_s)\Pi_R(p^2)\chi_p^*(x^\perp)B(p;y^\perp)e^{-ip\cdot(x-y)} \, . \tag{4.46}$$

Performing the y^0 integral, one obtains $2\pi\delta(p^0)$, which allows the integration of the p^0 component:

$$\eta_{\mu 0} q \int d^3\mathbf{y} \int \frac{d^3\mathbf{p}}{(2\pi)^3}\delta^3(\mathbf{y}-\mathbf{x}_s)\Pi_R(-\mathbf{p}^2)\chi_p^*(x^\perp)B(0,\mathbf{p};y^\perp)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \, .$$

Integration of the remaining delta yields

$$\eta_{\mu 0} q \int \frac{d^3\mathbf{p}}{(2\pi)^3}\Pi_R(-\mathbf{p}^2)\chi_p^*(x^\perp)B(0,\mathbf{p};x_s^\perp)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_s)} \, .$$

Here we notice that the $p^0 = 0$ restriction turns $\omega_\parallel$ into a purely imaginary number, i.e $\omega_\parallel = \sqrt{-\mathbf{p}_\parallel^2} = \pm i|\mathbf{p}_\parallel|$. Thus, $B(0,\mathbf{p},x_s^3)$ becomes multivalued:

$$B(0,\mathbf{p},x_s^\perp) = -\frac{i}{2}\frac{e^{ix_s^\perp p^\perp}}{\mathbf{p}^2}\left(e^{\mp|\mathbf{p}_\parallel|x_s^\perp} - \cos\big(p^\perp x_s^\perp\big)\right) \, . \tag{4.47}$$

However, the plus sign in the above yields a term that diverges as $x_s^\perp \to \infty$ and we will simply neglect it as a nonphysical contribution. In this sense, the contribution from the B term turns into

$$-\frac{1}{2}i\eta_{\mu 0} q \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\Pi_R(-\mathbf{p}^2)}{\mathbf{p}^2}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_e)}\left(e^{-|\mathbf{p}_\parallel|x_s^\perp}\left[e^{i\mathbf{p}_\perp\cdot\mathbf{x}} - e^{i\mathbf{p}_\perp\cdot\tilde{\mathbf{x}}}\right] - \cos(\mathbf{p}_\perp\cdot\mathbf{x}_s)[e^{i\mathbf{p}_\perp\cdot\mathbf{x}} - e^{-i\mathbf{p}_\perp\cdot\mathbf{x}}]\right) \, , \tag{4.48}$$

or, equivalently

$$\eta_{\mu 0} q \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\Pi_R(-\mathbf{p}^2)}{\mathbf{p}^2} e^{i\mathbf{p}_\parallel \cdot (\mathbf{x}-\mathbf{x}_e)} \left(e^{-|\mathbf{p}_\parallel| x_s^\perp} \sin(\mathbf{p}_\perp \cdot \mathbf{x}) - \cos(\mathbf{p}_\perp \cdot \mathbf{x}_s) \sin(\mathbf{p}_\perp \cdot \mathbf{x}) \right) , \quad (4.49)$$

where $\widetilde{\mathbf{x}} = (x^1, x^2, -x^3)$ and $\mathbf{p}_\perp = (0, 0, p^3)$.

A very similar story can be told for $C_{\beta\nu} = \eta_{\beta\nu} C(p; x)$ since its functional form is effectively the same. Specifically, after integration of the p^0 delta and substitution of $\mathbf{y}$ for $\mathbf{x}_s$, one can write

$$C(0, \mathbf{p}; x^3)\chi_p(x_e^3) = -\frac{i}{2} \frac{e^{-ix^\perp p^\perp}}{\mathbf{p}^2} \left(e^{\mp |\mathbf{p}_\parallel| x^\perp} - \cos\big(p^\perp x^\perp\big) \right) (1 - e^{2ip^\perp x_s^\perp}) , \quad (4.50)$$

which implies that the contribution from the C term becomes

$$\begin{aligned} &-\frac{1}{2} i\eta_{\mu 0} q \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\Pi_R(-\mathbf{p}^2)}{\mathbf{p}^2} e^{-ix^\perp p^\perp} \left(e^{-|\mathbf{p}_\parallel| x^\perp} - \cos\big(p^\perp x^\perp\big) \right) (1 - e^{2ip^\perp x_s^\perp}) e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_s)} \\ &= \eta_{\mu 0} q \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\Pi_R(-\mathbf{p}^2)}{\mathbf{p}^2} e^{i\mathbf{p}_\parallel \cdot (\mathbf{x}-\mathbf{x}_s)} \left(-e^{-|\mathbf{p}_\parallel| x^\perp} + \cos(\mathbf{p}_\perp \cdot \mathbf{x}) \right) \sin(\mathbf{p}_\perp \cdot \mathbf{x}_s) . \end{aligned} \quad (4.51)$$

The $D_{\mu\alpha\beta\nu} = \eta_{\mu\alpha}\eta_{\beta\nu} D(p; x^\perp, y^\perp)$ also has two possible values, once we write $\omega_\parallel = \pm i|\mathbf{p}_\parallel|$. Specifically, we have

$$\begin{aligned} D(0, \mathbf{p}; x^\perp, y^\perp) \;=\; & \frac{1}{\mathbf{p}^4} \Big(e^{\mp |\mathbf{p}_\parallel|(x^\perp + y^\perp)} + \cos\big(p^\perp x^\perp\big) \cos\big(p^\perp y^\perp\big) \\ & - \cos\big(p^\perp x^\perp\big) e^{\mp |\mathbf{p}_\parallel| y^\perp} - e^{\mp |\mathbf{p}_\parallel| x^\perp} \cos\big(p^\perp y^\perp\big) \Big) \\ & \times e^{-ip^\perp (x^\perp - y^\perp)} . \end{aligned}$$

The $\delta^3(\mathbf{y} - \mathbf{x}_s)$ delta together with the polarization scalar lead to the following expression for the fourth contribution

$$\begin{aligned} \eta_{\mu 0} q \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\Pi_R(-\mathbf{p}^2)}{\mathbf{p}^2} e^{i\mathbf{p}_\parallel \cdot (\mathbf{x}-\mathbf{x}_s)} & \Big\{ e^{-|\mathbf{p}_\parallel|(x^\perp + x_s^\perp)} + \cos\big(p^\perp x^\perp\big) \cos\big(p^\perp x_s^\perp\big) \\ & - \cos\big(p^\perp x^\perp\big) e^{-|\mathbf{p}_\parallel| x_s^\perp} - e^{-|\mathbf{p}_\parallel| x^\perp} \cos\big(p^\perp x_s^\perp\big) \Big\} . \end{aligned} \quad (4.52)$$

To further simplify the notation, we will take $\delta\overline{A}_\mu^{(1)} = \eta_{\mu 0} \delta\overline{\phi}^{(1)}$, since this is just the electrostatic potential. Then, the combined contributions from all of the above parts

yield

$$\begin{aligned}
\delta\overline{\phi}^{(1)}(\mathbf{x}) &= q\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\Pi(-\mathbf{p}^2)}{\mathbf{p}^2}\left\{\frac{1}{4}\left[e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_s)}-e^{i\mathbf{p}\cdot(\widetilde{\mathbf{x}}-\mathbf{x}_s)}-e^{i\mathbf{p}\cdot(\mathbf{x}-\widetilde{\mathbf{x}}_s)}+e^{i\mathbf{p}\cdot(\widetilde{\mathbf{x}}-\widetilde{\mathbf{x}}_e)}\right]\right.\\
&+\Big(e^{-|\mathbf{p}_\parallel|x_s^\perp}-\cos(\mathbf{p}_\perp\cdot\mathbf{x}_s)\Big)\sin(\mathbf{p}_\perp\cdot\mathbf{x})e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}\\
&-\Big(-e^{-|\mathbf{p}_\parallel|x^\perp}+\cos(\mathbf{p}_\perp\cdot\mathbf{x})\Big)\sin(\mathbf{p}_\perp\cdot\mathbf{x}_s)e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}\\
&+e^{-|\mathbf{p}_\parallel|(x^\perp+x_s^\perp)}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}+\cos\big(p^\perp x^\perp\big)\cos\big(p^\perp x_s^\perp\big)e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}\\
&\left.-\cos\big(p^\perp x^\perp\big)e^{-|\mathbf{p}_\parallel|x_s^\perp}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}-e^{-|\mathbf{p}_\parallel|x^\perp}\cos\big(p^\perp x_s^\perp\big)e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}\right\}\,.
\end{aligned}$$

Note that, as a consistency check, for $x^3=0$, $\delta\overline{\phi}^{(1)}$ vanishes.

Most of the above terms are known from previous calculations. The only terms which require special attention are the ones proportional to the decreasing exponential factors, since these are the ones where the perpendicular and parallel components of the momentum must be treated separately. However, even then, many of the above terms cancel. In particular, the two terms proportional to the sines vanish simply from the parity of the integrands.

Another simplification comes from the double cosine term in the fourth line. Writing the cosines as complex exponentials, one can absorb the parallel momentum phase and write

$$\frac{1}{4}\left\{e^{i\mathbf{p}\cdot(\mathbf{x}-\widetilde{\mathbf{x}}_s)}+e^{i\mathbf{p}\cdot(\widetilde{\mathbf{x}}-\mathbf{x}_s)}+e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_s)}+e^{i\mathbf{p}\cdot(\widetilde{\mathbf{x}}-\widetilde{\mathbf{x}}_s)}\right\}\,,$$

which can be combined with the terms appearing in the first line. Thus, so far, the expression has been simplified to

$$\begin{aligned}
\delta\overline{\phi}^{(1)}(\mathbf{x}) &= q\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\Pi(-\mathbf{p}^2)}{\mathbf{p}^2}\left\{\frac{1}{2}\left[e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_s)}+e^{i\mathbf{p}\cdot(\widetilde{\mathbf{x}}-\widetilde{\mathbf{x}}_s)}\right]+e^{-|\mathbf{p}_\parallel|(x^3+x_s^3)}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}\right.\\
&\left.-\cos\big(p^3x^3\big)e^{-|\mathbf{p}_\parallel|x_s^3}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}-e^{-|\mathbf{p}_\parallel|x^3}\cos\big(p^3x_s^3\big)e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}\right\}\,. \qquad (4.53)
\end{aligned}$$

As expected, in the limit $x_s^\perp\to\infty$ with $\mathbf{x}-\mathbf{x}_s$ fixed, we re-obtain the result in the absence of the plate given by Eq. (3.117). Moreover, when $x_s^\perp\to 0$ we obtain $\delta\overline{\phi}^{(1)}(\mathbf{x})\to 0$, as it

should. In this limit the grounded plate shields away completely any field produced by the point charge.

Note that the influence of the plate is given by the last three terms in Eq. (4.53)—precisely the terms which vanish in the limit $x_s^\perp \to \infty$. These terms cannot easily be analytically integrated and we discuss one of these in the appendix (B.3.3). For now, let us simply express the resulting contributions in a way which resembles the integral found in the standard Uehling potential. By following the steps outlined in the appendix (B.3.4) we can write the correction in terms of four terms, one of which is actually just the free plate Uehling correction. The full potential correction is

$$\begin{aligned}\delta\overline{\phi}^{(1)}(\mathbf{x}) &= q\frac{\alpha}{6\pi^2}\int_1^\infty d\xi\left(1+\frac{1}{2\xi^2}\right)\frac{\sqrt{\xi^2-1}}{\xi^2}\Bigg\{\frac{e^{-2m\xi|\mathbf{x}-\mathbf{x}_s|}}{|\mathbf{x}-\mathbf{x}_s|} \\ &\quad+\int_0^\infty d\rho\frac{\rho}{\sqrt{\rho^2+4m^2\xi^2}}J_0(\rho|\mathbf{x}_\parallel-\mathbf{x}_{e\parallel}|)\left[e^{-\rho(x_\perp+x_{s\perp})}\right. \\ &\quad\left.-e^{-(\rho x_\perp+x_{s\perp}\sqrt{\rho^2+4m^2\xi^2})}-e^{-(\rho x_{s\perp}+x_\perp\sqrt{\rho^2+4m^2\xi^2})}\right]\Bigg\}. \qquad (4.54)\end{aligned}$$

In order to further simplify the notation, we will label the integral appearing in the standard Uehling correction as $\mathcal{I}(r)$ ($r := |\mathbf{x}-\mathbf{x}_s|$) and the double integral appearing in Eq. (4.54) as $\mathcal{F}(\mathbf{x},\mathbf{x}_s)$. Specifically,

$$\begin{aligned}\mathcal{I}(r) &:= \int_1^\infty d\xi\left(1+\frac{1}{2\xi^2}\right)\frac{\sqrt{\xi^2-1}}{\xi^2}e^{-2m\xi r}\,, \\ \mathcal{F}(\mathbf{x},\mathbf{x}_s) &:= \int_1^\infty d\xi\left(1+\frac{1}{2\xi^2}\right)\frac{\sqrt{\xi^2-1}}{\xi^2}\int_0^\infty\frac{\rho J_0(\rho|\mathbf{x}-\mathbf{x}_s|)\,d\rho}{\sqrt{\rho^2+4m^2\xi^2}} \\ &\quad\times\left[e^{-\rho(x^\perp+x_s^\perp)}-e^{-(\rho x^\perp+x_s^\perp\sqrt{\rho^2+4m^2\xi^2})}-e^{-(\rho x_s^\perp+x^\perp\sqrt{\rho^2+4m^2\xi^2})}\right].\end{aligned}$$

Then, in terms of these two functions, the $\mathcal{O}(\alpha)$ correction in the presence of the plate is expressed as

$$\delta\overline{\phi}^{(1)}(\mathbf{x}) = \frac{q\alpha}{6\pi^2 r}\left[\mathcal{I}(r)+r\mathcal{F}(\mathbf{x},\mathbf{x}_s)\right]. \qquad (4.55)$$

Furthermore, the full potential is simply the above contribution added to the tree-level term, which we know is simply the image method for the Coulomb interaction. Explicitly,

one finds

$$\overline{\phi}^{(1)}(\mathbf{x}) = \frac{q}{4\pi}\left\{\frac{1}{r} - \frac{1}{\tilde{r}} + \frac{2\alpha}{3\pi}\left[\frac{\mathcal{I}(r)}{r} + \mathcal{F}(\mathbf{x}, \mathbf{x}_s)\right]\right\}, \tag{4.56}$$

where $\tilde{r} := |\mathbf{x} - \tilde{\mathbf{x}}_s|$, with $\tilde{\mathbf{x}}_s := \mathbf{x}_{s\parallel} - \mathbf{x}_{s\perp}$. In Fig. 4.1, we can see the cross sectional slices of the pure correction term $\delta\overline{A}_0^{(1)}$, for a given charge placement $z_s = L = 2\bar{\lambda}_c$, where $\bar{\lambda}_c$ is the reduced Compton wavelength ($\bar{\lambda}_c = mc/\hbar$). It can be seen that the plate clearly induces an anisotropic profile which becomes very pronounced in the zone between the charge and the plate at the origin.

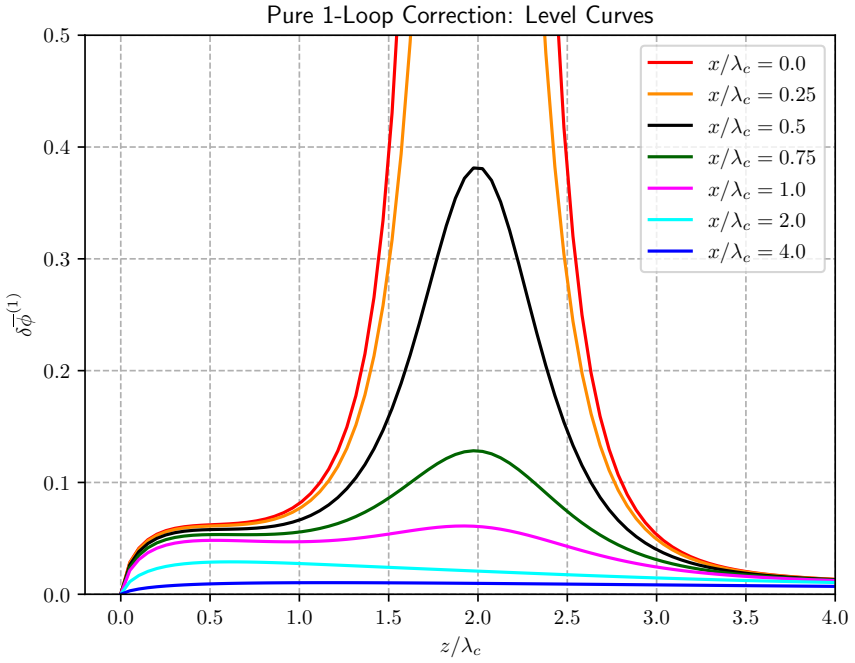


Figure 4.1: Transversal profile of the pure loop correction term, $\delta\overline{A}_0^{(1)}$, with a charge placed at $\mathbf{x}_s = (0, 0, 2\lambda_c)$. We normalize with the prefactor $\alpha e_R/6\pi^2$ and set $m = 1$.

From Eq. (4.56), we see that the classical interpretation of the potential from the method of images as a superposition of contributions from the charge and its image is lost at the quantum level (already at the first loop order). This distinction can be explicitly seen in Fig. 4.2, where the $\mathcal{O}(\alpha)$ correction obtained in Eq. (4.55) is plotted alongside the potential that would be obtained from a naive application of the image method, given

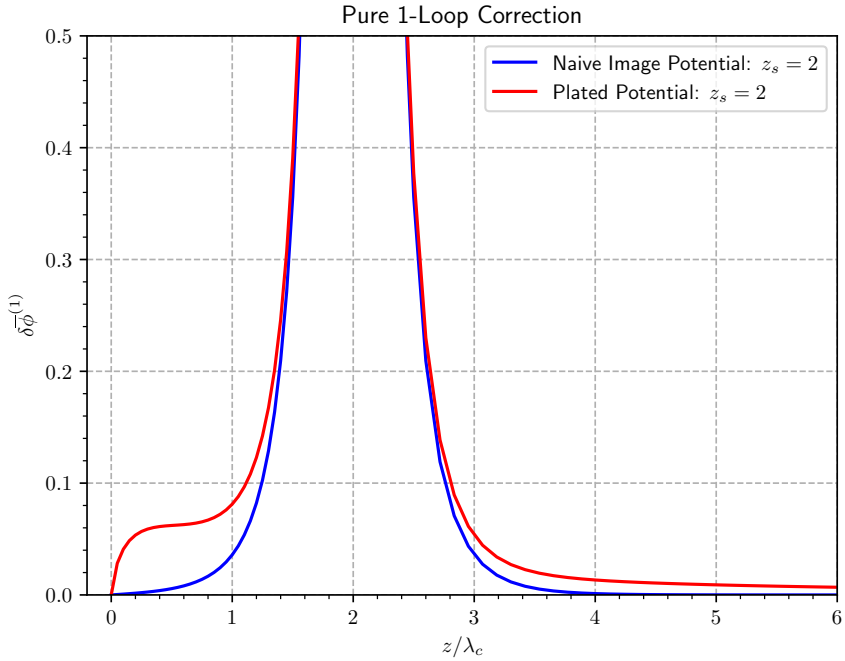
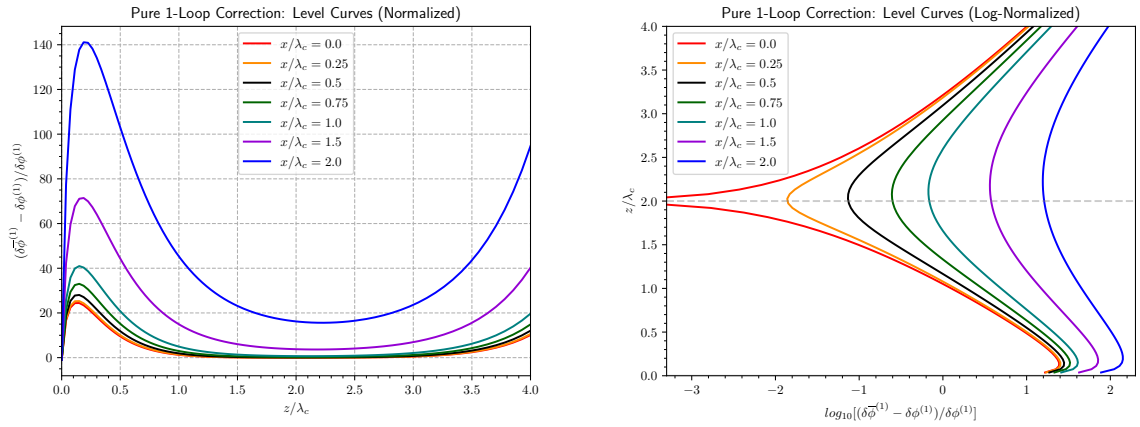
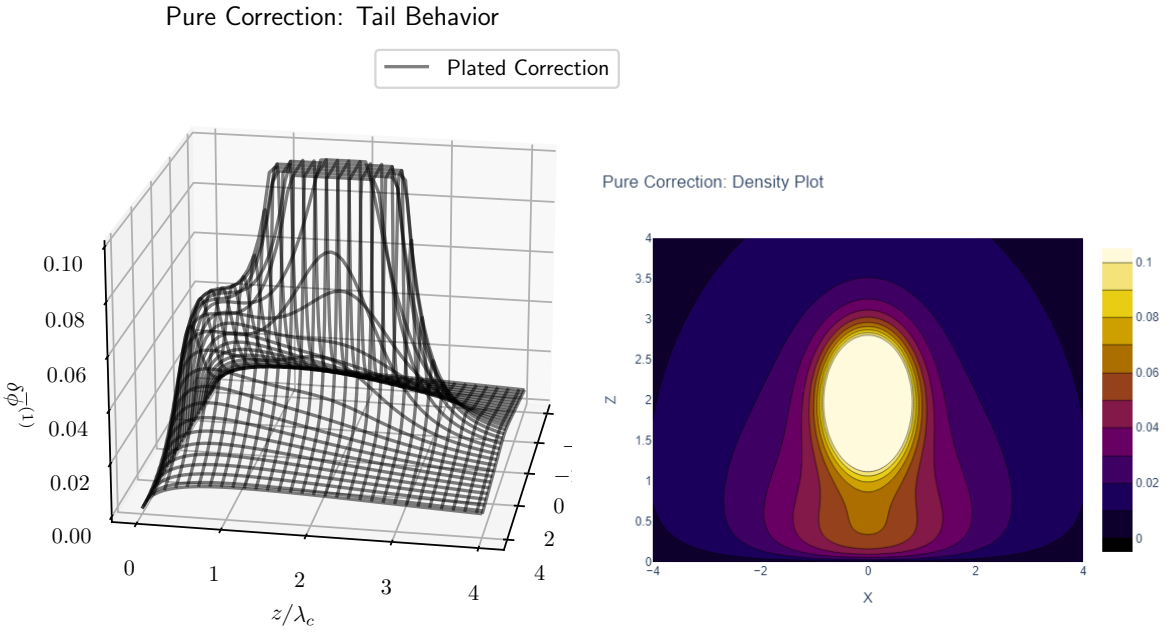


Figure 4.2: Plot of the linear-in-α correction to Coulomb's law for a charge located at $\mathbf{x}_s = (0, 0, 2\lambda_c)$, considering the potential found in expression (4.55) (red) and the naive prescription of the method of images (blue). Here we set $m = 1$ for the fermion mass in the loop.



(a) Standard Scale Plot (b) Log scale plot.

Figure 4.3: Plots of the normalized $\mathcal{O}(\alpha)$ contribution considering the presence of the plate ($\delta\overline{\phi}^{(1)}$), normalized by the standard potential without the plate ($\delta\phi^{(1)}$), with $\mathbf{x}_e = (0, 0, 2\lambda_c)$ ($\alpha = 1/137$, $e_R = 1$, $m = 1$, as before). The left corresponds to the standard scale while the right plot is log scaled and the axes have been swapped.



(a) 3D Surface corresponding to $\delta\overline{A}_0^{(1)}$. (b) Density plot for $\delta\overline{\phi}^{(1)}$.

Figure 4.4: A 3D surface plot and a density contour plot for the correction term for the charge placement $z_s = 2$ and the plate at $z = 0$. Here, we set $e_R = 1$, $\hbar = 1$, $c = 1$, $m = 1$.

by $\delta\overline{\phi}^{(1)\text{naive}}(\mathbf{x}) = \frac{e_R\alpha}{6\pi^2 r}(\mathcal{I}(r) + \mathcal{I}(\widetilde{r}))$. We see that the non-linearity enhances the Uehling correction for all distances. In particular, the Uehling correction rises much more steeply near the plate than what would follow from a direct image superposition.

4.4 Diagrammatic Interpretation

Let us now look more closely at the modified propagator we introduced before. Specifically, the pure loop modified piece was written as

$$\delta\overline{\Delta}^{(1)}_{\mu\nu}(x;y) = ie^2 \int d^4x_1 \int d^4x_2\, \overline{\Delta}^{(0)}_{\mu\alpha}(x;x_1) i\Pi^{\alpha\beta}(x_1 - x_2)\overline{\Delta}^{(0)}_{\beta\nu}(x_2;y)\,. \tag{4.57}$$

On the other hand, each tree propagator $\overline{\Delta}^{(0)}_{\mu\nu}$ is a combination of two propagators, one unreflected and another with either coordinate reflected in the plate's plane. This implies that the above expression can be rewritten as four different pieces. For the prescriptions

we had chosen above (equivalent to reflecting both outer points), this reads

$$\begin{aligned}
\delta\overline{\Delta}^{(1)}_{\mu\nu}(x;y) &= \int d^4x_1 \int d^4x_2\, \Delta^{(0)}_{\mu\alpha}(x-x_1) i\Pi^{\alpha\beta}(x_1-x_2)\Delta^{(0)}_{\beta\nu}(x_2-y) \\
&\quad - \int d^4x_1 \int d^4x_2\, \Delta^{(0)}_{\mu\alpha}(x-x_1) i\Pi^{\alpha\beta}(x_1-x_2)\Delta^{(0)}_{\beta\nu}(x_2-\widetilde{y}) \\
&\quad - \int d^4x_1 \int d^4x_2\, \Delta^{(0)}_{\mu\alpha}(\widetilde{x}-x_1) i\Pi^{\alpha\beta}(x_1-x_2)\Delta^{(0)}_{\beta\nu}(x_2-y) \\
&\quad + \int d^4x_1 \int d^4x_2\, \Delta^{(0)}_{\mu\alpha}(\widetilde{x}-x_1) i\Pi^{\alpha\beta}(x_1-x_2)\Delta^{(0)}_{\beta\nu}(x_2-\widetilde{y}) \qquad (4.58)
\end{aligned}$$

As we can see from the above, each of the four terms corresponds to a clear picture for the photons propagation. The first line, for example, is built of the standard propagators and corresponds to the photon simply ignoring the plate as it propagates from x to y. Similarly, the second and third lines, contain a single propagator with the outer point reflected about the plate, while the fourth has both photon trajectories reflected. This is the general spirit of trajectories in quantum mechanics: all possibilities must contribute to the total amplitude.

However, the image method propagator is more specific because it explicitly chooses one coordinate to reflect. This means that, for Eq. (4.58), the reflection of the outer points can be related to fictitious propagation of photons from beyond the plate as depicted in the diagram representation in Fig. 4.5.

In this sense, it is now natural to ask what happens if one changes the prescription used for each photon propagator. As we just discussed, the "real" diagram processes should not change, that is, the photon still can only propagate in four different ways. However, the choice of a different reflection for the space-time coordinates leads to a different interpretation in "image" diagrams. If, instead of reflecting x, y, we chose to reflect the inner points appearing in the loop, there are now three more possibilities (reflecting the first propagator's outer point while reflecting the loop vertex point of the second one, vice-versa, or both). Then, considering the "real" diagrams, this leads to 16 different combinations of the photon's "real" propagation and its associated image propagation.

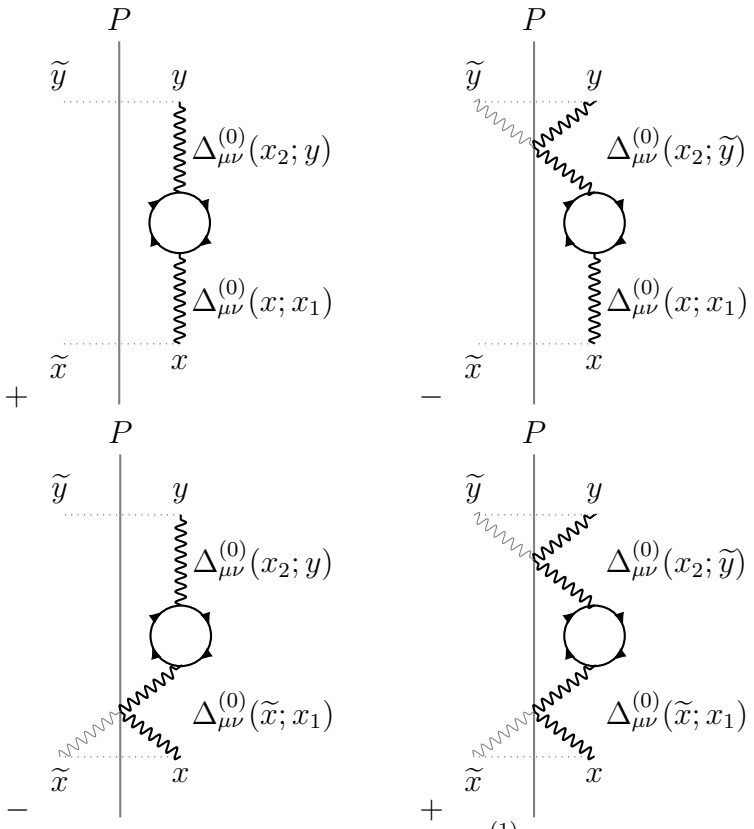


Figure 4.5: Diagrammatic representation of the $\delta\overline{\Delta}^{(1)}_{\mu\nu}$ correction under the prescription used in the calculations. Here P is the plate, and the gray photon lines represent fictitious propagation from the image coordinates $\widetilde{x}, \widetilde{y}$.

However, from the argument that the image propagators are actually equivalent (see eq. (4.9)), how we choose to represent these image diagrams is largely a matter of personal taste, that is, the calculations do not depend on the prescription choice. This is further illustrated with an explicit example in the appendix B.4.

In conclusion, the image propagator automatically introduces the alternate paths the particle can propagate on. These include the paths which interact with the boundary and the ones that do not.

Chapter 5

Applications III: Gravity

In this chapter, we outline the perturbative treatment of general relativity (GR) using quantum field theory approaches. It is important to emphasize that *there is currently no known theory of quantum gravity* which agrees with experiments. This is, in part, because of the fact that the energy scales at which quantum effects of gravity become important are far too extreme to be accessible. Nevertheless, one can still extract information from the perturbative treatment of GR, even at classical regimes. This has been of particular interest recently, with the advent of the LIGO gravitational wave detection [6, 8, 79, 7].

5.1 A Very Brief Outline of GR

Of course there is no way to discuss adequately the General theory of Relativity in a few pages, but, nevertheless, we must present the basic ingredients for the formulation and set the notation we are using. For a thorough introduction, see [80, 81, 82]. Here, we will simply introduce the metric and the field equations obtained by the variational principle.

To begin, we consider that space-time is a differentiable manifold $\mathcal{M}$ such that one can construct the notions of differentiation and integration. For this purpose, the standard procedure is to construct tangent spaces and cotangent spaces which cover point by point the manifold $\mathcal{M}$, and serve as "local vector spaces." This furnishes the manifold with vectors and their duals. Then, the existence of charts allow for a natural description

in terms of coordinates. The charts are then used to cover the manifold, allowing for the representation of vector fields $V^\mu(x)$ which are defined point by point. These vector fields can be covariant $V_\mu(x)$ or contra-variant, a property defined by their particular transformation properties when one subjects them to coordinate transformations $x \to x'$. In fact, a natural generalization is found in the introduction of tensor fields $T^{\mu_1\mu_2\cdots\mu_n}{}_{\nu_1\nu_2\ldots\nu_m}$ of rank (n, m), whose indices also change under coordinate transformations according to the general covariant or contra-variant character (whether they are up or down):

$$T^{\mu'_1\mu'_2\cdots\mu'_n}{}_{\nu'_1\nu'_2\ldots\nu'_m} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_n}}{\partial x^{\mu_n}} T^{\mu_1\mu_2\cdots\mu_n}{}_{\nu_1\nu_2\ldots\nu_m} \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \cdots \frac{\partial x^{\nu_m}}{\partial x^{\nu'_m}} \,. \tag{5.1}$$

The operation of raising and lowering indices is carried out by a metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, such that, for example, $T^\mu{}_\nu = g^{\mu\alpha} T_\alpha{}^\beta g_{\beta\nu}$. These second rank tensors are (in standard general relativity) symmetric with respect to their indices and naturally give a notion of line elements:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,. \tag{5.2}$$

Of course, if $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, the above naturally becomes the line element of Minkowski space in Cartesian coordinates.

From the metric $g_{\mu\nu}$, one can construct the notion of an affine connection $\Gamma^\lambda{}_{\mu\nu}$ which allows for the construction of covariant derivatives ∇ that are metric compatible, i.e. $\nabla g = 0$. Then, the basic fundamental theorem of Riemannian geometry states that there is a unique connection which is metric compatible:

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right) \,. \tag{5.3}$$

The above connection is often called the *Levi-Civita* connection and it can be used to construct the covariant derivative ∇. This derivative operator guarantees the covariant transformation properties of tensors and is of fundamental importance for, among other things, the definition of parallel transport along parametrized curves. The action of ∇ on

a generic tensor T is

$$\begin{aligned}\nabla_\lambda T^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} &= \partial_\lambda T^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} + \Gamma^{\mu_1}_{\lambda\alpha} T^{\alpha\ldots\mu_n}{}_{\nu_1\ldots\nu_m} + \cdots + \Gamma^{\mu_n}_{\lambda\alpha} T^{\mu_1\ldots\alpha}{}_{\nu_1\ldots\nu_m} \\ &\quad - \Gamma^{\alpha}_{\lambda\nu_1} T^{\mu_1\ldots\mu_n}{}_{\alpha\ldots\nu_m} - \cdots - \Gamma^{\alpha}_{\lambda\nu_m} T^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\alpha}\,. \end{aligned} \tag{5.4}$$

From the connection, one can construct other tensors which are of great importance for the study of the manifold $\mathcal{M}$ itself. The torsion is simply the anti-symmetric part of the connection, that is

$$T^{\beta}{}_{\mu\nu} := 2\Gamma^{\beta}{}_{[\mu\nu]} = (\Gamma^{\beta}{}_{\mu\nu} - \Gamma^{\beta}{}_{\nu\mu})\,. \tag{5.5}$$

Note that for the Levi-Civita connection, the torsion is automatically zero. The Riemann tensor is, in coordinate basis, defined by

$$R^{\alpha}{}_{\lambda\mu\nu} := \partial_\mu \Gamma^{\alpha}{}_{\nu\lambda} - \partial_\nu \Gamma^{\alpha}{}_{\mu\lambda} + \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\lambda} - \Gamma^{\alpha}{}_{\nu\beta}\Gamma^{\beta}{}_{\mu\lambda}\,. \tag{5.6}$$

The Ricci tensor is simply a contraction of the Riemann tensor (the only useful one):

$$R_{\mu\nu} := R^{\alpha}_{\mu\alpha\nu}\,. \tag{5.7}$$

Then, the Ricci scalar is given by the trace of the Ricci tensor, that is $R := R^{\mu}{}_{\mu}$.

The reason these tensors are important comes from the fact that the equations which govern the curvature of the manifold are all expressed in terms of these few quantities. These equations, in the context of general relativity, are the famous Einstein field equations and they describe the interplay between matter and the curvature of the manifold (space-time).

The fundamental utility of the action principle is further exemplified by the possibility of obtaining the Einstein equations from a fairly simple action, namely the action associated with the Einstein-Hilbert Lagrangian:

$$S_{EH} = \int R\sqrt{-g}\, d^4x \ , \tag{5.8}$$

where R is the curvature scalar. By using the action principle with a variation of the metric field $\delta g^{\mu\nu}$, and imposing $\delta S = 0$, one recovers the vacuum Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \,. \tag{5.9}$$

To include matter, we *define* the energy-momentum tensor as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\rm m}}{\delta g^{\mu\nu}} \,, \tag{5.10}$$

so that, for an action of the type $S = S_{EH} + S_{\rm m}$, one finds the full field equations of General Relativity:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N \, T_{\mu\nu} \,. \tag{5.11}$$

Actually, there is one term missing in the above; the famous cosmological constant Λ. It is of fundamental importance in cosmology, but for our discussion, it can be set aside.

5.2 Perturbative Expansion

Now, let us consider the approach of quantization first put forward by Feynman and deWitt [83, 84, 85]. As is well known, the resulting quantum theory is *non-renormalizable*, meaning that one formally needs an infinite number of counter terms to regulate the UV divergences of the theory. However, we once again recall the current understanding that *non-renormalizable* is not equal *to non-predictive* [86, 17, 87].

5.2.1 The Perturbative Approach

To begin, let us introduce the basic construction of the perturbative approach. Firstly, we separate the metric in two pieces, one of which will be the perturbation and another which will serve as a background.

In principle, one could choose an arbitrary background $\tilde{g}_{\mu\nu}$ and write

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + \kappa h_{\mu\nu} \tag{5.12}$$

where $h_{\mu\nu}$ is the perturbative field one is interested in and $\kappa \equiv \sqrt{32\pi G_N}$, with G_N being Newton's gravitational constant. The inverse $\tilde{g}^{\mu\nu}$ is used to formally raise indices, such as those of $h_{\mu\nu}$, i.e. $h^{\alpha\beta} = \tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu}h_{\mu\nu}$. From the fact that $g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\ \nu}$, one can find a formally infinite series in powers of $\kappa h_{\mu\nu}$:

$$g^{\mu\nu} = \tilde{g}^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu}_{\ \alpha}h^{\alpha\nu} - \kappa^3 h^{\mu}_{\ \alpha}h^{\nu}_{\ \beta}h^{\alpha\beta} + O(\kappa^4)\,. \tag{5.13}$$

With the aid of the above all occurrences of the metric can be expanded in powers of κ. For the determinant, a standard trick can be employed[82, 88, 80]. First, we recall that we are considering $-g = \det\{g_{\mu\nu}\} = \det(\tilde{g}_{\mu\nu} + \kappa h_{\mu\nu})$. Then, using the identity which relates the determinant of a matrix to its trace, we write

$$\sqrt{-g} = [\exp\{\mathrm{tr}\ln(\tilde{g}_{\mu\nu} + \kappa h_{\mu\nu})\}]^{1/2} = \exp\{\frac{1}{2}\ln\det\tilde{g} + \frac{1}{2}\kappa h - \frac{1}{2}\kappa^2\eta^{\mu\alpha}h_{\alpha}^{\ \lambda}h_{\lambda\mu} + O(\kappa^3)\} \tag{5.14}$$

where we have expanded the logarithm as a power series in $\kappa h_{\mu\nu}$. Then, expanding the exponential:

$$\sqrt{-g} = \sqrt{|\det\tilde{g}|}\left(1 + \frac{\kappa}{2}h + \frac{\kappa^2}{8}\left(h^2 - 2h^{\mu\nu}h_{\mu\nu}\right) + O(\kappa^3)\right)\,. \tag{5.15}$$

In this way, one can expand to the desired order in κh. For the EH action, the other important quantity appearing is the Ricci scalar R. The expansion is found by expressing R in terms of the Riemann tensor and writing the Riemann tensor in terms of the metric. Up to order κ^3, we quote the results(MATHEMATICA, [39]):

$$\begin{aligned} R &= \tilde{R} + \kappa(h^{\mu\ ;\nu}_{\ \mu\ ;\nu} - h^{\mu\ ;\nu}_{\ \nu\ ;\mu}) \\ &\quad +\kappa^2\left(\frac{1}{4}h^{\sigma\lambda;\mu}h_{\sigma\lambda;\mu} - \frac{1}{2}\left[h^{\lambda\sigma}\left(2h_{\sigma\mu}^{\ \ ;\mu} - h^{\mu}_{\ \mu;\sigma}\right)\right]_{;\lambda} + \frac{1}{2}h^{\tau}_{\ \rho;\beta}h^{\rho}_{\ \beta;\tau}\right. \\ &\quad \left. -\frac{1}{2}h^{\mu\nu}\left[h^{\alpha}_{\ \alpha;\mu\nu} - 2h^{\alpha}_{\ \mu;\nu\alpha}\right] - \frac{1}{4}h^{\mu}_{\ \mu;\tau}\left[2h^{\tau\ ;\alpha}_{\ \alpha} - h^{\nu\ ;\tau}_{\ \nu}\right] + h^{\alpha}_{\ \mu}h^{\mu\beta}\tilde{R}_{\alpha\beta}\right) + O(\kappa^3)\,, \end{aligned} \tag{5.16}$$

where the covariant derivatives are taken with respect to the background, i.e.: $A_{;\mu} = \tilde{\nabla}_{\mu}A$.

Often, however, we simply consider the background to be the flat Minkowski metric $\eta_{\mu\nu}$. This is also the starting point of construction for the linearized theory and, in fact, it is from this approach that gravitational waves are usually studied. With this choice, $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} = \pm\text{diag}(1,-1,-1,-1)$, and one writes the action up to κ^2 as

$$\begin{aligned} S_{EH} &= \int d^4x \frac{2}{\kappa^2}\left\{\kappa(\partial^2 h^{\mu}{}_{\mu} - \partial_\mu\partial_\nu h^{\mu\nu})\right. \\ &+\kappa^2\left(h^{\mu\nu}\partial_\mu\partial_\nu h - h^{\mu\nu}\partial_\nu\partial_\lambda h_\mu{}^\lambda - \frac{1}{4}\partial_\nu h\partial^\nu h - \partial_\mu h^{\mu\nu}\partial_\lambda h_\nu{}^\lambda + \partial^\nu h\partial_\lambda h_\nu{}^\lambda\right. \\ &-h^{\mu\nu}\partial_\nu\partial_\lambda h_\mu{}^\lambda + \frac{1}{2}h\partial_\nu\partial_\lambda h^{\nu\lambda} + h^{\mu\nu}\partial^2 h_{\mu\nu} - \frac{1}{2}h\partial^2 h \\ &\left.\left.-\frac{1}{2}\partial_\nu h_{\mu\lambda}\partial^\lambda h^{\mu\nu} + \frac{3}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu}\right) + O(\kappa^3)\right\}. \end{aligned} \tag{5.17}$$

In principle, from the above, one could go on to compute the propagator and graviton vertices. Note that higher order vertices exist and come from continuing the expansion to higher orders of κ.

5.2.2 Bosons Coupled to Gravity

To couple matter to gravity, one can follow the principle of minimal construction, that is, write the simplest (no inverse mass dimension parameters), covariant and local Lagrangian. First, we consider the simplest case of a real scalar field ϕ. The prescription is then to substitute the partial derivatives for covariant ones, and the d^4x measure for the $\sqrt{-g}d^4x$ measure of GR. From this, the action becomes

$$S = \int d^4x\,\sqrt{-g}\left(\frac{2R}{\kappa^2} + \frac{1}{2}\left(\nabla_\mu\phi\nabla^\mu\phi - m^2\phi^2\right)\right).$$

We see, from the above, that the coupling comes from the $\sqrt{-g}$ factor, which also contributes to graviton self interactions. The pure gravity component, once expressed up to order κ^2, is given by Eq.5.17. After integrating by parts and appropriately regrouping terms, we can write this piece as

$$S = \int d^4x\left(\frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\nu h^{\mu\nu}\partial^\lambda h_{\mu\lambda} - \partial_\nu h\partial_\mu h^{\mu\nu}\right). \tag{5.18}$$

This term yields the graviton propagator. However, since the graviton is a massless spin-2 gauge field, we have to add a gauge-fixing condition. The most common is the analogue of Lorenz gauge (often called de Donder gauge), where $\partial^\rho h_{\mu\rho} = \frac{1}{2}\partial_\mu h$. Writing the scalar field part as an interaction term of the form $h_{\mu\nu}T^{\mu\nu}$, one finds, in Lorenz gauge:

$$S = \int d^4x \left[-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \frac{1}{4}\partial^\mu h \partial_\mu h + \frac{\kappa}{2} h_{\mu\nu} T_{\mu\nu} \right] , \tag{5.19}$$

which is sometimes called the *Pauli-Fierz* action[82].

If one considers, instead of real scalar fields ϕ, complex scalar fields φ, the action is defined in an analogous way:

$$S = \int d^4x \sqrt{-g} \left(\frac{2R}{\kappa^2} + \nabla_\mu \varphi^* \nabla^\mu \varphi - m^2 \varphi^* \varphi \right) . \tag{5.20}$$

For nonzero spin bosons, the (minimal) coupling follows the same pattern. Thus, for a generic gauge boson with field strength $F^a_{\mu\nu}$ (group indices labelled with a), one writes

$$S = \int d^4x \sqrt{-g} \left(\frac{2R}{\kappa^2} - \frac{1}{4} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} \right) , \tag{5.21}$$

where, for clarity, $F^a_{\mu\nu}$ is written as

$$F^a_{\mu\nu} = \nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + igf^{abc} A^b_\mu A^c_\nu \, , \tag{5.22}$$

and the trace is performed over the group index a. For spin-1 photons, the structure constant is zero, and $F_{\mu\nu} = \nabla_{[\mu} A_{\nu]}$.

5.2.3 Fermions Coupled to Gravity

The case for fermions is more complicated. The reason for this resides in the fact that Fermions are part of semi-integer representations of the Lorentz group and thus transform in a different way from how Bosons transform. This leads to a problem in constructing a Lagrangian of Fermions coupled to gravity where gauge invariance is preserved, since gauge invariance for gravitons is intimately related to their own diffeomorphism invariance (Lorentz invariance).

To couple fermions, we introduce vierbeins $e^a_\mu(x)$ which are used to convert a coordinate basis to a local Lorentz basis. This allows one to construct locally inertial frames, point by point, and from them construct the representations. Thus, a vector is written as $A_\mu = e^a_\mu A_a$, and the metric itself as $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$. Then, one introduces covariant derivatives by use of the *spin connection* $\omega_{\mu ab}$:

$$D_\mu = \partial_\mu + \frac{1}{2}\sigma^{ab}\omega_{\mu ab}\,, \tag{5.23}$$

with $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$ corresponding to the generator of Lorentz transformations for the spinor representation.

In terms of the vierbeins, the objects in the EH action can be rewritten. The metric determinant, for example, is $\sqrt{-g} = \det(e^\mu_a) = e$. The Ricci scalar $R = \eta^{ab}R_{ab} := e^a_\mu e^b_\nu R^{\mu\nu}_{ab}$, such that the Lagrangian becomes

$$\mathcal{L}_{EH} = \frac{2}{\kappa^2} e R^{\mu\nu}_{ab} e^a_\mu e^b_\nu\,. \tag{5.24}$$

Thus, the spin 1/2 action (for a spinor field ψ) can be built:

$$\begin{aligned} S &= \int d^4x\; e\left(\frac{2}{\kappa^2}R^{\mu\nu}_{ab}e^a_\mu e^b_\nu + \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi\right)\,, \\ &= \int d^4x\; e\left(\frac{2}{\kappa^2}R^{\mu\nu}_{ab}e^a_\mu e^b_\nu + \overline{\psi}\left(ie^\mu_c\gamma^c\left[\partial_\mu + \frac{1}{2}\sigma^{ab}\omega_{\mu ab}\right] - m\right)\psi\right)\,. \end{aligned} \tag{5.25}$$

To pursue a perturbative treatment of the above, one can expand the vierbeins e^μ_a in terms of the perturbation $h_{\mu\nu}$ of a background $\tilde{g}_{\mu\nu}$ [89]. This can be converted into the vierbein language and, for example

$$e^a_\mu = \tilde{e}^a_\mu + \frac{\kappa}{2}h_{\mu\beta}(\tilde{g}^{-1})^{\beta\nu}\tilde{e}^a_\nu - \frac{\kappa^2}{8}\left(h_{\mu\beta}(\tilde{g}^{-1})^{\beta\nu}h_{\nu\lambda}(\tilde{g}^{-1})^{\lambda\alpha}\tilde{e}^a_\alpha\right) + O(\kappa^3)\,, \tag{5.26}$$

which, for the flat background $\eta_{\mu\nu}$, becomes

$$e^a_\mu = \delta^a_\mu + \frac{\kappa}{2}h^a_\mu - \frac{\kappa^2}{8}h_{\mu\nu}h^{\nu a} + O(\kappa^3)\,. \tag{5.27}$$

As a check, one can insert the above into $g_{\alpha\beta} = e^a_\alpha e^b_\beta \eta_{ab}$ and recover $g_{\alpha\beta} = \eta_{\alpha\beta} + \kappa h_{\alpha\beta}$.

5.2.4 Graviton Propagator

Having briefly discussed the different actions which couple gravitons to different matter fields, we could move on to attempt perturbative calculations. Before, however, let us explicitly obtain the free graviton propagator.

Since the propagator is effectively a 2-point function, for gravitons, we only need to consider the zero order terms in κ of the EH action. This leads to the linearized Einstein-Hilbert Lagrangian ($\mathcal{L}_{LEH}$), which we write as

$$\mathcal{L}_{LEH} = \frac{1}{2}\partial_\mu h \partial^\mu h - \partial_\nu h^{\mu\nu}\partial_\mu h + \partial_\lambda h_{\mu\nu}\partial^\nu h^{\mu\lambda} - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu}\,. \tag{5.28}$$

With the above, one could write the path integral for the $h_{\mu\nu}$ field and attempt to obtain the propagator. However, since this is a massless spin 2 field, the resulting operator is not invertible. This requires the addition of a gauge fixing term. For this purpose, we consider a generic gauge fixing functional

$$G^\nu[h_{\mu\nu}] := \partial_\mu h^{\mu\nu} + \theta\partial^\nu h - \omega^\nu\,, \tag{5.29}$$

and set $G^\nu[h_{\mu\nu}] = 0$ as a constraint (four, actually) for the path integral over $h_{\mu\nu}$. Then, the constrained path integral becomes

$$Z = V_{\text{diff}} \int \mathcal{D}h \left|\frac{\delta G^\nu[h_{\mu\nu}]}{\delta\xi_\mu}\right| \delta(G^\nu[h_{\mu\nu}]) e^{iS_{EH}}\,, \tag{5.30}$$

where V_{diff} refers to the volume of the gauge group which, for $h_{\mu\nu}$ is the group of diffeomorphisms over the base pseudo-Riemannian manifold. Explicitly, for the action (5.28), one can find that the transformations which leave the action invariant are

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_{(\mu}\xi_{\nu)}\,. \tag{5.31}$$

The invariance under the above term parametrized by ξ^μ is trivially true for Killing fields, which are generators of isometries. Thus, one has a class of vector fields over which

integration generates redundancies. Then, the Faddeev-Poppov determinant will turn into

$$\left|\frac{\delta G^{\nu}[h_{\mu\nu}]}{\delta\xi_{\mu}}\right| = \det\left(2\eta^{\mu\nu}\nabla^2 + 2\theta\nabla^{\nu}\nabla^{\mu}\right) = \int \mathcal{D}c\mathcal{D}\overline{c}\exp\left(i\int d^4x\, 2\overline{c}_{\mu}\left(\nabla^2 + \theta\nabla^{\mu}\nabla^{\nu}\right)c_{\nu}\right) \quad (5.32)$$

Thus, under this framework, the vectorial ghost fields $c^{\mu}, \overline{c}^{\nu}$ apparently do not decouple from the graviton field. For now, we are not concerned with the ghost part since the propagator itself does not depend on it.

Proceeding with the integration over the ω^{ν} functional, one finds that the Faddeev-Popov procedure will yield a gauge fixing term of the form

$$S_{gf} = -\int d^4x \left(\partial_{\mu}h^{\mu\nu} - \frac{1}{2\xi}\partial^{\nu}h\right)^2 . \quad (5.33)$$

With this generic gauge fixing function, one can rewrite the graviton action as

$$\begin{aligned} S &= S_{gf} + S_{LEH}\,, \\ &= \int d^4x \left(\frac{1}{2}\partial_{\mu}h\partial^{\mu}h - \frac{1}{2}\partial_{\lambda}h_{\mu\nu}\partial^{\lambda}h^{\mu\nu} + \partial_{\lambda}h_{\mu\nu}\partial^{\nu}h^{\mu\lambda} - \partial_{\nu}h^{\mu\nu}\partial_{\mu}h \right. \\ &\quad \left. -\partial_{\mu}h^{\mu\nu}\partial^{\lambda}h_{\lambda\nu} - \frac{1}{4\xi^2}\partial_{\mu}h\partial^{\mu}h + \frac{1}{\xi}\partial_{\mu}h^{\mu\nu}\partial_{\nu}h\right) . \end{aligned} \quad (5.34)$$

The third and fith term of the above are the same up to boundary terms as can be seen by two integration by parts. Thus, the action is simply

$$S = \int d^4x \left(\frac{1}{2}\left(1 - \frac{1}{2\xi^2}\right)\partial_{\mu}h\partial^{\mu}h - \partial_{\nu}h^{\mu\nu}\partial_{\mu}h\left(1 - \frac{1}{\xi}\right) - \frac{1}{2}\partial_{\lambda}h_{\mu\nu}\partial^{\lambda}h^{\mu\nu}\right) . \quad (5.35)$$

In the above considerations, the parameter ξ is parameterizing gauge families which give different expressions for the propagator. By far the most well known is found by choosing $\xi = 1$, which yields the harmonic (or de Donder) gauge, with

$$\partial_{\mu}h^{\mu\nu} = \frac{1}{2}\partial^{\nu}h\,. \quad (5.36)$$

This specific gauge choice leads to the most well known form for the graviton propagator, which in d dimensions is given by

$$\Delta_{\mu\nu\alpha\beta}(p) = \frac{i}{p^2 + i\epsilon}\frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\alpha\beta}\right) . \quad (5.37)$$

We will be mainly working with the above form. Note that it is customary in the literature to define the tensor part of the graviton propagator as

$$\mathcal{P}_{\mu\nu\alpha\beta} := \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\alpha\beta}\right), \tag{5.38}$$

which we will also adopt.

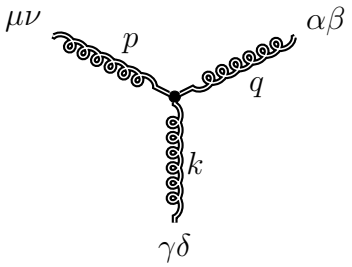
5.2.5 Diagrammatic Expansion

Having constructed the basic ingredients of the theory, we could proceed to attempt the perturbative treatment of the graviton and its interactions. This, however, is quickly seen to become very chaotic, since the infinite expansion in powers of $h_{\mu\nu}$ implies that the graviton has infinite self interaction vertices. This is to be compared and contrasted with Abelian gauge theories (where there are no self interactions, as we have seen for QED) and non Abelian gauge theories (where gluons can self interact in trivalent and quadrivalent vertices only).

Following Ref. [88], we quote here the main diagram terms which are of interest to us. To denote gravitons, we will use double curly lines, such that the graviton propagator becomes:

$\mu\nu$ $\alpha\beta$ $= \Delta_{\mu\nu\alpha\beta}(p) = \dfrac{i}{p^2 + i\epsilon}\mathcal{P}_{\mu\nu\alpha\beta}\,.$ (5.39)

So far so good. Now, let us consider the interaction vertices. We will follow a notation where τ^s corresponds to the trivalent vertex expression for gravitons interacting with spin s particles. Also, for $s = 1$ and $s = 2$ we suppose massless particles. With this notation in mind, the trivalent graviton vertex $\tau^2_{\mu\nu\alpha\beta\gamma\delta}$ is (according to [88]):



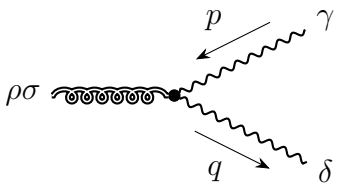
$$= \tau^2_{3\mu\nu\alpha\beta\gamma\delta}\,, \tag{5.40}$$

where

$$\begin{aligned}
\tau^2_{\mu\nu\alpha\beta\gamma\delta} = -\frac{i\kappa}{2}\Bigg(&\mathcal{P}_{\alpha\beta\gamma\delta}\left[p_\mu p_\nu + (p-q)^\mu(k-q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta_{\mu\nu}q^2\right] \\
&+2q_\lambda q_\sigma\left[I_{\alpha\beta}{}^{\sigma\lambda}I_{\gamma\delta\mu\nu} + I_{\gamma\delta}{}^{\sigma\lambda}I_{\alpha\beta\mu\nu} - I_{\alpha\beta\mu}{}^{\sigma}I_{\gamma\delta\nu}{}^{\lambda} - I_{\gamma\delta\mu}{}^{\sigma}I_{\alpha\beta\nu}{}^{\lambda}\right] \\
&+\Big[q_\lambda q_\mu\left(\eta_{\alpha\beta}I_{\gamma\delta\nu}{}^{\lambda} + \eta_{\gamma\delta}I_{\alpha\beta\nu}{}^{\lambda}\right) + q_\lambda q_\nu\left(\eta_{\alpha\beta}I_{\gamma\delta\mu}{}^{\lambda} + \eta_{\gamma\delta}I_{\alpha\beta\mu}{}^{\lambda}\right) \\
&-q^2\left(\eta_{\alpha\beta}I_{\gamma\delta\mu\nu} - \eta_{\gamma\delta}I_{\alpha\beta\mu\nu}\right) - \eta_{\mu\nu}q_\sigma q_\lambda\left(\eta_{\alpha\beta}I_{\gamma\delta}{}^{\sigma\lambda} + \eta_{\gamma\delta}I_{\alpha\beta}{}^{\sigma\lambda}\right)\Big] \\
&+\Big[2q_\lambda\left(I_{\alpha\beta}{}^{\lambda\sigma}I_{\gamma\delta\sigma\nu}(p-q)_\mu + I_{\alpha\beta}{}^{\lambda\sigma}I_{\gamma\delta\sigma\mu}(p-q)_\nu - I_{\gamma\delta}{}^{\lambda\sigma}I_{\alpha\beta\sigma\nu}p_\mu - I_{\gamma\delta}{}^{\lambda\sigma}I_{\alpha\beta\sigma\mu}p_\nu\right) \\
&+q^2\left(I_{\alpha\beta\sigma\mu}I_{\gamma\delta\nu}{}^{\sigma} + I_{\alpha\beta\nu}{}^{\sigma}I_{\gamma\delta\sigma\mu}\right) + \eta_{\mu\nu}q_\sigma q_\lambda\left(I_{\alpha\beta}{}^{\lambda\rho}I_{\gamma\delta\rho}{}^{\sigma} + I_{\gamma\delta}{}^{\lambda\rho}I_{\alpha\beta\rho}{}^{\sigma}\right)\Big] \\
&+\Bigg\{\left(p^2 + (p-q)^2\right)\left[I_{\alpha\beta\mu}{}^{\sigma}I_{\gamma\delta\sigma\nu} + I_{\gamma\delta\mu}{}^{\sigma}I_{\alpha\beta\sigma\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{P}_{\alpha\beta\gamma\delta}\right] \\
&-\left(I_{\gamma\delta\mu\nu}\eta_{\alpha\beta}p^2 + I_{\alpha\beta\mu\nu}\eta_{\gamma\delta}(p-q)^2\right)\Bigg\}\Bigg)\,,
\end{aligned} \tag{5.41}$$

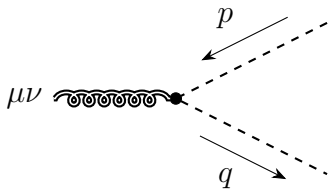
$$I_{\alpha\beta\mu\nu} := \frac{1}{2}\left(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu}\right)\,. \tag{5.42}$$

In general, self interaction diagrams are very difficult to calculate, and loop level calculations with graviton rings are very complicated. For more on this, see [90, 91, 88, 83, 92, 93]. Of more immediate interest are the scalar and photon interaction vertices. The photon vertex $\tau_3^{1\rho\sigma\gamma\delta}$ reads:



$$= \tau^{1\rho\sigma(\gamma\delta)} = i\kappa \Big[\mathcal{P}^{\rho\sigma(\gamma\delta)}(p\cdot q) + \frac{1}{2}\left(\eta^{\rho\sigma}p^{\delta}q^{\gamma} + \eta^{\gamma\delta}p^{(\rho}q^{\sigma)} - \left(q^{\gamma}p^{(\sigma}\eta^{\rho)\delta} + q^{\rho}p^{\delta}\eta^{\sigma\delta} + q^{\sigma}p^{\delta}\eta^{\rho\gamma}\right)\right)\Big] \ . \tag{5.43}$$

The scalar vertex $\tau^{0\mu\nu}$ is considerably simpler:

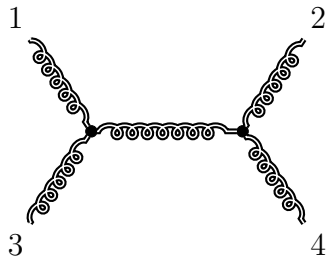


$$= \tau^{0\mu\nu} = -\frac{i\kappa}{2}\left[p^{(\mu}q^{\nu)} - \eta^{\mu\nu}[(p\cdot q) - m^2]\right] \ . \tag{5.44}$$

With the above, we can already discuss some of the applications of the perturbative methods to general relativity.

5.3 Pure Graviton Scattering

An immediate application where we can see the enormous difference in work by use of the newer amplitude-based methods is in pure graviton-graviton scattering. To be more specific, let us consider the following diagram calculation corresponding to the t channel of a tree-level self-interaction amplitude:



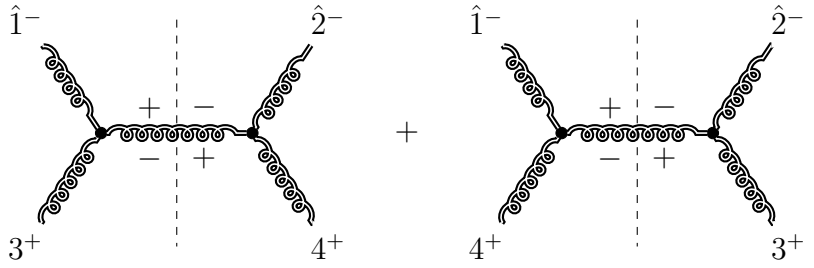
If one was interested in performing the calculation for the above t channel diagram using

the standard methods, the expression to be evaluated would be

$$\epsilon^{\mu_1\nu_1}(1)\epsilon^{\mu_3\nu_3}(3)\tau^2_{3\mu_1\nu_1\mu_3\nu_3\alpha\beta}(p_1,p_3)\frac{i\mathcal{P}^{\alpha\beta\sigma\rho}}{t}\tau^2_{3\rho\sigma\mu_2\nu_2\mu_4\nu_4}(p_2,p_4)\epsilon^{\mu_2\nu_2}(2)\epsilon^{\mu_4\nu_4}(4)\,. \tag{5.45}$$

A quick inspection of Eq. (5.40) reveals that the above becomes a tedious task. Furthermore, one would need to calculate this for the other two channels (which is admittedly not difficult, since the expressions are obtained by simply relabeling the particles).

Instead of brute forcing with the Feynman rules for the above, we will use the BCFW recursive method to obtain the graviton amplitudes from the simpler three particle amplitudes. The reason this is simpler is because we know how to construct these three graviton building blocks: little group scaling. Recalling the discussion in section 2.2.3, we can immediately write the expressions for the amplitudes given a set of helicity configurations. For the MHV amplitude $M_4(1^-,2^-,3^+,4^+)$, there are two channels (t, u) to consider in the $[1,2\rangle$ shift:



(5.46)

The four point amplitude becomes

$$M_4(1^-,2^-,3^+,4^+)=\sum_{h=\pm}M_3(\hat{1}^-,\hat{P}^h_{14},4^+)\frac{i}{P^2_{14}}M_3(-\hat{P}^{-h}_{14},\hat{2}^-,3^+)+(3\leftrightarrow 4)\,, \tag{5.47}$$

with the three particle subamplitudes given by

$$M_3(\hat{1}^-,\hat{P}^+_{14},4^+) = \kappa\frac{[\hat{P}_{14}4]^6}{[4\hat{1}]^2[\hat{1}\hat{P}_{14}]^2}\,; \qquad M_3(\hat{1}^-,\hat{P}^-_{14},4^+)=\kappa\frac{\langle\hat{1}\hat{P}_{14}\rangle^6}{\langle\hat{P}_{14}4\rangle^2\langle 4\hat{1}\rangle^2}\,; \tag{5.48}$$

$$M_3(-\hat{P}^-_{14},\hat{2}^-,3^+) = \kappa\frac{\langle-\hat{P}_{14}\hat{2}\rangle^6}{\langle\hat{2}3\rangle^2\langle 3(-\hat{P}_{14})\rangle^2}\,; \quad M_3(-\hat{P}^+_{14},\hat{2}^-,3^+)=\kappa\frac{[3(-\hat{P}_{14})]^6}{[-\hat{P}_{14}\hat{2}]^2[\hat{2}3]^2}\,. \tag{5.49}$$

Here, the on-shell condition is $\hat{P}_{14} = 0$ (or $\hat{P}_{13} = 0$ for the other channel). For the $[1,2\rangle$ shift, where $\left|\hat{1}\right\rangle = |1\rangle$, this implies that $[\hat{1}4] = 0$ (or $[\hat{1}3] = 0$, on the other channel), and thus the positive helicity contribution in the summation of Eq. (5.47) vanishes. The remaining contribution is simply

$$\begin{aligned} M_4(1^-,2^-,3^+,4^+) &= M_3(\hat{1}^-,\hat{P}_{14}^-,4^+)\frac{i}{P_{14}^2}M_3(-\hat{P}_{14}^+,\hat{2}^-,3^+) + (3 \leftrightarrow 4)\,, \\ &= i\kappa^2\frac{\langle\hat{1}\hat{P}_{14}\rangle^6}{\langle\hat{P}_{14}4\rangle^2\langle 4\hat{1}\rangle^2}\frac{1}{\langle 14\rangle\,[41]}\frac{[3(-\hat{P}_{14})]^6}{[-\hat{P}_{14}\hat{2}]^2[\hat{2}3]^2} + (3 \leftrightarrow 4)\,, \\ &= i\kappa^2\frac{(\langle 1|p_4|3])^6}{[12]^2\,\langle 41\rangle^2\,\langle 41\rangle^2\,[23]^2[41]\,\langle 41\rangle} + (3 \leftrightarrow 4)\,, \\ &= i\kappa^2\frac{[34]^6}{[12]^2}\left(\frac{\langle 14\rangle}{[14][23]^2} + \frac{\langle 13\rangle}{[13][24]^2}\right). \end{aligned} \tag{5.50}$$

One can massage the last expression in multiple different ways. By multiplying and dividing same factors, one can write:

$$\begin{aligned} M_4(1^-,2^-,3^+,4^+) &= i\kappa^2\frac{[34]^7}{[12]^2[13][14][23][24][34]}\Big(\overbrace{\langle 14\rangle\,[13]}^{=-\langle 42\rangle[23]}[24] + \overbrace{\langle 13\rangle\,[23]}^{=-\langle 14\rangle[42]}[14]\Big)\,, \\ &= i\kappa^2\frac{[34]^7}{[12]^2[13][14][23][24]}\left(\langle 24\rangle\,[24] + \langle 14\rangle\,[14]\right)\,, \\ &= i\kappa^2\frac{[34]^7\,\langle 43\rangle}{[12]^2[13][14][23][24]}\,. \end{aligned} \tag{5.51}$$

We also have, by parity, the flipped helicity amplitude $M_4(1^+,2^+,3^-,4^-)$:

$$M_4(1^+,2^+,3^-,4^-) = i\kappa^2\frac{\langle 34\rangle^7\,[43]}{\langle 12\rangle^2\,\langle 13\rangle\,\langle 14\rangle\,\langle 23\rangle\,\langle 24\rangle}\,. \tag{5.52}$$

Now, if one wants to calculate the full amplitude, one must still calculate the other helicity configurations such as $M_4(1^\pm 2^\pm 3^\pm 4^\pm)$, $M_4(1^\pm 2^\mp 3^\pm 4^\mp)$, $M_4(1^\pm 2^\pm 3^\pm 4^\mp)$ and so on. However, many of these can be obtained by simply exchanging particle labels or spinor variables.

5.4 Post Newtonian Expansion

Given the difficulty of the standard methods, one might wonder why the pursuit of gravity calculations using perturbation theory is relevant, given that the theory is known to be

non-renormalizable [85, 92, 90]. The reason for this is that one can still treat the theory under an effective-field-theory paradigm and obtain predictions valid for a given energy scale. This is the approach advocated in a number of works (see e.g. [94, 87, 88, 95, 96, 91]). Under this paradigm, quantum corrections are legitimate results of the perturbation expansion we have discussed above.

On the other hand, the use of the field theoretic formalism has proven useful in the recovering of classical observables, which do not depend on the renormalizability of the theory. In this context, there have been works that develop systematic approaches to extracting these observables (see for example [97, 98, 99]).

Important examples of the utilization of these methods is the Post Newtonian (PN) and Post Minkowskian (PM) expansions which we briefly discuss now. These are usually presented within the framework of standard General Relativity (see [82, 100]), but have recently been revisited with quantum field theory approaches (see, for example, [101, 102, 103, 104, 105, 106, 107]).

In essence, these expansions work under the perturbative framework with respect to two different quantities. In the post Newtonian expansions, one considers powers of the velocity of a given probe (v/c), which represent (special) relativistic corrections to Newton's law of gravitation. On the other hand, the post Minkowskian expansion is based on the expansion with respect to Newton's constant G_N. Therefore, the PM expansion is useful to quantify curvature perturbations generated by a given source. These expansions have proven an ideal framework to developing theoretical predictions for gravitational wave detectors, such as KAGRA-LIGO-Virgo [82, 100, 100, 106, 103, 108].

The extraction of a non-relativistic potential from S-matrix calculations in gravity is not trivial [88, 91, 94, 109]. There are, in fact, multiple approaches inspired by different areas. Here, we will briefly discuss two: the explicit amplitude matching using a non-relativistic prescription inspired by bound-state problems advocated in [86, 101, 88] and a Wilson loop inspired procedure originated in [110].

5.4.1 Lippmann-Schwinger Method

To begin, let us explicitly state the problem we are considering. We have two massive scalar fields ϕ_1, ϕ_2 with masses $m_1 = m_a$, $m_2 = m_b$ which interact via gravitons, such that the unexpanded action reads

$$S = \int d^4x \sqrt{-g} \left(\frac{2}{\kappa^2} R + \frac{1}{2} \sum_{i=1}^{2} (\partial_\mu \phi_i \partial^\mu \phi_i - m_i^2 \phi_i^2) \right) . \tag{5.53}$$

Upon expansion with $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, the above leads to the usual Feynman rules for the free graviton and the free scalars, supplemented by the interactions. The scalars will couple to the graviton field $h_{\mu\nu}$, yielding the vertex of Eq. (5.44). Now, we want to relate the amplitude calculation to the potential felt by each massive particle as the graviton is exchanged. To do this, we recall the discussion of the Lippmann-Schwinger iterative procedure which leads, in the non-relativistic scenario, to the Born series.

Consider the Hamiltonian for the two mass system we have been considering such that, in the relativistic regime, we write

$$H = H_0 + V = \sqrt{p_1^2 + m_a^2} + \sqrt{p_2^2 + m_b^2} + V \, . \tag{5.54}$$

Here, V is the unknown potential which we wish to compute. Furthermore, the free piece H_0 can be used to construct the free particle resolvent $G_0(z) = (z - H_0)^{-1}$. Then, considering that T corresponds to the Transfer matrix (the nontrivial part of the S-matrix $S = 1 - iT$), one can write the *Lippmann-Schwinger* equation (see [102, 111]):

$$T(z) = V + V G_0(z) T(z) \, . \tag{5.55}$$

On the other hand, by definition of the S-matrix, a given initial (in) state $|i\rangle$ and final (out) state $|f\rangle$ are related by $|f\rangle = S\,|i\rangle$. Then, if one takes the in and out states to be momentum states $|\mathbf{p}\rangle$, one can form the matrix elements

$$\langle \mathbf{p}'| \, T(z) \, |\mathbf{p}\rangle = \langle \mathbf{p}'| \, V |\mathbf{p}\rangle + \int \frac{d^3\mathbf{k}}{(2\pi)^3} \langle \mathbf{p}'| V \, |\mathbf{k}\rangle \, \frac{1}{z - E_k} \, \langle \mathbf{k}| \, T(z) \, |\mathbf{p}\rangle \ , \tag{5.56}$$

where we have used explicitly that $(z-H_0)|\mathbf{k}\rangle = |\mathbf{k}\rangle(z-E_k)$, with E_k being an eigenvalue of the free Hamiltonian. Now, we recall that the usual scattering amplitude $\mathcal{M}$ is related to T via a particular limit:

$$\mathcal{M}(\mathbf{p},\mathbf{p}') = \lim_{\epsilon\to 0} \langle \mathbf{p}| \, T(E_p + i\epsilon) \, |\mathbf{p}'\rangle \ . \tag{5.57}$$

Thus, keeping the limit implicit, one has that the scattering amplitude is related to the potential V by

$$\mathcal{M}(\mathbf{p}',\mathbf{p}) = \langle \mathbf{p}'| \, V \, |\mathbf{p}\rangle + \int \frac{d^3k}{(2\pi)^3} \, \langle \mathbf{p}'| \, V|\mathbf{k}\rangle \frac{1}{E_p - E_k + i\epsilon} \mathcal{M}(\mathbf{k},\mathbf{p}) \, . \tag{5.58}$$

The above equation can be used as an iterative expansion for the scattering amplitude $\mathcal{M}$ *or* the potential V, which is what we are interested in. Explicitly, one has to second iteration:

$$\begin{aligned} V(\mathbf{p}',\mathbf{p}) \ &= \ \mathcal{M}(\mathbf{p}',\mathbf{p}) - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathcal{M}(\mathbf{p}',\mathbf{k}) \frac{1}{E_p - E_k + i\epsilon} \mathcal{M}(\mathbf{k},\mathbf{p}) \\ &\quad + \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \mathcal{M}(\mathbf{p}',\mathbf{k}_1) \frac{1}{E_p - E_{k_1} + i\epsilon} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} \mathcal{M}(\mathbf{k}_1,\mathbf{k}_2) \frac{1}{E_p - E_{k_2} + i\epsilon} V(\mathbf{k}_2,\mathbf{p}) \, . \end{aligned} \tag{5.59}$$

If one begins from the simplest tree level amplitude which is of order G_N, it is not difficult to see that each iteration will give a new power of G_N, allowing the systematic, albeit perhaps difficult, evaluation of a *Post Minkowskian* expansion.

To see how the Post Newtonian expansion arises, it is simplest to calculate the trivial first order of G_N. At first order, all that we need is the scalar-graviton vertex, and, for now, let us carry out the calculation which corresponds to the following diagram:

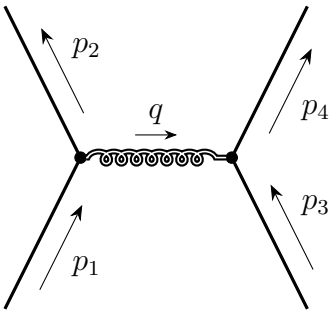


Figure 5.1: Classical tree level scattering of two masses mediated by a single graviton line. Here, $q = p_1 - p_2$.

By applying straightforward Feynman rules, in harmonic gauge, one has

$$\begin{aligned} i\mathcal{M} &= -\frac{i\kappa^2}{4}\left(p_1^{(\mu}p_2^{\nu)} - \eta^{\mu\nu}[(p_1\cdot p_2) - m_a^2]\right)\frac{\mathcal{P}_{\mu\nu\alpha\beta}}{q^2}\left(p_3^{(\alpha}p_4^{\beta)} - \eta^{\alpha\beta}[(p_3\cdot p_4) - m_b^2]\right), \\ &= -\frac{i\kappa^2}{4q^2}\Big\{2(p_1\cdot p_3)(p_2\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3) - 2(p_1\cdot p_2)(p_3\cdot p_4) \\ &\quad + 2m_b^2(p_1\cdot p_2) + 2m_a^2(p_3\cdot p_4) + \frac{2d}{2-d}m_a^2m_b^2\Big\}, \end{aligned} \tag{5.60}$$

which in $d = 4$ reduces to

$$\begin{aligned} \mathcal{M} &= -\frac{16\pi G_N}{q^2}\Big\{(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_2)(p_3\cdot p_4) \\ &\quad + m_b^2(p_1\cdot p_2) + m_a^2(p_3\cdot p_4) - 2m_a^2m_b^2\Big\}. \end{aligned} \tag{5.61}$$

The rest of the calculation can be considerably simplified by changing to the center of mass frame, where $p_1 = (E_1, \mathbf{p})$, $p_3 = (E_2, -\mathbf{p})$, $p_2 = (E_1, \mathbf{p}')$, $p_4 = (E_2, -\mathbf{p}')$, which implies that $q^\mu = (0, \mathbf{p} - \mathbf{p}')$ and $|\mathbf{p}| = |\mathbf{p}'|$. In this frame, the amplitude $\mathcal{M}$ calculated in (5.61) reduces to [87]

$$\mathcal{M}(\mathbf{q}) = -\frac{16\pi G_N}{\mathbf{q}^2}\left[\mathbf{q}^2(p_1\cdot p_3) + 2(p_1\cdot p_3)^2 - m_a^2m_b^2\right], \tag{5.62}$$

which, when Fourier transformed, yields the first order potential. To proceed, we pick a specific normalization for the amplitude, corresponding to the non-relativistic external states of the scalar particles. This amounts to the substitution

$$\mathcal{M}(p_1,p_2,p_3,p_4) \to \frac{1}{\sqrt{2E_{p_1}2E_{p_2}2E_{p_1}2E_{p_4}}}\mathcal{M}(p_1,p_2,p_3,p_4), \tag{5.63}$$

which in the CM frame corresponds to a factor of $1/4E_aE_b$. Thus:

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4E_aE_b}\int \frac{d^3\mathbf{q}'}{(2\pi)^3}\mathcal{M}(\mathbf{q}')e^{i\mathbf{q}'\cdot\mathbf{r}} \\ &= \frac{4\pi G_N}{E_aE_b}\left\{\delta^3(\mathbf{r}) - \frac{1}{4\pi r}\left[2(p_1\cdot p_3)^2 - m_a^2m_b^2\right]\right\} . \end{aligned} \tag{5.64}$$

A first feature to note in the above is the delta function which encodes an ultra-local behavior reminiscent of the point particle description. The more interesting part arises from the second term which we define as the 1PM potential. Specifically:

$$V_{\rm 1PM}(r) = \frac{G_N}{r}\frac{1}{E_aE_b}\left[m_a^2m_b^2 - 2(p_1\cdot p_3)^2\right] . \tag{5.65}$$

If one expands the energies in the denominator as powers of the momentum $\mathbf{p}^2$, it is apparent that the above will produce a series of powers in v^2, the speed of the particle in CM frame. Therefore, inside of this first 1PM term, we have PN terms. To be more explicit, denoting the numerator of Eq. (5.65) as $c_{\rm 1pm}$,

$$\begin{aligned} V_{\rm 1PM}(r) &= \frac{G_Nc_{\rm 1pm}}{m_am_br}\left(\frac{\mathbf{p}^2}{m_a^2}+1\right)^{-1/2}\left(\frac{\mathbf{p}^2}{m_a^2}+1\right)^{-1/2} \\ &= \frac{G_Nc_{\rm 1pm}}{m_am_br}\sum_{n,m=0}^{\infty}\frac{(-1)^{n+m}(2n-1)!!(2m-1)!!}{2^{n+m}n!m!}\frac{\mathbf{p}^{2(m+n)}}{m_a^{2n}m_b^{2m}} , \end{aligned} \tag{5.66}$$

which, upon writing $c_{\rm 1pm}(p) = -2(\mathbf{p}^2+E_aE_b)^2+m_a^2m_b^2$, leads to a series in powers of v^2, as advertised. It is worth mentioning that, of course, the zeroth order PN term corresponds to the Newtonian potential between the two masses. Furthermore, the complete 1PN potential also requires considerations of loop contributions arising from triangle graviton exchange diagrams. The detailed discussion can be seen in [87, 112].

So far, we have relied on the straightforward but tedious evaluation of Feynman diagrams using the standard methods. While simple enough to be performed in this case, the expressions obtained in Eqs. (5.60) and (5.61) can, nevertheless, be directly obtained using more modern approaches. For this, we recall the discussion in section 2.2.3, where we constructed the three point amplitudes involving a single massless leg and two massive ones (with the same mass). Then, the minimal coupling of those amplitudes for the

gravitational case is simply

$$M(\mathbf{1},\mathbf{2},k^{++}) = \alpha x_{12}^2 \frac{\langle \mathbf{12}\rangle^{2S}}{m^{2S-1}}\,; \qquad M(\mathbf{1},\mathbf{2},k^{--}) = \alpha x_{12}^{-2} \frac{[\mathbf{12}]^{2S}}{m^{2S-1}}\,, \tag{5.67}$$

where S corresponds to the massive particle's spin and α is the coupling, whose specific form depends on the spin of the massive particle. Furthermore, the x factor satisfies

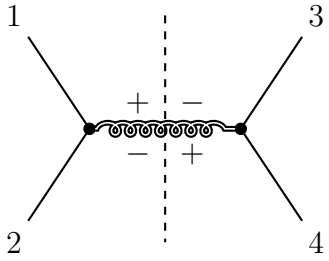
$$x_{12}|k] = \frac{1}{m} p_1 \,|k\rangle \ , \qquad x_{12}\,|k\rangle = \frac{1}{m} p_1 |k]\,. \tag{5.68}$$

For this specific case of scalar particle scattering by gravitons, considering the assignments in the diagram, these amplitudes become

$$M(\mathbf{1},\mathbf{2},k^{++}) = \frac{1}{2}\kappa m_a^2 x_{12}^2\,; \qquad M(\mathbf{1},\mathbf{2},k^{--}) = \frac{1}{2}\kappa m_a^2 x_{12}^{-2}\,, \tag{5.69}$$

where the massess are squared to match the dimensions of the coupling ($[\kappa^2] \sim m^{-2}$).

To construct the four point amplitude, we glue the three point amplitudes considering that the internal mediator is the massless graviton. This corresponds to



$$= \frac{\kappa^2 m_a^2 m_b^2}{4t}\left(\frac{x_{12}^2}{x_{34}^2} + \frac{x_{34}^2}{x_{12}^2}\right)\,, \tag{5.70}$$

with $t = (p_1 - p_2)^2 = q^2$. Note that the dashed line represents the sum of both possibilities for the helicity assignments of the inner graviton. To simplify the above, we first analyze the following:

$$\frac{x_{12}}{x_{34}}|k]^{\dot\alpha} = \frac{p_3^{\dot\alpha\alpha}}{m_b} x_{12}\,|k\rangle_\alpha = \frac{p_3^{\dot\alpha\alpha} p_{1\alpha\dot\beta}}{m_a m_b}|k]^{\dot\beta}\,, \tag{5.71}$$

such that

$$\begin{aligned}
\left(\frac{x_{12}}{x_{34}} + \frac{x_{34}}{x_{12}}\right)|k]^{\dot\alpha} &= \frac{1}{m_a m_b}\left(p_3^{\dot\alpha\alpha} p_{1\alpha\dot\beta} + p_1^{\dot\alpha\alpha} p_{3\alpha\dot\beta}\right)|k]^{\dot\beta} \\
&\Downarrow \\
\left(\frac{x_{12}}{x_{34}} + \frac{x_{34}}{x_{12}}\right)\delta^{\dot\alpha}_{\ \dot\beta} &= \frac{1}{m_a m_b}\left(p_3^{\dot\alpha\alpha} p_{1\alpha\dot\beta} + p_1^{\dot\alpha\alpha} p_{3\alpha\dot\beta}\right)\,.
\end{aligned} \tag{5.72}$$

Consequently, we find that

$$\left(\frac{x_{12}}{x_{34}}+\frac{x_{34}}{x_{12}}\right) = \frac{2p_1\cdot p_3}{m_a m_b}\,, \tag{5.73}$$

$$\left(\frac{x_{12}^2}{x_{34}^2}+\frac{x_{34}^2}{x_{12}^2}\right) = \left\{\left(\frac{x_{12}}{x_{34}}+\frac{x_{34}}{x_{12}}\right)^2-2\right\} = \frac{4(p_1\cdot p_3)^2-2m_a^2m_b^2}{m_a^2m_b^2}\,. \tag{5.74}$$

Thus, the factorization immediately yields

$$\mathcal{M}(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}) = \frac{16\pi G_N}{t}\left[2(p_1\cdot p_3)^2-m_a^2m_b^2\right]\,, \tag{5.75}$$

which agrees with Eq. (5.61), up to the local term which originates the delta function in Eq. (5.64), and upon exchanging the t for the momentum transfer in the CM frame, i.e.: $t\to-\vec{\mathbf{q}}^2$. The fact that this term escapes this factorization argument is natural, after all, it does not correspond to a pole (it is a "contact term").

5.5 Light Bending

The phenomenon of the deviation of a null geodesic by the curvature of space-time induced by a given source has been paramount to the development of General Relativity as many sources point out [113, 80, 81]. Here, we will see how this same phenomenon can be described in the context of the scattering amplitudes paradigm.

5.5.1 Light by Light

As a first warm up, let us consider the interaction between two photons mediated by a graviton exchange. For another aproach to this, see For other sources discussing this, see [114]. The tree level diagram for the process is shown below.

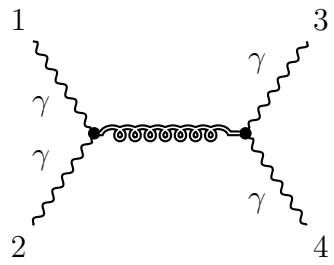


Figure 5.2: Tree level scattering of two photons mediated by a single graviton line.

First, recall that we could, in principle, carry out the calculation using the Feynman rule for the photon-graviton vertex we obtained above. On the other hand, the fact that this scattering process only involves massless particles renders the recursive method discussed in 2.3.2 very fitting. In fact, once again, all of the building blocks are immediate: the three point massless amplitudes! Consider the three point amplitude with two photons and a single graviton. Then, by recalling the little group scailing argument, there are 8 possibilities for the amplitude with different helicity combinations:

$$M_3(1^+2^+3^{++}) = \begin{cases} \kappa \langle 23\rangle^{-2} \langle 31\rangle^{-2} ; \\ \kappa [23]^2 [31]^2 ; \end{cases} \qquad M_3(1^-2^-3^{--}) = \begin{cases} \kappa \langle 23\rangle^{2} \langle 31\rangle^{2} ; \\ \kappa [23]^{-2} [31]^{-2} ; \end{cases} \tag{5.76}$$

$$M_3(1^-2^+3^{++}) = \begin{cases} \kappa \frac{\langle 12\rangle^2}{\langle 23\rangle^4 \langle 31\rangle^2} ; \\ \kappa \frac{[23]^4}{[12]^2} ; \end{cases} \qquad M_3(1^+2^-3^{--}) = \begin{cases} \kappa \frac{\langle 23\rangle^4}{\langle 12\rangle^2} ; \\ \kappa \frac{[12]^2}{[23]^4 [31]^2} ; \end{cases} \tag{5.77}$$

$$M_3(1^-2^-3^{++}) = \begin{cases} \kappa \frac{\langle 12\rangle^4}{\langle 23\rangle^2 \langle 31\rangle^2} ; \\ \kappa \frac{[23]^2 [31]^2}{[12]^4} ; \end{cases} \qquad M_3(1^+2^+3^{--}) = \begin{cases} \kappa \frac{\langle 23\rangle^2 \langle 31\rangle^2}{\langle 12\rangle^4} ; \\ \kappa \frac{[12]^4}{[23]^2 [31]^2} ; \end{cases} \tag{5.78}$$

$$M_3(1^+2^-3^{++}) = \begin{cases} \kappa \frac{\langle 12\rangle^2}{\langle 31\rangle^4} ; \\ \kappa \frac{[31]^4}{[12]^2} ; \end{cases} \qquad M_3(1^-2^+3^{--}) = \begin{cases} \kappa \frac{\langle 31\rangle^4}{\langle 12\rangle^2} ; \\ \kappa \frac{[12]^2}{[31]^4} ; \end{cases} \tag{5.79}$$

Now, to filter out the terms that don't contribute to the theory of general relativity, we can look at the mass dimensions of the above and compare to the mass dimensions of the photon-graviton three vertex of Eq.(5.43). Since $[G_N] \sim m^{-2}$, $[\kappa] \sim m^{-1}$, while the vertex has two powers of momentum, such that the overall mass dimension is 1. Recalling that the spinor products have mass dimension 1, we see that the only terms above which match the counting of the three vertex obtained from perturbation theory are

$$M_3(1^+, 2^-, 3^{++}) = \kappa \frac{[31]^4}{[12]^2} , \qquad M_3(1^-2^+3^{--}) = \kappa \frac{\langle 31\rangle^4}{\langle 12\rangle^2} , \tag{5.80}$$

$$M_3(1^-, 2^+, 3^{++}) = \kappa \frac{[23]^4}{[12]^2} , \qquad M_3(1^+2^-3^{--}) = \kappa \frac{\langle 23\rangle^4}{\langle 12\rangle^2} , \tag{5.81}$$

which all have mass dimension 1, as required. Thus, the other possibilities vanish and we find an amusing fact that gravitons do not couple to same helicity photons.

Using the above, we can construct the amplitude for a photon-photon scattering mediated by gravitons using, for example, appropriate BCFW shifts. Since the three point

amplitudes will be our building blocks, there are four point amplitudes we can already rule out. In fact, from the fact that the three point amplitudes with same helicity photons vanish, we immediately find that

$$M_4(1^+2^+3^+4^+) = M_4(1^+2^+3^-4^-) = M_4(1^-2^-3^+4^+) = M_4(1^-2^-3^-4^-) = 0 \quad (5.82)$$

$$M_4(1^+2^-3^+4^+) = M_4(1^+2^-3^-4^-) = M_4(1^-2^+3^+4^+) = M_4(1^-2^+3^-4^-) = 0 \quad (5.83)$$

$$M_4(1^+2^+3^+4^-) = M_4(1^+2^+3^-4^+) = M_4(1^-2^-3^+4^-) = M_4(1^-2^-3^-4^+) = 0\,. \quad (5.84)$$

Thus, all we need are the four amplitudes $M_4(1^\pm 2^\mp 3^\pm 4^\mp), M_4(1^\mp 2^\pm 3^\pm 4^\mp)$. Since they are related by parity, one only needs to calculate two of them.

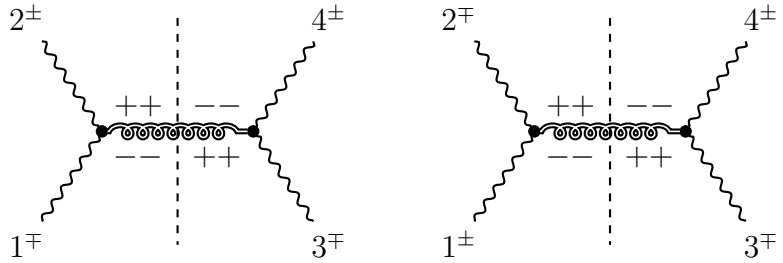


Figure 5.3: The only four helicity configurations needed for the tree level scattering of two photons mediated by a single graviton line.

Let us first, evaluate $M_4(1^-2^+3^-4^+)$ using a $[1,3\rangle$ shift. Then:

$$\begin{aligned} iM_4(1^-2^+3^-4^+) &= \sum_{h=\pm\pm} M_3(\hat{1}^-2^+\hat{P}_{12}^h)\frac{i}{P_{12}^2}M_3(-\hat{P}_{12}^{-h}4^+\hat{3}^-)\,, \\ &= i\kappa^2\left\{\frac{[2\hat{P}_{12}]^4}{[\hat{1}2]^2}\frac{1}{\langle 12\rangle[21]}\frac{\langle\hat{3}(-\hat{P}_{12})\rangle^4}{\langle 4\hat{3}\rangle^2} + \frac{\langle\hat{P}_{12}\hat{1}\rangle^4}{\langle\hat{1}2\rangle^2}\frac{1}{\langle 12\rangle[21]}\frac{[(-\hat{P}_{12})4]^4}{[4\hat{3}]^2}\right\}. \end{aligned} \quad (5.85)$$

Now, the on-shell momentum $\hat{P}_{12}$ implies $\hat{P}_{12}^2 = 0 \implies \langle\hat{1}2\rangle[2\hat{1}] = 0$. Thus, $[2\hat{1}]$ must vanish, which kills the first term above. We are then left with

$$M_4(1^-2^+3^-4^+) = \kappa^2\frac{\langle 12\rangle\,[24]^4}{[21][43]^2}\,, \quad (5.86)$$

which automatically leads to

$$M_4(1^+2^-3^+4^-) = \kappa^2\frac{[12]\,\langle 42\rangle^4}{\langle 21\rangle\,\langle 43\rangle^2}\,. \quad (5.87)$$

For $M_4(1^-2^+3^+4^-)$, one can perform a $[1,4\rangle$ shift, which leads to the same development as the one for $M_4(1^-2^+3^-4^+)$, only interchanging $3 \leftrightarrow 4$. We thus, have

$$M_4(1^-2^+3^+4^-) = \kappa^2 \frac{\langle 12\rangle\,[23]^4}{[21][43]^2}\,, \tag{5.88}$$

which gives, by parity,

$$M_4(1^+2^-3^-4^+) = \kappa^2 \frac{[12]\,\langle 23\rangle^4}{\langle 21\rangle\,\langle 43\rangle^2}\,. \tag{5.89}$$

In principle, these are all of the amplitudes one needs to compute if the external photons are distinguishable. By summing their squares, we find that the full transition amplitude is

$$\sum_{\{h_1,h_2,h_3,h_4\}} |M_4(1^{h_1}2^{h_2}3^{h_3}4^{h_4})|^2 = \frac{8\kappa^4}{(p_3\cdot p_4)^2}\Big[(p_2\cdot p_3)^4 + (p_2\cdot p_4)^4\Big] = 4\kappa^4\frac{u^4+t^4}{s^2} \tag{5.90}$$

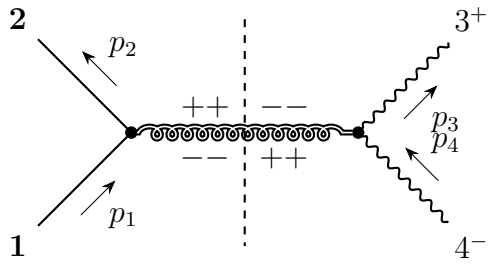
Note that the amplitude is suppressed by a factor $8\kappa^4 = 8192\pi^2 G_N^2 \approx 3.6\times 10^{-16}(SI)$ which is indeed very small, as expected for gravitational interactions of elementary particles.

5.5.2 Light by Scalar

Now, let us consider the case of a massive scalar interacting with a photon via graviton exchange. If we consider the same approach of gluing together the three point amplitudes, we now have to consider that on one side we have a photon-graviton-photon amplitude while on the other side we have a scalar-graviton-scalar one. As we saw before, the graviton will only couple to opposite helicity photons, which means that there are only two possibilities for each graviton helicity assignment.

At first, let us consider the amplitude $A(\mathbf{1},\mathbf{2},3^+,4^-)$. Then, the gluing procedure for

the three point amplitudes would be depicted by



$$\begin{aligned} &= \sum_{h=\pm} A_3(\mathbf{1},\mathbf{2},k^h)\frac{1}{q^2}A_3(k^{-h},3^+,4^-) \\ &= \frac{m^2\kappa^2}{2q^2}\left(x_{12}^2\frac{\langle q4\rangle^4}{\langle 34\rangle^2}+x_{12}^{-2}\frac{[q3]^4}{[34]^2}\right), \end{aligned}$$

with $q^2=(p_1-p_2)^2=-t$. The above can still be simplified by the kinematical properties of the 4 and 3 point amplitudes. Considering that $q=p_3-p_4$, we have $[q3]^2=[43]^2$ and also $\langle q4\rangle^2=\langle 34\rangle^2$, such that the amplitude can be written as

$$A(\mathbf{1},\mathbf{2},3^+,4^-)=\frac{m^2\kappa^2}{2q^2}\left(x_{12}^2\langle q4\rangle^2+\frac{1}{x_{12}^2}[q3]^2\right).$$

Furthermore, we recall that the x factor above is defined as the proportionality coefficient between the massless spinors associated with the massless q (graviton) leg and one of the massive momenta, either p_1 or p_2. In this sense, we can write

$$x_{12}\,|q\rangle=\frac{p_1}{m}|q]\,;\quad \frac{1}{x_{12}}|q]=\frac{p_1}{m}\,|q\rangle\ . \tag{5.91}$$

Consequently, the above expression could be rewritten as

$$\begin{aligned} A(\mathbf{1},\mathbf{2},3^+,4^-) &= \frac{\kappa^2}{2q^2}\left(\langle 4|p_1|q]^2+[3|p_1\,|q\rangle^2\right), \\ &= \frac{\kappa^2}{2q^2}\left([\langle 4|p_1|q]+[3|p_1\,|q\rangle]^2-2\,\langle 4|\,p_1|q]\,\langle q|\,p_1|3]\right). \end{aligned}$$

Now, note that the first term inside the parenthesis above can be expressed without reference to the spinors:

$$\begin{aligned} [\langle 4|p_1|q]+[3|p_1\,|q\rangle]^2 &= [\langle 4\mathbf{1}\rangle[\mathbf{1}|(|3]-|4])+[3\mathbf{1}]\langle\mathbf{1}|(|3\rangle-|4\rangle)]^2 \\ &= [\langle 4\mathbf{1}\rangle\,[\mathbf{1}3]-\langle 4\mathbf{1}\rangle\,[\mathbf{1}4]+[3\mathbf{1}]\,\langle\mathbf{1}3\rangle-[3\mathbf{1}]\,\langle\mathbf{1}4\rangle]^2 \\ &= (-\langle 4\mathbf{1}\rangle\,[\mathbf{1}4]+[3\mathbf{1}]\,\langle\mathbf{1}3\rangle)^2 \\ &= 4\,[(p_1\cdot p_4)-(p_1\cdot p_3)]^2\ . \end{aligned} \tag{5.92}$$

Similarly, for the second, one can use conservation of momentum in the form $|q]\langle q| = -q^{\dot{\alpha}\alpha} = -(p_3^{\dot{\alpha}\alpha} - p_4^{\dot{\alpha}\alpha}) = |3]\langle 3| - |4]\langle 4|$. Then, since $\langle 4|\, p_1|3] = [3|p_1\,|4\rangle$, we have

$$\begin{aligned} A(\mathbf{1},\mathbf{2},3^+,4^-) &= \frac{2\kappa^2}{q^2}\Big([(p_1\cdot p_4)-(p_1\cdot p_3)]^2 - [(p_1\cdot p_3)-(p_1\cdot p_4)]\,\langle 4|\, p_1|3]\Big) \\ &= \frac{2\kappa^2(p_1\cdot q)}{q^2}\Big((p_1\cdot q) - \langle 4|\, p_1|3]\Big) \end{aligned} \tag{5.93}$$

The opposite helicity amplitude $A(\mathbf{1},\mathbf{2},3^-,4^+)$ is now given by the parity conjugate of the above. Upon summing their moduli squared, we would obtain the full transition amplitude as before.

By using the amplitude above, one can recover some of the standard results of classical general relativity, notably, the scattering angle which, in GR, corresponds to a simple model of gravitational lensing. This calculation dives into field of describing classical theories and phenomena in terms of amplitudes, something which as we mentioned has garnered much attention lately. For this, there are many formalisms which have been developed and their discussion unfortunately goes beyond the scope of our (fairly) brief exposition (see [97, 115, 116]).

Chapter 6

Conclusions

In this dissertation we have studied some of the theoretical tools which constitute the basis for the perturbative description of quantum field theory. In this regard, we have presented the standard way in which these calculations are carried out via Feynman diagrams and also contrasted them with the on-shell approach which has been paramount in the scattering amplitudes community.

As we have discussed and exemplified, the philosophy behind this on-shell approach is deeply rooted in the representation theory of the Poincaré group. As far as some authors ([25, 17]) would put, the very definition of what is an *elementary particle* can be rephrased as *what are the irreducible representations of the Poincaré group*. As Wigner *et al* [117, 118] studied, the inhomogeneous Lorentz group is directly associated with the spin and momenta (thus, helicity) of a given state and this is precisely what leads one to the description of scattering amplitudes themselves in terms of helicity-spinors. To some extent, this is the Lego™ of fundamental particles: arbitrary spin states can be simply constructed from the smallest non-trivial irreducible representations, the 1/2 spinors. Thus, the power of the group theory description of fundamental physics remains unquestioned, allowing for investigations which elucidate the mathematical richness behind the interplay between relativity and quantum mechanics (see [119] for a discussion of how the massless little group for photons is related to the cylindrical group).

This story allows one to study the perturbative behavior of quantum field theories without needing to invoke the full apparatus one is usually acquainted with in a standard graduate course. This is particularly clear in the applications we discussed in chapter 5 to gravity. All of the introduction and the construction of the perturbative expansion of the Einstein-Hilbert Lagrangian was done in order to show the reader how the usual theory is developed and ultimately suffers from complexity in even the simplest tree-level processes of pure graviton scattering. On the other hand, as we discussed later in that chapter, the entire construction could be bypassed if one accepts that the field which should describe gravity is a massless spin 2 irreducible representation of the Poincaré group. Once this is done, the little group bootstrap is able to reproduce the helicity amplitude expressions in an almost self contained way, only requiring the appropriate identification and matching of the dimensions of the resulting expressions with the correct ones from the original Lagrangian. Furthermore, these methods have been combined with techniques (such as generalized unitarity) which allow one to push the precision of the theoretical calculations beyond the standard realm of feasibility (see, for example, [120, 38, 41, 121]). This, coupled to the increasing computational power of numerical algorithms could, perhaps, allow the complete calculation of the relativistic S-matrix of some theories, something thought of as almost impossible even in the simplest cases.

This reinstatement of the perturbative calculations has also drawn attention to other results which would probably have gone unnoticed without the new methods. Specifically, the interconnection between the amplitudes of theories of apparently completely different particle content has been brought to the spotlight (see the seminal work in [122, 123] and multiple others such as [124, 125, 126, 127]). Now, the so-called gauge gravity dualities have been reinstated as theories with *double copies* which allow one to fuse together the amplitudes of one theory (or two different theories) and arrive at the amplitudes of another theory. By now, the number of theories which satisfy these relations and the mapping of these double copies has been intensively studied (see the review [128] and [129, 130]). This

has also been explored at the Lagrangian level, instead of the perturbative one, in what is now called the *classical double copy* (see [131, 132, 133, 134, 135, 136, 137] for classical double copies and [138, 139] for the relation to twistors). These perturbative relations pave the way to deeper understandings of the underlying structure of gauge and gravity theories which seem to be the most complete mathematical descriptions the community has been able to construct so far. Furthermore, this avenue of work has benefited from conversations with other areas (see [140] for celestial double copies), including string theory, which was actually where these doubling relations were first discovered in the context of open and closed string scattering, the so-called KLT relations [141] (see also, newer developments such as [142, 30, 143] and the relation between string theory amplitudes and Compton gravitational ones [43]).

While we discussed these new investigations into the underlying mathematics of quantum field theory, we have also discussed how the standard framework can still yield novel results which have not been appreciated. Specifically, as we presented in chapter 4, the interplay between macroscopic structures—realized as non-asymptotic boundary conditions—and the quantum vacuum is a rich sea whose exploration is surely worthwhile. As we mentioned before, this fact is made undeniable by phenomena such as the Casimir effect, which yields the famous attractive force between neutral conducting plates. On the other hand, even with a single plate, the effect of the boundary conditions on some of the most basic results of undergraduate physics (namely the method of images for the Coulomb potential near a perfect mirror) yields unexpected behaviors at the quantum relativistic level. As we showed, the intensity of the leading α correction is much stronger than one would initially anticipate. Pragmatically speaking, while such effect is small, since no experiment is done in the absence of a boundary, such effects could surely become relevant as our technological capabilities become more and more precise. Ultimately, however, what will and will not be relevant is left for the future to decide!

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Appendix A

Formalism

A.1 Conventions

Throughout the text, we alternate between the mostly positive $\eta = \text{diag}(-1, +1, +1, +1)$ and mostly negative $\eta = \text{diag}(-1, +1, +1, +1)$ metrics for the sake of comparison with the results in the literature. Usually there will be a warning if such a signature change is carried out.

We use the natural system of units throughout, meaning that $c = \hbar = 1$.

For the spinor-helicity formalism, we always adapt the conventions so that square brackets are related to positive helicities and angled brackets are related to negative helicities. This means that the relationship between the dotted and undotted indices and the respective bracket type will depend on the convention for all incoming or all outgoing particles. To summarize, the relationship for each set is respectively:

$$|\lambda],\ [\lambda| \quad = \quad \begin{cases} \lambda^{\dot{\alpha}},\ \lambda_{\dot{\alpha}}\,, & \text{All Incoming} \\ \lambda_{\alpha},\ \lambda^{\alpha}\,, & \text{All Outgoing} \end{cases} \tag{A.1}$$

$$|\lambda\rangle,\ \langle\lambda| \quad = \quad \begin{cases} \lambda_{\alpha},\ \lambda^{\alpha}\,, & \text{All Incoming} \\ \lambda^{\dot{\alpha}},\ \lambda_{\dot{\alpha}}\,, & \text{All Outgoing} \end{cases} \tag{A.2}$$

For spatial three-vectors, we use bold notation (e.g $\mathbf{q}$) or arrows (e.g $\vec{q}$).

A.2 Spinor Indices

In this appendix we give a quick introduction to the spinor indices which appear throughout the text. More details can be found in [14, 15, 144].

To begin, consider the generators of Lorentz transformations Λ which we label as $M^{\mu\nu}$. Then, infinitesimal transformations are constructed from the generators via

$$U(1+\omega) = 1 + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\,, \tag{A.3}$$

where $\omega_{\mu\nu}$ is a matrix of infinitesimal parameters which is anti-symmetric. From the fact that these transformations form a group, they satisfy composition properties which allow for the calculation of the commutation relations between these generators:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}) - i(g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho})\,. \tag{A.4}$$

Now, these generators can be decomposed into the angular momentum operators J^i and the boost operators K^i by $J_i = \epsilon_{ijk}M^{jk}/2$ and $K^i = M^{0i}$. Expressing the commutators in terms of these operators, we find three relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k\,, \tag{A.5}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k\,, \tag{A.6}$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k\,. \tag{A.7}$$

The first of these relations can be identified with the familiar Lie algebra of the groups $SU(2)$ and $SO(3)$. Now, in order to simplify these commutation relations to a form which is more suggestive, we define

$$N_i := \frac{1}{2}(J_i + iK_i)\,, \quad N_i^\dagger := \frac{1}{2}(J_i - iK_i)\,. \tag{A.8}$$

If we use this mapping to re-express the commutation relations for the Lorentz group, we

find that the corresponding algebra factorizes into *two* $su(2)$ algebras:

$$[N_i, N_j] = i\epsilon_{ijk}N_k\,, \tag{A.9}$$

$$[N_i, N_j^\dagger] = 0\,, \tag{A.10}$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger\,. \tag{A.11}$$

What this tells us, is that the representations of the proper Lorentz group $SO^+(1,3)$ can be decomposed into two copies of $SU(2)$ related by hermitian conjugation that do not mix–that is, two elements of each commute with one another.

Since $SU(2)$ has elements which we can label according to integer of half-integer numbers j, one can refer to representations of the Lorentz group by a set (j, j') with $j = 0, 1/2, 1, 3/2 \ldots$. Then we refer to $(1/2, 0)$ representations as *left-handed* 1/2 spinors, whose two components we index with two dimensional un-dotted spinor indices, while $(0, 1/2)$ would correspond to the *right-handed* ones, whose components we index with dotted indices. These two irreducible representations are also referred to as $Weyl$ (left-handed or right-handed) spinors. For a field $\psi^{\dot\alpha}(x)$ transforming in the right-handed Weyl spinor representation, for example, one would have

$$U(\Lambda)\psi^{\dot\alpha}(x)U^{-1}(\Lambda) = L^{\dot\alpha}_{\ \dot\beta}\psi^{\dot\beta}(\Lambda^{-1}x) \tag{A.12}$$

with $L^{\dot\alpha}_{\ \dot\beta}$ being an element of the $SU(2)$ group in the $(0, 1/2)$ representation. Just as for the usual way one introduces the metric as an invariant object under Lorentz transformations, one can introduce a metric-like object which will serve to raise and lower the spinor indices. This is actually the anti-symmetrical tensor: $\epsilon_{\alpha\beta}$ or $\epsilon_{\dot\alpha\dot\beta}$ depending on the representation.

As we discussed in the text, there is also a natural relationship between the groups SO(3,1) and SU(2, $\mathbb{C}$) which can be used to map any four vector into a complex unitary matrix which transforms in the representation (1/2,1/2). The existence of such a relationship should not come as a surprise since the (1/2,1/2) representation has $2 \times 2 = 4$ components, just like a four vector. This property is in fact the starting point of twistor

theory, where one maps the space-time coordinates themselves. As noted in the text, the simplest example of this mapping is to take a given four vector v^μ and contract it with the Pauli 4-vector $\sigma^\mu_{\alpha\dot{\beta}} = (1_{\alpha\dot{\beta}}, \sigma_\alpha^{\ \dot{\beta}})^\mu$. However, this is very simply generalized to higher rank tensors by simply adding a contraction for each Lorentz index. As an example, one could contract the Minkowski metric itself:

$$\eta_{\alpha\dot{\alpha}\beta\dot{\beta}} := \eta_{\mu\nu}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} = -2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\,, \tag{A.13}$$

showing that there is indeed a mapping between the metrics of both spaces and the dotted and undotted basis naturally decouples from the mixed object in the left handed side of the first equality.

A.3 Quantization of Spinor Fields

Here, we will give a brief outline of the quantization of spinor fields, including their basic expansion in terms of annihilation-creation operators.

To begin, let us restate the Dirac Lagrangian for the Dirac field and its two chiral representations.

$$\mathcal{L}_{\text{Dirac}} = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi\,, \tag{A.14}$$

$$\Psi^{\text{T}} = \begin{pmatrix}\chi_a & \xi^{\dot{a}}\end{pmatrix}\,, \qquad \Psi^\dagger = \begin{pmatrix}\chi_{\dot{a}} & \xi^a\end{pmatrix} \tag{A.15}$$

This equation can also be written in terms of the two Weyl equations for each of the fields, which can be obtained by recalling the form of the gamma matrices (which we restate for reference):

$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix}\,,$$

$$\implies \overline{\Psi} = \Psi^\dagger\beta = \begin{pmatrix}\chi_{\dot{\alpha}} & \xi^\alpha\end{pmatrix}\begin{pmatrix}0 & \delta^{\dot{\alpha}}_{\ \dot{c}} \\ \delta_\alpha^{\ c} & 0\end{pmatrix} = \begin{pmatrix}\xi^c & \chi_{\dot{c}}\end{pmatrix}\,. \tag{A.16}$$

The canonical momentum associated to Ψ is given by

$$\Pi = \frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial\partial_0\Psi} = i\overline{\Psi}\gamma^0\,. \tag{A.17}$$

In order to quantize these fermion fields, we impose anticommutation relations between the field Ψ and its momentum Π. Thus, we write

$$\{\Psi_A(x), \Psi_B(y)\} = 0\,, \tag{A.18}$$

$$\{\Pi_A(x), \Pi_B(y)\} = 0\,, \tag{A.19}$$

$$\{\Psi_A(x), \Pi_B(y)\} = i\delta_{AB}\delta^3(\mathbf{x}-\mathbf{y})\,, \tag{A.20}$$

where we have labelled the spinor indices with the generic A, B. Upon using the definition of Π and the fact that $(\gamma^0)^2 = 1$, we can also write

$$\{\Psi(x), \overline{\Psi}(y)\} = \gamma^0\delta^3(\mathbf{x}-\mathbf{y})\,, \tag{A.21}$$

which makes it clear that the anticommutation is between 4 component matrices.

Now, we can introduce the Fourier decomposition of the Dirac field. For this, however, let us note that the equations of motion for the field Ψ are

$$\frac{\partial\mathcal{L}}{\partial\overline{\Psi}} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\overline{\Psi}} = (-m + i\partial\!\!\!/)\Psi = 0\,.$$

Multiplying by $(m + i\partial\!\!\!/)$ from the left and using that $\partial\!\!\!/\partial\!\!\!/ = -\partial^2$ (valid for the mostly positive signature), we see that the Dirac field also satisfies the Klein-Gordon equation

$$(m^2 - \partial^2)\Psi = 0\,.$$

Since the Klein-Gordon equation admits the usual Fourier expansion, we are led to consider

$$\Psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left(a(\mathbf{p})u(\mathbf{p})e^{ix\cdot p} + c^\dagger(\mathbf{p})v(\mathbf{p})e^{-ix\cdot p}\right)\,. \tag{A.22}$$

A few comments about (A.22) are noteworthy. Firstly, $a, b^\dagger$ are creation-annihilation operators, just as in the complex scalar case. Furthermore, u and v are momentum dependent four-component spinors. In fact, they satisfy the following equations

$$(p\!\!\!/ + m)u(\mathbf{p}) = 0\,, \tag{A.23}$$

$$(p\!\!\!/ - m)v(\mathbf{p}) = 0\,, \tag{A.24}$$

which are, in fact, just the momentum space Weyl equations. Each possess two solutions. It is more common to express Ψ in terms of these two solutions which are the components of u, v. Labeling the components with the index $s = \pm$, we write

$$\Psi(x) = \sum_{s=\pm} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ix\cdot p} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ix\cdot p} \right) . \tag{A.25}$$

We can describe the properties of the base spinors u_s, v_s by solving the Weyl equations. For massive particles, one can go to the rest frame where $\mathbf{k} = 0$ ($k^0 = m$) and further simplify the calculations. Specifically:

$$(-\gamma^0 + \mathbf{1}) m u_s(\mathbf{0}) = 0 , \tag{A.26}$$

$$\implies \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1^+ \\ a_1^- \\ a_2^+ \\ a_2^- \end{pmatrix} = 0 , \tag{A.27}$$

which implies two solutions:

$$u_+^T(\mathbf{0}) = a^+ \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} , \tag{A.28}$$

$$u_-^T(\mathbf{0}) = a^- \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix} . \tag{A.29}$$

For the appropriate conjugate $\overline{u}_s(\mathbf{0}) = u_s^\dagger(\mathbf{0})\gamma^0$, we have

$$\overline{u}_+(\mathbf{0}) = a^+ \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} , \tag{A.30}$$

$$\overline{u}_-(\mathbf{0}) = a^- \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix} . \tag{A.31}$$

We can now normalize these expressions as we see fit. As [15] we take $a^+ = a^- = \sqrt{m}$ such that

$$\overline{u}_{s'}(\mathbf{0}) u_s(\mathbf{0}) = 2m\delta_{ss'} . \tag{A.32}$$

A similar discussion holds for $v_s(\mathbf{0})$ and $\overline{v}_s(\mathbf{0})$, under opposite sign for the gamma matrix in equation (A.26). Thus

$$v_+(\mathbf{0}) = \sqrt{m} \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix} , \tag{A.33}$$

$$v_-(\mathbf{0}) = \sqrt{m} \begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix} . \tag{A.34}$$

Note that for the opposite metric signature, that is, $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$, we have an exchange of $u_\pm \to v_\pm$ and vice versa.

To obtain the spinors in a generic frame, we can apply the Lorentz boosts to $u_s(\mathbf{0}), v_s(\mathbf{0})$. However, the orthogonality of them is unaffected by this, such that

$$\overline{u}_{s'}(\mathbf{p})\, u_s(\mathbf{p}) = 2m\delta_{ss'}\,, \tag{A.35}$$

$$\overline{v}_{s'}(\mathbf{p})\, v_s(\mathbf{p}) = -2m\delta_{ss'}\,. \tag{A.36}$$

To explicitly write u_s, v_s, let us recall how they transform under the Lorentz boosts. Denoting the unitary transformation by $U(\Lambda)$, for a general 1/2 spinor, we have

$$U(\Lambda) = \exp(i\theta_{\mu\nu}S^{\mu\nu})\,, \tag{A.37}$$

where $S^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/4$ and $\theta_{\mu\nu}$ are the parameters. For the boost operators, we have $K^i = S^{i0} = i[\gamma^i, \gamma^0]/4 = i\gamma^i\gamma^0/2$. Thus, for a boost with rapidity $\boldsymbol{\eta} = \hat{\mathbf{p}}\eta = \hat{\mathbf{p}}\sinh(|\mathbf{p}|/m)$, the spinor $u_s(\mathbf{p})$ will be given by

$$u_s(\mathbf{p}) \;=\; \exp(i\boldsymbol{\eta}\cdot\mathbf{K})u_s(\mathbf{0})\,.$$

Another set of useful identities are the Gordon identities. In their general form, we have

$$2m\overline{u}_{s'}(\mathbf{p}')\gamma^\mu u_s(\mathbf{p}) \;=\; \overline{u}_{s'}(\mathbf{p}')\left[p^\mu + p'^\mu + 4iS^{\mu\nu}(p_\nu - p'_\nu)\right] u_s(\mathbf{p})\,, \tag{A.38}$$

$$-2m\overline{v}_{s'}(\mathbf{p}')\gamma^\mu v_s(\mathbf{p}) \;=\; \overline{v}_{s'}(\mathbf{p}')\left[p^\mu + p'^\mu + 4iS^{\mu\nu}(p_\nu - p'_\nu)\right] v_s(\mathbf{p})\,, \tag{A.39}$$

But the more useful form comes from the case $\mathbf{p}' = \mathbf{p}$:

$$\overline{u}_{s'}(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) \;=\; 2p^\mu\delta_{ss'}\,, \tag{A.40}$$

$$\overline{v}_{s'}(\mathbf{p})\gamma^\mu v_s(\mathbf{p}) \;=\; 2p^\mu\delta_{ss'}\,. \tag{A.41}$$

Also important are the spin sums. Following ([15]), we can calculate

$$\sum_{s=\pm} u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) \;=\; -\not{p} + m\,, \tag{A.42}$$

$$\sum_{s=\pm} v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) \;=\; -\not{p} - m\,. \tag{A.43}$$

Now, let us deduce the anti-commutation relations between the creation and annihilation operators for the spinor Ψ. To this end, it is useful to find the creation-annihilation operators in terms of the Dirac spinors. First, let us consider

$$\begin{aligned}\int d^3\mathbf{x}\,\Psi(x)e^{-ix\cdot p'} &= \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3}\int d^3\mathbf{x}\,\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(b_s(\mathbf{p})u_s(\mathbf{p})e^{ix\cdot(p-p')} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ix\cdot(p+p')}\Big)\,,\\ &= \sum_s \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(b_s(\mathbf{p})u_s(\mathbf{p}) + d_s^\dagger(-\mathbf{p})v_s(-\mathbf{p})e^{2ix^0\omega_{\mathbf{p}}}\Big)\,. \qquad \text{(A.44)}\end{aligned}$$

Using $\overline{u}_{s'}(\mathbf{p})\gamma^0 u_s(\mathbf{p}) = 2\omega_{\mathbf{p}}\delta_{ss'}$ and $\overline{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) = 0$ we obtain

$$b_s(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\int d^3\mathbf{x}\,\overline{u}_s(\mathbf{p})\gamma^0\Psi(x)e^{-ix\cdot p}\,. \qquad \text{(A.45)}$$

If, on the other hand, we used

$$\begin{aligned}\int d^3\mathbf{x}\,\Psi(x)e^{ix\cdot p'} &= \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3}\int d^3\mathbf{x}\,\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(b_s(\mathbf{p})u_s(\mathbf{p})e^{ix\cdot(p+p')} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ix\cdot(p-p')}\Big)\,,\\ &= \sum_s \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(b_s(-\mathbf{p})u_s(-\mathbf{p})e^{-2ix^0\omega_{\mathbf{p}}} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})\Big)\,. \qquad \text{(A.46)}\end{aligned}$$

Using $\overline{v}_{s'}(\mathbf{p})\gamma^0 v_s(\mathbf{p}) = 2\omega_{\mathbf{p}}\delta_{ss'}$ and $\overline{v}_{s'}(\mathbf{p})\gamma^0 u - s(-\mathbf{p}) = 0$, we can isolate $d_s^\dagger(\mathbf{p})$:

$$d_s^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\int d^3\mathbf{x}\,\overline{v}_s(\mathbf{p})\gamma^0\Psi(x)e^{ix\cdot p}\,. \qquad \text{(A.47)}$$

If the metric signature was the opposite, the only difference in the above expressions would be the phase $e^{ix\cdot p} \to e^{-ix\cdot p}$ and vice versa.

With the expressions for the operators, we can find the anticommutators of each pair of operators. We must take care to impose equal times, such that $x^\mu = (t,\mathbf{x}), y^\mu = (t,\mathbf{y})$. For $d_s, d_s^\dagger$:

$$\begin{aligned}\{d_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} &= \frac{1}{\sqrt{4\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}\int d^3\mathbf{x}\int d^3\mathbf{y}\,e^{i\mathbf{x}\cdot\mathbf{p}-i\mathbf{y}\cdot\mathbf{p}'}\overline{v}_s(\mathbf{p})\gamma^0\{\Psi(t,\mathbf{x}),\overline{\Psi}(t,\mathbf{y})\}\gamma^0 v_{s'}(\mathbf{p}')\,,\\ &= \frac{(2\pi)^3}{\sqrt{4\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}\int d^3\mathbf{x}\,e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')}\overline{v}_s(\mathbf{p})(\gamma^0)^3 v_{s'}(\mathbf{p}')\,,\\ &= (2\pi)^3\delta_{ss'}\delta^3(\mathbf{p}-\mathbf{p}')\,. \qquad \text{(A.48)}\end{aligned}$$

Similarly, we can find

$$\{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} = (2\pi)^3\delta_{ss'}\delta^3(\mathbf{p}-\mathbf{p}'). \qquad \text{(A.49)}$$

If we chose the normalization factor of the integration measure as $1/2\omega_{\mathbf{p}}$ instead of $\sqrt{2\omega_{\mathbf{p}}}$, the above anticommutation relationships would gain a multiplicative factor of $2\omega_{\mathbf{p}}$, as in [15].

A.4 Ward-Takahashi Identity

A very important result which is used in calculations of QED processes, is the Ward-Takahashi identity. It can be derived by using the gauge invariance of the fermion field in the form

$$\psi \to e^{-i\alpha(x)}\psi\,. \tag{A.50}$$

As we know, under this transformation, the Dirac term of the Lagrangian changes as

$$i\overline{\psi}\partial\!\!\!/\psi \to i\overline{\psi}\partial\!\!\!/\psi + \overline{\psi}\gamma^{\mu}\psi\partial_{\mu}\alpha\,.$$

Now, consider, for example, the correlation function $\langle\psi(x_1)\overline{\psi}(x_2)\rangle$. Under this transformation, to first order, we have $\psi \to (1 - i\alpha)\psi$, such that

$$\begin{aligned}\langle\psi(x_1)\overline{\psi}(x_2)\rangle &= \int\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\left\{i\int d^4x\left[\overline{\psi}(i\partial\!\!\!/ + m)\psi + \cdots\right]\right\}\psi(x_1)\overline{\psi}(x_2)\,,\\ &= \int\int \mathcal{D}\psi\mathcal{D}\overline{\psi}(1 - i\alpha(x_1))\psi(x_1)(1 + i\alpha(x_2))\overline{\psi}(x_2)e^{iS}\left(1 + i\int d^4x\overline{\psi}\gamma^{\mu}\psi\partial_{\mu}\alpha\right)\,,\end{aligned}$$

and, upon canceling the first term, we arrive at

$$\int \mathcal{D}\psi\mathcal{D}\overline{\psi}e^{iS}\left[i\int d^4x\overline{\psi}\gamma^{\mu}\psi\partial_{\mu}\alpha\psi(x_1)\overline{\psi}(x_2) + i\alpha(x_2)\psi(x_1)\overline{\psi}(x_2) - i\alpha(x_1)\psi(x_1)\overline{\psi}(x_2)\right] = 0\,.$$

If we integrate by parts the first term and factor out the space-time integral from the path integrals on the fermion fields, we can write

$$\int d^4x\alpha(x)\int \mathcal{D}\psi\mathcal{D}\overline{\psi}e^{iS}\left[-i\partial_{\mu}\left(\overline{\psi}\gamma^{\mu}\psi\psi(x_1)\overline{\psi}(x_2)\right) + i\psi(x_1)\overline{\psi}(x_2)(\delta^4(x - x_2) - \delta^4(x - x_1))\right] = 0\,,$$

or, identifying the QED current as $j^{\mu}(x) = \overline{\psi}\gamma^{\mu}\psi$, we obtain

$$\partial_{\mu}\left\langle j^{\mu}(x)\psi(x_1)\overline{\psi}(x_2)\right\rangle = \left\langle\psi(x_1)\overline{\psi}(x_2)\right\rangle(\delta^4(x - x_2) - \delta^4(x - x_1))\,. \tag{A.51}$$

As these four Schwinger-Dyson equations, we can obtain multiple differential equations for the correlation functions and the Noether currents associated with the specific transformations under study. On the other hand, let us Fourier transform the equation above to momentum space. To this end, we define

$$M^{\mu}(p,q_1,q_2) \equiv \int d^4x_1\, d^4x_2 d^4x e^{ip\cdot x} e^{iq_1\cdot x_1} e^{-iq_2\cdot x_2} \left\langle j^{\mu}(x)\psi(x_1)\overline{\psi}(x_2)\right\rangle ,$$
$$G(q_1,q_2) = \int d^4x_1 d^4x_2 e^{iq_1\cdot x_1} e^{-iq_2\cdot x_2} \left\langle \psi(x_1)\overline{\psi}(x_2)\right\rangle .$$

Now, we can observe that

$$\int d^4x_1\, d^4x_2 d^4x e^{ip\cdot x} e^{iq_1\cdot x_1} e^{-iq_2\cdot x_2} \partial_{\mu} \left\langle j^{\mu}\psi(x_1)\overline{\psi}(x_2)\right\rangle$$
$$= -ip_{\mu} \int d^4x_1\, d^4x_2 d^4x e^{ip\cdot x} e^{iq_1\cdot x_1} e^{-iq_2\cdot x_2} \left\langle j^{\mu}\psi(x_1)\overline{\psi}(x_2)\right\rangle ,$$

while

$$\int d^4x_1\, d^4x_2 d^4x e^{ip\cdot x} e^{iq_1\cdot x_1} e^{-iq_2\cdot x_2} \left\langle \psi(x_1)\overline{\psi}(x_2)\right\rangle (\delta^4(x-x_2) - \delta^4(x-x_1))$$
$$= \int d^4x_1\, d^4x_2 e^{iq_1\cdot x_1} e^{-iq_2\cdot x_2} \left\langle \psi(x_1)\overline{\psi}(x_2)\right\rangle (e^{ip\cdot x_2} - e^{ip\cdot x_1}) ,$$

or, in terms of the definitions of M^{μ} and G, we find that

$$ip_{\mu}M^{\mu}(p,q_1,q_2) = G(q_1+p,q_2) - G(q_1,q_2-p) . \tag{A.52}$$

This is a so-called Ward-Takahashi identity for QED. For S-matrix elements, say $M^{\mu}(p)$, this identity implies $p_{\mu}M^{\mu}(p) = 0$, which is referred to as the Ward identity. Note that the definition of G is just the two point Green's function in momentum space.

A.5 Generators in SHF

As we have discussed, one can choose to study spinors and other representations of the Lorentz group in terms of spinor helicity variables or twistors. In this sense, a natural question would be how the Lorentz generators look like in the $SU(2)\otimes SU(2)$ representations. Recall that, in general, the Lorentz generators which we will denote $J^{\mu\nu}$ are given

by

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}\,, \tag{A.53}$$

where $S^{\mu\nu}$ corresponds to the intrinsic spin of the field and $L^{\mu\nu} = x^{[\mu}p^{\nu]}$ is the "orbtial" part. For a scalar, one only considers the first part, and so

$$J^{\mu\nu} = L^{\mu\nu} = x^{[\mu}p^{\nu]} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu}\,. \tag{A.54}$$

As an operator acting on a scalar field $\phi(x)$, the above can be written using the momentum operator, namely $p^{\nu} = -i\partial^{\nu}$. If one is studying the momentum space representation of the field, i.e.: $\tilde{\phi}(p)$, the position operator can be written $x^{\mu} = i\partial^{\mu}_{p}$, such that $\tilde{L}$ (the generator acting on the momentum space field $\tilde{\phi}(p)$) becomes

$$\tilde{L}^{\mu\nu} = ip^{[\mu}\frac{\partial}{\partial p_{\nu]}}\,. \tag{A.55}$$

To see how $J^{\mu\nu}$ would look like if it was acting on helicity spinors, one must recall that, depending on the chirality, the spin generator $S^{\mu\nu}$ in (A.53) will be given by

$$(S^{\mu\nu}_{L})_{a}{}^{b} = \frac{i}{4}\sigma^{[\mu}_{a\dot{a}}\overline{\sigma}^{\nu]\dot{a}b}\,, \tag{A.56}$$

for left handed Weyl spinors, or

$$(S^{\mu\nu}_{R})^{\dot{a}}_{\ \dot{b}} = \frac{i}{4}\overline{\sigma}^{[\mu|\dot{a}a}\sigma^{\nu]}_{a\dot{b}}\,, \tag{A.57}$$

for right handed ones. In this sense, one can consider introducing the SL(2,$\mathbb{C}$) representation by projecting $\tilde{L}^{\mu\nu}$ onto each representation, i.e.: (1/2,0) or (0,1/2). Let us denote the left handed by m_{ab} and the right handed by $\overline{m}_{\dot{a}\dot{b}}$. For m, one has

$$m_{ab} = (S^{\mu\nu}_{L})_{ab}\tilde{L}_{\mu\nu} = -\frac{1}{4}\varepsilon_{bc}\sigma^{[\mu}_{a\dot{a}}\overline{\sigma}^{\nu]\dot{a}c}p_{[\mu}\frac{\partial}{\partial p^{\nu]}}\,. \tag{A.58}$$

By expanding, we find

$$\begin{aligned} m_{ab} &= -\frac{1}{4}\varepsilon_{bc}(\sigma^{\mu}_{a\dot{a}}\overline{\sigma}^{\nu\dot{a}c} - \sigma^{\nu}_{a\dot{a}}\overline{\sigma}^{\mu\dot{a}c})(p_{\mu}\frac{\partial}{\partial p^{\nu}} - p_{\nu}\frac{\partial}{\partial p^{\mu}})\,, \\ &= -\frac{1}{2}\varepsilon_{bc}p_{a\dot{a}}\overline{\sigma}^{\mu\dot{a}c}\frac{\partial}{\partial p^{\mu}} + \frac{1}{2}\varepsilon_{bc}\sigma^{\mu}_{a\dot{a}}p^{\dot{a}c}\frac{\partial}{\partial p^{\mu}}\,. \end{aligned} \tag{A.59}$$

Now, in order to express the derivatives with respect to the momentum p^μ, we can use the following for the bi-spinors:

$$p^\mu = \frac{1}{2}\sigma^\mu_{a\dot{a}} p^{\dot{a}a} = \frac{1}{2}\lambda^a \sigma^\mu_{a\dot{a}} \widetilde{\lambda}^{\dot{a}} \,, \tag{A.60}$$

$$p^\mu = \frac{1}{2}\overline{\sigma}^{\mu\dot{a}a} p_{a\dot{a}} = \frac{1}{2}\widetilde{\lambda}_{\dot{a}} \overline{\sigma}^{\mu\dot{a}a} \lambda_a \,, \tag{A.61}$$

so that, for example:

$$\frac{\partial}{\partial \lambda^a} = \frac{\partial p^\mu}{\partial \lambda^a}\frac{\partial}{\partial p^\mu} = \frac{1}{2}\sigma^\mu_{a\dot{a}} \tilde{\lambda}^{\dot{a}} \frac{\partial}{\partial p^\mu} \,, \tag{A.62}$$

$$\frac{\partial}{\partial \lambda_a} = \frac{\partial p^\mu}{\partial \lambda_a}\frac{\partial}{\partial p^\mu} = \frac{1}{2}\tilde{\lambda}_{\dot{a}} \overline{\sigma}^{\mu\dot{a}a} \frac{\partial}{\partial p^\mu} \,. \tag{A.63}$$

With the above, equation (A.59) becomes

$$m_{ab} = -\varepsilon_{bc}\lambda_a \frac{\partial}{\partial \lambda_c} + \lambda_b \frac{\partial}{\partial \lambda^a} \,. \tag{A.64}$$

Since $\lambda_c = \varepsilon_{cb}\lambda^b$, we have $\frac{\partial \lambda_c}{\partial \lambda^b} = \varepsilon_{cb} = -\varepsilon_{bc}$, such that the first term can be reexpressed as

$$\lambda_a \frac{\partial \lambda_c}{\partial \lambda^b}\frac{\partial}{\partial \lambda_c} = \lambda_a \frac{\partial}{\partial \lambda^b} \,,$$

where the last equality is just the chain rule. In conclusion,

$$m_{ab} = \lambda_a \frac{\partial}{\partial \lambda^b} + \lambda_b \frac{\partial}{\partial \lambda_a} \,. \tag{A.65}$$

The same analysis is valid for $\overline{m}_{\dot{a}\dot{b}}$. The only difference, is that one exchanges $\lambda \to \widetilde{\lambda}$, and we have

$$\overline{m}_{\dot{a}\dot{b}} = \widetilde{\lambda}_{\dot{a}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{b}}} + \widetilde{\lambda}_{\dot{b}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{a}}} \,. \tag{A.66}$$

The reason this construction is actually useful can be seen in the analysis of Lorentz invariant quantities. One example of this is $[ij], \langle ij \rangle, s_{ij}$ or any general function of the $\lambda, \widetilde{\lambda}$ spinors. For a function of N particles, where each of them carries momentum p_i, $i = 1, 2 \ldots N$, the appropriate generators are constructed by summing over each label:

$$m_{ab} = \sum_{i=1}^{N} \left(\lambda_a \frac{\partial}{\partial \lambda^b} + \lambda_b \frac{\partial}{\partial \lambda_a} \right) \,, \tag{A.67}$$

$$\overline{m}_{\dot{a}\dot{b}} = \sum_{i=1}^{N} \left(\widetilde{\lambda}_{\dot{a}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{b}}} + \widetilde{\lambda}_{\dot{b}} \frac{\partial}{\partial \widetilde{\lambda}_{\dot{a}}} \right) \,. \tag{A.68}$$

Thus, one can apply (A.67) to say $[ij]$ and (A.68) to $\langle ij \rangle$ and check that the result is zero. Then, a mixed object such as $s_{ij} = [ij] \langle ji \rangle$ is invariant under the action of each of them. A consistency requirement, comes from the fact that directly applying $M_{\mu\nu}$ in the momentum representation also gives zero, as expected.

Another relevant operator is the helicity generator. In our conventions, $\tilde{\lambda}$ is associated with *positive helicity* (square brackets) and λ is associated with *negative helicity* (angled brackets). In this sense, one can define the operator h which measures helicity by

$$h = s \sum_{i=1}^{N} \left(-\lambda_i^a \frac{\partial}{\partial \lambda_i^a} + \widetilde{\lambda}_i^{\dot{a}} \frac{\partial}{\partial \widetilde{\lambda}_i^{\dot{a}}} \right) , \tag{A.69}$$

where s is the spin and i labels each particle (for a single particle, there is no sum). Thus, for a particle with negative helicity, we have $h\lambda^a = -s\lambda^a$, while $h\widetilde{\lambda}^{\dot{a}} = s\widetilde{\lambda}^{\dot{a}}$.

Appendix B

Useful Formulae

B.1 Feynman parameters

We outline the derivation of the Feynman parameter representation for fractions with polynomial denominators. We first, consider the relation

$$\frac{\Gamma(\alpha)}{A^\alpha} = \int_0^\infty t^{\alpha-1} e^{-At} dt\,, \tag{B.1}$$

which is obtainable by a simple substitution $t \to At$. Here, A is a function of x. For multiple different functions $A_1,\ A_2 \ldots A_n$ with associated powers $\alpha_1,\ \alpha_2 \ldots \alpha_n$, we can multiply the respective equations (B.1) to obtain

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_n^{\alpha_n}} = \prod_{i=1}^n \frac{1}{\Gamma(\alpha_i)} \int_0^\infty t_i^{\alpha_i - 1} e^{-A_i t_i} dt_i\,.$$

Since all the t's are positive, we can insert the following identity

$$1 = \int_0^\infty ds\, \delta(s - \sum_j t_j)\,,$$

$$\implies \frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_n^{\alpha_n}} = \prod_{i=1}^n \left(\frac{1}{\Gamma(\alpha_i)} \int_0^\infty dt_i\, t_i^{\alpha_i - 1} e^{-A_i t_i} \right) \int_0^\infty ds\, \delta(s - \sum_j t_j)\,,$$

by exchanging the order of integration:

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_n^{\alpha_n}} = \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty ds \int_0^\infty dt_1\, t_1^{\alpha_1 - 1} e^{-A_1 t_1} \ldots \delta(s - \sum_j t_j)\,,$$

and upon changing variables to $t_j \to s x_j$ in the time integrals:

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \dots A_n^{\alpha_n}} = \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^\infty ds \int_0^1 s\, dx_1\, (s x_1)^{\alpha_1 - 1} e^{-A_1 s x_1} \dots \delta(s[1 - \sum_j x_j])\,.$$

Now, we use the scaling property of the delta to take s out of the argument: $\delta(s(1 - \Sigma_j x_j)) = s^{-1}\delta(1 - \Sigma_j x_j)$, and also exchange the order of integration once again:

$$\begin{aligned}
\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \dots A_n^{\alpha_n}} =& \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^1 dx_1\, x_1^{\alpha_1 - 1} \dots \delta([1 - \sum_j x_j]) \int_0^\infty ds\, s^{\Sigma_k \alpha_k - 1} e^{-s \sum_\ell A_\ell x_\ell}\,, \\
=& \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^1 dx_1\, x_1^{\alpha_1 - 1} \dots \delta([1 - \sum_j x_j]) \frac{\Gamma\left(\sum_k \alpha_k\right)}{\left(\sum_\ell x_\ell A_\ell\right)^{\sum_k \alpha_k}}\,, \\
=& \frac{\Gamma\left(\sum_k \alpha_k\right)}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^1 dx_1 \cdots \int_0^1 dx_n\, \delta([1 - \sum_j x_j]) \frac{\prod_{j=1} x_j^{\alpha_j - 1}}{\left(\sum_\ell x_\ell A_\ell\right)^{\sum_k \alpha_k}}\,. \qquad \text{(B.2)}
\end{aligned}$$

To simplify the expression, we define a measure dF_n such that

$$\int dF_n \equiv \int_0^1 dx_1 \cdots \int_0^1 dx_n (n-1)! \delta(1 - \sum_{j=1}^n x_j)\,, \qquad \text{(B.3)}$$

$$\implies \frac{1}{\prod_{i=1}^n A_i^{\alpha_i}} = \frac{\Gamma\left(\sum_k \alpha_k\right)}{\prod_{i=1}^n \Gamma(\alpha_i)} \frac{1}{(n-1)!} \int dF_n \frac{\prod_j x_j^{\alpha_j - 1}}{\left(\sum_\ell x_\ell A_\ell\right)^{\sum_j \alpha_j}}\,. \qquad \text{(B.4)}$$

Specifically, for $\alpha_i = 1$, $\Gamma(1) = 1$, $\Gamma(n) = (n-1)!$ and equation(B.2) yields

$$\frac{1}{A_1 A_2 \dots A_n} = \int dF_n \left(\sum_{i=1}^n x_i A_i\right)^{-n} \qquad \text{(B.5)}$$

The usefulness of Feynman parameters comes from the possibility of writting denominators as polynomials. As an example, consider the case for $n = 2$:

$$\frac{1}{AB} = \int_0^1 dx \int_0^1 dx' \frac{\delta(1 - x' - x)}{[xA + x'B]^2} = \int_0^1 dx \frac{1}{[Ax + (1-x)B]^2}\,. \qquad \text{(B.6)}$$

This form can be used in loop calculations, as, for example, with a scalar field loop. In this case, the integrand takes the form

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k-p)^2 - m^2 + i\epsilon}\,.$$

If we consider the A, B terms as each of the propagators inside, we can insert an integral in x and write

$$\begin{aligned} I &= -\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[x(k^2-m^2+i\epsilon)+(1-x)((k-p)^2-m^2+i\epsilon]^2}\,, \\ &= -\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[x(k^2-m^2+i\epsilon)+(1-x)((k-p)^2-m^2+i\epsilon]^2}\,, \\ &= -\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{([k-p(1-x)]^2+xp(1-x)-m^2)^2}\,, \\ &= -\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{(q^2-\Delta)^2}\,, \end{aligned} \tag{B.7}$$

where we define $\Delta = m^2 - xp^2(1-x)$ and used $q = k - p(1-x)$. This equation is now in a suitable form to apply regularisation methods, such as Pauli-Villars regularization or dimensional regularization.

B.2 Dimensional Regularization

As discussed in the last section, the calculation of loop integrals often revolves around integration of fractions of polynomials in the momentum. These integrals are often divergent, but not necessarily useless or inconsistent, that is, they still allow us to extract information regarding the behaviour of the theory under study. To do such extraction involves obtaining information about the *way* with which the divergence occurs, that is, whether the functions scale logarithmically, with inverse powers and so on. Here, we give a rough outline of a procedure called *dimensional regularization*.

The main gist of dimensional regularization is considering the following general equality in D dimensions:

$$I_D = \int \frac{d^Dq}{(2\pi)^D} \frac{(q^2)^a}{(q^2+\Delta)^b} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(b-a-D/2)\Gamma(a+D/2)}{\Gamma(b)\Gamma(D/2)} \Delta^{-(b-a-D/2)}\,. \tag{B.8}$$

The proof can be established by considering the definition of the *Beta* function $B(z_1, z_2)$:

$$\beta(z_1,z_2) \equiv \int_0^1 dx\, x^{z_1-1}(1-x)^{z_2-1} = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}\,. \tag{B.9}$$

Firstly, let us decompose the D-dimensional integral into hyperspherical coordinates, such that $d^D q = q^{D-1} d\Omega_D \, dq$, where q is just the radial variable, ranging from 0 to ∞. Then, we can write

$$\begin{aligned} I_D &= \frac{1}{(2\pi)^D} \int d\Omega_D \int_0^\infty q^{D-1} \, dq \, \frac{(q^2)^a}{(q^2+\Delta)^b} \,, \\ &= \frac{\Omega_D}{(2\pi)^D} \int_0^\infty dq \, \frac{q^{D-1}(q^2)^a}{(q^2+\Delta)^b} \,, \end{aligned}$$

where Ω_D is the hypersphere surface area in D dimensions. Now, make the substitution $q = \sqrt{\Delta}\tan\theta$. We obtain

$$\begin{aligned} I_D &= \frac{\Omega_D}{(2\pi)^D} \Delta^{D/2+a-b} \int_0^{\pi/2} d\theta \, \frac{\sec^2\theta \tan^{2ae+D-1}\theta}{(\tan^2\theta+1)^b} \,, \\ &= \frac{\Omega_D}{(2\pi)^D} \Delta^{D/2+a-b} \int_0^{\pi/2} d\theta \sec^{2(1-b)}\theta \tan^{2a+D-1}\theta \,, \\ &= \frac{\Omega_D}{(2\pi)^D} \Delta^{D/2+a-b} \int_0^{\pi/2} d\theta \sin^{2a+D-1}\theta \cos^{2(b-a)-D-1}\theta \,. \\ &= \frac{\Omega_D}{(2\pi)^D} \Delta^{D/2+a-b} \int_0^1 dx \, (1-x^2)^{a-1+D/2} x^{2(b-a)-D-1} \,. \\ &= \frac{\Omega_D}{2(2\pi)^D} \Delta^{D/2+a-b} \int_0^1 dz \, (1-z)^{a+D/2-1} z^{b-a-D/2-1} \,. \end{aligned}$$

Thus, we see that the integral can be written as a beta function $B(a+D/2, b-a-D/2)$. Therefore, using identity (B.9), and writing the explicit form of the factor Ω_D, we obtain

$$\begin{aligned} I_D &= \frac{2\pi^{D/2}}{2(2\pi)^D \Gamma(D/2)} \Delta^{D/2+a-b} B(a+D/2, b-a-D/2) \,, \\ &= \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(a+D/2)\Gamma(b-a-D/2)}{\Gamma(D/2)\Gamma(b)} \Delta^{D/2+a-b} \,, \end{aligned} \tag{B.10}$$

which establishes the desired result.

To see the use in such a formula, consider the integral which appears in the simplest loop calculation, after the introduction of Feynmann parameters:

$$I = -\int_0^1 dx \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2-\Delta)^2} \,. \tag{B.11}$$

Here, we are omitting an important factor which is contained inside the Δ variable, namely the $+i\epsilon$. This factor pushes the poles apart from the real axis in such a way

that we are allowed to rotate counter-clockwise the integration limits of q_0 from $-\infty, \infty$ to $-i\infty, i\infty$. In doing so, we are making $q_0' = iq_0$, which turns the inner product into $q^2 = q_0^2 - \mathbf{q}^2 = -(q_0'^2 + \mathbf{q}^2) = -q_0'^2$. After performing such a *Wick rotation*, the integral becomes

$$I = -i \int_0^1 dx \int \frac{d^4q'}{(2\pi)^4} \frac{1}{(q'^2 + \Delta)^2} \,. \tag{B.12}$$

In this form, we can already identify the applicability of the result derived above for the last integral. However, note that in $D = 4$, the above integral diverges, since the denominator receives a factor $\Gamma(0)$, which consists of a pole of the Gamma function. In order to extract the behaviour of this integral, we consider $D = 4 - \varepsilon$ and expand the resulting expression in power series around $\varepsilon = 0$. This procedure will allow us to find the type of singularity that we are dealing with. Since we will be expanding the functions, we list here the proper expansions of Gamma and a^x functions around $x = 0$:

$$\Gamma(x) = \frac{1}{x} - \gamma_E + \mathcal{O}(x) \,, \tag{B.13}$$

$$a^x = 1 + x \ln a + \mathcal{O}(x^2) \,, \tag{B.14}$$

where γ_E is the Euler-Mascheroni constant. Thus, such an expansion will lead us to

$$\begin{aligned} \int \frac{d^{4-\varepsilon}q}{(2\pi)^{4-\varepsilon}} \frac{1}{(q'^2 + \Delta)^2} &= \frac{1}{(4\pi)^{2-\varepsilon/2}} \frac{\Gamma(2 - \varepsilon/2)\Gamma(\varepsilon/2)}{\Gamma(2 - \varepsilon/2)\Gamma(2)} \Delta^{-\varepsilon/2} \,, \\ &= \frac{1}{16\pi^2} \left[1 + \frac{\varepsilon}{2} \ln \frac{4\pi}{\Delta} + \dots \right] \left[\frac{2}{\varepsilon} - \gamma_E + \dots \right] \,, \\ &= \frac{1}{16\pi^2} \left[\frac{2}{\varepsilon} + \ln \frac{4\pi e^{-\gamma_E}}{\Delta} + \mathcal{O}(\varepsilon) \right] \,. \end{aligned} \tag{B.15}$$

Here, it becomes apparent that the divergence structure of the amplitude comes from the single pole produced by the first term in the sum, while the second term survives *independently* of ε. After having isolated the divergent factor, one proceeds to subtract it by adding a suitable counter term to the Lagrangian which reproduces this divergence with the opposite sign. Depending on how this subtraction is performed, we say that a specific *subtraction scheme* has been adopted.

B.3 Integrals

B.3.1 Sokhotski-Plemelj Theorem

In the discussion of the restricted boundary integration of chapter 4, we ran into the following integral

$$I := \int_{\mathbb{R}^{2,1}\times\mathbb{R}_+} d^4x\, e^{ix\cdot p}\,. \tag{B.16}$$

Here, the boundary is given by the full real numbers in x^0, x^1, x^2 and only the positive semi-axis for x^3. From the definition of the Dirac delta, the integrals over the full reals yield deltas for each component. Thus, the only nontrivial part is the x^3 integral:

$$I \;=\; (2\pi)^3\delta(p^0)\delta^2(\mathbf{p}_\parallel)\int_0^\infty e^{ix^3p^3}\,dx^3\;.$$

This last integral is not well defined and must be regularized in some way. The most common is to redefine it as the limit of a decreasing exponential:

$$\int_0^\infty e^{ix^3p^3}\,dx^3 := \lim_{\mu\to 0^+}\int_0^\infty e^{ix^3(p^3+i\mu)}\,dx^3\;. \tag{B.17}$$

One can integrate directly, which yields

$$\int_0^\infty e^{ix^3p^3}\,dx^3 = \lim_{\mu\to 0^+}\frac{i}{p^3+i\mu}\,. \tag{B.18}$$

To evaluate the limit above, one must be careful. The Sokhotski-Plemelj theorem states that, counter to naive intuition, the proper value (in the distributional sense) of the above limit is given by

$$\lim_{\mu\to 0^+}\frac{1}{p^3+i\mu} = -i\pi\delta(p^3)+\mathcal{P}\frac{1}{p^3}\,, \tag{B.19}$$

where $\mathcal{P}$ is the Cauchy principal value. With the above, the value of I is

$$I = (2\pi)^3\delta(p^0)\delta^2(\mathbf{p}_\parallel)\left[\pi\delta(p^3)+i\mathcal{P}\frac{1}{p^3}\right]\,. \tag{B.20}$$

B.3.2 Complex Integrals

As discussed in chapter 4, when restricting the integration domain of the vertex for each side of the fermion loop, one encounters the following integral

$$I := \mathcal{P}\int_{-\infty}^{\infty} \frac{\sin(xy)}{(x^2-\omega^2-i\epsilon)(x-p)}\,dx\ . \tag{B.21}$$

One way of calculating the above integral is separating it in two pieces, namely:

$$I := \frac{I_+ - I_-}{2i}\,, \tag{B.22}$$

where

$$I_\pm := \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{\pm ixy}}{(x^2-\omega^2-i\epsilon)(x-p)}\,dx\ . \tag{B.23}$$

At first, let us consider I_+. As the Cauchy principal value entails, the above integral can be analyzed by interpreting it as a section of the integral

$$\oint_C f(z)\,dz\ ,$$

where

$$f(z) := \frac{e^{izy}}{(z^2-\omega^2-i\epsilon)(z-p)}\,, \tag{B.24}$$

and C is the closed contour defined by four curves C_1, C_2, C_3, C_4, depicted below (B.1). The poles of f (also depicted in the figure) are given by $z = p$, $z = z_+ = \omega + i\epsilon$ and $z_- = -\omega - i\epsilon$.

By the definition of the Cauchy principal value:

$$\mathcal{P}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{R\to\infty\\ r\to 0}} \int_{-R}^{p-r} f(x)\,dx + \lim_{\substack{R\to\infty\\ r\to 0}} \int_{p+r}^{R} f(x)\,dx = \lim_{\substack{R\to\infty\\ r\to 0}} \int_{C_1\cup C_3} f(z)\,dz\ ,$$

while the integral over the upper hemisphere C_R vanishes in the $R\to\infty$ limit. The integral over the small arc C_2 is given by

$$\begin{aligned}\lim_{r\to 0}\int_{C_2} f(z)\,dz &= \lim_{r\to 0}\int_{\pi}^{0} \frac{ire^{i\varphi}e^{iy(p+re^{i\varphi})}}{(p+re^{i\varphi}-z_-)(p+re^{i\varphi}-z_+)((p+re^{i\varphi})-p)}\,d\varphi \\ &= -i\pi\frac{e^{iyp}}{p^2-\omega^2}\,.\end{aligned} \tag{B.25}$$

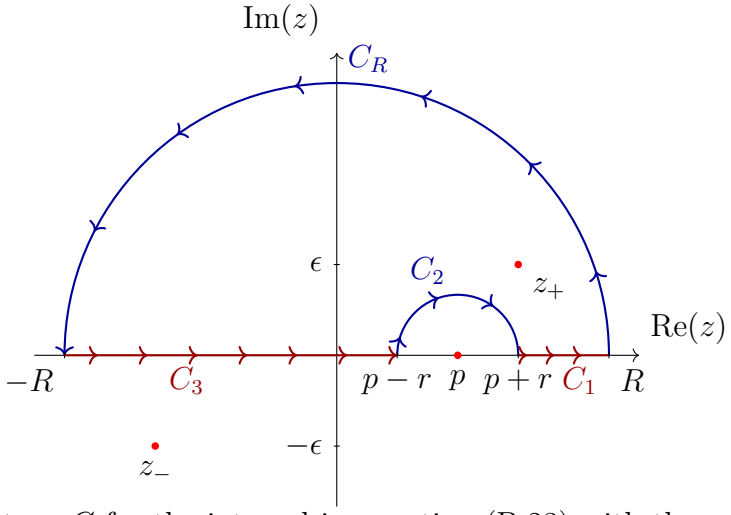

Figure B.1: Contour C for the integral in equation (B.23) with the positive sign (I_+).

With the above, we have

$$\oint_C f(z)\,dz = \mathcal{P}\int_{-\infty}^{\infty} f(x)\,dx - \frac{i\pi e^{iyp}}{p^2-\omega^2}\,. \tag{B.26}$$

The integral on the left hand side can be evaluated using the Residue theorem. Thus, we can isolate the desired part and write

$$\begin{aligned}\mathcal{P}\int_{-\infty}^{\infty} f(x)\,dx &= i\pi\frac{e^{iyp}}{p^2-\omega^2} + 2\pi i\sum_k \mathrm{Res}[f(z), z_k]\,,\\ &= i\pi\frac{e^{iyp}}{p^2-\omega^2} + \frac{i\pi}{\omega(\omega-p)}e^{i\omega y}\,.\end{aligned} \tag{B.27}$$

With this, we write

$$\lim_{\epsilon\to 0^+}\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{ixy}}{(x^2-\omega^2-i\epsilon)(x-p)}\,dx = i\pi\frac{e^{iyp}}{p^2-\omega^2} + \frac{i\pi}{\omega(\omega-p)}e^{i\omega y}\,. \tag{B.28}$$

Let us now consider I_- given by equation (B.23) with the negative sign chosen for the argument of the exponential. Because of this minus sign, the contour of integration must be chosen so that the bigger arc with radius R lies in the lower imaginary axis, to ensure that in the limit $R\to\infty$, the integral over C_R vanishes. This leads us to use the contour of integration depicted in figure (B.2).

For this integral the discussion is effectively the same, with the main difference coming from the signs of the integrations. Specifically, following the orientation of the paths in

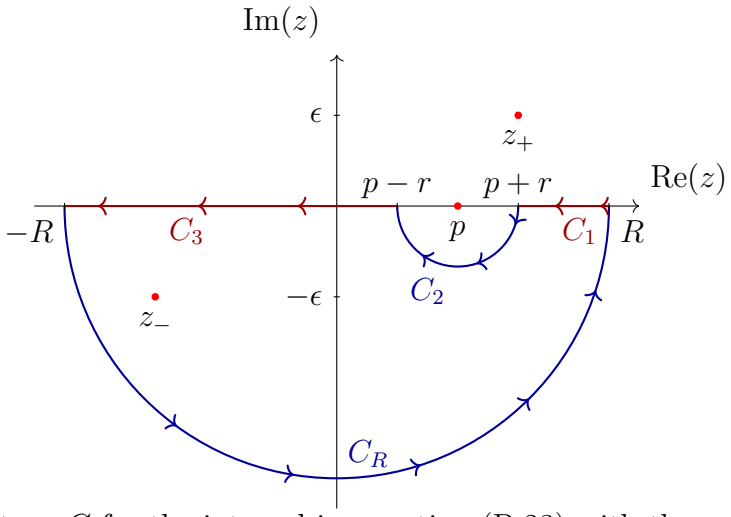

Figure B.2: Contour C for the integral in equation (B.23) with the negative sign (I_-).

figure (B.2), we find

$$\oint_C f(z)\,dz = \mathcal{P}\int_{\infty}^{-\infty} f(x)\,dx - \frac{i\pi e^{-iyp}}{p^2-\omega^2}\,, \tag{B.29}$$

which can also be evaluated using the Residue theorem on the integral in the left hand side. The result is

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{-izy}}{(z^2-\omega^2-i\epsilon)(z-p)} = -\frac{i\pi e^{-iyp}}{p^2-\omega^2} - \frac{i\pi}{\omega(\omega+p)}e^{i\omega y}\,. \tag{B.30}$$

With this, we conclude that

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{\sin(xy)}{(x^2-\omega^2-i\epsilon)(x-p)}\,dx \;=\; \frac{\pi}{\omega^2-p^2}\left(e^{i\omega y}-\cos(yp)\right)\,. \tag{B.31}$$

B.3.3 Potential integrals: Analytical Expressions

As noted in the text, the potential integrals appearing in the plated vacuum polarization correction can be explicitly evaluated.

The term containing $\exp\big(-|\mathbf{p}_\parallel|(x^\perp + x_s^\perp)\big)$, for example, can be written as

$$\begin{aligned}
& \frac{\alpha}{4\pi m^2}\int_0^1 v^2(1-v^2/3)\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{1+\mathbf{p}^2(1-v^2)/4m^2}e^{-|\mathbf{p}_\parallel|(x_{e\perp}+x_\perp)}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)} \\
= \;& \frac{\alpha}{4\pi m^2}\int_0^1 v^2(1-v^2/3)\int \frac{d^2\mathbf{p}_\parallel}{(2\pi)^2}e^{-|\mathbf{p}_\parallel|(x_{s\perp}+x_\perp)}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)} \\
& \times\int_{-\infty}^{\infty}\frac{dp_\perp}{2\pi}\frac{1}{1+(a(v)\mathbf{p}_\parallel)^2+(a(v)\mathbf{p}_\perp)^2}\,, \\
= \;& \frac{\alpha}{4\pi m^2}\int_0^1 v^2(1-v^2/3)\int \frac{d^2\mathbf{p}_\parallel}{(2\pi)^2}e^{-|\mathbf{p}_\parallel|(x_{s\perp}+x_\perp)}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}\frac{1}{\pi a(v)\sqrt{1+(a(v)\mathbf{p}_\parallel)^2}}\int_0^\infty\frac{d\xi}{1+\xi^2}\,, \\
= \;& \frac{\alpha}{4\pi m}\int_0^1\frac{v^2(1-v^2/3)}{\sqrt{1-v^2}}\int\frac{d^2\mathbf{p}_\parallel}{(2\pi)^2}\frac{e^{-|\mathbf{p}_\parallel|(x_{s\perp}+x_\perp)}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}}{\sqrt{1+(a(v)\mathbf{p}_\parallel)^2}}\,, \qquad \text{(B.32)}
\end{aligned}$$

where we defined $a^2(v) = (1-v^2)/4m^2$ to simplify the notation. Using polar coordinates,

$$I_{e++} \;=\; \frac{\alpha}{4\pi m}\int_0^1\frac{v^2(1-v^2/3)}{\sqrt{1-v^2}}\frac{1}{4\pi^2}\int_0^\infty\frac{\rho}{\sqrt{1+(a(v)\rho)^2}}e^{-\rho(x_{s\perp}+x_\perp)}\,d\rho\int_0^{2\pi}e^{i\rho|\mathbf{x}_\parallel-\mathbf{x}_{s\parallel}|\cos(\theta)}\,d\theta\,. \qquad \text{(B.33)}$$

From the Jacobi-Anger expansion, the last term can be represented in terms of the zeroth order Bessel function of the first kind $J_0(z)$:

$$\begin{aligned}
I_{e++} &= \frac{\alpha}{8\pi^2 m}\int_0^1\frac{v^2(1-v^2/3)}{\sqrt{1-v^2}}\,dv\int_0^\infty\frac{\rho}{\sqrt{1+(a(v)\rho)^2}}J_0(\rho|\mathbf{x}_\parallel-\mathbf{x}_{e\parallel}|)e^{-\rho(x_{s\perp}+x_\perp)}\,d\rho \\
&= \frac{\alpha m}{2\pi^2}\int_0^1\frac{v^2(1-v^2/3)}{(1-v^2)^{3/2}}\,dv\int_0^\infty\frac{\rho}{\sqrt{1+\rho^2}}J_0(\rho A)e^{-\rho B}\,d\rho\,, \qquad \text{(B.34)}
\end{aligned}$$

where we defined

$$A := \frac{|\mathbf{x}_\parallel-\mathbf{x}_{s\parallel}|}{a(v)}\,, \qquad\qquad B := \frac{x_\perp + x_{s\perp}}{a(v)}\,. \qquad \text{(B.35)}$$

The last integral can be explicitly calculated. To do so, we first note that

$$\frac{1}{\sqrt{1+\rho^2}} = \int_0^\infty J_0(\tau)e^{-\tau\rho}\,d\tau\,. \qquad \text{(B.36)}$$

Inserting the above into the integral we have

$$\int_0^\infty d\tau\,J_0(\tau)\int_0^\infty d\rho\,\rho J_0(\rho A)e^{-\rho(B+\tau)} \;=\; \frac{2\Gamma(3/2)}{\sqrt{\pi}}\int_0^\infty\frac{J_0(\tau)(\tau+B)}{(A^2+(\tau+B)^2)^{3/2}}\,d\tau \qquad \text{(B.37)}$$

where the ρ integral has been performed using Eq.(WA422(6)) from reference ([145]). This last equation can also be integrated in terms of multiple special functions (B.38)

$$\begin{aligned}
\int_0^\infty \frac{J_0(\tau)(B+\tau)}{(A^2+(B+\tau)^2)^{3/2}}\,d\tau &= -\frac{\partial}{\partial B}\int_0^\infty \frac{J_0(\tau)}{\sqrt{A^2+(B+\tau)^2}}\,d\tau \\
&= \frac{1}{iA}\frac{\partial}{\partial B}\left\{ {}_0F_1^{(1,0)}\left(1,\frac{1}{4}(A+iB)^2\right) - {}_0\,F_1^{(1,0)}\left(1,\frac{1}{4}(A-iB)^2\right)\right. \\
&\quad + \frac{\pi}{2}\boldsymbol{H}_0(iA-B) + \frac{\pi}{2}\boldsymbol{H}_0(B+iA) \\
&\quad + \ln 2(-J_0(iA-B)+J_0(B+Ai)) \\
&\quad \left. - \ln\left(\frac{i}{A+iB}\right) I_0(A+Bi) + \ln\left(-\frac{i}{A-iB}\right) I_0(A-iB)\right\} \\
&= -\frac{i}{2A}\left(i(A+iB)\,{}_0F_1^{(1,1)}\left(1,\frac{1}{4}(A+iB)^2\right) + \right. \\
&\quad + (B+iA)\,{}_0F_1^{(1,1)}\left(1,\frac{1}{4}(A-iB)^2\right) \\
&\quad - \pi\boldsymbol{H}_{-1}(iA-B) + \pi\boldsymbol{H}_{-1}(B+Ai) \\
&\quad + \frac{2I_0(A+Bi)}{B-iA} - \frac{2I_0(A-iB)}{B+iA} \\
&\quad - \log(4)(J_1(iA-B)+J_1(B+Ai)) \\
&\quad \left. -2i\left(\log\left(\frac{1}{B-iA}\right) I_1(A+Bi) + \log\left(\frac{1}{B+iA}\right) I_1(A-iB)\right)\right)
\end{aligned} \tag{B.38}$$

Here, ${}_0F_1$ is the confluent hypergeometric function, $\boldsymbol{H}_0$ is the Struve function of first kind, I_ν is the modified Bessel function of the second kind and J_ν is the Bessel function of the first kind. As noted in the text, the above form is not very enlightening, justifying our choice to resort to the numerical analysis.

B.3.4 Potential Integrals: Derivation of Eq. (4.54)

Here we outline the steps to rewrite the correction in the presence of the idealized conducting plate in the form found in (4.54). We can start with the term which was explicitly evaluated in the preceding section, i.e. Eq. (B.34).

Changing variables from v to $\xi = (1-v^2)^{-1/2}$, we have

$$\begin{aligned} & \frac{m\alpha q\eta_{\mu 0}}{2\pi^2}\int_0^1 \frac{v^2(1-v^2/3)}{(1-v^2)^{3/2}}\,dv\int_0^\infty \frac{\rho}{\sqrt{1+\rho^2}}J_0(\rho A)e^{-\rho B}\,d\rho\ , \\ = \ & \frac{\eta_{\mu 0}q\alpha}{6\pi^2}\int_1^\infty d\xi\left(1+\frac{1}{2\xi^2}\right)\frac{\sqrt{\xi^2-1}}{\xi^2}\int_0^\infty \frac{\rho}{\sqrt{\rho^2+4m^2\xi^2}}J_0(\rho|\mathbf{x}_\parallel-\mathbf{x}_{s\parallel}|)e^{-\rho(x_\perp+x_{s\perp})}\ . \end{aligned} \tag{B.39}$$

which is the form we will use.

Since the last two terms of (4.53) are effectively the same (up to exchanging x^3 and x_s^3) we can focus on a single one. The integrals are very similar:

$$\begin{aligned} & -\frac{q\alpha}{4\pi m^2}\int_0^1 v^2(1-v^2/3)\,dv\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\cos(p^3x_s^3)e^{-|\mathbf{p}_\parallel|x^3}}{1+\mathbf{p}^2(1-v^2)/4m^2}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}\ , \\ = \ & -\frac{e_R\alpha}{4\pi m^2}\int_0^1 v^2(1-v^2/3)\,dv\int\frac{d^2\mathbf{p}_\parallel}{(2\pi)^2}e^{-|\mathbf{p}_\parallel|x^3}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)} \\ & \times\int_{-\infty}^\infty\frac{dp^3}{2\pi}\frac{\cos(p^3x_s^3)}{1+(a(v)\mathbf{p}_\parallel)^2+(a(v)p^3)^2}\ , \\ = \ & -\frac{\eta_{\mu 0}q\alpha}{4\pi m^2}\int_0^1 v^2(1-v^2/3)\,dv\int\frac{d^2\mathbf{p}_\parallel}{(2\pi)^2}e^{-|\mathbf{p}_\parallel|x^3}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)} \\ & \times\frac{1}{2\pi a(v)\sqrt{1+(a(v)\mathbf{p}_\parallel)^2}}\int_{-\infty}^\infty\frac{\cos\left(\xi x_s^3\sqrt{\mathbf{p}_\parallel^2+4m^2/(1-v^2)}\right)d\xi}{1+\xi^2}\ , \\ = \ & -\frac{q\alpha}{4\pi m^2}\int_0^1 v^2(1-v^2/3)\,dv\int\frac{d^2\mathbf{p}_\parallel}{(2\pi)^2}e^{-|\mathbf{p}_\parallel|x^3}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)} \\ & \times\frac{2m^2}{1-v^2}\frac{e^{-x_s^3\sqrt{\mathbf{p}_\parallel^2+4m^2/(1-v^2)}}}{\sqrt{\mathbf{p}_\parallel^2+4m^2/(1-v^2)}}\ , \\ = \ & -\frac{q\alpha}{2\pi}\int_0^1\frac{v^2(1-v^2/3)}{1-v^2}\,dv\int\frac{d^2\mathbf{p}_\parallel}{(2\pi)^2}\frac{e^{-x^3|\mathbf{p}_\parallel|-x_s^3\sqrt{\mathbf{p}_\parallel^2+4m^2/(1-v^2)}}}{\sqrt{\mathbf{p}_\parallel^2+4m^2/(1-v^2)}}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}\ . \end{aligned} \tag{B.40}$$

The last integral can also be written in terms of Bessel functions, using polar coordinates:

$$\int\frac{d^2\mathbf{p}_\parallel}{(2\pi)^2}\frac{e^{-x^3|\mathbf{p}_\parallel|-x_s^3\sqrt{\mathbf{p}_\parallel^2+4m^2/(1-v^2)}}}{\sqrt{\mathbf{p}_\parallel^2+4m^2/(1-v^2)}}e^{i\mathbf{p}_\parallel\cdot(\mathbf{x}-\mathbf{x}_s)}=\int_0^\infty\frac{d\rho}{2\pi}\frac{\rho J_0\left(\rho|\mathbf{x}_\parallel-\mathbf{x}_{s\parallel}|\right)}{\sqrt{\rho^2+4m^2/(1-v^2)}}e^{-x^3\rho-x_s^3\sqrt{\rho^2+4m^2/(1-v^2)}}\ , \tag{B.41}$$

which allows us to write the contribution as

$$
\begin{aligned}
&-\frac{q\alpha}{4\pi^2}\int_0^1 \frac{v^2(1-v^2/3)}{1-v^2}\,dv\int_0^\infty d\rho\,\frac{\rho J_0\left(\rho|\mathbf{x}_\parallel-\mathbf{x}_{s\parallel}|\right)}{\sqrt{\rho^2+4m^2/(1-v^2)}}e^{-x^3\rho-x_s^3\sqrt{\rho^2+4m^2/(1-v^2)}}\,,\\
&=-\frac{q\alpha}{6\pi^2}\int_1^\infty d\xi\left(1+\frac{1}{2\xi^2}\right)\frac{\sqrt{\xi^2-1}}{\xi^2}\int_0^\infty \frac{\rho J_0\left(\rho|\mathbf{x}_\parallel-\mathbf{x}_{s\parallel}|\right)}{\sqrt{\rho^2+4m^2\xi^2}}e^{-x^3\rho-x_s^3\sqrt{\rho^2+4m^2\xi^2}}\,d\rho\,.
\end{aligned}
\tag{B.42}
$$

The last contribution is the same as (B.42) but with x_s^3 and x^3 interchanged. By adding the latter to (B.42), (B.39) and the first piece which corresponds to the standard Uehling correction, we arrive at the expression (4.54).

B.4 Boundary Diagram Image Prescriptions

As discussed in section 4.4, the choice of prescription for the image propagators does not change the final result of the calculation. Here, we give an explicit illustration of the argument.

Suppose that we reflected the x_2 coordinate in the second propagator of eq. (4.58). Then, the resulting loop expression would contain four terms, the first of which is the same for all prescriptions, namely no reflection. Furthermore, one of the terms is the same as the third line of Eq. (4.58), since the first propagator is not changed. The only differences would arise from the following two contributions

$$
\begin{aligned}
&-\int_{\mathcal{R}} d^4x_1\int_{\mathcal{R}} d^4x_2\,\Delta^{(0)}_{\mu\alpha}(x-x_1)i\Pi^{\alpha\beta}(x_1-x_2)\Delta^{(0)}_{\beta\nu}(\widetilde{x}_2-y)\\
&+\int_{\mathcal{R}} d^4x_1\int_{\mathcal{R}} d^4x_2\,\Delta^{(0)}_{\mu\alpha}(\widetilde{x}-x_1)i\Pi^{\alpha\beta}(x_1-x_2)\Delta^{(0)}_{\beta\nu}(\widetilde{x}_2-y)\,.
\end{aligned}
\tag{B.43}
$$

The above contributions have corresponding image diagrams given in figure B.3, where the signs are written explicitly.

At first, let us consider the first diagram with the minus sign. In momentum space,

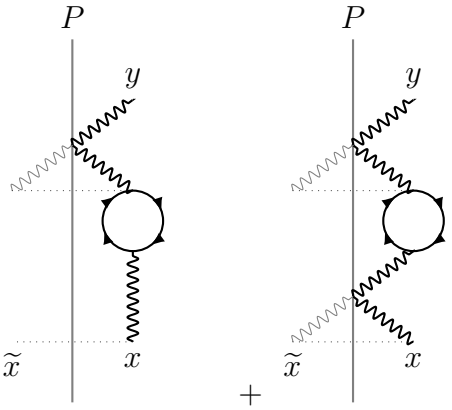

Figure B.3: Representation of the second prescription.

one can write

$$-\int\left(\prod_{i=1}^{4}\frac{d^4p_i}{(2\pi)^4}\right)\Delta^{(0)}_{\mu\alpha}(p_1)i\Pi^{\alpha\beta}(p_2,p_3)\Delta^{(0)}_{\beta\nu}(p_4)e^{-ip_1\cdot x}e^{ip_4\cdot y}$$
$$\times\int_{\mathcal{R}}d^4x_1\,e^{ix_1\cdot(p_1+p_3-p_2)}\int_{\mathcal{R}}d^4x_2\,e^{ix_2\cdot(p_2-p_3)}e^{-ip_4\cdot\widetilde{x}_2}\,. \tag{B.44}$$

Here, we see that the main difference comes from the presence of the reflected coordinate $\widetilde{x}_2$ inside of the boundary integration. However, $p_4\cdot\widetilde{x}_2 = p_4^{\parallel}\cdot x_2^{\parallel}+p_4^{\perp}x_2^{\perp} = \widetilde{p}_4\cdot x_2$. Therefore, the last term can simply be written as

$$\int_{\mathcal{R}}d^4x_1\,e^{ix_1\cdot(p_1+p_3-p_2)}\int_{\mathcal{R}}d^4x_2\,e^{ix_2\cdot(p_2-p_3-\widetilde{p}_4)}$$
$$=(2\pi)^8\delta^{2,1}_{\parallel}(p_1+p_3-p_2)\delta^{2,1}_{\parallel}(p_2-p_3-\widetilde{p}_4)$$
$$\times\left[\frac{1}{2}\delta^{\perp}(p_1+p_3-p_2)+\frac{i}{2\pi}\mathcal{P}\frac{1}{p_2^{\perp}-p_3^{\perp}-p_1^{\perp}}\right]$$
$$\times\left[\frac{1}{2}\delta^{\perp}(p_2-p_3+p_4)+\frac{i}{2\pi}\mathcal{P}\frac{1}{-p_2^{\perp}+p_3^{\perp}-p_4^{\perp}}\right].$$

For comparison, if we go back to the calculation of the propagator with the prescription of shifting the outside vertex, we see that the piece which corresponds to the same diagram but with the different reflection prescription is given by

$$-\int\left(\prod_{i=1}^{4}\frac{d^4p_i}{(2\pi)^4}\right)\Delta^{(0)}_{\mu\alpha}(p_1)i\Pi^{\alpha\beta}(p_2,p_3)\Delta^{(0)}_{\beta\nu}(p_4)e^{-ip_1\cdot x}e^{ip_4\cdot\widetilde{y}}$$
$$\times\int_{\mathcal{R}}d^4x_1\,e^{ix_1\cdot(p_1+p_3-p_2)}\int_{\mathcal{R}}d^4x_2\,e^{ix_2\cdot(p_2-p_3-p_4)}\,, \tag{B.45}$$

with the boundary integrals evaluating to

$$\begin{aligned}
&\int_{\mathcal{R}} d^4x_1\, e^{ix_1\cdot(p_1+p_3-p_2)} \int_{\mathcal{R}} d^4x_2\, e^{ix_2\cdot(p_2-p_3-p_4)} \\
&= (2\pi)^8 \delta_{\parallel}^{2,1}(p_1+p_3-p_2)\delta_{\parallel}^{2,1}(p_2-p_3-p_4) \\
&\times \left[\frac{1}{2}\delta^{\perp}(p_1+p_3-p_2) + \frac{i}{2\pi}\mathcal{P}\frac{1}{p_2^{\perp}-p_3^{\perp}-p_1^{\perp}}\right] \\
&\times \left[\frac{1}{2}\delta^{\perp}(p_2-p_3-p_4) + \frac{i}{2\pi}\mathcal{P}\frac{1}{-p_2^{\perp}+p_3^{\perp}+p_4^{\perp}}\right].
\end{aligned}$$

Comparing the integration pieces, we see that the effect is just to reverse the momentum factor associated with the reflected coordinate. This might be intuitive from the classical idea that the plate originates an impulse, changing the photon's perpendicular component.

From this, the question of whether the prescription matters is easily answered. If we extract the d^4p_4 integration in expression (B.44) we can collect the p_4 dependent terms, namely:

$$\int d^4p_4\, e^{-ip_4\cdot\widetilde{x}_2} e^{ip_4\cdot y}\Delta^{(0)}_{\beta\nu}(p_4)\,. \tag{B.46}$$

Then, using $p_4\cdot\widetilde{x}_2 = \widetilde{p}_4\cdot x_2$, we can transfer the minus sign to the momentum p_4, and upon a change of variables $p_4^{\perp}\to -p_4^{\perp}$ in the integration measure of the perpendicular component, we have $\widetilde{p}_4\to p_4$ and vice versa. Consequently, this piece is just

$$\int d^4p_4\, e^{-ip_4\cdot x_2} e^{i\widetilde{p}_4\cdot y}\Delta^{(0)}_{\beta\nu}(\widetilde{p}_4) = \int d^4p_4\, e^{-ip_4\cdot x_2} e^{ip_4\cdot\widetilde{y}}\Delta^{(0)}_{\beta\nu}(\widetilde{p}_4)\,. \tag{B.47}$$

Now, the propagator $\Delta^{(0)}_{\mu\nu}(\widetilde{p}_4)$, in the Feynman gauge, is left unchanged by the minus sign since it only depends on p_4^2. Thus, $\Delta^{(0)}_{\mu\nu}(\widetilde{p}_4) = \Delta^{(0)}_{\mu\nu}(p_4)$, and we have

$$\int d^4p_4\, e^{-ip_4\cdot\widetilde{x}_2} e^{ip_4\cdot y}\Delta^{(0)}_{\beta\nu}(p_4) = \int d^4p_4\, e^{-ip_4\cdot x_2} e^{ip_4\cdot\widetilde{y}}\Delta^{(0)}_{\beta\nu}(p_4) \tag{B.48}$$

Therefore, Eqs. (B.44) and (B.45) are equal. From a physical point of view, the thing that matters is that the photon's perpendicular momentum is reflected by the plate.